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CONTINUOUS DIVISION OF DIFFERENTIAL OPERATORS

by H. HAUSER and L. NARVÁEZ-MACARRO (*)

Introduction.

The classical Weierstrass Division Theorem for convergent power series was extended by Hironaka and Grauert to the case of division by several series. Galligo put these extensions on firm ground by providing a systematic treatment and proving the continuous dependence of quotients and remainder of the division on the given data. The formal case being straightforward, the proof of convergence and continuity requires to work with suitable polycylinders and to establish intricate norm estimates for the involved series. The general idea that a power series can be interpreted as an arbitrarily small perturbation of its initial monomial is disguised by the technicalities of the proof. This defect was overcome by Hauser and Müller [HM] by approximating instead of the series directly the $O_n$-linear map between power series spaces $O^n_p$ defined by the given series. A short and direct proof became thus available. This quite general strategy is valid for different situations and can be applied equally in the polynomial context or for solutions of certain differential equations [HRT], [H]. The only ingredient from analysis is the convergence of the geometric series of a linear map between Banach spaces, provided this map has norm $< 1$. As such the argument is, though comparable in spirit, much less difficult than

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the proof of the Constant Rank Theorem for analytic maps between power series spaces or the Nash-Moser Implicit Function Theorem.

Briançon-Maisonobe in one variable and Castro in general first proved division theorems for differential operators [BM], [C]. Full details appear in Castro’s thesis but have not been published. Recently, Mebkhout and Narváez [MN] considered a canonical topology on the rings of linear differential operators with analytic coefficients and proved the continuity of division. Their ad-hoc proof is rather technical and long. They give lots of applications and show how instrumental continuous division of differential operators is in the theory of $D$-modules. In particular, they obtain a short proof of the faithfully flatness of the ring of infinite order differential operators over the ring of finite order differential operators.

The method of proof of [HM] is taken up in the present paper to establish the continuity of division by differential operators with analytic coefficients in a simple and transparent way. Due to the non-commutativity the same routine requires additional efforts to get the correct norm estimates. Actually, the pseudo-norms on the ring of differential operators have to be chosen with particular care in order to make the arguments work. Slight deviations from the setting immediately cause serious obstructions.

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1. Topological structure on the ring of differential operators.

Let $\mathcal{D}_C^n$ be the sheaf of linear differential operators with coefficients in $\mathcal{O}_{C^n}$, the sheaf of holomorphic functions on $\mathbb{C}^n$. Denote by $\mathcal{O}_n$ and by $\mathcal{D}_n$ the stalk at the origin of the sheaves $\mathcal{O}_{C^n}$ and $\mathcal{D}_{C^n}$ respectively.

In [MN], §2, Mebkhout and Narváez have introduced a natural metrizable locally convex structure on $\mathcal{D}_C^n$. For instance, if $U \subset \mathbb{C}^n$ is an open poly-cylinder of polyradius $\sigma = (\sigma_i)$ and $E = \sum_\beta a_\beta D^\beta = \sum_\alpha a_\alpha x^\alpha D^\beta$ is a differential operator on $U$, where $x = (x_1, \ldots, x_n)$, $D = (D_1, \ldots, D_n)$ is the vector of partial derivatives $D_i = \partial_i$ and the $a_\beta = \sum_\alpha a_\alpha x^\alpha$ are holomorphic functions on $U$, the topology of $\mathcal{D}_{C^n}(U)$
is determined by the norms

\[ |E|_\rho^L = \sum_{d=0}^{\infty} dt \left( \sum_{|\beta|=d} |a_\beta|_\rho L^\beta \right) = \sum_{\alpha, \beta} |\beta|! |a_{\alpha, \beta}|_\rho^\alpha L^\beta, \]

where \( \rho \) is a polyradius with \( \rho_i < \sigma_i \) and \( L = (L_1, \ldots, L_n) > 0 \). It turns out that this topology does not depend on the local coordinates and that the completion of \( \mathcal{D}_C(U) \) is the space of infinite order differential operators on \( U \), see loc. cit.

In this paper we shall consider another way of indexing the norms above. For \( s, t > 0 \), weights \( \lambda, \mu \in \mathbb{Q}_+^n \) and \( E = \sum_{\alpha, \beta} a_{\alpha, \beta} x^{\alpha} D^\beta \in \mathcal{D}_n \) we set

\[ |E|_{st} = \sum_{\alpha, \beta} |a_{\alpha, \beta}| \cdot |\beta|! \cdot s^{\lambda, \alpha} \cdot t^{-\mu, \beta}, \]

where we omit any reference to the weights, often fixed, for simplicity. We have obviously \( |E|_{st} = |E|_{t^{-\mu}} \).

The factorial of \( |\beta| \) is crucial, as in [MN], and appears also in the present proof. For \( \lambda \) fixed, denote by \( \mathcal{D}_n(s) \) the subspace of \( \mathcal{D}_n \) of operators of finite norm w.r.t. \( s \) (and w.r.t. some or any \( t, \mu \)), filtered by the Banach spaces \( \mathcal{D}_n(s,d) \) of operators of order \( \leq d \). The induced topology on \( \mathcal{D}_n \) does not depend on the choice of \( \lambda \) and \( \mu \).

For \( U \subset \mathbb{C}^n \) an open poly-cylinder of polyradius \( \sigma = s_0^\lambda \) (\( \lambda \) fixed), the topology of \( \mathcal{D}_C(U) \) is determined by any family of norms \( | - |_{st} \), with \( 0 < s < s_0 \) and \( t^{-\mu} \gg 0 \) w.r.t. the componentwise order (for instance, \( 0 < t < t_0 \) and \( \mu \) fixed, or \( t \in [0,1[ \) fixed and all \( \mu \gg 0 \)).

2. Continuity of division.

Any monomial order in \( \mathbb{N}^{2n} \) allows to associate to a differential operator its initial monomial as the monomial of minimal exponent occurring in its symbol [C]. For finitely many given differential operators \( M_i \) there exists \( (\lambda, \mu) \in \mathbb{N}_+^{2n} \) such that for each \( i \) there is a unique \( (\alpha_i, \beta_i) \) in the support of \( M_i \) with \( \lambda \alpha_i - \mu \beta_i \) minimal and such that \( x^{\alpha_i} D^\beta_i \) is the initial monomial of \( M_i \) w.r.t. the chosen monomial order, see the addendum for a proof. In fact, taking weights \( (\sigma, \rho) \in \mathbb{N}_+^{2n} \) which induce the monomial order on a sufficiently large subset of \( \mathbb{N}^{2n} \) one may choose \( \lambda = \sigma \) and \( \mu = k - \rho \) where \( k \) denotes the vector \( (k, \ldots, k) \) in \( \mathbb{N}^n \) for some large \( k \).
Write $M_i = x^\alpha D^\beta_i + \sum_{\alpha \beta} c_{\alpha \beta} x^\alpha D^\beta = x^\alpha D^\beta_i + M'_i$. Then $\lambda(\alpha_i - \alpha) < \mu(\beta_i - \beta)$ for all $(\alpha, \beta)$ in the support of $M_i$.

Let $T = (T_1, \ldots, T_n)$ be the shift operator given by $T_i x^\alpha D^\beta = x^\alpha D^{\beta + e_i}$. For $M_1, \ldots, M_p \in D_n$ define subspaces $L$ of $D^{p_n}$ and $J$ of $D_n$ as follows. For $P = \sum_{\gamma} c_{\gamma} x^\gamma D^\delta \in D_n$, let $P^o = \sum_{\gamma} c_{\gamma} x^\gamma T^\delta$ and $P' = P - P^o$ be the induced operators on $D_n$. Consider the map $f : D^{p_n} \rightarrow D_n$ given by $f(A) = \sum A_i^o x^\alpha D^\beta_i$. The kernel and the image of $f$ admit direct complements $L$ and $J$ in $D^{p_n}$ and $D_n$ given by support conditions: Set $F = N_2^n + N_2^n$ and let $F = \bigcup_i F_i$ be a partition of $F$ with $(\alpha_i, \beta_i) \in F_i$. Then

$$L = \{ A \in D^{p_n} : \text{supp } A_i \subset F_i - (\alpha_i, \beta_i) \}$$

$$J = \{ B \in D_n : \text{supp } B \subset N_2^n \setminus F \}.$$

We shall show that $f$ is a sufficiently precise approximation of the $O_n$-linear map $D^{p_n} \rightarrow D_n$ given by $A \rightarrow \sum A_i M_i$.

**Theorem.** — For $(\lambda, \mu)$ as above with $\mu \gg \lambda$ componentwise, the map

$$u : L \oplus J \rightarrow D_n, \ (A, B) \rightarrow \sum A_i M_i + B$$

is a bicontinuous compatible isomorphism. Assume that $\mu \beta_i > \mu \beta$ for all $\beta$ in the support of $M_i$. Then there are constants $s_0 > 0$ and $C > 0$ such that for all $E \in D_n(s)$ the unique elements $A \in L(s)$ and $B \in J(s)$ with $E = \sum A_i M_i + B$ satisfy

$$\sum |A_i|_{st} |M_i|_{st} + |B|_{st} \leq C |E|_{st}$$

for $0 < s < s_0$ and $0 < t \leq s^2$.

Compatible means that $u$ respects the filtrations of $D_n$ by $D_n(s)$ for sufficiently small $s$. Observe that the opposed map $(A, B) \rightarrow \sum M_i A_i + B$ is not compatible since derivations alter the radius of convergence of a series (however, it can be shown that the corresponding division is continuous).

The assumption on $\mu$ is satisfied in case that $(\alpha, \beta) < (\alpha', \beta')$ implies $(0, \ldots, 0, \beta') < (0, \ldots, 0, \beta')$ w.r.t. the monomial order on $N_2^n$. Without that assumption on $\mu$, the estimate of the theorem holds for $0 < s < s_0$, $t = s$, $\lambda = \sigma$ and $\mu = k - \rho$, $k \gg 0$, with $C$ independent of $k$. Varying $k$, this case covers in particular the estimate of Theorem 3.1.2 of [MN].

**Example.** — We show that the assumption on $\mu$ cannot be omitted if $t$ is allowed to approach 0 for fixed $s$. Take $M = \partial_x \partial_y - x \partial_x^2 - x \partial_y^2$ with
(λ, μ) = (σ_1, σ_2, k−ρ_1, k−ρ_2). Assume that ρ_1 ≠ ρ_2 and that |ρ_1−ρ_2| < σ_1. Then A = ∂_x ∂_y is the initial monomial of M, and dividing A by M yields the remainder R = x∂_x^2 + x∂_y^2 with norm |R|_{st} = s^{λ_1}(t^{−μ_1} + t^{−μ_2}). There is no constant C such that this norm remains bounded by C · |A|_{st} = C · t^{−μ_1−μ_2} for fixed s and t going to 0.

Proof. — We show that u is bicontinuous by interpreting it as a perturbation of a purely combinatorial and bicontinuous bijective map v. As such u is shown to be bijective and its inverse will be given as v⁻¹ times a geometric series in v − u. Bounding uniformly the norm of the restrictions of v − u to L(s) ⊕ J(s) for s > 0 sufficiently small yields the required estimate for u⁻¹.

The map u is Oₙ-linear and decomposes into u = v + w₁ + w₂ where

\[ v(A, B) = \sum A_0^α x_α D_β, \]
\[ w_1(A, B) = \sum A_1^α x_α D_β, \]
\[ w_2(A, B) = \sum A_2^α D_β. \]

The definition of L and J is chosen so that v is a bicontinuous compatible isomorphism. We shall show that there are positive constants s₀, r and C₁, C₂ such that for all 0 < s < s₀, t > 0, d ∈ N and E ∈ Dₙ(s)_d we have

\[ |(w_1 v⁻¹)E|_{st} \leq C₁ · |E|_{st} · s⁻^r t^r \]

and

\[ |(w_2 v⁻¹)E|_{st} \leq C₂ · |E|_{st} · s⁻^r t^r. \]

Choosing s₀ sufficiently small and 0 < s < s₀, 0 < t ≤ s² these inequalities imply that the geometric series (Id+(w₁+w₂)v⁻¹)⁻¹ converges to a map on the space of differential operators of infinite order Dₙ(s)∞. As w₁v⁻¹ and w₂v⁻¹ do not increase the order of a differential operator the series defines an automorphism of the Banach spaces (Dₙ(s)_d, |−|_{st}) for each d and t, and in fact an automorphism of the locally convex space Dₙ(s). It follows that u = (Id+(w₁+w₂)v⁻¹)v⁻¹ is an isomorphism from the spaces v⁻¹Dₙ(s) to Dₙ(s) whose inverse has norm ≤ C w.r.t. s and t for some constant C > 0. This is just the assertion of the theorem.

Choose s₀ > 0 such that Mᵢ and all its derivatives Dᵦ Mᵢ, δ ∈ Nⁿ, belong to Dₙ(s) for all i and all 0 < s < s₀. Let E ∈ Dₙ. Decompose E into E = \( \sum_{i,γ,δ} a_{i,γ,δ}^i x^{\alpha_0+i+γ} D^{δ_0+i+δ} + B \) with B ∈ J and A = (A₁, ..., A_p) ∈ L,
\[ A_i = \sum_{\gamma \delta} a_{\gamma \delta}^i x^{\alpha_\gamma + \gamma} D^{\beta_\delta + \delta}. \]

Then

\[ |E|_{st} \geq \left| \sum_{\gamma \delta} a_{\gamma \delta}^i x^{\alpha_\gamma + \gamma} D^{\beta_\delta + \delta} \right|_{st} = \sum_{\gamma \delta} |a_{\gamma \delta}^i| \cdot |\beta_i + \delta|! \cdot s^{\lambda(\alpha_i + \gamma)} \cdot t^{-\mu(\beta_i + \delta)}. \]

The equality follows from the fact that all exponents are distinct. Set

\[ B_{\delta \gamma} = |a_{\gamma \delta}^i| \cdot |\beta_i + \delta|! \cdot s^{\lambda(\alpha_i + \gamma)} \cdot t^{-\mu(\beta_i + \delta)}. \]

(a) We estimate first the norm of \( w_1 v^{-1} \). Denoting by \( \alpha \leq \beta \) the componentwise order in \( \mathbb{N}^n \), we have

\[
|w_1 v^{-1} E|_{st} = \left| w_1 v^{-1} \left( \sum_{\gamma \delta} a_{\gamma \delta}^i x^{\alpha_\gamma + \gamma} D^{\beta_\delta + \delta} + B \right) \right|_{st}
\]

\[
= \left| w_1 \left( \left( \sum_{\gamma \delta} a_{\gamma \delta}^i x^{\gamma} D^{\delta} \right)_{i} + B \right) \right|_{st}
\]

\[
= \sum_{\gamma \delta} a_{\gamma \delta}^i x^{\gamma} \left( D^{\delta} - T^{\delta} \right) x^{\alpha_\gamma} D^{\beta_\delta} \bigg|_{st}
\]

\[
\leq \sum_{\gamma \delta} \sum_{0 < \varepsilon < \alpha_\gamma} \sum_{0 < \varepsilon < \alpha_\delta} |a_{\gamma \delta}^i| \cdot \frac{\alpha_\gamma! d!}{(\alpha_i - \varepsilon)! (\delta - \varepsilon)!} \cdot x^{\alpha_\gamma - \varepsilon} D^{\beta_\delta - \varepsilon - \varepsilon} \bigg|_{st}
\]

\[
\leq \sum_{\gamma \delta} \sum_{0 < \varepsilon < \alpha_\gamma} \sum_{0 < \varepsilon < \alpha_\delta} |a_{\gamma \delta}^i| \cdot 2^{\alpha_\gamma} \frac{\delta!}{(\delta - \varepsilon)!} \cdot |\beta_i + \delta - \varepsilon|! \cdot s^{\lambda(\alpha_i + \gamma - \varepsilon)} \cdot t^{-\mu(\beta_i + \delta - \varepsilon)}. \]

Set

\[ A_{i \gamma \delta \varepsilon} = |a_{\gamma \delta}^i| \cdot 2^{\alpha_\gamma} \frac{\delta!}{(\delta - \varepsilon)!} \cdot |\beta_i + \delta - \varepsilon|! \cdot s^{\lambda(\alpha_i + \gamma - \varepsilon)} \cdot t^{-\mu(\beta_i + \delta - \varepsilon)}. \]

Then

\[
\frac{|w_1 v^{-1} E|_{st}}{|E|_{st}} \leq \sum_{i \varepsilon} \sum_{\gamma \delta \varepsilon} A_{i \gamma \delta \varepsilon} \frac{B_{i \delta \gamma}}{\sum_{\gamma \delta} B_{i \delta \gamma}} \leq \sum_{i \varepsilon} \frac{\sum_{\gamma \delta \varepsilon} A_{i \gamma \delta \varepsilon}}{\sum_{\gamma \delta} B_{i \delta \gamma}}.
\]

We have

\[
\frac{\sum_{\gamma \delta \varepsilon} A_{i \gamma \delta \varepsilon}}{\sum_{\gamma \delta} B_{i \delta \gamma}} = \sum_{\gamma \delta} \frac{|a_{\gamma \delta}^i| \cdot 2^{\alpha_\gamma} \frac{\delta!}{(\delta - \varepsilon)!} \cdot |\beta_i + \delta - \varepsilon|! \cdot s^{\lambda(\alpha_i + \gamma - \varepsilon)} \cdot t^{-\mu(\beta_i + \delta - \varepsilon)}}{\sum_{\gamma \delta} |a_{\gamma \delta}^i| \cdot |\beta_i + \delta|! \cdot s^{\lambda(\alpha_i + \gamma)} \cdot t^{-\mu(\beta_i + \delta)}}
\]

\[
= 2^{\alpha_\gamma} \cdot s^{-\lambda \varepsilon} \cdot t^{\mu \varepsilon} \cdot \frac{\sum_{\gamma \delta \varepsilon} |a_{\gamma \delta}^i| \frac{\delta!}{(\delta - \varepsilon)!} \cdot |\beta_i + \delta - \varepsilon|! \cdot s^{\lambda \gamma} \cdot t^{-\mu \delta}}{\sum_{\gamma \delta} |a_{\gamma \delta}^i| \cdot |\beta_i + \delta| \cdot s^{\lambda \gamma} \cdot t^{-\mu \delta}}
\]

\[
\leq 2^{\alpha_\gamma} \cdot s^{-\lambda \varepsilon} \cdot t^{\mu \varepsilon}.
\]
The last inequality follows from
\[
\frac{\delta! \cdot |\beta_i + \delta - \varepsilon|!}{(\delta - \varepsilon)! \cdot |\beta_i + \delta|!} \leq \frac{\delta! \cdot (\beta_i + \delta - \varepsilon)!}{(\delta - \varepsilon)! \cdot (\beta_i + \delta)!} = \left( \frac{\delta}{\varepsilon} \right) \cdot \left( \frac{\beta_i + \delta}{\varepsilon} \right)^{-1} \leq 1.
\]
Here we use that for \( \alpha, \beta \in \mathbb{N}^n \) with \( \alpha \leq \beta \) one has \( \frac{|\alpha|!}{|\alpha|!} \leq \frac{|\beta|!}{|\beta|!} \). Combining the above gives
\[
\left| \frac{(w_1v^{-1})E}{|E|} \right|_{st} \leq C_1 \cdot \sum_i \sum_{0 < \varepsilon \leq \alpha_i} 2^{\alpha_i} \cdot s^{-\lambda \varepsilon} \cdot t^{\mu \varepsilon} \leq C_1' \cdot \sum_i \sum_{0 < \varepsilon \leq \alpha_i} s^{-\lambda \varepsilon} \cdot t^{\mu \varepsilon}
\]
for some constant \( C_1' > 0 \). Choose \( r \) such that \( \lambda \varepsilon \leq r \leq \mu \varepsilon \) for all \( 0 < \varepsilon \leq \alpha_i \) and all \( i \). This is possible since \( \mu \gg \lambda \). We obtain for suitable \( C_1 > 0 \) that
\[
\left| \frac{(w_1v^{-1})E}{|E|} \right|_{st} \leq C_1' \cdot \sum_i \sum_{0 < \varepsilon \leq \alpha_i} s^{-\lambda \varepsilon} \cdot t^{\mu \varepsilon} \leq C_1 \cdot s^{-r} \cdot t^r.
\]
(b) We estimate the norm of \( w_2v^{-1} \).

\[
\left| (w_2v^{-1})E \right|_{st} = \left| (w_2v^{-1}) \left( \sum_{i \gamma \delta} a_i^i x^\gamma a^\alpha + \gamma D_i + B \right) \right|_{st}
\]
\[
= \left| w_2 \left( \left( \sum_{i \gamma \delta} a_i^i x^\gamma D^\delta \right)_1, B \right) \right|_{st}
\]
\[
= \left| \sum_{i \gamma \delta} (a_i^i x^\gamma D^\delta) M^i \right|_{st}
\]
\[
= \left| \sum_{i \gamma \delta} (a_i^i x^\gamma D^\delta) \left( \sum_{\alpha \beta} c_i^\alpha x^\alpha D^\beta \right) \right|_{st}
\]
\[
= \sum_{i \gamma \delta} \sum_{\alpha \beta} a_i^i c_i^\alpha x^\gamma D^\delta (x^\alpha D^\beta)
\]
\[
= \sum_{i \gamma \delta} \sum_{\alpha \beta} \sum_{\alpha \beta} a_i^i c_i^\alpha x^\gamma D^\delta (x^\alpha D^\beta + \delta - \varepsilon)
\]
\[
\leq \sum_{i \gamma \delta} \sum_{\alpha \beta} \sum_{\alpha \beta} |a_i^i| \cdot |c_i^\alpha| \cdot |\gamma| \cdot |\beta + \delta - \varepsilon| \cdot s^{\lambda(\alpha + \gamma - \varepsilon)} \cdot t^{-\mu(\beta + \delta - \varepsilon)}
\]

The inequality need not be an equality since certain \( \alpha + \gamma - \varepsilon \), respectively \( \beta + \delta - \varepsilon \), may coincide. Set
\[
C_{1\alpha\beta\gamma\delta\varepsilon} = \frac{|a_i^i|}{(\delta - \varepsilon)!} \cdot |\beta + \delta - \varepsilon| \cdot s^{\lambda(\alpha + \gamma - \varepsilon)} \cdot t^{-\mu(\beta + \delta - \varepsilon)}.
\]
Then
\[
|\frac{(w_2v^{-1})E}{E}|_{st}
\leq \sum_{i} \sum_{\alpha, \beta} \frac{|c_{i\alpha\beta}| \cdot (\alpha^\gamma) \cdot \sum_{\delta \geq \varepsilon} \sum_{\gamma} C_{i\alpha\beta\gamma\delta}}{\sum_{i} \sum_{\delta \geq 0} \sum_{\gamma} B_{i\delta\gamma}}
\leq \sum_{i} |c_{i\alpha\beta}| \cdot (\alpha^\gamma) \cdot \frac{\sum_{\gamma, \delta \geq \varepsilon} C_{i\alpha\beta\gamma\delta}}{\sum_{\gamma, \delta \geq \varepsilon} A_{i\gamma\delta}}
\leq \sum_{i} |c_{i\alpha\beta}| \cdot (\alpha^\gamma) \cdot \frac{\sum_{\gamma, \delta \geq \varepsilon} |a_{i\gamma\delta}| \cdot |r_{i\gamma\delta}| \cdot |\beta + \delta - \varepsilon| \cdot \lambda^{(\alpha + \gamma - \varepsilon)} \cdot t^{-\mu(\beta + \delta - \varepsilon)}}{\sum_{\gamma, \delta \geq \varepsilon} |a_{i\gamma\delta}| \cdot |\beta_i + \delta| \cdot \lambda^{(\alpha_i + \gamma)} \cdot t^{-\mu(\beta_i + \delta)}}
\leq C_2' \sum_{i} |c_{i\alpha\beta}| \cdot 2^{|\alpha|} \cdot s^{\lambda(\alpha - \alpha_i - \varepsilon)} \cdot t^{-\mu(\beta - \beta_i - \varepsilon)} =: R
\]

for some constant $C_2' > 0$ and using the assumption $|\beta| \leq |\beta_i|$. For each $\beta$ which is an exponent of $D$ in the monomials of $M_i'$ let $(\alpha_i\beta, \beta_i)$ be an element of the support of $M_i'$ with minimal first component: $A_{\alpha\beta}$ for all $\alpha$ with $(\alpha, \beta) \in \text{supp} M_i'$. Then

\[
R \leq C_2' \sum_{i} \sum_{\alpha, \beta} |c_{i\alpha\beta}| \cdot 2^{|\alpha|} \cdot s^{-\lambda \varepsilon} \cdot t^{\mu \varepsilon} \cdot s^{\lambda(\alpha - \alpha_i\beta)} \cdot s^{\lambda(\alpha_i\beta - \alpha_i)} \cdot t^{-\mu(\beta - \beta_i)}
\leq C_2' \cdot s^{-r} \cdot t^r \cdot \sum_{\alpha, \beta} |c_{i\alpha\beta}| \cdot 2^{|\alpha|} \cdot s^{\lambda(\alpha - \alpha_i\beta)}
\]

where $r > 0$ is chosen such that $\lambda(\alpha_i - \alpha_i\beta) \leq r \leq \mu(\beta_i - \beta)$ for every $i$ and $\beta$ (here the assumption $\mu(\beta_i - \beta) > 0$ is used). The last series converges and remains bounded for $s$ sufficiently small because of $\lambda(\alpha - \alpha_i\beta) \geq 0$. This shows

\[
\frac{|(w_2v^{-1})E|_{st}}{|E|_{st}} \leq C_2 \cdot s^{-r} \cdot t^r
\]

for $C_2 > 0$ suitable and concludes the proof of the theorem.

In case $\mu(\beta_i - \beta)$ is arbitrary but $t$ equals $s$, the first estimate for $R$ holds again, and the second inequality is

\[
\leq C_2' \cdot s^a \sum_{i} s^{\mu \varepsilon - \lambda \varepsilon} \sum_{|\alpha|} |c_{i\alpha\beta}| \cdot 2^{|\alpha|} \cdot s^{\lambda(\alpha - \alpha_i\beta)}
\]

where $a$ is a positive constant such that

\[
0 < a \leq \lambda(\alpha_i\beta - \alpha_i) - \mu(\beta - \beta_i)
\]

for all $i$ and $\beta$. Then

\[
\frac{|(w_2v^{-1})E|_{ss}}{|E|_{ss}} \leq C_2 \cdot s^a
\]
for $0 < s < s_0$ and $C_2 > 0$ suitable but independent of $k \gg 0$ (since the series $\sum_{s} s^{k}e^{-\lambda s} = \sum_{s} s^{k}\epsilon_{s}^{-\epsilon}(\rho + \sigma)^{-\epsilon}$ remains bounded for $k \gg 0$). \qed

As in [MN], 4.1, we can follow Serre’s idea in the case of the completion of a local noetherian commutative ring and deduce from the theorem above a very short proof of the faithfully flatness of the extension $D_{n} \subset D_{n}^{\infty}$, originally proven by Sato-Kashiwara-Kawai by using microlocal techniques. 

**Addendum.**

Fix a monomial order $\epsilon$ in $\mathbb{N}^{2n}$. We prove that for finitely many differential operators $M_i$ there exists $(\lambda, \mu) \in \mathbb{Q}^{2n}_{+}$, with $\lambda \ll \mu$, such that for each $i$ there is a unique $(\alpha_i, \beta_i)$ in the support of $M_i$ with $\lambda \alpha_i - \mu \beta_i$ minimal and such that $x^{\alpha_i}D^{\beta_i}$ is the initial monomial of $M_i$ w.r.t. $\epsilon$.

Recall that on any finite subset of $\mathbb{N}^{n}$ every monomial order can be defined by an $\mathbb{N}$-linear map $\eta : \mathbb{N}^{n} \rightarrow \mathbb{N}$ by pulling back the natural order on $\mathbb{N}$. Identify $\eta$ with a vector in $\mathbb{N}^{n}$. Let $(\sigma, \rho) \in \mathbb{N}^{2n}_{+}$ be weights which induce the same order as $\epsilon$ on a sufficiently big subset of $\mathbb{N}^{2n}$ [including the finitely many monomials of $M_i$ which have minimal $x$-exponent and maximal $D$-exponent w.r.t. the componentwise order on $\mathbb{N}^{n}$]. Set $\lambda = \sigma$ and $\mu = k - \rho$, where $k$ is a positive integer and $k = (k, \ldots, k) \in \mathbb{N}^{n}$. Then $\lambda \alpha - \mu \beta = -k \cdot |\beta| + (\sigma \alpha + \rho \beta)$. For $k$ sufficiently large and for each $i$ the pair $(\alpha_i, \beta_i)$ in the support of $M_i$ with $\lambda \cdot \alpha - \mu \cdot \beta$ minimal satisfies

$|\beta|$ is maximal, i.e. $(\alpha, \beta)$ belongs to the support of the symbol of $M_i$.

$(\alpha, \beta)$ is minimal in the support of the symbol of $M_i$ w.r.t. $(\sigma, \rho)$.

This implies that the monomial of $M_i$ with exponent $(\alpha, \beta)$ minimal w.r.t. $\lambda \cdot \alpha - \mu \cdot \beta$ coincides with the minimal monomial of the symbol of $M_i$ w.r.t. $\epsilon$. By taking $k$ sufficiently large we obtain $\lambda \ll \mu$. For $\epsilon$ the inverse graded order as in [C] let $(\pi, \tau) \in \mathbb{N}^{2n}_{+}$ be weights which define in the chosen finite subset of $\mathbb{N}^{2n}$ the inverse lexicographic order (not the graded one). Then

$\lambda = k + \pi$ and $\mu = k^{2} - \tau$

will work for $k$ sufficiently large.
BIBLIOGRAPHY


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