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Finite determinacy of dicritical singularities in \((\mathbb{C}^2, 0)\)


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FINITE DETERMINACY OF DICRITICAL SINGULARITIES IN \((\mathbb{C}^2, 0)\)

by Gabriel CALSAMIGLIA-MENDLEWICZ (*)

Abstract. — For germs of singularities of holomorphic foliations in \((\mathbb{C}^2, 0)\) which are regular after one blowing-up we show that there exists a functional analytic invariant (the transverse structure to the exceptional divisor) and a finite number of numerical parameters that allow us to decide whether two such singularities are analytically equivalent. As a result we prove a formal-analytic rigidity theorem for this kind of singularities.

Résumé. — Nous montrons l’existence d’un invariant analytique fonctionnel (la structure transverse au diviseur exceptionnel) et d’un nombre fini de paramètres numériques associés aux germes de feuilletages holomorphes dans \((\mathbb{C}^2, 0)\) qui ne présentent pas de singularités après un éclatement. Ceux-ci permettent de décider si deux telles singularités sont analytiquement équivalentes. On dérive ensuite un théorème de rigidité formelle-analytique pour ce type de singularité.

1. Introduction

Given a holomorphic germ of 1-form \(\omega\) in \((\mathbb{C}^2, 0)\) with an isolated zero at the origin we can define its associated singular foliation by holomorphic curves \(F_\omega\): the origin 0 is the singular set and its leaves are the integral curves of \(\omega\) outside 0. Let \(E : \tilde{\mathbb{C}}^2 \to (\mathbb{C}^2, 0)\) denote the quadratic blow up at the origin expressed in coordinates by \(E(t, x) = (x, tx) = (X, Y)\), and \(E_0\) its exceptional divisor corresponding to the set \(\{x = 0\}\) in the chart \((t, x)\).

It is well known that \(E^* (\omega)\) defines a regular foliation in the complement of \(E_0\) which can be uniquely extended to a holomorphic foliation \(\tilde{F}_\omega\) in a neighborhood of \(E_0\) in \(\tilde{\mathbb{C}}^2\) with a finite set of isolated singularities on \(E_0\). Let \(D_0\) denote the set of foliations \(F_\omega\) such that \(\tilde{F}_\omega\) is a regular foliation. We

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are interested in describing the space of analytic equivalence classes of $D_0$. The index theorem in ([3], p.592) forces $E_0$ to be generically transverse to $\tilde{F}_\omega$, so $F_\omega$ has a dicritical singularity. However, there is a finite set of points $\Sigma_{\tilde{F}_\omega} \subset E_0$ corresponding to the points $p \in E_0$ where the leaf of $\tilde{F}_\omega$ through $p$ is tangent to the curve $E_0$ with contact order $r(p) + 1$. A result by Klughertz [7] asserts that topologically there aren’t any other invariants:

**Theorem 1.** — Given $\mathcal{F}, \mathcal{F}' \in D_0$, there exists a homeomorphism $\Psi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ sending leaves of $\mathcal{F}$ to leaves of $\mathcal{F}'$ if and only if there exists a bijection $\psi : \Sigma_\mathcal{F} \to \Sigma_{\mathcal{F}'}$ such that $r(\psi(p)) = r(p)$ for all $p \in \Sigma_\mathcal{F}$.

In other words the partition of $D_0$ into subsets whose elements are topologically equivalent can be described as $D_0(n; r_1, \ldots, r_n)$ where $n \in \mathbb{N}$ denotes the number of points of tangency, and $r_1, \ldots, r_n \in \mathbb{N}^*$ their orders of tangency when $n \neq 0$. The case $n = 0$ is solved by Poincaré’s linearization theorem: every $F \in D_0(0)$ is analytically equivalent to the radial foliation $\mathcal{F}_{YdX-XdY}$. Suzuki’s example (see Section 2 below or [13]) shows that there are two elements in $D_0(1; 1)$ which are not analytically equivalent. The obstruction is related to the following analytic invariant: given $F \in D_0(n; r_1, \ldots, r_n)$, consider for each $p_i \in \Sigma_F$ a local holomorphic first integral $F_i : U_i \to \mathbb{C}$ of $\tilde{F}$ in a neighborhood $U_i$ of $p_i$ in $\tilde{\mathbb{C}}^2$.

Define $f_i = F_i|_{U_i \cap E_0}$. The group of invariance of $F$ at $p_i$ is

$$H(\mathcal{F}, p_i) = \{ h \in \text{Diff}(E_0, p_i) \mid f_i \circ h = f_i \}.$$ 

It is a cyclic group of germs of order $r(p_i) + 1$. We define the transverse structure of $\mathcal{F}$ as

$$H(\mathcal{F}) = \bigcup_{p \in \Sigma_\mathcal{F}} H(\mathcal{F}, p).$$

Observe that if $\Psi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is a biholomorphism then $\Psi^*(\mathcal{F}) \in D_0$ and the restriction $\psi \in \text{Aut}(E_0)$ of $\Psi$ to $E_0$ defines a bijection

$$h \mapsto \psi^{-1} \circ h \circ \psi$$

from $H(\mathcal{F})$ to $H(\Psi^*(\mathcal{F}))$. We thus define the projective class of the transverse structure $H[\mathcal{F}]$ as the conjugacy class of $H(\mathcal{F})$ by holomorphic automorphisms of $E_0$, which are just Möbius transformations. The previous argument shows that $H[\mathcal{F}]$ depends only on the analytic class $[\mathcal{F}]$ of $\mathcal{F}$, and also that if $\Psi^*(\mathcal{F}) = \mathcal{F}'$, then up to linear conjugacy we can suppose $H(\mathcal{F}') = H(\mathcal{F})$. On the other hand, the fact that elements in $D_0$ can be constructed using foliated surgery techniques (and Grauert’s Theorem)
allows us to realize any finite union of cyclic groups of germs of diffeomorphisms of \((E_0, p)\) at points \(p \in E_0\) of finite order as the transverse structure of an element of \(D_0\).

A natural question is to decide whether the projective class of the transverse structure determines the analytic class of the foliation completely. In the case of \(D_0(1; 1)\) the answer is positive:

**Theorem 2** (Cerveau). — Given \(\mathcal{F}, \mathcal{F}' \in D_0(1; 1)\), there exists a germ of biholomorphism \(\Psi : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)\) with \(\Psi^*(\mathcal{F}') = \mathcal{F}\) if and only if \(H[\mathcal{F}] = H[\mathcal{F}']\).

In the remaining cases we provide examples showing that there are elements \(\mathcal{F}, \mathcal{F}' \in D_0\) with \(H(\mathcal{F}) = H(\mathcal{F}')\) and \(#H(\mathcal{F}) > 2\) arbitrary which are not analytically equivalent. Our main result states that, apart from the projective class of the transverse structure, there are at most a finite number of analytic invariants of numerical nature in each topological class \(D_0(n; r_1, \ldots, r_n)\):

**Theorem 3.** — Let \(\omega, \omega'\) be two holomorphic 1-forms in \((\mathbb{C}^2, 0)\) defining foliations \(\mathcal{F}, \mathcal{F}' \in D_0(n; r_1, \ldots, r_n)\) respectively. Define

\[
N := r_1 + \cdots + r_n, \quad \kappa := (N + 1) + \max\{r_i\}(3N - 2).
\]

Suppose

(i) \(H(\mathcal{F}) = H(\mathcal{F}')\);

(ii) the jets of \(\omega\) and \(\omega'\) at 0 satisfy \(j_0^\kappa(\omega) = j_0^\kappa(\omega')\).

Then there exists a biholomorphism \(\Psi : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)\) such that

\[
\Psi^*(\mathcal{F}') = \mathcal{F} \quad \text{and} \quad d\Psi(0, 0) = \text{Id}.
\]

As a corollary we get a theorem of formal-analytic rigidity in \(D_0\):

**Corollary 4.** — Two elements in \(D_0\) are formally equivalent if and only if they are analytically equivalent.

Since the algebraic multiplicity of the elements in \(D_0(n; r_1, \ldots, r_n)\) is \(N + 1\), we can state Theorem 3 in terms of the algebraic multiplicity instead of \(\kappa\). This theorem can be reinterpreted in the following way: fix a 1-form \(\omega\) such that \(\mathcal{F}_\omega \in D_0\). Consider the set

\[
(1.1) \quad D_0[\mathcal{F}_\omega] := \{\omega' \mid \mathcal{F}_\omega' \in D_0 \text{ and } H(\mathcal{F}_\omega) = H(\mathcal{F}_\omega')\}.
\]

The assertion is that each fiber of the map

\[
j_0^\kappa : D_0[\mathcal{F}_\omega] \longrightarrow \mathbb{C}^M \quad \text{where} \quad M := \binom{\kappa + 1}{2} - \binom{N + 1}{2}
\]
defines a unique equivalence class in $\mathcal{D}_0$. Different fibers might define the same class or not. Nevertheless we have at most $C^M$ different analytic classes with the same transverse structure.

A theorem of finite determinacy of a similar type was proven by Klughertz [7] in her doctoral thesis. Our approach improves the order of the jet involved (in her statement the dependence is quadratic on $N$). On the other hand the methods used for the proof differ. We will geometrically construct a biholomorphism by choosing adequate generically transverse auxiliary foliations, whereas Klughertz used the cohomological methods developped in [8] to find a biholomorphism which is tangent to the identity up to a certain order.

Ortiz-Bobadilla, Rosales-Gonzalez and Voronin [10] have recently proved a formal-analytic rigidity theorem in $\mathcal{D}_0(n; 1, \ldots, 1)$, after what formal normal forms are constructed for the analytic classes in $\mathcal{D}_0(n; 1, \ldots, 1)$, and formal invariants are identified from this normal form. Again, the process lacks of a geometric interpretation, and we hope that our approach will shed some light on the problem of identifying the invariants and giving them a geometrical meaning, eventually enabling us to construct normal forms for every analytic class in $\mathcal{D}_0$.

Finally, we want to remark that a finite determinacy theorem for generalized cusps can be proven as a consequence of Theorem 3 and that the proof we present of the latter can be generalized to prove a theorem of finite determinacy for regular germs of holomorphic foliations defined in a neighborhood of the zero section of a Hopf component of negative auto-intersection having the same transverse structure (for a proof of these results see [1]). Similarly, Theorem 2 can be generalized to give the analytic classification of regular germs of foliations in a neighborhood of the zero section of a line bundle $L \to E$ over a rational or elliptic Riemann surface $E$ having a single simple tangency with the curve $E$, provided that $c_1(L) < -1$. Both generalizations, and the method for the proof of these results, are inspired by the paper [2].

The content of the article is organized as follows: in Section 2 we provide examples of singularities in $\mathcal{D}_0$ that are not analytically equivalent; in Section 3 we prove Theorem 3 and in Section 4 we prove Corollary 4.

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2. Examples

Let us first establish some definitions that will be used throughout this article. For \( \mathcal{F} \in \mathcal{D}_0 \) we define \( \Sigma_{\mathcal{F}} \) as the set of points \( p \in E_0 \) where the leaf \( \tilde{L}_p \) of \( \tilde{\mathcal{F}} \) is tangent to \( E_0 \) at \( p \). The order of contact at \( p \) will be denoted by \( r(p) \) + 1; remark that \( r(p) \) is the order of the zero of the normal component to \( E_0 \) of the local 1-form defining \( \tilde{\mathcal{F}} \) at \( p \). Define

\[
\text{Sep}(\tilde{\mathcal{F}}) = \bigcup_{p \in \Sigma_{\mathcal{F}}} \tilde{L}_p
\]
as the set of \textit{isolated separatrices}. The blow down of \( \text{Sep}(\tilde{\mathcal{F}}) \) by \( E \) will be denoted as \( \text{Sep}(\mathcal{F}) \). The latter is a union of germs of generalized cusps. Observe that the foliation \( \tilde{\mathcal{F}} \) is the minimal resolution of \( \mathcal{F} \). Nevertheless we will repeatedly use a different resolution \( S_{\mathcal{F}} \) that will call extended resolution which corresponds to the resolution of \( \text{Sep}(\mathcal{F}) \) in the sense of (reducible) curves. It is the result of composing \( E \) with \( S_{(p,r(p)+1)} \) at each point \( p \in \Sigma_{\mathcal{F}} \), where \( S_{(p,r(p)+1)} \) is defined inductively by the rules: \( S_{(p,1)} \) is the blowing-up of the point \( p \), and

\[
S_{(p,i)} = S_{(p,i-1)} \circ S_{(p_1,1)} \quad \text{where} \quad \hat{p}_i = S_{(p,i-1)}^{-1}(p) \cap S_{(p,i-1)}^{-1}(E_0 \setminus p).
\]

For each \( p \) we have \( r(p) + 1 \) irreducible components \( E_1^p, \ldots, E_{r(p)+1}^p \) of the divisor \( \mathcal{D}_{\mathcal{F}} \) associated to \( S_{\mathcal{F}} \). The strict transform of each irreducible component of \( \text{Sep}(\mathcal{F}) \) by \( S_{\mathcal{F}} \) intersects transversely exactly one irreducible component of \( \mathcal{D}_{\mathcal{F}} \). We call \( \tilde{\mathcal{F}} = S^*_S(\mathcal{F}) \) the pull back of \( \mathcal{F} \) by \( S_{\mathcal{F}} \).

Next observe that for a point \( p \in E_0 \) we can find a neighborhood \( U_p \subset \mathbb{C}^2 \) and a local biholomorphism \( \Phi_p : (U_p, p) \rightarrow (\mathbb{C}^2, 0) \) which call normalizing chart of \( \tilde{\mathcal{F}} \) at \( p \) such that \( (u, v) = \Phi_p(t, x) = (\Phi_1(t, x), \Phi_2(t, x)) \) with \( \Phi_2(t, 0) \equiv 0 \) and such that \( (\Phi_p^{-1})^*(\tilde{\mathcal{F}}|_{U_p}) \) can be described as the levels of the function \( f_p(u, v) = v - u^{r(p)+1} \). It is important to remark that the change of coordinates is local. With this at hand it is obvious that for a point \( p \in \Sigma_{\mathcal{F}} \) the group of invariance \( H(\mathcal{F}, p) \) is cyclic of order \( r(p) + 1 \); in fact \( \Phi_1(t, 0) \) conjugates it with the group of rotations of order \( r(p) + 1 \).

In particular if we choose any two elements \( \mathcal{F}, \mathcal{F}' \in \mathcal{D}_0(n; r_1, \ldots, r_n) \) with \( \Sigma_{\mathcal{F}} = \{p_1, \ldots, p_n\} \) and \( \Sigma_{\mathcal{F}'} = \{p'_1, \ldots, p'_n\} \), \( r(p_i) = r(p'_i) \), we can find germs of biholomorphism \( \psi_i : (E_0, p_i) \rightarrow (E_0, p'_i) \) conjugating \( H(\mathcal{F}, p_i) \) with \( H(\mathcal{F}', p'_i) \). However, in general, there does not exist an automorphism \( \psi \) of \( E_0 \) whose restriction to a neighborhood of \( p_i \) is \( \psi_i \) for \( i = 1, \ldots, n \), even in the case \( n = 1 \). In these cases \( \mathcal{F} \) and \( \mathcal{F}' \) cannot be analytically conjugated, for the existence of an equivalence would imply the existence...
of a $\psi$ with the said properties. Suzuki’s example is an instance of this phenomenon: define

$$\omega = (2Y^2 + X^3) dX - 2XY dY,$$
$$\omega' = (Y^3 + Y^2 - XY) dX - (2XY^2 + XY - X^2) dY.$$ 

They have first integrals

$$f(X,Y) = \frac{Y^2 - X^3}{X^2}, \quad f'(X,Y) = \frac{X}{Y} e^{Y(Y+1)/X}$$

respectively, and define foliations $\mathcal{F}_\omega, \mathcal{F}_{\omega'} \in \mathcal{D}_0[1; 1]$. In the $(t,x)$ chart of $\tilde{\mathbb{C}}^2$ we have $\Sigma_{\mathcal{F}_\omega} = \{(0,0)\}$ and $\Sigma_{\mathcal{F}_{\omega'}} = \{(1,0)\}$; $H(\mathcal{F}_\omega, (0,0)) = \langle h \rangle$ with $h(t) = -t$ and $H(\mathcal{F}_{\omega'}, (1,0)) = \langle h' \rangle$. Consider the maps

$$H : (\mathbb{C},0) \to \mathbb{C}^2, \quad t \mapsto (t, h(t)),$$
$$H' : (\mathbb{C},1) \to \mathbb{C}^2, \quad t \mapsto (t, h'(t))$$

and define $C = \text{Im} H$, $C' = \text{Im} H'$. $C$ is algebraic in $\mathbb{C}^2$. Suppose there exists $\psi \in \text{Aut}(E_0)$ such that $H'(t + 1) = (\psi, \psi) \circ H(t)$. Recall that $\psi$ is a rational function of $t$, so $C'$ should also be algebraic. However $C'$ is not algebraic (see [6] or [1]). Using this kind of argument it is possible to give necessary and sufficient conditions to decide which elements in $\mathcal{F} \in \mathcal{D}_0$ admit a meromorphic first integral (see [12]). In fact the conditions depend only on $H[\mathcal{F}]$.

When $\# H(\mathcal{F}) = 2$ the correspondence $[\mathcal{F}] \mapsto H[\mathcal{F}]$ is injective (see Theorem 2). In the remaining cases this is no longer true. Mattei [8] showed that locally (in the sense of unfoldings) there exists a vector space of dimension $\frac{1}{2} N(N - 1)$ of analytic classes once we have fixed a transverse structure.

In next paragraph we construct some explicit families of counterexamples which give a clear idea of the kind of obstructions that appear.

The first family is related to the fact that $H(\mathcal{F})$ does not determine the analytic class of $\text{Sep}(\mathcal{F})$. Fix $n \geq 2$. In the $(t,x)$ chart of $\tilde{\mathbb{C}}^2$ choose points $p_i = (t_i, 0)$ and $r_i \in \mathbb{N}^*$ for $i = 1, \ldots, n$. Define

$$P(t) = \int (t - t_1)^{r_1} \cdots (t - t_n)^{r_n} \, dt$$

with $P(0) = 0$. In the same chart consider foliations

$$(2.1) \quad \tilde{\mathcal{F}} = \{ P(t) + x = C^{te} \}, \quad \tilde{\mathcal{F}}' = \{ P(t) + x(1 + (t - t_1)) = C^{te} \}.$$ 

They extend to regular holomorphic foliations in $\tilde{\mathbb{C}}^2$, and we call $\mathcal{F} = E^{-1*}(\tilde{\mathcal{F}})$ and $\mathcal{F}' = E^{-1*}(\tilde{\mathcal{F}}')$ the singular foliations they define in $(\mathbb{C}^2,0)$ after implosion. They admit meromorphic first integrals in $(X,Y)$ with common denominator $X^{r_1 + \cdots + r_n + 1}$. From (2.1) we deduce that $\mathcal{F}, \mathcal{F}' \in \mathcal{D}_0[0; 1]$.
Let $D_0(n;r_1,\ldots,r_n)$ with $\Sigma_F = \Sigma_{F'} = \{p_1,\ldots,p_n\}$, and by the definition of the transverse structure $H(F) = H(F')$.

Lemma 2.1. — One has $[F] \neq [F']$ for a generic choice of $p_i$'s.

Proof. — If $n > 2$, generically in the choice of $p_i$'s we have that
\begin{equation}
\text{Aut}(H(F)) := \{ \varphi \in \text{Aut}(E_0) \mid \varphi H(F) \varphi^{-1} = H(F) \} = \{\text{Id}\}.
\end{equation}
Suppose $\Psi : (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ is a biholomorphism such that $\Psi^*(F') = F$. From $H(F) = H(F')$ and (2.2) we deduce that there exists $\lambda \in \mathbb{C}^*$ such that $d\Psi(0,0) = \lambda \text{Id}$. We also have that $\Psi(\text{Sep}(F)) = \text{Sep}(F')$, but by the choices made in (2.1), it is easily seen by studying the action of $\Psi$ on the divisor of the extended resolution of $F$ and $F'$ and the position of the points of intersection of $\text{Sep}(F)$ with it that there is no possible value for $\lambda$. In the case $n = 2$, $\text{Aut}(H(F))$ consists of two elements, but generically the case where $\Psi_0 \neq \text{Id}$ is excluded by a similar argument.

Motivated by this proof we establish the following definition:

Definition 2.2. — Given $F \in D_0(n;r_1,\ldots,r_n)$ and two families $P,Q$ of $n$ points in $D_F$ we say that $P \sim Q$ if and only if there exists a biholomorphism $\Psi : (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ whose lifting $\hat{\Psi}$ to a neighborhood of $D_F$ satisfies $\hat{\Psi}|_{E_0} = \text{Id}$ and $\hat{\Psi}(P) = Q$. Define $Q_F \subset D_F$ as the set of $n$ singularities of $\hat{F}$ which are not corners of $D_F$ and $q(F) := [Q_F]$ its class by the equivalence relation $\sim$.

Observe that, although $q(F)$ is not an analytic invariant of $F$, it is invariant by the subgroup of biholomorphisms which fix every point in $E_0$. By using coordinates it is easily seen that the space of classes of points of type $Q_F$ is isomorphic to a subset of $\mathbb{CP}^{n-1}$. On the other hand, $q(F)$ depends only on $\text{Sep}(F)$.

The second family of examples shows that even fixing $H(F)$ and $\text{Sep}(F)$ there are analytically different elements in $D_0$. Fix $r \geq 3$ and consider
\begin{align*}
F &= \{ f := (X^{r+1} + Y^r)/X^r = C^{\text{te}} \}, \\
F' &= \{ f(X,Y) \cdot (1 + X) = C^{\text{te}} \}
\end{align*}
contained in $D_0(1;r-1)$. After one blowing up we have
\begin{align*}
\hat{F} &= \{ \hat{f} := x + t^r = C^{\text{te}} \}, \\
\hat{F}' &= \{ \hat{f}' := (x + t^r)(1 + x) = C^{\text{te}} \}.
\end{align*}
Clearly $H(F) = H(F')$, $\text{Sep}(F) = \text{Sep}(F')$ and $\text{Aut}(H(F))$ is the set of nonzero homotetias in the $t$ variable. Suppose there exists a biholomorphism $\Psi : (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ such that $\Psi^*(F) = F'$. From the previous
facts we get that after blowing up \( \mathbb{C}^2 \) at 0, \( \Psi \) lifts to
\[
\tilde{\Psi}(t, x) = (\lambda t + x\phi_1(t, x), x(\mu + \phi_2(t, x)))
\]
for some holomorphic functions \( \phi_1, \phi_2 \) defined in a neighborhood of \( E_0 \setminus \infty \) and \( \lambda, \mu \in \mathbb{C}^* \). Since \( \Psi \) conjugates the foliations we have
\[
\tilde{f}(\tilde{\Psi}(t, x)) = \lambda^r \cdot \tilde{f}'(t, x).
\]
From this last equation we get
\[
(2.3) \quad x\phi_2(t, x) = \lambda^r(x + t^r)(1 + x) - (\lambda t + x\phi_1(t, x))^r - \mu x.
\]
Now from the fact that \( \tilde{\Psi} \) is the lifting of \( \Psi \) we have that the left hand side of equation (2.3) must be a series of the form \( \sum_{i \geq 1} A_i(t)x^i \) where \( A_i \) are polynomials in \( t \) and \( \deg A_i \leq i \). Thus, to eliminate the \( t^r x \) term we need \( \phi_1(t, 0) = \lambda t \), but this will produce a nonzero term in the \( t^r x^2 \)-monomial which cannot be cancelled with any other term of the right hand side of equation (2.3), producing a contradiction. Hence \([F] \neq [F']\).

We are thus interested in determining other analytic invariants. This is quite a difficult problem even for the reducible curves \( \text{Sep}(F) \) associated to \( F \in D_0 \), for which, except in some cases (see [15], [14]), a complete list of analytic invariants is unknown.

In the examples above a short calculation shows that, denoting by \( \omega \) and \( \omega' \) the forms defining the foliations \( F \) and \( F' \) we have that
\[
j^{N+1}(\omega) = j^{N+1}(\omega') \quad \text{but} \quad j^{N+2}(\omega) \neq j^{N+2}(\omega').
\]
In the case of germs of curves we know that there is finite determinacy (see [6]): if two equations of such germs coincide up to a sufficiently high order, they are analytically equivalent. Our approach is to prove a theorem of finite determinacy in \( D_0(n; r_1, \ldots, r_n) \) (see Theorem 3) whose proof is given in the following section.

3. Proof of Theorem 3

Consider \( F = F_\omega \in D_0(n; r_1, \ldots, r_n) \) and suppose, without loss of generality, that \( \Sigma_F = \{p_1, \ldots, p_n\} \) is contained in the \((t, x)\) chart of \( \tilde{\mathbb{C}}^2 \). Denote by \( p_\infty = (\infty, 0) \) the point at infinity in this chart. Let \( C_i \) be the separatrix whose strict transform \( \tilde{C}_i \) passes through \( p_i \). Take irreducible Weierstrass polynomials in \( Y, f_i(X, Y) \) such that \( C_i = \{f_i = 0\} \) for \( i = 1, \ldots, n, \infty \), and a unit \( \phi \in \mathcal{O}^*_\omega(\mathbb{C}^2, 0) \). Define
\[
N = r_1 + \cdots + r_n, \quad F := \prod_{i=1}^n f_i
\]
and the meromorphic function in \((\mathbb{C}^2, 0)\)
\[
g = \frac{f_{\infty}^N + n + 1}{F} \cdot \phi.
\]
It defines a germ of holomorphic foliation \(\mathcal{G} = \{g = \text{Cte}\}\) with a dicritical singularity at 0 whose blowing up
\[
\tilde{\mathcal{G}} := E^*(\mathcal{G}) = \{\tilde{g} := g \circ E = \text{Cte}\}
\]
has the following properties:

**Lemma 3.1.** —
(i) \(E_0, \tilde{C}_1, \ldots, \tilde{C}_n, \tilde{C}_\infty\) are invariant by \(\tilde{G}\);
(ii) \(\text{Sing}(\tilde{G}) = \{p_1, \ldots, p_n, p_\infty\}\);
(iii) \(\tilde{G}\) is dicritic at \(p_1, \ldots, p_n\) and has a saddle with local holomorphic first integral and index \(-1/(N + n + 1)\) at \(p_\infty\);
(iv) the holonomy of \(\tilde{G}\) at \(p_i\) along \(E_0\) is trivial.

Moreover, for a generic choice of unit \(\phi\), we have that the set \(\text{tang}(\tilde{F}, \tilde{G})\) of tangencies between \(\tilde{F}\) and \(\tilde{G}\) satisfies
\[
\text{tang}(\tilde{F}, \tilde{G}) = \tilde{C}_\infty + \sum_{i=1}^{n} (\tilde{C}_i + \tilde{T}_i)
\]
where \(\tilde{T}_i\) is a regular irreducible analytic set tangent to \(E_0\) at \(p_i\) with order \(r_i = r(p_i)\) of contact, when \(\tilde{C}_i\) and \(E_0\) are tangent with contact \(r_i + 1\) at \(p_i\).

In this case we will say that \((\mathcal{F}, \mathcal{G})\) (or \((\tilde{\mathcal{F}}, \tilde{\mathcal{G}})\)) are companion foliations (see Figure 3.1 for diagrams).

![Figure 3.1. Companion foliations \((\tilde{\mathcal{F}}, \tilde{\mathcal{G}})\) with \(r(p_1) = 1, r(p_2) > 1\).](image)

**Proof.** — For the proofs of (i)–(iv) it suffices to say that
\[
\text{div}(\tilde{g}) = (N + n + 1)\tilde{C}_\infty + E_0 - \sum_{i=1}^{n} \tilde{C}_i.
\]

For the proof of (3.1) observe that \(\tilde{F}\) and \(\tilde{G}\) are transverse at all \(p \in E_0 \setminus \{p_1, \ldots, p_n, p_\infty\}\). At \(p_\infty\), \(\tilde{C}_\infty\) is a common separatrix of \(\tilde{F}\) and \(\tilde{G}\) and
since \( p_{\infty} \) is a saddle for \( \tilde{G} \) there are no other components of \( \text{tang}(\tilde{F}, \tilde{G}) \) there.

For \( i = 1, \ldots, n \), consider a normalizing chart for \( \tilde{F} \) around \( p_i : (t, x) \), where \( p_i = (0, 0) \). In this chart \( \tilde{F} \) is expressed as the levels of \( f(t, x) = x - t^{r_i+1} \) and \( \tilde{G} \) as the levels of

\[
g(t, x) = \frac{x}{x - t^{r_i+1}} \cdot \nu(t, x)
\]

where \( \nu \) is a unit depending on \( \phi \) and the normalizing chart. Hence the components of tangency are described in this chart by the expression

\[
0 = df \wedge ((x - t^{r_i+1})^2 dg)
= (x - t^{r_i+1})[x\partial_t \nu + (r_i + 1)t^{r_i}(\nu + x\partial_x \nu)]dx \wedge dt.
\]

For generic values of \( j^1(\nu) \) (which depend on generic values of \( j^1(\phi) \)) the set \( \{x\partial_t \nu + (r_i + 1)t^{r_i}(\nu + x\partial_x \nu) = 0\} \) is regular at \( (0, 0) \) and has contact \( r_i \) with the set \( \{x = 0\} \). Since the inverse of the normalizing chart sends \( \{x = 0\} \) to \( E_0 \) and preserves orders of tangency we get, after (3.2), two components of \( \text{tang}(\tilde{F}, \tilde{G}) \) at \( p_i, \tilde{C} \); and \( \tilde{T} \), with the required properties.

Let \( (F, G) \) and \( (F', G') \) be two pairs of companion foliations where \( G = \{g = C^{\text{te}}\} \) and \( G' = \{g' = C^{\text{te}}\} \) associated to \( F, F' \in D_0(n; r_1, \ldots, r_n) \) with \( \Sigma_F = \Sigma_{F'} \). For a point \( p \in E_0 \setminus \{\Sigma_F, p_{\infty}\} \), consider holomorphic first integrals \( F_p, F'_p : (U_p, p) \to (\mathbb{C}, 0) \) of \( \tilde{F} \) and \( \tilde{F}' \) respectively defined in a small neighborhood \( U_p \) of \( p \). By transversality and Lemma 3.1, for each \( q \in U_p \) there is a unique point \( \Psi_p(q) \in U_p \) such that

\[
F'_p(\Psi_p(q)) = F_p(q), \quad g'(\Psi_p(q)) = g(q).
\]

By holomorphicity of all the foliations under consideration, \( \Psi_p : (U_p, p) \to (U_p, p) \) defines a germ of biholomorphism. If \( p, p' \in E_0 \setminus \{\Sigma_F, p_{\infty}\} \) and \( U_p \cap U_{p'} \neq \emptyset \) then \( \Psi_p \equiv \Psi_{p'} \) on \( U_p \cap U_{p'} \). Thus we have a biholomorphism \( \Psi : U \to U \) from a neighborhood \( U \) of \( E_0 \setminus \{\Sigma_F, \infty\} \) in \( \mathbb{C}^2 \) to itself. The properties of \( G \) and \( G' \) at \( p_{\infty} \) and its relations with \( F \) and \( F' \) allow us to extend \( \Psi \) to a neighborhood of \( p_{\infty} \) as a biholomorphism by using a theorem of Mattei and Moussu (see [9], p. 482). The following lemma provides necessary and sufficient conditions for \( \Psi \) to extend to neighborhoods of all points of \( \Sigma_F \):

**Lemma 3.2.** — \( \Psi \) extends to a neighborhood of \( E_0 \) as a biholomorphism if and only if the following conditions are fulfilled:

(a) \( H(F) = H(F') \);
(b) \( q(F) = q(F') \);
(c) there exist homeomorphisms $\psi_i : T_i \rightarrow T'_i$ between the irreducible components of $\text{tang}(\mathcal{F}, \mathcal{G})$ and $\text{tang}(\mathcal{F}', \mathcal{G}')$ not invariant by any of the foliations passing through $p_i \in \Sigma_\mathcal{F}$ such that

$$F'_i(\psi_i(q)) = F_i(q), \quad g'(\psi_i(q)) = g(q)$$

for all $q \in T_i$ and local first integrals $F_i$ and $F'_i$ of $\hat{\mathcal{F}}$ and $\hat{\mathcal{F}}'$ respectively around $p_i$ whose restriction to $E_0$ coincide.

Proof. — Suppose first that $\Psi$ extends to a neighborhood of $E_0$ as a biholomorphism. Then it is the identity on $E_0$ and we have already seen that this implies (a) and (b). For the proof of (c) it suffices to observe that $\Psi(T_i) = T'_i$, take $\psi_i \equiv \Psi|_{T_i}$ and apply the equations (3.3) defining $\Psi$ in the neighborhood of $p_i$.

For the converse, we have to consider the extended resolution $S_\mathcal{F}$ of $\mathcal{F}$ and $\mathcal{F}'$. Call $\hat{\mathcal{F}} = S^*_\mathcal{F}(\mathcal{F})$, $\hat{\mathcal{F}}' = S^*_\mathcal{F}(\mathcal{F}')$, $\hat{\mathcal{G}} = S^*_\mathcal{F}(\mathcal{G})$, $w\hat{\mathcal{G}}' = S^*_\mathcal{F}(\mathcal{G}')$ and $D_\mathcal{F}$ the exceptional divisor associated to $S_\mathcal{F}$. $D_\mathcal{F}$ consists of $N + n + 1$ irreducible components: one of them is transverse to $\hat{\mathcal{F}}$ and $\hat{\mathcal{F}}'$, and we still call it $E_0$ (by abuse of language). From each $p_i$ there is a chain of $r_i + 1$ irreducible components that we will call $E^i_1, \ldots, E^i_{r_i+1}$ where $E^i_j$ and $E^i_{j+1}$ intersect transversally at a corner for $j = 1, \ldots, r_i$ and $p_i = E_0 \cap E^i_{r_i+1}$.

Call $\hat{C}_i$ and $\hat{T}_i$ the strict transforms of $C_i$ and $T_i$ respectively by $S_\mathcal{F}$ for $i = 1, \ldots, n$. The relevant properties of $\hat{\mathcal{F}}$ are:

1) $E^i_1, \ldots, E^i_{r_i+1}, \hat{C}_i$ and $\hat{C}_\infty$ are invariant by $\hat{\mathcal{F}}$ for $i = 1, \ldots, n$.
2) $\text{Sing}(\hat{\mathcal{F}}) = \{\text{corners}, Q_1, \ldots, Q_n\}$ where $Q_i \in E^i_{r_i+1}$ is not a corner.
All singularities are reduced saddles.
3) For $j = 1, \ldots, r_i + 1$ the holonomy of $\hat{\mathcal{F}}$ along $E^i_j$ of the singularities in $E^i_j$ have degree $j$.

For the companion foliation $\hat{\mathcal{G}}$ the relevant properties are:

4) $\hat{\mathcal{G}}$ is regular and transversal to $E^i_{r_i+1}$ in all its points. $E_0, E^i_1, \ldots, E^i_r, \hat{C}_i$ and $\hat{C}_\infty$ are invariant by $\hat{\mathcal{G}}$ for $i = 1, \ldots, n$.
5) $\text{Sing}(\hat{\mathcal{G}}) = \{Q^i_1, Q^i_2, p_\infty, \text{corners not contained in } E^i_{r_i+1} : i = 1, \ldots, n\}$
where $Q^i_1 \in E^i_1$ and $Q^i_2 \in E^i_{r_i}$ are not corners. All singularities are reduced saddles with holomorphic first integral.
6) From Lemma 3.1, $\hat{T}_i$ is a regular disc transverse to $E^i_{r_i}$

All these properties are also satisfied by $\hat{\mathcal{F}}'$ and $\hat{\mathcal{G}}'$.

Condition (b) implies that after a diagonal linear change of coordinates in the original foliations we can suppose $\text{Sing}(\hat{\mathcal{F}}) = \text{Sing}(\hat{\mathcal{F}}') = \{Q_1, \ldots, Q_n\}$. Condition (a) implies that the holonomy maps of $Q_i$ along $E^i_{r_i+1}$, and the index of $Q_i$ are the same for $\hat{\mathcal{F}}$ and $\hat{\mathcal{F}}'$. As before we can use the theorem.
of Mattei and Moussu to extend \( \widehat{\Psi} \), the lifting of \( \Psi \), to the separatrix \( \widehat{C}_i \) for \( i = 1, \ldots, n \).

Now fix \( i \in \{1, \ldots, n\} \). Condition (a) implies the existence of the first integrals \( F_i \) and \( F'_i \) with the properties stated in (c). Let \( L, L' \) be leaves of \( \widehat{F} \) and \( \widehat{F}' \) respectively with \( L \cap E_0 = L' \cap E_0 \neq \{p_i\} \) sufficiently close to \( p_i \). The construction requires that \( \Psi(L) = L' \). Observe that \( L \cap \widehat{T}_i = \{W_1, \ldots, W_{r_i}\} \), by properties 3) and 6). Observe that these points correspond to the critical points of the restriction \( \widehat{g}_{1L} \) of \( \widehat{g} := S_{\infty}(g) \) to \( L \). Take a disc \( D \subset L \) containing \( L \cap \widehat{T}_i \), and such that the image of \( \partial D \) by \( \Psi \) has already been defined. Let \( D' \subset L' \) be the disc containing \( L' \cap \widehat{T}'_i = \{W'_1, \ldots, W'_{r_i}\} \) such that \( \partial D' = \Psi(\partial D) \). If \( L \) is sufficiently close to \( E^{i}_{r_i} \) then all the multiplicities \( v(W_j) \) of the critical points of \( \widehat{g}_{1D} \) coincide and are equal to some \( v > 1 \). Applying Hurwitz’s formula applied to the pasting of two copies of \( \widehat{g}_{1D} \) we get

\[
2 = 2 \cdot (r_i + 1) - 2 \left( \sum_{q \in D} (v(q) - 1) \right)
\]

which means that \( v = 2 \) and that \( v(q) = 1 \) for all \( q \in D \setminus \{W_1, \ldots, W_{r_i}\} \).

The second equation in (3.4) means that by renaming the points we can suppose \( \widehat{g}'(W'_j) = \widehat{g}(W_j) =: w_j \in \mathbb{C} \) for \( j = 1, \ldots, r_i \). Observe that \( \widehat{g}^{-1}(w_j) \) contains \( r_i \) points. Define \( V = \{w_1, \ldots, w_{r_i}\} \). Suppose \( D \) is big enough to contain \( \widehat{g}^{-1}(V) \cap L =: W \). Define \( W' = \widehat{g}'^{-1}(V) \cap L' \). Thus \( \widehat{g}_{1D \setminus W} : D \setminus W \to \widehat{g}(D) \setminus V \) is a \((r_i + 1):1\) holomorphic covering. We can copy the construction with \( \widehat{g}' \) and observe that the image of both coverings is the same. Thus, using a covering argument, we can find a unique topological extension \( \widehat{\Psi}_D \) making the following diagram commutative

\[
\begin{array}{ccc}
D \setminus W & \xrightarrow{\widehat{\Psi}_D} & D' \setminus W' \\
\widehat{g} \downarrow & & \downarrow \widehat{g}' \\
\widehat{g}(D) \setminus V & = & \widehat{g}'(D') \setminus V
\end{array}
\]

which is holomorphic and extends holomorphically to \( W \) because we can interpret \( \widehat{\Psi}_D \) as a holomorphic map between subsets of discs. This can be done for each leaf \( L \) not containing \( p_i \).

The holomorphicity in the transverse direction to \( \widehat{F} \) comes from (3.4). Thus after blowing \( E^j_1 \) down for \( j = 1, \ldots, r_i + 1 \), \( \Psi \) extends biholomorphically to \( U_i \setminus p_i \) where \( U_i \) is a neighborhood of \( p_i \) in \( \widetilde{\mathbb{C}}^2 \), and by Hartogs’ theorem (see [11], p. 341) we extend it to \( p_i \). \( \square \)

The following lemma finishes the proof of Theorem 3:
Lemma 3.3. — The hypotheses in Theorem 3 imply the existence of functions \( g, g' \) satisfying (a), (b) and (c) of Lemma 3.2.

Proof: Suppose \( \mathcal{F} = \mathcal{F}_\omega \) and \( \mathcal{F}' = \mathcal{F}_{\omega'} \in \mathcal{D}_0(n; r_1, \ldots, r_n) \) with \( H(\mathcal{F}) = H(\mathcal{F}') \). Thus (a) is already satisfied. Recall \( N = r_1 + \cdots + r_n \) and that the algebraic multiplicity of \( \omega \) and \( \omega' \) is \( N + 1 \). After a linear change of coordinates we can suppose \( \Sigma_\mathcal{F} = \{ p_1, \ldots, p_n \} \) with coordinates \( p_i = (t_i, 0) \) in the chart \((t, x) \) of \( \mathbb{C}^2 \) such that \( |t_i| \neq |t_j| \) for \( i \neq j \). A direct calculation shows that \( j^{N+2}(\omega) = j^{N+2}(\omega') \) implies (b). For the proof of (c), consider a companion foliation \( \mathcal{G} = \{ g = C^{\text{te}} \} \) for \( \mathcal{F} \). Recall that \( C_i = \{ f_i = 0 \} \) (resp. \( C'_i = \{ f'_i = 0 \} \)) is the Weierstrass polynomial of the separatrix of \( \mathcal{F} \) (resp. \( \mathcal{F}' \)) whose strict transform by \( E \) passes through \( p_i \) for \( i = 1, \ldots, n, \infty \). Let \( F = \prod_{i=1}^n f_i, F' = \prod_{i=1}^n f'_i \). We need to find a unit \( u \in \mathcal{O}^*_0(\mathbb{C}^2) \) such that

\[
\mathcal{G}' = \left\{ g' := \frac{f_{\infty}^{N+n+1}}{F'} \cdot u = C^{\text{te}} \right\}
\]

is a companion foliation for \( \mathcal{F}' \) and

\((*)\) \( \text{tang}(\mathcal{F}, \mathcal{G}) = \text{tang}(\mathcal{F}', \mathcal{G}') \);

\((**)\) \( g \) and \( g' \) satisfy (3.4) on each component of type \( T_i \) of \( \text{tang}(\mathcal{F}, \mathcal{G}) \), where \( \psi_i : T_1 \rightarrow T_i \) is defined by the first equation of (3.4).

We know \( C_i = \{ f_i = 0 \} \) is invariant by \( \mathcal{F}_\omega \) so

\[
df_i \wedge \omega = f_i \cdot H_i dX \wedge dY
\]

for some holomorphic function \( H_i \in \mathcal{O}^*_0(\mathbb{C}^2) \). The divisor \( \text{tang}(\mathcal{F}, \mathcal{G}) \) is defined by

\[
0 = (F^2 dg) \wedge \omega
\]

\[(3.5)\]

\[
f_{\infty}^{N+n+1} F \left( H_{\infty} - \sum_{i=1}^n H_i \right) dX \wedge dY =: f_{\infty}^{N+n+1} F H dX \wedge dY.
\]

As we saw in Lemma 3.1 the divisor \( T = \{ H = 0 \} \) has \( n \) irreducible components \( T_1, \ldots, T_n \) with multiplicity one. Each \( T_i \) is a generalized cusp of type \( (r_i, r_i + 1) \) (when \( r_i = 1 \) it is just a regular disc). We will decompose the problem of constructing \( u \) in two steps by finding functions \( \phi \in \mathcal{O}^*_0(\mathbb{C}^2) \) and \( \varphi \in \mathcal{O}(\mathbb{C}^2) \) such that \( u = \phi + H \cdot \varphi \). The idea is that conditions \((*)\) and \((**)\) define the values of \( \phi \) and \( \varphi \) on the analytic subset of dimension one \( T \). We then need to find holomorphic functions defined in the whole neighborhood of the origin in \( \mathbb{C}^2 \) taking the same values on \( T \). For this purpose we need to construct the \( \psi_i \)'s appearing in (3.4) first:
Using Lemma 3.4 (i) consider \( F_i, F'_i \) local holomorphic first integrals of \( \mathcal{F} \) and \( \mathcal{F}' \) respectively around \( p_i \) such that \( F_i|_{V_i} = F'_i|_{V_i} \) in a neighborhood \( V_i \subset E_0 \) of \( p_i \). Consider the following diagram:

\[
\begin{align*}
\tilde{T}_i \setminus p_i \quad &\quad \tilde{T}_i \setminus p_i \\
\pi_i \downarrow \quad &\quad \pi'_i \downarrow \\
(\tilde{T}_i \setminus p_i)/F_i \quad &\quad (\tilde{T}_i \setminus p_i)/F'_i \\
\cong \quad &\quad \cong (V_i \setminus p_i)/F_i = (V_i \setminus p_i)/F'_i
\end{align*}
\]

where \( \pi_i \) and \( \pi'_i \) are \( r_i \) : 1 holomorphic coverings corresponding to the projections onto the local leave spaces. Hence we can construct a homeomorphism \( \psi_i : \tilde{T}_i \setminus p_i \to \tilde{T}_i \setminus p_i \) such that \( \pi'_i \circ \psi_i = \pi_i \). In fact there are \( r_i \) different possibilities for constructing \( \psi_i \). After blowing down we can consider \( \psi_i : T_i \to T_i \) as a homeomorphism.

Define the function \( \phi_i := \phi|_{T_i} : (T_i, p_i) \to (\mathbb{C}, 1) \) using condition (**):

\[
\phi_i(q) = \left( \frac{f_{\infty} \circ \psi_i^{-1}}{f'_\infty} \right)^{N+n+1} \left( \prod_{i=1}^{n} \frac{f'_i}{f_i \circ \psi_i^{-1}} \right)^{(q)}
\]

for \( q \in T_i \). From now on we will suppose \( j^s(\omega) = j^s(\omega') \) and find bounds for \( s \) to insure that the construction can be done.

**Claim 1.** — If \( s \geq (N + 1) + \max\{r_i\}(N - 1) \) there exists \( \phi \in \mathcal{O}^*_r(\mathbb{C}^2, 0) \) such that \( \phi|_{T_i} = \phi_i \) for \( i = 1, \ldots, n \). Moreover,

\[
d\phi(X, Y) = X^{(s-(N+1))/\max\{r_i\}} \nu(X, Y)
\]

for some holomorphic germ of 1-form \( \nu \).

**Proof of Claim 1.** — We need to analyse the relations between the jets of the 1-form \( \omega = \sum_{j \geq N+1} (P_j dX + Q_j dY) \) where \( P_j(X, Y), Q_j(X, Y) \) are homogeneous polynomials of degree \( j \) and the form defining \( \check{\mathcal{F}}_\omega \) in the \((t, x)\) chart of \( \mathbb{C}^2 \):

\[
\check{\omega}(t, x) := \frac{E^*\omega(t, x)}{x^{N+2}} = \sum_{j \geq 0} x^j [Q_{j+N+1}(1, t) dt + R_{j+N+2}(t) dx]
\]

where \( R_{j+N+2}(t) := P_{j+N+2}(1, t) + tQ_{j+N+2}(1, t) \).

**Lemma 3.4.** — If \( \mathcal{F}_\omega, \mathcal{F}_{\omega'} \in \mathcal{D}_0(n; r_1, \ldots, r_n) \) satisfy \( j^s_0(\omega) = j^s_0(\omega') \), then:

(i) \( \check{\omega}'(t, x) = \check{\omega}(t, x) + x^{s-(N+1)}\omega_2(t, x) \) for some holomorphic 1-form \( \omega_2 \). We define \( K(s) := s - (N + 1) \in \mathbb{N} \).
(ii) If moreover, $H(\widetilde{F}_ω, p) = H(\widetilde{F}_ω', p)$ for $p ∈ E₀$, given a local holomorphic first integral $f$ of $\widetilde{F}_ω$ there exists a local holomorphic first integral $f'$ of $\widetilde{F}_ω'$ in a neighborhood of $p$ such that

$$f'(t, x) = f(t, x) + x^{K(s)+1}h(t, x)$$

for some holomorphic function $h$ defined in a neighborhood of $p$.

(iii) Equation (3.7) implies that $j^{K(s)+r_i}(f_i) = j^{K(s)+r_i}(f_i')$ for the Weierstrass polynomials $f_i, f_i'$ of $C_i$ and $C_i'$ respectively.

Proof. — To prove (i), observe that in (3.7), the terms in the $j$-th member of the sum depend on the $(j+1)+N+1$ jet of $ω$. For the proof of (ii) we can construct $f'$ by extending the function $f(t, 0)$ along the leaves of $\widetilde{F}_ω'$ in a neighborhood of $p = (t₀, 0)$. We can assume $∂f/∂x(p) ≠ 0$. Write $h(t, x) = f(t, x) - f'(t, x) = \sum_{i≥0}h_i(t)x^i$ with $h_i$ holomorphic functions of $t$. From item (i) we get

$$x^{K(s)}(ω₂ \wedge df') + \widetilde{ω} \wedge dh = \widetilde{ω}' \wedge df' ≡ 0$$

Since $\widetilde{ω} = Adt + Bdx$ is regular at $p$, and $dh = \sum_{i≥1}∂h_i/∂t x^i dt + ix^i-1h_i(t)dx$ we get, by comparing jets:

$$0 ≡ \sum_{i=0}^{K(s)} x^{i-1}Ah_i - \sum_{i=0}^{K(s)-1} Bx^i∂h_i/∂t$$

By hypothesis $h_0 ≡ 0$. Inductively we get $h_1(t) ≡ \ldots ≡ h_{K(s)}(t) ≡ 0$.

For the proof of (iii) we take Puiseux parametrizations $w ↦ (w^{r_i+1}, Q_i(w))$ and $w ↦ (w^{r_i+1}, Q'_i(w))$ of $C_i$ and $C_i'$ respectively. By (3.7) we get, by comparing terms after the blowing up, $j^{K(s)+r_i}(Q_i) = j^{K(s)+r_i}(Q'_i)$ which implies the assertion in (iii).

Now consider a Puiseux parametrization

$$τ_i : D → T_i, \quad w ↦ (w^{r_i}, \hat{P}_i(w))$$

from a small disc $D$ to $T_i$. Hence there exists a homeomorphism $b_i : D → D$ such that

$$ψ_i(w^{r_i}, \hat{P}_i(w)) = (b_i(w))^{r_i}, \hat{P}_i(b_i(w)))$$

In fact $b_i$ is holomorphic outside 0, which implies that it is also holomorphic there. Suppose $b_i(w) = \sum_{j≥0}b_j^i w^j$ and that we have chosen $ψ_i$ by imposing $b_1^i = 1$. Using (3.7) and the fact that $h \circ E^{-1} \circ τ_i(w) = a_iw^{r_i} + \ldots$ with $a_i ≠ 0$ we see inductively that $b_2^i = b_3^i = \ldots = b_{K(s)}^i = 0$. This together with (3.6) and Lemma 3.4 (iii) implies that

$$φ_i(w^{r_i}, \hat{P}_i(w)) = 1 + w^{K(s)}φ_i(w)$$
for some holomorphic function \( \tilde{\phi}_i \). Since \( \tilde{P}_i(w) = ti, w^{r_i}, \ldots \) and \( |t_i| \neq |t_j| \) if \( i \neq j \) we can apply the following interpolation result due to Cartan (see [5], p. 102), which has been adapted to our situation:

For each \( i \in \{1, \ldots, n\} \) consider a germ of analytic irreducible set

\[
T_i = \left\{ (X, Y) \in \mathbb{C}^2 \mid Y^{r_i} \sum_{j=0}^{r_i-1} \alpha_j(X)Y^j = 0 \right\}
\]

with its Puiseux parametrization \( x \mapsto (x^{r_i}, \tilde{P}_i(x)) \). Suppose that \( \tilde{P}_i(x) = A_ix^{r_i} + \cdots \) for \( A_i \in \mathbb{C} \) with \( A_i^{r_k} \neq A_j^{r_k} \) if \( i \neq j \) and \( i, j, k \in \{1, \ldots, n\} \).

**Lemma 3.5.** — Let \( \nu_i : T_i \to \mathbb{C} \) be a continuous function such that \( \nu_i(0, 0) = a \in \mathbb{C} \) (independently of \( i \)) and \( \nu_i(x^{r_i}, \tilde{P}_i(x)) = a + c_i x^{\ell_i} + \cdots \) is holomorphic with \( c_i \neq 0 \). If \( \ell := \min \ell_i/r_i \geq (\sum_{i=1}^{n} r_i) - 1 =: N - 1 \), then there exists a holomorphic function \( \nu \in \mathcal{O}(\mathbb{C}^2, 0) \) such that

(i) \( \nu|_{T_i} = \nu_i; \)

(ii) \( d\nu(X, Y) = X^{\ell-N} \eta(X, Y) \) for a holomorphic 1-form \( \eta \).

In other words, \( \nu \) extends holomorphically to a neighborhood of \( (0, 0) \) in \( \mathbb{C}^2 \) the functions \( \nu_i \) defined on the analytic subsets \( T_i \) (of dimension 1).

**Proof of Lemma 3.5.** — For each \( i \in \{1, \ldots, n\} \) choose a branch \((\cdot)^{1/r_i}\) of the \( r_i \)-th root and a primitive \( r_i \)-th root of unity \( \zeta_i \). We index with \( j \) the set of all branches of the union \( T \) of all the \( T_i \)'s: let \( r_0 := 0 \) and for \( j \in \{1, \ldots, N\} \) define \( P_j(X) := \tilde{P}_i(\zeta_i^{[j]} X^{1/r_i}) \) where \( [j] = j - (r_0 + \cdots + r_{i-1}) \) if \( j \in \{r_0 + \cdots + r_i - 1, \ldots, r_0 + \cdots + r_i\} \). We claim that the expression

\[
\nu(X, Y) := \sum_{i=1}^{N} \left( \prod_{j \neq i} \frac{Y - P_j(X)}{P_i(X) - P_j(X)} \right) \nu_i(X, P_i(X))
\]

defines an univaluated, continuous function in \( (\mathbb{C}^2, 0) \) with \( \nu(p) = \nu_i(p) \) for \( p \in T_i \). Outside \( \{X = 0\} \) this is a consequence of the symmetry of the expression, for when we follow a loop around the origin in the \( X \)-plane, we exchange the order of the members of the sum, leaving its value unchanged.

Define \( \mathcal{K} := \mathcal{M}(X) \) the field of meromorphic functions in \( X \), and \( \overline{\mathcal{K}} \) its algebraic closure. Obviously \( P_j \in \overline{\mathcal{K}} \) and \( P_i \neq P_j \) if \( i \neq j \). Hence for any \( a \in \mathbb{C} \) the polynomial in \( \overline{\mathcal{K}}[Y] \) of degree \( N - 1 \) defined by

\[
\left( \sum_{i=1}^{N} \left( \prod_{j \neq i} \frac{Y - P_j(X)}{P_i(X) - P_j(X)} \right) a \right) - a
\]

has \( N \) different roots, and is therefore the zero polynomial. From \( A_i^{r_k} \neq A_j^{r_k} \) if \( i \neq j \) we obtain \( |P_i(X) - P_j(X)| = |X| \cdot |h_{ij}(X)| \) for a continuous function \( h_{ij} \) such that \( h_{ij}(0) \neq 0 \). From the hypotheses on \( \nu_i \) we have...
\[ |\nu_i(X, P_i(X)) - a| \leq |X|^{N-1} h_i(|X|) \] for real continuous functions \( h_i \) defined in the neighborhood of 0. Therefore,

\[ (3.11) \quad |\nu(X, Y) - a| \leq \left( \sum_{i=1}^{N} \left( \prod_{j \neq i} \frac{|Y - P_j(X)|}{|h_{ij}(X)|} \right) h_i(|X|) \right) |X|^{N-1} \rightarrow 0 \]

when \((X, Y) \rightarrow (0, 0)\). A similar argument can be used to prove continuity on the remaining points of \( \{X = 0\} \). Hence \( \nu \) is continuous on \((\mathbb{C}^2, 0)\) and trivially holomorphic on \((\mathbb{C}^2, 0) \setminus \{X = 0\} \), hence holomorphic on \((\mathbb{C}^2, 0)\).

For the proof of (ii) write \( \nu(X, Y) = \sum_{j=0}^{N-1} a_j(X) Y^j \) where \( a_0(X) - a \) and \( a_j(X) \) are holomorphic functions of \( X \). By (3.11) they are zero up to order \( \ell - (N - 1) \) for \( j = 1, \ldots, N - 1 \), and taking derivatives we have that \( X^{\ell-N} \) divides all terms of \( d\nu(X, Y) \).

This finishes the proof of Claim 1. Let us continue with the proof of Lemma 3.3. Condition (*) is satisfied if \( \varphi_i := \varphi|_{T_i} \) satisfies

\[ 0 = (F'^2 dg') \wedge \omega'|_{T_i} \]

\[ (3.12) \quad F' \cdot f_i^{N+n+1}(\phi H' dX \wedge dY + d\phi \wedge \omega' + \varphi_i dH \wedge \omega'(q))|_{T_i} \]

where \( H' \) is obtained by a process similar to equation (3.5) but using the \( f_i' \)'s. In fact, from Lemma 3.4 (iii) we have

\[ (3.13) \quad j^{K(s)+N}(H) = j^{K(s)+N}(H'). \]

Equation (3.12) is equivalent to

\[ (3.14) \quad \phi(q)H'(q)dX \wedge dY + d\phi \wedge \omega'(q) + \varphi_i(q)(dH \wedge \omega'(q)) = 0 \]

for each point \( q \in T_i \). The expression in (3.14) defines the values of \( \varphi_i \).

**Claim 2.** — If \( s \geq (N + 1) + \max\{r_i\}(3N - 2) =: \kappa \) there exists \( \varphi \in \mathcal{O}(\mathbb{C}^2, 0) \) such that \( \varphi|_{T_i} = \varphi_i \) for \( i = 1, \ldots, n \).

**Proof of Claim 2.** — To use Lemma 3.5 for interpolation, we need to analyse the order of \( \varphi_i(w^{r_i}, \hat{P}(w)) \). This can be done using (3.14):

- \( H'(w^{r_i}, \hat{P}(w)) \) has order \( r_i(K(s) + N) \), from (3.13) and the fact that \( \hat{P}_i \) has order \( r_i \);
- \( dH(w^{r_i}, \hat{P}(w)) \) has order \( r_i N - 1 \), since the tangent cone of \( H \) is \( \prod_{i=1}^{n} (Y - t_i X)^{r_i} \);
- \( \omega'(w^{r_i}, P'_i(w)) \) has order \( r_i(N + 2) \), since

\[ j^{N+1}(\omega') = \prod_{i=1}^{n} (Y - t_i X)^{r_i}(Y dX - X dY); \]

- \( d\phi(w^{r_i}, P'_i(w)) \) has order \( r_i(K(s)/\max r_i - N) \), by the the second part of Claim 1.
Hence, if
\begin{equation}
\begin{aligned}
& r_i(K(s) + N) - (r_iN - 1) - r_i(N + 2) \geq r_i(N - 1), \\
& r_i\left(\frac{K(s)}{\max r_i} - N\right) - (r_iN - 1) \geq r_i(N - 1) \quad \text{for } i = 1, \ldots, n
\end{aligned}
\end{equation}
then we have \( \varphi_i(w^r_i, P_i^r(w)) = w^r_i(N-1)\tilde{\varphi}_i(w) \) for some holomorphic function \( \tilde{\varphi}_i \). The hypothesis \( s \geq \kappa \) guarantees that the hypothesis of Claim 1 and the inequalities in (3.15) are satisfied. Applying Lemma 3.5 we obtain the desired function \( \varphi \).
\[\square\]

4. Proof of Corollary 4

Proof of Corollary 4. — Take \( F_\omega \) and \( F_{\omega'} \) in \( D_0 \), and suppose there exists a formal equivalence \( \hat{\phi} \) in \( (\mathbb{C}^2,0) \) and a formal power series \( \hat{h} \) such that \( \hat{\phi}^*\omega' = \hat{h}\omega \). After a linear change of coordinates we can suppose \( \Sigma_{F_\omega} = \Sigma_{F_{\omega'}} \), with the same orders of tangency and \( \hat{\phi} \) tangent to the identity. Given \( \ell \in \mathbb{N} \) there exist 1-forms \( \omega_\ell \in \omega'_\ell \) such that \( F_{\omega_\ell} = F_\omega, F_{\omega'_\ell} \) is analytically equivalent to \( F_{\omega'}, j^\ell(\omega_\ell) = j^\ell(\omega'_\ell), H(\omega) = H(\omega_\ell) \) and \( H(\omega') = H(\omega'_\ell) \). Let us prove \( H(\omega') = H(\omega) \). Let \( h_p \in H(\tilde{F}_\omega, p) \) (resp. \( h'_p \in H(\tilde{F}_{\omega'}, p) \)) be a generator, where \( p \in \Sigma_\omega \). Given \( s \in \mathbb{N} \), there exists a big enough \( \ell(s) \in \mathbb{N} \) such that \( j^\ell(s)(\omega_{\ell(s)}) \) determines \( j^s(h_p) \) uniquely where the jet of \( h_p \) is taken in a global coordinate \( t \) of \( E_0 \). Therefore we have \( j^s(h_p) = j^s(h'_p) \) for each \( s \in \mathbb{N} \) and \( h_p = h'_p \). Now apply Theorem 3 to \( \omega_\ell \) and \( \omega'_\ell \), for a big \( \ell \) and we get a biholomorphism from \( F_\omega \) to \( F_{\omega'} \).
\[\square\]

5. Addendum

In section 3, when \( n = 1 \) we need to define the companion foliation in a different manner to be able to construct the biholomorphism. The problem is that in this case the set of tangencies between \( F \) and \( G \) is not a regular curve after applying a blow-up. To avoid this we consider the function defining the companion foliation \( \omega \) to be a product \( g = \frac{f_{r+1}^2}{f_1 : f_2} \) where \( f_1 = 0 \) is the isolated separatrix and \( f_2 = 0 \) is a (regular) separatrix tangent to some other direction \( p_2 \). This produces a radial singularity at the point \( p_2 \) in \( E_0 \) for the companion foliation \( \omega \). The construction of the biholomorphism between two elements \( F, F' \in D_0(1;r) \) by using the values of their companion foliations extends without any extra conditions to a neighborhood of \( p_2 \), as can be seen by blowing this point once and using the same argument as at \( p_\infty \).
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