Jean-Louis VERGER-GAUGRY

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ON GAPS IN RÉNYI $\beta$-EXPANSIONS OF UNITY FOR $\beta > 1$ AN ALGEBRAIC NUMBER

by Jean-Louis VERGER-GAUGRY

Abstract. — Let $\beta > 1$ be an algebraic number. We study the strings of zeros ("gaps") in the Rényi $\beta$-expansion $d_\beta(1)$ of unity which controls the set $\mathbb{Z}_\beta$ of $\beta$-integers. Using a version of Liouville’s inequality which extends Mahler’s and Gütting’s approximation theorems, the strings of zeros in $d_\beta(1)$ are shown to exhibit a "gappiness" asymptotically bounded above by $\log(M(\beta))/\log(\beta)$, where $M(\beta)$ is the Mahler measure of $\beta$. The proof of this result provides in a natural way a new classification of algebraic numbers $> 1$ with classes called $Q_{\beta}^j$ which we compare to Bertrand-Mathis’s classification with classes $C_1$ to $C_5$ (reported in an article by Blanchard). This new classification relies on the maximal asymptotic “quotient of the gap” value of the “gappy” power series associated with $d_\beta(1)$. As a corollary, all Salem numbers are in the class $C_1 \cup Q_{0}^{(1)} \cup Q_{0}^{(2)} \cup Q_{0}^{(3)}$; this result is also directly proved using a recent generalization of the Thue-Siegel-Roth Theorem given by Corvaja.

Résumé. — Soit $\beta > 1$ un nombre algébrique. Nous étudions les plages de zéros ("lacunes") dans le $\beta$-développement de Rényi $d_\beta(1)$ de l’unité qui contrôle l’ensemble $\mathbb{Z}_\beta$ des $\beta$-entiers. En utilisant une version de l’inégalité de Liouville qui étend des théorèmes d’approximation de Mahler et de Gütting, on montre que les plages de zéros dans $d_\beta(1)$ présentent une "lacunarité" asymptotiquement bornée supérieurement par $\log(M(\beta))/\log(\beta)$, où $M(\beta)$ est la mesure de Mahler de $\beta$. La preuve de ce résultat fournit de manière naturelle une nouvelle classification des nombres algébriques $> 1$ en classes appelées $Q_{\beta}^j$ que nous comparons à la classification de Bertrand-Mathis avec les classes $C_1$ à $C_5$ (reportée dans un article de Blanchard). Cette nouvelle classification repose sur la valeur asymptotique maximale du "quotient de lacune" de la série "lacunaire" associée à $d_\beta(1)$. Comme corollaire, tous les nombres de Salem sont dans la classe $C_1 \cup Q_{0}^{(1)} \cup Q_{0}^{(2)} \cup Q_{0}^{(3)}$; ce résultat est également obtenu par un théorème récent qui généralise le théorème de Thue-Siegel-Roth donné par Corvaja.

Keywords: Beta-integer, beta-numeration, PV number, Salem number, Perron number, Mahler measure, Diophantine approximation, Mahler’s series, mathematical quasicrystal.

Math. classification: 11B05, 11Jxx, 11J68, 11R06, 52C23.
1. Introduction

The exploration of the links between symbolic dynamics and number theory of $\beta$-expansions, when $\beta > 1$ is an algebraic number or more generally a real number, started with Bertrand-Mathis [6] [7]. Bertrand-Mathis, in Blanchard [8], reported a classification of real numbers according to their $\beta$-shift, using the properties of the Rényi $\beta$-expansion $d_\beta(1)$ of 1. A lot of questions remain open concerning the distribution of the algebraic numbers $\beta > 1$ in this classification. The Rényi $\beta$-expansion of 1 is important since it controls the $\beta$-shift [38] and the discrete and locally finite set $\mathbb{Z}_\beta \subset \mathbb{R}$ of $\beta$-integers [13] [18] [25] [26]. The aim of this note is to give a new Theorem (Theorem 1.1) on the gaps (strings of 0’s) in $d_\beta(1)$ for algebraic numbers $\beta > 1$, and investigate how it provides (partial) answers to some questions of [8], in particular for Salem numbers (Corollary 1.2).

Theorem 1.1 provides an upper bound on the asymptotic quotient of the gap of $d_\beta(1)$ and is obtained by a version of Liouville’s inequality extending Mahler’s and Güting’s approximation theorems. The proof of Theorem 1.1 turns out to be extremely instructive in itself since it leads to a new classification of the algebraic numbers $\beta$ as a function of the asymptotics of the gaps in $d_\beta(1)$ and “intrinsic features”, namely the Mahler measure $M(\beta)$, of $\beta$ (the definition of $M(\beta)$ is recalled in Section 3). The existence of this double parametrization, symbolic and algebraic, was guessed in [8] p 137. This new classification complements Bertrand-Mathis’s (Blanchard [8] pp 137–139) and both are recalled below for comparison’s sake. The question whether an algebraic number $\beta > 1$ is contained in one class or another has already been discussed by many authors [5] [6] [7] [8] [9] [10] [11] [12] [17] [22] [32] [33] [38] [39] [41] [42] and depends at least upon the distribution of the conjugates of $\beta$ in the complex plane. Only the conjugates of $\beta$ of modulus strictly greater than unity intervene in Theorem 1.1 via the Mahler measure of $\beta$. Corollary 1.2 is readily deduced from this remark. We deduce that Salem numbers belong to $C_1 \cup C_2 \cup \mathbb{Q}_0$, whereas the Pisot numbers are in $C_1 \cup C_2$ [45].

Another proof of Corollary 1.2 consists of controlling the gaps of $d_\beta(1)$ by stronger Theorems of Diophantine Geometry which allow suitable collections of places of the number field $\mathbb{Q}(\beta)$ associated with the conjugates of $\beta$ and the properties of $d_\beta(1)$ to be taken into account simultaneously. This alternative proof of Corollary 1.2, just sketched in Section 4, is obtained using the Theorem of Thue-Siegel-Roth given by Corvaja [1] [15].
Theorem 1.1. — Let $\beta > 1$ be an algebraic number and $M(\beta)$ be its Mahler measure. Denote by $d_\beta(1) := 0.t_1t_2t_3\ldots$, with $t_i \in A_\beta := \{0,1,2,\ldots, [\beta - 1]\}$, the Rényi $\beta$-expansion of 1. Assume that $d_\beta(1)$ is infinite and gappy in the following sense: there exist two sequences $\{m_n\}_{n\geq 1}$, $\{s_n\}_{n\geq 0}$ such that

$$1 = s_0 \leq m_1 < s_1 \leq m_2 < s_2 \leq \ldots \leq m_n < s_n \leq m_{n+1} < s_{n+1} \leq \ldots$$

with $(s_n - m_n) \geq 2$, $t_{m_n} \neq 0$, $t_{s_n} \neq 0$ and $t_i = 0$ if $m_n < i < s_n$ for all $n \geq 1$. Then

$$\limsup_{n \to +\infty} \frac{s_n}{m_n} \leq \frac{\log(M(\beta))}{\log(\beta)}. \quad (1.1)$$

Moreover, if $\liminf_{n \to +\infty} (m_{n+1} - m_n) = +\infty$, then

$$\limsup_{n \to +\infty} \frac{s_{n+1} - s_n}{m_{n+1} - m_n} \leq \frac{\log(M(\beta))}{\log(\beta)}. \quad (1.2)$$

As in Ostrowski [37] the quotient $s_n/m_n \geq 1$ is called the quotient of the gap, relative to the $n$th-gap (assuming $t_j \neq 0$ for all $s_n \leq j \leq m_{n+1}$ to characterize uniquely the gaps). Note that the term "lacunary" has often other meanings in literature and is not used here to describe "gappiness". Denote by $L(S_\beta)$ the language of the $\beta$-shift [8] [23] [24] [34]. Bertrand-Mathis’s classification ([8] pp 137–139) is as follows:

- $C_1 : d_\beta(1)$ is finite.
- $C_2 : d_\beta(1)$ is ultimately periodic but not finite.
- $C_3 : d_\beta(1)$ contains bounded strings of 0’s, but is not ultimately periodic.
- $C_4 : d_\beta(1)$ does not contain some words of $L(S_\beta)$, but contains strings of 0’s with unbounded length.
- $C_5 : d_\beta(1)$ contains all words of $L(S_\beta)$.

Present classes of algebraic numbers, with the notations of Theorem 1.1:

- $Q_0^{(1)} : 1 < \limsup_{n \to +\infty} \frac{s_n}{m_n}$ with $(m_{n+1} - m_n)$ bounded.
- $Q_0^{(2)} : 1 = \lim_{n \to +\infty} \frac{s_n}{m_n}$ with $(s_n - m_n)$ bounded and $\lim_{n \to +\infty} (m_{n+1} - m_n) = +\infty$.
- $Q_0^{(3)} : 1 = \lim_{n \to +\infty} \frac{s_n}{m_n}$ with $\limsup_{n \to +\infty} (s_n - m_n) = +\infty$.
- $Q_1 : 1 < \limsup_{n \to +\infty} \frac{s_n}{m_n} < \frac{\log(M(\beta))}{\log(\beta)}$.
- $Q_2 : \limsup_{n \to +\infty} \frac{s_n}{m_n} = \frac{\log(M(\beta))}{\log(\beta)}$.

What are the relative proportions of each class in the whole set $\overline{Q}_{>1}$ of algebraic numbers $\beta > 1$? Comparing $C_2$, $C_3$ and $Q_0^{(1)}$, what are the relative proportions in $Q_0^{(1)}$ of those $\beta$ which give ultimate periodicity in
$d_\beta(1)$ and those for which $d_\beta(1)$ is not ultimately periodic? Schmeling ([41] Theorem A) has shown that the class $C_3$ (of real numbers $\beta > 1$) has Hausdorff dimension one. We have:

- $\overline{\mathbb{Q}}_{>1} \cap C_2 \subset Q_0^{(1)}$,
- $\overline{\mathbb{Q}}_{>1} \cap C_3 \subset Q_0^{(1)} \cup Q_0^{(2)}$, with $C_3 \cap Q_0^{(3)} = \emptyset$,
- $\overline{\mathbb{Q}}_{>1} \cap C_4 \subset Q_0^{(3)} \cup Q_1 \cup Q_2$.

The Pisot numbers $\beta$ are contained in $C_1 \cup Q_0^{(1)}$ since they are such that $d_\beta(1)$ is finite or ultimately periodic (Parry [38], Bertrand-Mathis [5]). Recall that a Perron number is an algebraic integer $\beta > 1$ such that all the conjugates $\beta^{(i)}$ of $\beta$ satisfy $|\beta^{(i)}| < \beta$. Conversely, as shown in Lind [32], Denker, Grillenberger, Sigmund [17] and Bertrand-Mathis [7], if $\beta > 1$ is such that $d_\beta(1)$ is ultimately periodic (finite or not), then $\beta$ is a Perron number. Not all Perron numbers are attained in this way: a Perron number which possesses a real conjugate greater than 1 cannot be such that $d_\beta(1)$ is ultimately periodic ([8] p 138). Parry numbers belong to $C_1 \cup C_2$. Let $Q_0 = Q_0^{(1)} \cup Q_0^{(2)} \cup Q_0^{(3)}$.

**Corollary 1.2. — Let $\beta > 1$ be a Salem number which does not belong to $C_1$. Then $\beta$ belongs to the class $Q_0$.**

The attribution of Salem numbers to $C_1$, $Q_0^{(1)}$, $Q_0^{(2)}$ and $Q_0^{(3)}$ is an open problem in general, except in low degree. Boyd [9] [12] has shown that Salem numbers of degree 4 belong to $C_2$, hence to $Q_0^{(1)}$. It is also the case of some Salem numbers of degree 6 and $\geq 8$ in the framework of a probabilistic model [11] [12]. In Section 5 we ask the question whether Corollary 1.2 could still be true for Perron numbers.

The definition of the class $Q_0$ does not make any allusion to $\beta$, i.e. to $M(\beta)$, to the conjugates of $\beta$, to the minimal polynomial of $\beta$ or to its length, etc, but takes only into account the quotients of the gaps in $d_\beta(1)$. Hence this class $Q_0$ can be applied to real numbers $\beta > 1$ in full generality instead of only to algebraic numbers $> 1$. The question whether there exist transcendental numbers $\beta > 1$ which belong to the class $Q_0$ was asked in [8]; what proportion appears in each subclass? Examples of transcendental numbers (Komornik-Loreti constant [2] [29], Sturmian numbers [14]) in $Q_0$ are given in Section 5.

In the present note, we deal with the algebraicity of values of “gappy” series, deduced from $d_\beta(1)$, at the algebraic point $\beta^{-1}$. In a related context, more related to transcendency, Nishioka [36] and Corvaja Zannier [16] have followed different paths and applied the Subspace Theorem [43] to deduce different results.
2. Definitions

For \( x \in \mathbb{R} \) the integer part of \( x \) is \( \lfloor x \rfloor \) and its fractional part \( \{x\} = x - \lfloor x \rfloor \). The smallest integer larger than or equal to \( x \) is denoted by \( \lceil x \rceil \). For \( \beta > 1 \) a real number and \( z \in [0,1) \) we denote by \( T_\beta(z) = \beta z \mod 1 \) the \( \beta \)-transform on \([0,1]\) associated with \( \beta \) [38] [40], and iteratively, for all integers \( j \geq 0 \), \( T_\beta^{j+1}(z) := T_\beta(T_\beta^j(z)) \), where by convention \( T_\beta^0 = \text{Id} \).

Let \( \beta > 1 \) be a real number. A beta-representation (or \( \beta \)-representation, or representation in base \( \beta \)) of a real number \( x \geq 0 \) is given by an infinite sequence \( (x_i)_{i \geq 0} \) and an integer \( k \in \mathbb{Z} \) such that \( x = \sum_{i=0}^{+\infty} x_i \beta^{-i+k} \), where the digits \( x_i \) belong to a given alphabet \((\subset \mathbb{N}) [23] [24] [34] \). Among all the beta-representations of a real number \( x \geq 0 \), \( x \neq 1 \), there exists a particular one called Rényi \( \beta \)-expansion, which is obtained via the greedy algorithm: in this case, \( k \) satisfies \( \beta^k \leq x < \beta^{k+1} \) and the digits

\[
(2.1) \quad x_i := \lfloor \beta T_\beta^i \left( \frac{x}{\beta^{k+1}} \right) \rfloor \quad i = 0, 1, 2, \ldots \, ,
\]

belong to the finite canonical alphabet \( \mathbb{A}_\beta := \{0,1,2,\ldots,\lfloor \beta - 1 \rfloor \} \). If \( \beta \) is an integer, then \( \mathbb{A}_\beta := \{0,1,2,\ldots,\beta - 1 \} \); if \( \beta \) is not an integer, then \( \mathbb{A}_\beta := \{0,1,2,\ldots,\lfloor \beta \rfloor \} \). We denote by

\[
(2.2) \quad (x)_\beta := x_0 x_1 x_2 \ldots x_k x_{k+1} x_{k+2} \ldots
\]

the couple formed by the string of digits \( x_0 x_1 x_2 \ldots x_k x_{k+1} x_{k+2} \ldots \) and the position of the dot, which is at the \( k \)-th position (between \( x_k \) and \( x_{k+1} \)). By definition the integer part (in base \( \beta \)) of \( x \) is \( \sum_{i=0}^{k} x_i \beta^{-i+k} \) and its fractional part (in base \( \beta \)) is \( \sum_{i=k+1}^{+\infty} x_i \beta^{-i+k} \). If a Rényi \( \beta \)-expansion ends in infinitely many zeros, it is said to be finite and the ending zeros are omitted. If it is periodic after a certain rank, it is said to be eventually periodic (the period is the smallest finite string of digits possible, assumed not to be a string of zeros); for the substitutive approach see [19] [39].

The Rényi \( \beta \)-expansion which plays an important role in the theory is the Rényi \( \beta \)-expansion of unity, denoted by \( d_\beta(1) \) and defined as follows: since \( \beta^0 \leq 1 < \beta \), the value \( T_\beta(1/\beta) \) is set to 1 by convention. Then using the formulae (2.1)

\[
(2.3) \quad t_1 = \lfloor \beta \rfloor, t_2 = \lfloor \beta \{ \beta \} \rfloor, t_3 = \lfloor \beta \{ \beta \{ \beta \} \} \rfloor, \ldots
\]

The writing

\[
d_\beta(1) = 0.t_1 t_2 t_3 \ldots
\]

corresponds to

\[
1 = \sum_{i=1}^{+\infty} t_i \beta^{-i}.
\]
Links between the set \( \mathbb{Z}_\beta \) of beta-integers and \( d_\beta(1) \) are evoked in [18] [21] [27] [26] [46]. A real number \( \beta > 1 \) such that \( d_\beta(1) \) is finite or eventually periodic is called a beta-number or more recently a Parry number (this recent terminology appears in [18]). In particular, it is called a simple beta-number or a simple Parry number when \( d_\beta(1) \) is finite. Beta-numbers (Parry numbers) are algebraic integers ([38]) and all their conjugates lie within a compact subset which looks like a fractal in the complex plane [44]. The conjugates of beta-numbers are all bounded above in modulus by the golden mean \( \frac{1}{2}(1 + \sqrt{5}) \) ([20] [44]).

### 3. Proof of Theorem 1.1

Since algebraic numbers \( \beta > 1 \) for which the Rényi \( \beta \)-expansion \( d_\beta(1) \) of 1 is finite are excluded, we may consider that \( \beta \) does not belong to \( \mathbb{N} \). Indeed, if \( \beta = h \in \mathbb{N} \), then \( d_h(1) = 0.h \) is finite (Lothaire [34], Chap. 7). If \( \beta \not\in \mathbb{N} \), then \( \lceil \beta - 1 \rceil = \lfloor \beta \rfloor \) and the alphabet \( A_\beta \) equals \( \{0, 1, 2, \ldots, \lfloor \beta \rfloor\} \), where \( \lceil x \rceil \) denotes the greatest integer smaller than or equal to \( x \).

Let \( f(z) := \sum_{i=1}^{+\infty} t_i z^i \) be the “gappy” power series deduced from the representation \( d_\beta(1) = 0.t_1t_2t_3 \ldots \) associated with the \( \beta \)-shift (gappy in the sense of Theorem 1.1). Since \( d_\beta(1) \) is assumed to be infinite, its radius of convergence is 1. By definition, it satisfies

\[
(3.1) \quad f(\beta^{-1}) = 1,
\]

which means that the function value \( f(\beta^{-1}) \) is algebraic, equal to 1, at the real algebraic number \( \beta^{-1} \) in the open disk of convergence \( D(0, 1) \) of \( f(z) \) in the complex plane. This fact is a general intrinsic feature of the Rényi expansion process which leads to the following important consequence by the theory of admissible power series of Mahler [35].

**Proposition 3.1.** —

\[
(3.2) \quad \limsup_{n \to +\infty} \frac{s_n}{m_n} < +\infty.
\]

**Proof.** — This is a consequence of Theorem 1 in [35]. Indeed, if we assume that there exists a sequence of integers \( (n_i) \) which tends to infinity such that \( \lim_{i \to +\infty} s_{n_i}/m_{n_i} = +\infty \), then \( f(z) \) would be admissible in the sense of [35]. Since \( f(z) \) is a power series with nonnegative coefficients, which is not a polynomial, the function value \( f(\beta^{-1}) \) should not be algebraic. But it equals 1. Contradiction. \( \Box \)
Let us improve Proposition 3.1. Assume that
\begin{equation}
\limsup \frac{s_n}{m_n} > \frac{\log(M(\beta))}{\log(\beta)}
\end{equation}
and show the contradiction with (1.1) and (1.2). Recall that, if
\[ P_\beta(X) = \sum_{i=0}^{d} \alpha_i X^i = \alpha_d \prod_{i=0}^{d-1} (X - \beta^{(i)}) \]
with \( d \geq 1, \alpha_0 \alpha_d \neq 0 \), denotes the minimal polynomial of \( \beta = \beta^{(0)} > 1 \), having \( \beta^{(1)}, \beta^{(2)}, \ldots, \beta^{(d-1)} \) as conjugates, the Mahler measure of \( \beta \) is by definition
\[ M(\beta) := |\alpha_d| \prod_{i=0}^{d-1} \max\{1, |\beta^{(i)}|\}. \]
Güting [28] has shown that the approximation of algebraic numbers by algebraic numbers is fairly difficult to realize by polynomials. In the present proof, we use a version of Liouville’s inequality which generalizes approximation theorems obtained by Güting [28], and apply it to the values of the “polynomial tails” of the power series \( f(z) \) at the algebraic number \( \beta^{-1} \), to obtain the contradiction. Let us write
\begin{equation}
f(z) = \sum_{n=0}^{+\infty} Q_n(z)
\end{equation}
with
\begin{equation}
Q_n(z) := \sum_{i=s_n}^{m_n+1} t_i z^i, \quad n = 0, 1, 2, \ldots.
\end{equation}
By construction the polynomials \( Q_n(z) \), of degree \( m_{n+1} \), are not identically zero and \( Q_n(1) > 0 \) is an integer for all \( n \geq 1 \).
Denote by \( S_n(z) = -1 + \sum_{i=1}^{m_n} t_i z^i \) the \( m_n \)th-section polynomial of the power series \( f(z) - 1 \) for all \( n \geq 1 \). Recall that, for \( R(X) = \sum_{i=0}^{v} \alpha_i X^i \in \mathbb{Z}[X] \), \( L(R) := \sum_{i=0}^{v} |\alpha_i| \) denotes the length of the polynomial \( R(X) \). We have: \( L(S_n) = 1 + \sum_{i=1}^{m_n} t_i = 1 + \sum_{j=0}^{n-1} Q_j(1) \). From Theorem 5 in [28] we deduce that only one of the following cases (G-i) or (G-ii) holds, for all \( n \geq 1 \):
\begin{align*}
(3.6) \quad & (G - i) \quad S_n(\beta^{-1}) = 0, \\
(3.7) \quad & (G - ii) \quad |S_n(\beta^{-1})| \geq \frac{1}{\left(1 + \sum_{j=0}^{n-1} Q_j(1)\right)^{d-1} \left(L(P_\beta^*)\right)^{m_n}},
\end{align*}
where $P_\beta^*(X) = X^d P_\beta(1/X)$ is the reciprocal polynomial of the minimal polynomial of $\beta$, for which $L(P_\beta) = L(P_\beta^*) \in \mathbb{N} \setminus \{0, 1\}$.

Case (G-i) is impossible for any $n$. Indeed, if there exists an integer $n_0 \geq 1$ such that (G-i) holds, then, since all the digits $t_i$ are positive and that $\beta^{-1} > 0$, we would have $t_i = 0$ for all $i \geq s_{n_0}$. This would mean that the Rényi expansion of 1 in base $\beta$ is finite, which is excluded by assumption. Contradiction. Therefore, the only possibility is (G-ii), which holds for all integers $n \geq 1$. From Lemma 3.10 and Liouville’s inequality (Proposition 3.14) in Waldschmidt [47] the inequality (G-ii) can be improved to

$$\left| S_n(\beta^{-1}) \right| \geq \frac{1}{\left(1 + \sum_{j=0}^{n-1} Q_j(1)^{d-1} (M(\beta))^{m_n}\right)}.$$

This improvement may be important; recall the well-known inequalities:

$$M(\beta) \leq L(P_\beta) \leq 2^{\text{deg}(\beta)} M(\beta)$$

and see [47] p. 113 for comparison with different heights. On the other hand, since $\left| S_n(\beta^{-1}) \right| = \sum_{i=s_n}^{+\infty} t_i \beta^{-i}$ for all integers $n \geq 1$, we deduce

$$\left| S_n(\beta^{-1}) \right| \leq \frac{[\beta]}{1 - \beta^{-1} \beta^{-s_n}} \quad n = 1, 2, \ldots.$$

Putting together (3.8) and (3.9), we deduce that

$$\beta^{s_n} \left(1 + \sum_{j=0}^{n-1} Q_j(1)\right)^{d-1} (M(\beta))^{m_n} \leq \frac{[\beta]}{1 - \beta^{-1}}$$

should be satisfied for $n = 1, 2, 3, \ldots$. Denote

$$u_n := \frac{\beta^{s_n}}{(1 + \sum_{j=0}^{n-1} Q_j(1))^{d-1} (M(\beta))^{m_n}} \quad \text{for all } n \geq 1.$$

**Proof of (1.1):** from (3.3) assumed to be true there exists a sequence of integers $(n_i)$ which tends to infinity and an integer $i_0$ such that

$$\frac{s_{n_i}}{m_{n_i}} > \frac{\log(M(\beta))}{\log(\beta)} \quad \text{for all } i \geq i_0.$$
Now,
\begin{equation}
(3.11) \quad \left(1 + \left\lfloor \beta \right\rfloor m_{n_i} \right)^{d-1} \left( \frac{\beta^{s_{n_i}}}{M(\beta)} \right)^{m_{n_i}} \leq \frac{1}{\left(1 + \sum_{j=0}^{n_i-1} Q_j(1)\right)^{d-1}} \left( \frac{\beta^{s_{n_i}}}{M(\beta)} \right)^{m_{n_i}} \leq u_{n_i}.
\end{equation}

For \(i \geq i_0\) the inequality
\begin{equation}
(3.12) \quad 1 = \frac{\beta^{\log(M(\beta))}}{M(\beta)} \frac{\beta^{s_{n_i}}}{M(\beta)} < \beta^{s_{n_i}}
\end{equation}
holds. This implies that the left-hand side member of (3.11) tends exponentially to infinity when \(i\) tends to infinity. By (3.11) this forces \(u_{n_i}\) to tend to infinity. The contradiction now comes from (3.10) since the sequence \((u_n)\) should be uniformly bounded.

Proof of (1.2): for \(n = 1, 2, \ldots\), let us rewrite the \(n\)-th quotient
\begin{equation}
(3.13) \quad \frac{u_{n+1}}{u_n} = \frac{\beta^{s_{n+1} - s_n}}{M(\beta)^{m_{n+1} - m_n}} \frac{\left(1 + \sum_{j=0}^{n-1} Q_j(1)\right)^{d-1}}{\left(1 + \sum_{j=0}^{n} Q_j(1)\right)^{d-1}}
\end{equation}
as
\begin{equation}
(3.14) \quad \frac{u_{n+1}}{u_n} = \left( \frac{\beta^{m_{n+1} - m_n}}{M(\beta)} \right)^{m_{n+1} - m_n} \left[\frac{\left(1 + \sum_{j=0}^{n-1} Q_j(1)\right)^{d-1}}{\left(1 + \sum_{j=0}^{n} Q_j(1)\right)^{d-1}}\right]
\end{equation}
and denote
\begin{equation}
(3.15) \quad U_n := \frac{1}{(m_{n+1} - m_n + 1)^{(d-1)}} \left( \frac{\beta^{s_{n+1} - s_n}}{M(\beta)} \right)^{m_{n+1} - m_n}
\end{equation}
and
\begin{equation}
(3.16) \quad W_n := (m_{n+1} - m_n + 1)^{(d-1)} \left( \frac{1 + \sum_{j=0}^{n-1} Q_j(1)}{1 + \sum_{j=0}^{n} Q_j(1)} \right)^{d-1}
\end{equation}
so that \(u_{n+1}/u_n = U_n W_n\).
Lemma 3.2. —

(3.17) \[ 0 < \liminf_{n \to +\infty} W_n \]

Proof. — Assume the contrary. Then there exists a subsequence \((n_i)\) of integers which tends to infinity such that \(\lim_{i \to +\infty} W_{n_i} = 0\). In other terms, for all \(\epsilon > 0\), there exists \(i_1\) such that \(i \geq i_1\) implies \(W_{n_i} \leq \epsilon\), equivalently

(3.18) \[ (m_{n_i+1} - m_{n_i} + 1)(1 + \sum_{j=0}^{n_i-1} Q_j(1)) \leq \epsilon \frac{1}{\epsilon} \times (1 + \sum_{j=0}^{n_i} Q_j(1)). \]

Since, by hypothesis, \(t_{s_n} \geq 1\) and \(t_{m_{n+1}} \geq 1\) for all \(n \geq 1\), we have: \(n_i \leq 1 + \sum_{j=0}^{n_i-1} Q_j(1)\). On the other hand, \(Q_{n_i}(1) \leq \lfloor \beta \rfloor (m_{n_i+1} - m_{n_i} + 1)\). Then, from (3.18) with \(\epsilon\) taken equal to 1, we would have

(3.19) \[ n_i \leq 1 + \sum_{j=0}^{n_i-1} Q_j(1) \]

\[ \leq \frac{Q_{n_i}(1)}{(m_{n_i+1} - m_{n_i} + 1) - 1} \leq \lfloor \beta \rfloor \times \frac{m_{n_i+1} - m_{n_i} + 1}{m_{n_i+1} - m_{n_i}} \leq \frac{3}{2} \lfloor \beta \rfloor. \]

But the left-hand side member of (3.19) tends to infinity which is impossible. Contradiction.

Let us assume that (1.2) does not hold and show the contradiction; that is, assume that \(\liminf_{n \to +\infty} (m_{n+1} - m_n) = +\infty\) and \(\limsup_{n \to +\infty} (s_{n+1} - s_n)/(m_{n+1} - m_n) > \log(M(\beta))/\log(\beta)\) hold. Then

(3.20) \[ 1 = \beta \frac{\log(M(\beta))}{\log(\beta)} < \beta \frac{s_{n_i+1} - s_n}{m_{n_i+1} - m_{n_i}}. \]

for some sequence of integers \((n_i)\) which tends to infinity. This proves that \(\limsup_{n \to +\infty} U_n = +\infty\) since \(\lim_{i \to +\infty} U_{n_i} = +\infty\) exponentially, by (3.15) and (3.20).

By Lemma 3.2 there exists \(r > 0\) such that \(W_n \geq r\) for all \(n\) large enough. Therefore, \(u_{n+1}/u_n = U_n W_n \geq r U_n\) for all \(n\) large enough. Since \(\limsup_{n \to +\infty} U_n = +\infty\) we conclude that \(\limsup u_{n+1}/u_n = +\infty\), hence that \(\limsup u_n = +\infty\). This contradicts (3.10) and proves (1.2).

4. A direct proof of Corollary 1.2

Let \(\beta > 1\) be a Salem number such that \(\beta \notin C_1\). Using the notations of Theorem 1.1 we show that the assumption

(4.1) \[ \limsup_{n \to +\infty} \frac{s_n}{m_n} > 1 \]
leads to a contradiction.

Denote by $\mathbb{K}$ the algebraic number field $\mathbb{Q}(\beta)$, considered as a multivalued field with the product formula [15] [43] (see also [30]).

The present proof is merely an adaptation of that of Theorem 1 in [1], though the aims are different, and therefore does not merit publication. We simply point out a few hints for the interested reader.

The main result which is used is Corollary 1 of the Main Theorem in [15], as in [1]. This is a version of the Thue-Siegel-Roth Theorem given by Corvaja which is stronger than Roth Theorem for number fields [31] [43] to the extent it allows us to introduce a missing proportion of places of $\mathbb{K}$ by considering the projective approximation of the point at infinity in $\mathbb{P}^1(\mathbb{K})$. Since $\beta$ is a Salem number, it is a unit [4]. Hence, this missing proportion has just to be chosen among the pairwise distinct Archimedean places of $\mathbb{K}$.

5. On the class $Q_0$

5.1. Perron numbers

Let us give, after Solomyak ([44], p 483), the example of a Perron number which is not a beta-number and therefore which is not in the class $C_2$, without knowing whether it is in the class $Q_0$. This example allows us to estimate the sharpness of the upper bound $\log(M(\beta))/\log(\beta)$ in (1.1).

Recall that a real number $\beta > 1$ is a beta-number if the orbit of $x = 1$ under the transformation $T_\beta : x \to \beta x \pmod{1}$ is finite [34] [39]. The set of all conjugates of all beta-numbers is the union of the closed unit disc in the complex plane and the set of reciprocals of zeros of the function class $\{f(z) = 1 + \sum a_j z^j \mid 0 \leq a_j \leq 1\}$. The closure of this domain, say $\Phi$, is compact and was studied by Flatto, Lagarias and Poonen [20] and Solomyak [44]. After [44], the Perron number $\beta = \frac{1}{2}(1 + \sqrt{13})$, dominant root of $P_\beta(X) = X^2 - X - 3$, is not a beta-number, though its only conjugate $\beta' = \frac{1}{2}(1 - \sqrt{13})$ lies in the interior $\text{int}(\Phi)$. We have $M(\beta) = 3$. By Theorem 1.1 the quotients of the gaps are asymptotically bounded above by $\log(3)/\log(\beta) = 1.3171\ldots$, a much better bound than the degree $d = 2$ of $\beta$ (see Lemma 5.1). This does not suffice to conclude that $\frac{1}{2}(1 + \sqrt{13})$ belongs to $Q_0$.

Do all Perron numbers belong to $Q_0$? Let $\beta > 1$ be a Perron number of degree $d \geq 2$ and denote by $\beta^{(1)}, \beta^{(2)}, \ldots, \beta^{(d-1)}$ the conjugates of $\beta = \beta^{(0)}$, roots of the minimal polynomial $P_\beta(X)$ of $\beta$. Let $K_\beta := \max\{|\beta^{(i)}| \mid i = 1, 2, \ldots, d - 1\}$. 

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Lemma 5.1. — Let \( n = n_\beta \) (with \( 2 \leq n_\beta \leq d \)) be the number of conjugates of \( \beta \) of modulus strictly greater than unity (including \( \beta \)). Then

\[
\frac{\log(M(\beta))}{\log(\beta)} \leq n - \frac{n - 1}{(d\beta)^{6d^3} \log \beta}.
\]

Proof. — Obvious since (Lemma 2 in [33]): \( K_\beta < \beta (1 - \frac{1}{(d\beta)^{6d^3}}) \). \( \square \)

The upper bound (5.1) does not allow us to give a positive answer to the question and has probably to be improved.

5.2. Transcendental numbers

Let us show that the Komornik-Loreti constant \( [2] [29] \) belongs to \( Q^{(1)}_0 \).

Theorem 5.2. — There exists a smallest \( q \in (1, 2) \) for which there exists a unique expansion of 1 as \( 1 = \sum_{n=1}^{\infty} \delta_n q^{-n} \), with \( \delta_n \in \{0, 1\} \). Furthermore, for this smallest \( q \), the coefficient \( \delta_n \) is equal to 0 (respectively, 1) if the sum of the binary digits of \( n \) is even (respectively, odd). This number \( q \) can then be obtained as the unique positive solution of \( 1 = \sum_{n=1}^{\infty} \delta_n q^{-n} \). It is equal to 1.787231650... .

This constant \( q \) is named Komornik-Loreti constant. Allouche and Cosnard [2] have shown the following result.

Theorem 5.3. — The constant \( q \) is a transcendental number, where the sequence of coefficients \( (\delta_n)_{n \geq 1} \) is the Prouhet-Thue-Morse sequence on the alphabet \( \{0, 1\} \).

The uniqueness of the development of 1 in base \( q \) given by Theorem 5.2 allows us to write

\[
d_q(1) = 0.\delta_1 \delta_2 \delta_3 \ldots ,
\]
the coefficients \( \delta_n \) being the digits of the Rényi \( q \)-expansion of 1. Since the strings of zeros and 1’s in the Prouhet-Thue-Morse sequence are known (Thue, 1906/1912; [3]) and uniformly bounded, the constant \( q \) belongs to the class \( Q^{(1)}_0 \).

As second example, let us show that Sturmian numbers in the interval (1, 2) (in the sense of [14]) belong to \( Q^{(1)}_0 \).

A real number \( \beta > 1 \) is called a Sturmian number if \( d_\beta(1) \) is a Sturmian word over a binary alphabet \( \{a, b\} \), with \( 0 \leq a < b = \lfloor \beta \rfloor \). Chi and Kwon [14] have shown the following theorem.

Theorem 5.4. — Every Sturmian number is transcendental.
Let us consider all the Sturmian numbers $\beta \in (1, 2)$ for which the two-letter alphabet is $\{0, 1\}$. For such numbers gappiness appears in $d_\beta(1)$ (in the sense of Theorem 1.1). By Theorem 3.3 in [14] strings of zeros, resp. of 1’s, cannot be arbitrarily long. This gives the claim.

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BIBLIOGRAPHY


Jean-Louis VERGER-GAUGRY
Université de Grenoble I
Institut Fourier
UMR CNRS 5582
BP 74 - Domaine Universitaire,
38402 Saint-Martin d’Hères (France)
jlverger@ujf-grenoble.fr