Karl OELJEKLAUS & Matei TOMA

Non-Kähler compact complex manifolds associated to number fields

<http://aif.cedram.org/item?id=AIF_2005__55_1_161_0>
NON-KÄHLER COMPACT COMPLEX MANIFOLDS ASSOCIATED TO NUMBER FIELDS

by Karl OELJEKLAUS & Matei TOMA

1. Notations, construction and first properties.

Consider an algebraic number field $K$, that is a finite extension field of the field of rational numbers $\mathbb{Q}$. Let $n := (K : \mathbb{Q})$ be its degree. The field $K$ admits precisely $n = s + 2t$ distinct embeddings $\sigma_1, \ldots, \sigma_n$ into $\mathbb{C}$, where we suppose that $\sigma_1, \ldots, \sigma_s$ are the real embeddings, $\sigma_{s+1}, \ldots, \sigma_n$ are the complex ones and that $\sigma_{s+i} = \overline{\sigma_{s+i}}$ for $1 \leq i \leq t$. We shall suppose throughout the paper that both $s$ and $t$ are strictly positive. Furthermore, let $\mathcal{O}_K$ denote the ring of algebraic integers of $K$. This is a free $\mathbb{Z}$-module of rank $n$. In fact our construction works also for arbitrary orders $\mathcal{O}$ of $K$, i.e. for subrings $\mathcal{O}$ of $\mathcal{O}_K$ which have rank $n$ as $\mathbb{Z}$-modules.

Set now $m := s + t$ and consider the ”geometric representation” of $K$:

$$\sigma : K \to \mathbb{C}^m, \quad \sigma(a) := (\sigma_1(a), \ldots, \sigma_m(a)).$$

It is known that the image $\sigma(\mathcal{O}_K)$ of $\mathcal{O}_K$ through $\sigma$ is a lattice of rank $n$ in $\mathbb{C}^m$, cf. [1], 2.3.1, p. 95ff. We thus get a properly discontinuous action of $\mathcal{O}_K$...
on $\mathbb{C}^m$ by translations. Consider furthermore the following multiplicative action of $K$ on $\mathbb{C}^m$: for $a \in K$ and $z \in \mathbb{C}^m$ set
$$az := (\sigma_1(a)z_1, ..., \sigma_m(a)z_m).$$
For $a \in \mathcal{O}_K$, $a\sigma(\mathcal{O}_K)$ is contained in $\sigma(\mathcal{O}_K)$. Let $\mathcal{O}_K^*$ denote the group of units in $\mathcal{O}_K$ and
$$\mathcal{O}_K^{*, +} := \{a \in \mathcal{O}_K^* \mid |\sigma_i(a)| > 0 \text{ for all } 1 \leq i \leq s\}.$$  
Since for $s > 0$ the only torsion elements of $\mathcal{O}_K^{*, +}$ are 1 and $-1$, Dirichlet’s Units Theorem allows us to write $\mathcal{O}_K^{*} = G \cup (-G)$, where $G$ is a free abelian (multiplicative) group of rank $m - 1$. One may choose $G$ so that it contains $\mathcal{O}_K^{*, +}$, automatically with finite index. We denote by $\mathbb{H}$ the upper complex half-plane, $\mathbb{H} := \{z \in \mathbb{C} \mid \text{Im } z > 0\}$. Combining the additive action of $\mathcal{O}_K$ with the induced multiplicative action of $\mathcal{O}_K^{*, +}$ we get an action of $\mathcal{O}_K^{*, +} \ltimes \mathcal{O}_K$ on $\mathbb{C}^m$ which is free on the invariant domain $\mathbb{H}^s \times \mathbb{C}^t$. We shall now choose a subgroup $U$ of rank $s$ of $\mathcal{O}_K^{*, +}$ such that the action of $U \ltimes \mathcal{O}_K$ on $\mathbb{H}^s \times \mathbb{C}^t$ becomes properly discontinuous, thus yielding a smooth quotient which will be shown to be compact. In order to do this we consider the logarithmic representation of units
$$l : \mathcal{O}_K^* \rightarrow \mathbb{R}^m, \ l(u) := (ln|\sigma_1(u)|, ..., ln|\sigma_s(u)|, 2ln|\sigma_{s+1}(u)|, ..., 2ln|\sigma_m(u)|),$$
cf. [1] 2.3.3. Dirichlet’s Units Theorem implies that $l(\mathcal{O}_K^{*, +})$ is a full lattice in the subspace $L := \{x \in \mathbb{R}^m \mid \sum_{i=1}^m x_i = 0\}$ of $\mathbb{R}^m$. Since $t > 0$, the projection $pr : L \rightarrow \mathbb{R}^s$ given by the first $s$ coordinate functions is surjective. Thus there exist subgroups $U$ of rank $s$ of $\mathcal{O}_K^{*, +}$ such that $pr(l(U))$ is a full lattice in $\mathbb{R}^s$. Such a subgroup will be called admissible for $K$.

Take now $U$ admissible for $K$. The quotient $\mathbb{H}^s \times \mathbb{C}^t / \sigma(\mathcal{O}_K)$ is clearly diffeomorphic to a trivial torus bundle $(\mathbb{R}_{>0})^s \times (S^1)^n$ and $U$ operates properly discontinuously on it since it induces a properly discontinuous action on the base $(\mathbb{R}_{>0})^s$ by our choice. Differentiably the quotient of this action is a fiber bundle over $(S^1)^s$ with $(S^1)^n$ as fiber. We thus get an $m$-dimensional compact complex affine manifold
$$X = X(K, U) := (\mathbb{H}^s \times \mathbb{C}^t)/(U \ltimes \mathcal{O}_K).$$
This paper is devoted to the description of these complex manifolds.

**Remark 1.1.** — For every choice of natural numbers $s$ and $t$, algebraic number fields with precisely $s$ real and $2t$ complex embeddings exist.
Since we don’t know of any source for this observation we include here an argument we owe to Ph. Eyssidieux.

Proof. — Consider the non-empty open set $D$ of points $a = (a_1, ..., a_n) \in \mathbb{Q}^n$ such that the polynomials $P = X^n + a_1 X^{n-1} + ... + a_n$ admit exactly $s$ real distinct and $2t$ complex non-real roots. The open set $D$ will contain arbitrarily large open balls since the map $(a_1, ..., a_n) \mapsto (ba_1, b^2a_2, ..., b^n a_n)$ leaves it invariant for any choice of rational numbers $b$.

Choose now a prime number $p$ and $\tilde{P} = X^n + \tilde{a}_1 X^{n-1} + ... + \tilde{a}_n \in \mathbb{Z}[X]$ an Eisenstein polynomial with respect to $p$, that is $p|\tilde{a}_i$ for all $i$ but $p^2 \nmid \tilde{a}_n$. Then the set $\tilde{a} + p^2\mathbb{Z}^n$ intersects $D$ and consists only of Eisenstein hence irreducible polynomials. 

Remark 1.2. — For $s = 1$, $t = 1$ and $U = \mathcal{O}^*_K$, $X(K, U)$ is an Inoue-Bombieri surface $S_M$; cf. [3].

Remark 1.3. — When $s = 1$ or $t = 1$ all subgroups $U$ of rank $s$ of $\mathcal{O}^*_K$ are admissible for $K$. But this need not be the case in general as the following example shows. Take two field extensions $K'$ and $K''$ of $\mathbb{Q}$ with corresponding numbers of real and complex embeddings $s' = 1$, $t' = 2$, $s'' = 2$, $t'' = 1$ and $K$ the composite of $K'$ and $K''$. Then $s = 2$ but $\mathcal{O}^*_K$ is not admissible for $K$.

Lemma 1.4. — Let $U$ be a subgroup of $\mathcal{O}^*_K$ not contained in $\mathbb{Z}$. Then the following are equivalent:

- The action of $U$ on $\mathcal{O}_K$ admits a proper non-trivial invariant submodule of lower rank.
- There exists some proper intermediate field extension $\mathbb{Q} \subset K' \subset K$ with $U \subset \mathcal{O}^*_K$.

Proof. — Suppose $M$ is a proper $\mathbb{Z}$-submodule of $\mathcal{O}_K$ which is invariant under $U$ and with $0 < \operatorname{rank} M = r < n$. We consider the coefficient ring of $M$, $\mathcal{O}_M := \{a \in K \,|\, aM \subset M\}$. We have $U \subset \mathcal{O}_M$, hence $\mathcal{O}_M$ is not contained in $\mathbb{Q}$. Let now $K'$ be the field of fractions of $\mathcal{O}_M$. We have to show that $K' \neq K$. Let $x \in K'$ be a primitive element for $K'/\mathbb{Q}$ with $x = a/b$, $a, b \in \mathcal{O}_M$. Then the action of $x$ on $M$ is described by an $r \times r$ matrix with rational coefficients in terms of a basis of $M$. If $K'$ and $K$ coincided, then the characteristic polynomial of $x$ would allow
a factor of degree $r$ over $\mathbb{Q}$. This proves the lemma in one direction. The converse is clear. \hfill \Box

**Definition 1.5.** — We shall call the manifold $X(K, U)$ of **simple type** if $U$ does not satisfy the equivalent conditions of the previous lemma.

**Lemma 1.6.** — Let $\mathbb{Q} \subset K' \subset K$ be a proper intermediate extension and $U \subset O_{K'}^*$ an admissible subgroup for $K$. Let $s'$, $2t'$ be the numbers of distinct real and respectively complex embeddings of $K'$. Then $s = s'$, $t' > 0$ and $U$ is admissible for $K'$.

**Proof.** — The restrictions to $K'$ of two different real embeddings of $K$ cannot coincide since $U \subset K'$ and $U$ is admissible for $K$. Thus $s' \geq s$. We show now that $s \geq s'$ as well.

Let $k := (K : K')$. The restriction to $K'$ of a real embedding of $K$ will have to coincide with the restrictions of exactly $k - 1$ complex embeddings of $K$. In particular since these restrictions are real these $k - 1$ complex embeddings occur in complex conjugate pairs. So $k - 1$ is even.

Suppose now that there is a real embedding of $K'$ which is not the restriction of any real embedding of $K$. Such an embedding has then to be the restriction of exactly $k$ complex embeddings of $K$ and $k$ would then be even by the same reason as above. Thus $s = s'$.

By Dirichlet’s Units Theorem and since $U$ has rank $s$, $t'$ has to be strictly positive. It is clear now that $U$ is admissible for $K'$.

\hfill \Box

**Remark 1.7.** — If $X(K, U)$ is not of simple type with $\mathbb{Q} \subset K' \subset K$ as intermediate extension and $U \subset O_{K'}^*$, then there exists a holomorphic foliation of $X(K, U)$ with a leaf isomorphic to $X(K', U)$. Just look at the foliation of $\mathbb{C}^m$ defined by the translates $V_{K'} + v, v \in \mathbb{C}^m$ of the complex vector subspace $V_{K'}$ of $\mathbb{C}^m$ spanned by $\sigma(O_{K'})$. Its restriction to $\mathbb{H}^s \times \mathbb{C}^t$ is invariant under the action of $U \times O_K$ and thus descends to $X(K, U)$. It is clear that $(V_{K'} \cap (\mathbb{H}^s \times \mathbb{C}^t))/U \times O_{K'}$ is a leaf of this foliation which is compact since $U$ is admissible for $K'$.

**Remark 1.8.** — Not all manifolds $X = X(K, U)$ are of simple type. Keeping the notations of Remark 1.3 take for instance $K', K''$ with $s' = s'' = t' = t'' = 1$ but non isomorphic and $K$ their composite. Then $s = 1$, $t = 4$ and this time $O_{K'}^*$ is admissible for $K$. Note however that...
for any choice of $K$ there are infinitely many $X(K,U)$ of simple type, since the number of intermediate extensions of $K$ is finite.

2. Invariants and metrics.

We investigate here some properties of the varieties $X(K,U)$, where $K$ is a number field as before and $U$ is admissible for $K$.

We start with some preparations for the computation of the first Betti numbers of $X(K,U)$.

**Remark 2.1.** — Let $a \in \mathcal{O}_K^*$ and consider its action on $\text{Hom}_\mathbb{Z} (\mathcal{O}_K, \mathbb{C})$ by $(af)(x) := f(a^{-1}x)$ for $f \in \text{Hom}_\mathbb{Z} (\mathcal{O}_K, \mathbb{C})$ and $x \in \mathcal{O}_K$. Then the restrictions to $\mathcal{O}_K$ of the embeddings $\sigma_1, ..., \sigma_n$ of $K$ give a basis of $\text{Hom}_\mathbb{Z} (\mathcal{O}_K, \mathbb{C})$ of eigenvectors for this action with associated eigenvalues $\sigma_1(a^{-1}), ..., \sigma_n(a^{-1})$.

**Lemma 2.2.** — Let $\theta = \sum_{i=1}^n a_i \sigma_i$, $a_i \in \mathbb{C}$ be a 1-form which is $\mathbb{Q}$-valued on $\mathcal{O}_K$. Then either all coefficients $a_i$ are non-zero or they all vanish.

*Proof.* — It is easy to see that there exists a primitive element $\alpha$ for $K/\mathbb{Q}$ in $\mathcal{O}_K$. Then $\theta(\alpha^k) \in \mathbb{Q}$ for $0 \leq k \leq n - 1$. Let $\alpha_i := \sigma_i(\alpha)$ be the roots of the characteristic polynomial of $\alpha$.

The rationality condition for $\theta$ can be written as a linear system of equations for the coefficients $a_i$:

$$\sum_{i=1}^n a_i \alpha_i^k = b_k, \quad 0 \leq k \leq n - 1,$$

for some rational numbers $b_0, ..., b_{n-1}$. Let $A$ be the coefficient matrix $(\alpha_i^k)_{1 \leq i \leq n, 0 \leq k \leq n-1}$ of this system. By the choice of $\alpha$ we have $\det A \neq 0$.

We suppose now that $\theta \neq 0$, so not all $b_i$-s vanish, and that one of the $a_i$-s is zero, say $a_n = 0$. This means that the determinant of the matrix obtained from $A$ by replacing the last column with the free vector $b = (b_0, ..., b_{n-1})$ shall vanish. Hence expanding this determinant after its last column gives us the following linear dependency relation over $\mathbb{Q}$ among the coefficients of the polynomial $\Pi_{1 \leq i \leq n-1} (X - \alpha_i)$:

$$b_{n-1} + b_{n-2}s_1 + ... + b_0s_{n-1} = 0.$$
Here we denoted by \( s_i \) the \( i \)-th elementary symmetric function in \( a_1, \ldots, a_{n-1} \).

Now we express inductively the elementary symmetric functions in \( a_1, \ldots, a_{n-1} \) in terms of those in \( a_1, \ldots, a_n \) and the powers of \( \alpha_n \):

\[
s_1(a_1, \ldots, a_{n-1}) = s_1(a_1, \ldots, a_n) - \alpha_n,
\]

\[
s_2(a_1, \ldots, a_{n-1}) = s_2(a_1, \ldots, a_n) - \alpha_n s_1(a_1, \ldots, a_{n-1})
\]

\[= s_2(a_1, \ldots, a_n) - \alpha_n s_1(a_1, \ldots, a_n) + \alpha_n^2, \ldots\]

This leads to a non-trivial relation over \( \mathbb{Q} \) among \( 1, \alpha_n, \ldots, \alpha_n^{n-1} \) which contradicts the choice of \( \alpha \). \( \Box \)

**Proposition 2.3.** — For all \( X = X(K,U) \) the first Betti number is \( b_1 = s \). When \( X \) is of simple type one also has \( b_2 = \binom{s}{2} \).

**Proof.** — The cohomology groups of \( X \) with coefficients in \( \mathbb{Q} \) are isomorphic to those of its fundamental group. We thus have to compute \( H^1(U \times \mathcal{O}_K; \mathbb{Q}) \) and \( H^2(U \times \mathcal{O}_K; \mathbb{Q}) \). The Lyndon-Hochschild-Serre spectral sequence associated to the short exact sequence

\[
0 \to \mathcal{O}_K \to U \times \mathcal{O}_K \to U \to 0
\]

gives

\[
E_2^{pq} = H^p(U; H^q(\mathcal{O}_K; \mathbb{Q})) \Longrightarrow H^{p+q}(U \times \mathcal{O}_K; \mathbb{Q})
\]

and an exact sequence of low degree terms:

\[
0 \to H^1(U; \mathbb{Q}\mathcal{O}_K) \to H^1(U \times \mathcal{O}_K; \mathbb{Q}) \to H^1(\mathcal{O}_K; \mathbb{Q})U \to H^2(U; \mathbb{Q}\mathcal{O}_K)
\]

\[
\to H^2(U \times \mathcal{O}_K; \mathbb{Q});
\]

cf. [4], 6.8. Here \( \mathbb{Q} \) is seen as a trivial \( U \times \mathcal{O}_K \)-module. Then \( H^1(\mathcal{O}_K; \mathbb{Q}) \simeq \text{Hom}(\mathcal{O}_K; \mathbb{Q}) \) is a non-trivial \( U \)-module via:

\[(uf)(x) := f(u^{-1}x), \text{ for all } u \in U, \ f \in \text{Hom}(\mathcal{O}_K, \mathbb{Q}), \ x \in \mathcal{O}_K;\]

cf. [4] 6.8.1. Thus \( H^1(\mathcal{O}_K; \mathbb{Q})U := \{f \in \text{Hom}(\mathcal{O}_K, \mathbb{Q})| uf = f, \text{ for all } u \in U\} \) and this last space is trivial by Remark 2.1. Thus \( H^1(U \times \mathcal{O}_K; \mathbb{Q}) \simeq H^1(U; \mathbb{Q}\mathcal{O}_K) \simeq H^1(U; \mathbb{Q}) \simeq H^1(\mathbb{Z}^*; \mathbb{Q}) \simeq \mathbb{Q}^* \).

Moreover, the map \( H^2(U; \mathbb{Q}\mathcal{O}_K) \to H^2(U \times \mathcal{O}_K; \mathbb{Q}) \) is injective. We only need to prove that it is surjective as well when \( X \) is of simple type. To see this it is enough to check that in this case the terms \( E_2^{0,2} \) and \( E_2^{1,1} \) of the spectral sequence vanish.

Consider first

\[
E_2^{0,2} = H^0(U; H^2(\mathcal{O}_K; \mathbb{Q})) = H^2(\mathcal{O}_K; \mathbb{Q})U \simeq \text{Alt}^2(\mathcal{O}_K; \mathbb{Q})U.
\]
This is the space of alternating 2-forms on $\mathcal{O}_K$ which are fixed by $U$. Let
$$
\gamma = \sum_{1 \leq i < j \leq n} a_{ij} \sigma_i \wedge \sigma_j \in \text{Alt}^2(\mathcal{O}_K; \mathbb{Q})^U
$$
with $a_{ij} \in \mathbb{C}$. The fact that $\gamma$ is invariant under the action of some $u \in U$ means that $\sigma_i(u)\sigma_j(u) = 1$ whenever $a_{ij} \neq 0$; cf. Remark 2.1. From this we get $a_{ij} = 0$ for all $1 \leq i < j \leq s$ since $U$ is admissible for $K$. The relation $\sigma_i(u)\sigma_j(u) = 1$ for all $u \in U$ and the fact that $X$ is of simple type imply moreover that $a_{ij} = 0$ whenever $1 \leq i \leq s$ and that for each $i > s$ there exists at most one $j = j(i) > i$ with $a_{ij} \neq 0$. (Otherwise we would get two equal embeddings $\sigma_j = \sigma_j$.)
Thus $\gamma = \sum_{s<i<n} a_{ij(i)} \sigma_i \wedge \sigma_{j(i)}$. Let $\alpha \in \mathcal{O}_K$ be a primitive element for $\mathcal{O}_K$. Then $\gamma(\alpha^k, 1) \in \mathbb{Q}$ for all $k \in \mathbb{Z}$, that is $\sum_{s<i<n} a_{ij(i)}(\sigma_i(\alpha^k) - \sigma_{j(i)}(\alpha^k)) \in \mathbb{Q}$ for all $k \in \mathbb{Z}$. But then we get a rational 1-form $\sum_{s<i<n} a_{ij(i)}(\sigma_i - \sigma_{j(i)})$ which by Lemma 2.2 has to vanish.

We now check that $E_2^{1,1} = H^1(U; H^1(\mathcal{O}_K; \mathbb{Q}))$ is trivial. Since $U$ is free abelian we reduce ourselves by the Lyndon-Hochschild-Serre spectral sequence for $0 \to \mathbb{Z} \to \mathbb{Z}^s \to \mathbb{Z}^{s-1} \to 0$ to the computation of $H^1(\mathbb{Z}; H^1(\mathcal{O}_K; \mathbb{Q}))$ where $\mathbb{Z}$ here is the subgroup generated by some $u \in U$. Now $H^1(\mathbb{Z}; H^1(\mathcal{O}_K; \mathbb{Q})) \cong H^1(\mathcal{O}_K; \mathbb{Q})_\mathbb{Z} \cong H^1(\mathcal{O}_K; \mathbb{Q})/\langle uf - f \mid f \in H^1(\mathcal{O}_K; \mathbb{Q}) \rangle$; cf. [4] 6.1.4. But the action of $u - id$ is invertible by Remark 2.1 hence $H^1(\mathbb{Z}; H^1(\mathcal{O}_K; \mathbb{Q}))$ vanishes.

**Lemma 2.4.** — Every holomorphic function on $\mathbb{H}^s \times \mathbb{C}^t/\sigma(\mathcal{O}_K)$ is constant.

**Proof.** — Take any element $v \in \mathbb{H}^s$. We shall first prove the following

**Claim.** — The image of $\{v\} \times \mathbb{C}^t$ in $(v + \mathbb{R}^s) \times \mathbb{C}^t/\sigma(\mathcal{O}_K)$ is dense in this space.

We shall just check that $0 \times \mathbb{C}^t$ has a dense image in $\mathbb{R}^s \times \mathbb{C}^t/\sigma(\mathcal{O}_K)$.
For this it is enough to prove that the image of $\mathcal{O}_K$ through $\sigma' = (\sigma_1, \ldots, \sigma_s): \mathcal{O}_K \to \mathbb{R}^s$ is dense in $\mathbb{R}^s$.

Consider the connected component $V$ of 0 of the topological closure of $\sigma'(\mathcal{O}_K)$ in $\mathbb{R}^s$ and the $\mathbb{Z}$-submodule $M := \sigma'^{-1}(V)$ of $\mathcal{O}_K$. If $V \neq \mathbb{R}^s$ we would have rank $M < n$. Take now $\alpha \in \mathcal{O}_K$ a primitive element for $K$. On $\mathcal{O}_K$ we have a multiplicative action of $\alpha$. The submodule $\alpha M$ of $\mathcal{O}_K$ has finite index so the induced linear action of $\alpha$ on $\mathbb{R}^s$ will leave $V$ invariant. Thus $M$ also remains invariant under the action of $\alpha$. But this would imply that the characteristic polynomial of $\alpha$ admits a factor of degree rank $M$ over $\mathbb{Q}$, which is absurd.
Take now a holomorphic function $f$ on $\mathbb{H}^s \times \mathbb{C}^t / \sigma(O_K)$ and $v \in \mathbb{H}^s$. Since $f$ is bounded on $(v + \mathbb{R}^s) \times \mathbb{C}^t / \sigma(O_K) \simeq (S^1)^n$ its lift $\tilde{f}$ to $\mathbb{H}^s \times \mathbb{C}^t$ will be bounded on each $(v + \mathbb{R}^s) \times \mathbb{C}^t$ hence constant on $\{v\} \times \mathbb{C}^t$. By our Claim it follows now that $\tilde{f}$ is constant on $(v + \mathbb{R}^s) \times \mathbb{C}^t$. But then $\tilde{f}$ must be constant on $\mathbb{H}^s \times \mathbb{C}^t$ by the identity principle.

**Proposition 2.5.** — The following vector bundles on $X = X(K, U)$ are flat and admit no non-trivial global holomorphic sections:

$$\Omega^1_X, \Theta_X, K^\otimes_k_X, \text{ for all } k \neq 0.$$ 

Moreover $\dim H^1(X, O_X) \geq s$. In particular $\kappa(X) = -\infty$ and $X$ is non-Kähler.

**Proof.** — Let $z_1, ..., z_m$ be the standard complex coordinate functions on $\mathbb{H}^s \times \mathbb{C}^t$. A section $\omega$ of $K^\otimes_k_X$ lifted to $\mathbb{H}^s \times \mathbb{C}^t$ will have the form $\tilde{\omega} = f(dz_1 \wedge ... \wedge dz_m)^\otimes_k$. Since this section descends to $\mathbb{H}^s \times \mathbb{C}^t / \sigma(O_K)$ it follows from Lemma 2.4 that $f$ is constant on $\mathbb{H}^s \times \mathbb{C}^t$. Moreover if $f \neq 0$, the invariance of $\tilde{\omega}$ with respect to $U$ gives $(\Pi_{i=1}^m \sigma_i(u))^k = 1$ for all $u \in U$. Multiplying this by $(\Pi_{i=1}^m \bar{\sigma}_i(u))^k = 1$ and using the fact that $(\Pi_{i=1}^m \sigma_i(u))^k = 1$ we get $(\Pi_{i=1}^s \sigma_i(u))^k = 1$ which contradicts the admissibility of $U$.

In the case of $\Omega^1_X$ the automorphy factors are $\sigma_i(u), i = 1, ..., m$ and it is clear that none of them equals 1. An analogous argument works for $\Theta_X$ using the vector fields $\partial/\partial z_i$. The flatness of these bundles is evident.

The statement on $\dim H^1(O_X)$ follows now from Proposition 2.3 and the exact sequence of sheaves on $X$:

$$0 \rightarrow \mathbb{C} \rightarrow \mathcal{O} \rightarrow d\mathcal{O} \rightarrow 0.$$ 

**Remark 2.6.** — The above proof also shows that the embeddings of $U$ by $\sigma_1, ..., \sigma_m$ are determined by the complex structure of $X(K, U)$ through the automorphy factors of $\Omega^1_X$. In particular when $X$ is of simple type its complex structure determines both $K$ and $U$.

**Corollary 2.7.** — The group of holomorphic automorphisms of $X$ is discrete. It is infinite when $t > 1$ since the elements of $O_{X,K}^* / U$ induce automorphisms of $X(K, U)$.

It is known that the Inoue-Bombieri surfaces $S_M$ admit locally conformally Kähler metrics. This means that there is a representation...
\[ \rho : \pi_1(S_M) \to \mathbb{R}_{>0} \] and a closed strongly positive \((1,1)\)-form \(\omega\) on the universal cover of \(S_M\) such that \(g^*\omega = \rho(g)\omega\) for all \(g \in \pi_1(S_M)\); cf. [2]. We now investigate the existence of locally conformally Kähler metrics more generally on the manifolds \(X(K,U)\).

**Example.** — When \(t = 1\) all manifolds \(X(K,U)\) admit locally conformally Kähler metrics. Consider indeed the following potential \(F\):

\[ F : \mathbb{H}^s \times \mathbb{C} \to \mathbb{R}, \quad F(z) := \frac{1}{\prod_{j=1}^s (i(z_j - \bar{z}_j))} + |z_m|^2. \]

Then \(\omega := i\partial\bar{\partial}F\) gives the desired Kähler metric on \(\mathbb{H}^s \times \mathbb{C}\).

**Remark 2.8.** — The manifolds \(X(K,U)\) with \(s = 2\) and \(t = 1\) give counterexamples to a conjecture of I. Vaisman, according to which a compact locally conformally Kähler manifold admitting even Betti numbers with odd index and non-zero Betti numbers with even index should already be Kähler; (see [2], p. 8).

**Proof.** — We have the following Betti numbers for \(X(K,U)\): \(b_0 = b_6 = 1, b_1 = b_5 = 2, b_2 = b_4 = 1\) and \(b_3 = 0\). In fact, here \(X(K,U)\) is of simple type and therefore we can apply Proposition 2.3 to get \(b_1\) and \(b_2\). For \(b_3\) note that the Euler characteristic equals \(c_3(X(K,U)) = 0\), since \(\Theta\) is flat.

**Proposition 2.9.** — When \(s = 1\) and \(t > 1\) there exists no locally conformally Kähler metric on \(X(K,U)\).

**Proof.** — Let \(s = 1, \omega = \sum_{1 \leq i, j \leq m} g_{ij}dz_i \wedge d\bar{z}_j\) a closed strictly positive \((1,1)\)-form on \(\mathbb{H} \times \mathbb{C}^t\) and \(\rho : U \times \mathcal{O}_K \to \mathbb{R}_{>0}\) a representation such that \(g^*\omega = \rho(g)\omega\) for all \(g \in U \times \mathcal{O}_K\). We shall show that \(t = 1\).

It is clear that \(\rho\) factorizes through a representation of \(U\) which we denote again by \(\rho\). Since \(\omega\) descends to \((\mathbb{H} \times \mathbb{C}^t) / \sigma(\mathcal{O}_K) \simeq \mathbb{R}_{>0} \times (S^1)^n\), we may assume by averaging over \((S^1)^n\) that the coefficients \(g_{ij}\) are constant in the directions of \(\sigma(\mathcal{O}_K)\). In particular they are constant on the subspaces \(\{v\} \times \mathbb{C}^t\) for each \(v \in \mathbb{H}\). Since \(d\omega = 0\), this implies that for \(i, j > 1\) the coefficients \(g_{ij}\) are constant on the whole of \(\mathbb{H} \times \mathbb{C}^t\). By the compatibility of \(\omega\) with \(\rho\) we thus get

\[ \rho(u) = |\sigma_2(u)|^2 = |\sigma_3(u)|^2 \ldots = |\sigma_m(u)|^2, \quad \forall u \in U. \]

Consider now a non-trivial element \(u\) of \(U\) and its characteristic polynomial \(X^n - a_1X^{n-1} + \ldots + a_{2t}X - 1\). This polynomial must be
irreducible, otherwise there would exist some \( i > 1 \) such that \( \sigma_1(u) = \sigma_i(u) \) \( \forall u \in U \). But this would imply \( \sigma_1(u) = 1 \) which is impossible.

We have
\[
\sigma_1(u) = \frac{1}{\rho(u)^t},
\]
\[
a_1 = \frac{1}{\rho(u)^t} + \sum_{j=2}^{m}(\sigma_j(u) + \bar{\sigma}_j(u)),
\]
\[
a_{2t} = \sum_{j=1}^{m} \frac{1}{\sigma_j(u)} = \rho(u)^t + \frac{\sum_{j=2}^{m}(\sigma_j(u) + \bar{\sigma}_j(u))}{\rho(u)} = \rho(u)^t + \frac{a_1}{\rho(u)} - \frac{1}{\rho(u)^{t+1}}.
\]

Thus \( \rho(u) \) satisfies the following equation:
\[
\rho(u)^n - a_{2t}\rho(u)^{t+1} + a_1\rho(u)^t - 1 = 0.
\]

Since \( \mathbb{Q}[\sigma_1(u)] \subset \mathbb{Q}[\rho(u)] \) these field extensions must be equal, hence \( \rho(u) \) is a non-torsion unit in \( \mathcal{O}_K \) having the same property as \( u \), namely that its images through the complex embeddings of \( K \) have the same absolute value: \( \rho(u)^{-1/t} = \sigma_1(u)^{1/t^2} \). But the same argument as before yields a new non-torsion unit \( \rho(u)^{-1/t} \) which for \( t > 1 \) satisfies the equation \( X^n - 1 = 0 \). This is a contradiction! \( \Box \)

---

**BIBLIOGRAPHY**


Karl Oeljeklaus,  
Université d’Aix-Marseille I  
CMI - LATP-UMR(CNRS) 6632  
39, rue Joliot-Curie  
13453 Marseille Cedex 13, (France).  
karloelj@cmi.univ-mrs.fr

Matei Toma,  
Institute of Mathematics of the Romanian Academy  
Bucharest, 014700 (Romania).  
matei@mathematik.uni-osnabrueck.de