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A linear extension operator for Whitney fields on closed o-minimal sets

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A LINEAR EXTENSION OPERATOR FOR WHITNEY FIELDS ON CLOSED O-MINIMAL SETS

by Wiesław PAWŁUCKI (*)

Dedicated to my wife Jolanta

Abstract. — A continuous linear extension operator, different from Whitney’s, for $C^p$-Whitney fields ($p$ finite) on a closed o-minimal subset of $\mathbb{R}^n$ is constructed. The construction is based on special geometrical properties of o-minimal sets earlier studied by K. Kurdyka with the author.

1. Introduction

In 1997 K. Kurdyka and the author gave in [6] the following o-minimal version of the Whitney extension theorem:

Theorem 1.1 ([6]). — Given any o-minimal structure on the ordered field of real numbers $\mathbb{R}$, a compact definable subset $E \subset \mathbb{R}^n$, a definable $C^p$-Whitney field $F$ on $E$, where $p \in \mathbb{N} \setminus \{0\}$, then for any integer $q \geq p$, there exists a definable $C^p$-extension $f : \mathbb{R}^n \to \mathbb{R}$ of $F$ which is $C^q$ on $\mathbb{R}^n \setminus E$.

However, the extension operator $F \mapsto f$ from [6] is not linear and it was not clear how the construction from [6] based on o-minimal geometry could be adapted to get an extension operator for all Whitney fields on

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any compact (or more generally closed) $o$-minimal subset $E$ of $\mathbb{R}^n$. The present paper is devoted to this question. The main goal here is to prove the following

**Theorem 1.2.** — Let $E$ be a closed $o$-minimal subset of $\mathbb{R}^n$ and $p \in \mathbb{N}$. Let $\mathcal{E}^p(E)$ denote the Fréchet algebra of all $C^p$-Whitney fields on $E$.

Then there exists a continuous linear extension operator $L : \mathcal{E}^p(E) \to C^p(\mathbb{R}^n)$ which has the following properties

1. $L$ is a finite composition of operators each of which either preserves definability or (only if $p > 0$) is an integration with respect to a parameter;
2. operators preserving definability in (1) are only of the following five types: substituting with a definable mapping; taking a linear combination with definable coefficients; differentiation; restriction to a definable subset and extending by zero;
3. there exists a constant $M > 0$ such that if $\omega$ is a modulus of continuity of a field $F$, then $M\omega$ is a modulus of continuity of $LF$.

Since $L$ involves integration, it may not preserve definability in the initial $o$-minimal structure where $E$ is definable. For example, if $F$ is a (globally) subanalytic $C^p$-Whitney field, then $LF$ can a priori involve the function $t \mapsto t \log t$, not subanalytic at 0. By a result of Lion and Rolin [7], we get in this case the following

**Corollary 1.3.** — Let $A$ denote the algebra of real functions generated by (globally) subanalytic functions and their logarithms; i.e. $A$ consists of all functions of the form $P(h_1, \ldots, h_m, \log h_1, \ldots, \log h_m)$, where $h_i : \mathbb{R}^n \to \mathbb{R}$ ($i = 1, \ldots, m$) are subanalytic, $m \in \mathbb{N}\setminus\{0\}$, $P \in \mathbb{R}[Y_1, \ldots, Y_{2m}]$, and where we adopt the convention: $\log t = 0$, for $t \leq 0$. Let $E$ be a closed subanalytic subset of $\mathbb{R}^n$ and $p \in \mathbb{N}$.

Then there exists a continuous linear extension operator $L : \mathcal{E}^p(E) \to C^p(\mathbb{R}^n)$ which has the following properties:

1. if $F$ is a $C^p$-Whitney field on $E$ all derivatives of which $F^\kappa$ are (restrictions to $E$ of) functions in $A$, then $LF \in A$;
2. there exists a constant $M > 0$ such that if $\omega$ is a modulus of continuity of a field $F$, then $M\omega$ is a modulus of continuity of $LF$.

The case $p = 0$ in Theorem 1.2, when integration is not used seems worth being stated separately

**Corollary 1.4.** — Let $E$ be a closed $o$-minimal subset of $\mathbb{R}^n$ and let $\mathcal{C}(E)$ denote the Fréchet space of all real continuous functions on $E$
Then there exists a continuous linear extension operator $L : C(E) \rightarrow C(\mathbb{R}^n)$ preserving definability and such that there exists $M > 0$ such that, if $\omega$ is a modulus of continuity for $F \in C(E)$, then $M \omega$ is a modulus of continuity for $LF$.

By an o-minimal subset of an Euclidean space $\mathbb{R}^n$ we mean a subset definable in any o-minimal structure on the ordered field of real numbers $\mathbb{R}$ (see [2, 3] for the definition and fundamental properties).

We refer the reader to [13], [4], [8], [11] or/and [12] for basic facts on Whitney fields. It will be convenient for us to adopt the following definition of a Whitney field.

Let $p \in \mathbb{N} \setminus \{0\}$ and let $A$ be a locally closed subset of $\mathbb{R}^n$; i.e. contained and closed in some open subset $G \subset \mathbb{R}^n$. A $C^p$-Whitney field on $A$ is a polynomial

$$F(u, X) = \sum_{|\alpha| \leq p} \frac{1}{\alpha!} F^{\alpha}(u) X^\alpha \in C(A)[X] = C(A)[X_1, \ldots, X_n],$$

which fulfills the following condition

\begin{equation}
(*) \quad \text{for each } c \in A \text{ and each } \alpha \in \mathbb{N}^n \text{ such that } |\alpha| \leq p \quad D_X^\alpha F(a, 0) - D_X^\alpha F(b, a - b) = o(|a - b|^{p - |\alpha|}), \quad \text{when } A \ni a \rightarrow c, A \ni b \rightarrow c,
\end{equation}

or equivalently (see [8], Chapter I, Theorem 2.2) - the condition

\begin{equation}
(**) \quad \text{for each } c \in A \quad F(a, x - a) - F(b, x - b) = o(|x - a|^p + |x - b|^p),
\end{equation}

uniformly with respect to $x \in \mathbb{R}^n$, when $A \ni a \rightarrow c, A \ni b \rightarrow c$.

We will denote by $\mathcal{E}^p(A)$ the real algebra of all $C^p$-Whitney fields on $A$. It is a Fréchet algebra with the topology defined by the following system of seminorms

$$||F||_p^K = |F|_p^K + \sup_{\alpha \in K, a \neq b} \frac{|D_X^\alpha F(a, 0) - D_X^\alpha F(b, a - b)|}{|a - b|^{p - |\alpha|}},$$

where $K$ is a compact subset of $A$ and $| . |_p^K$ is a seminorm defined by

$$|F|_p^K = \sup_{\alpha \in K} |F^\alpha(a)|.$$

Let $C^p(G)$ denote the usual Fréchet algebra of real functions of class $C^p$ ($C^p$-functions) on $G$. Then we have the following homomorphism of Fréchet
algebras

\[ T : \mathcal{C}^p(G) \longrightarrow \mathcal{E}^p(A), \quad T f(a, X) = T^p_a f(X) = \sum_{|\alpha| \leq p} \frac{1}{\alpha!} D^\alpha f(a) X^\alpha, \]

and the Whitney extension theorem [13] says that there exists a linear continuous mapping

\[ W : \mathcal{E}^p(A) \longrightarrow \mathcal{C}^p(G) \quad \text{such that} \quad T \circ W = id_{\mathcal{E}^p(A)}, \]

called an extension operator.

A subset \( E \) of \( \mathbb{R}^n \) is said to be 1-regular (with a constant \( C \geq 1 \)) if any two points \( a, b \) of \( E \) can be joined in \( E \) by a rectifiable arc \( \gamma : [0,1] \longrightarrow E \) of length \( |\gamma| \leq C|a-b| \).

If \( F \in \mathcal{E}^p(A) \) and \( K \) is a compact 1-regular subset of \( A \) with a constant \( C \), then

\[ |F|^K_p \leq ||F||^K_p \leq 2n^2 C^p |F|^K_p \]  

(See [12], p.76, (2.5.1)). Consequently, if every compact subset \( L \) of \( A \) is contained in a 1-regular compact subset \( K \) of \( A \), then the topology of \( \mathcal{E}^p(A) \) is defined by the system of seminorms \( |.|^K_p \).

As was shown by Glaeser [4] (see also [8], [12] or [11]) it is convenient to use a notion of a modulus of continuity in connection with Whitney fields. By a modulus of continuity we will understand any continuous, increasing and concave function \( \omega : [0, +\infty) \longrightarrow [0, +\infty) \), vanishing at 0. By a modulus of continuity of a \( \mathcal{C}^p \)-Whitney field

\[ F(u, X) = \sum_{|\alpha| \leq p} \frac{1}{\alpha!} F^\alpha(u) X^\alpha \]
on a subset \( A \) of \( \mathbb{R}^n \) we will understand such a modulus of continuity \( \omega \) that

\[ |D^\alpha_X(a,0) - D^\alpha_X(b,a-b)| \leq \omega(|a-b|)|a-b|^{p-|\alpha|}, \]

whenever \( |\alpha| \leq p \) and \( a, b \in A \). For a \( \mathcal{C}^p \)-function \( f \in \mathcal{C}^p(G) \) on an open subset \( G \), by its modulus of continuity we will understand a modulus of continuity of the \( \mathcal{C}^p \)-Whitney field \( Tf \) on \( G \).

Every \( \mathcal{C}^p \)-Whitney field on a compact subset of \( \mathbb{R}^n \) admits a modulus of continuity. If a \( \mathcal{C}^p \)-Whitney field \( F \) on a subset \( A \) has a modulus of continuity \( \omega \), then it is easily seen that \( F \) extends by uniform continuity to a \( \mathcal{C}^p \)-Whitney field on \( \overline{A} \) with the same modulus of continuity. Whitney’s extension operator [13] has the following property (see [4]):
There exists a constant $M$ depending only on $p$ and $n$ such that, for every $F \in E^p(A)$ admitting a modulus of continuity $\omega$, $M\omega$ is a modulus of continuity for $WF$. (In fact a localization by a partition of unity is necessary.)

We have also the following

**Proposition 1.5.** — Let $F$ be a $C^p$-Whitney field on a (locally) closed 1-regular with constant $C$ subset $A$.

(1) If $\omega$ is a modulus of continuity of $F$ on $A$, then $|F^\alpha(a) - F^\alpha(b)| \leq \omega(|a-b|)$, whenever $|\alpha| = p$, $a, b \in A$.

(2) If $\omega$ is a modulus of continuity such that $|F^\alpha(a) - F^\alpha(b)| \leq \omega(|a-b|)$, whenever $|\alpha| = p$, $a, b \in A$, then $n^{\frac{p}{2}}C^p\omega$ is a modulus of continuity of $F$ on $A$.

**Proof.** — (1) being trivial, for (2) see again [12], (2.5.1), p.76. □

Shortly, our construction of the extension operator $L$ is as follows. First we show how to extend $C^p$-Whitney fields from a linear subspace $\mathbb{R}^k \times 0$ of $\mathbb{R}^n$. Then we generalize the construction to the set of the form $\Omega \times 0$, where $\Omega$ is open in $\mathbb{R}^k$ for fields flat on $\partial \Omega \times 0$, simply by Hestenes Lemma. Using induction on dimension of $A$, this gives an extension operator for $A = \Gamma$, where $\Gamma = \Omega \times 0$ assuming we have it already built for the boundary $\partial \Gamma = \overline{\Gamma} \setminus \Gamma$ of $\Gamma$ which in this case is $\partial \Omega \times 0$. The next generalization is by taking $A = \Gamma$, where $\Gamma$ is a $\Lambda_p$-regular leaf of dimension $k$ in the sense of [6], and again assuming the fields are flat on $\partial \Gamma$. Additionally, the extension can be chosen vanishing outside a conical neighbourhood of $\Gamma$; i.e. the set $\{x \in \Omega \times \mathbb{R}^{n-k} : d(x, \Gamma) < \varepsilon d(x, \partial \Gamma)\}$, where $\Omega$ is the orthogonal projection of $\Gamma$ to $\mathbb{R}^k \times 0$ and $\varepsilon$ is a positive arbitrary constant. The next generalization is to the closure of a finite tower of $\Lambda_p$-regular leaves lying over a common open $\Lambda_p$-regular cell in $\mathbb{R}^k$. To finish the construction we will prove that every closed definable $k$-dimensional subset $A$ admits a finite decomposition $A = M_0 \cup \cdots \cup M_s$ such that each $M_i$ is a finite tower of definable $\Lambda_p$-regular leaves in a suitable linear coordinate system and for any $i, j \in \{0, \ldots, s\}$, where $i \neq j$, $M_i$ and $\overline{M_j}$ are simply separated relative to $\partial M_i$; i.e. $d(x, M_j) \geq Cd(x, \partial M_i)$, for each $x \in M_i$, with some positive constant $C$. (The proof of this $\Lambda_p$-regular Decomposition Theorem is based on [6] and [10].)
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2. Extension operator for a linear subspace

Observe that if $\Omega$ is an open subset of $\mathbb{R}^k$ and $A = \Omega \times 0 \subset \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n$, then the algebra $E^p(A)$ can be identified with the algebra of polynomials
\[
F(u, W) = \sum_{|\alpha| \leq p} \frac{1}{\alpha!} F^\alpha(u) W^\alpha = \sum_{|\alpha| \leq p} \frac{1}{\alpha!} F^\alpha(u) W_1^{\alpha_1} \ldots W_l^{\alpha_l},
\]
where $l = n - k$ and $F^\alpha \in \mathcal{C}^{p-|\alpha|}(\Omega)$, for each $\alpha \in \mathbb{N}^l$ such that $|\alpha| \leq p$ (cf. [4], Chap.III, (8.4)).

Let us now consider the case $k = n - 1$ and $A = \mathbb{R}^k \times 0$. Then the extension operator will be produced using regularization of functions $F^\alpha$ by convolution. Strictly, we have the following

**Proposition 2.1.** — Let $\sigma \in \{0, \ldots, p\}$, $g \in \mathcal{C}^{p-\sigma}(\mathbb{R}^k)$, $\varphi \in \mathcal{C}^p(\mathbb{R}^k)$. Assume that $\operatorname{supp} \varphi$ is compact and put
\[
\varphi_w(v) = \frac{1}{w^k} \varphi\left(\frac{v}{w}\right)
\]
and
\[
G(u, w) = \frac{1}{\sigma!} (g \ast \varphi_w)(u) w^\sigma = \frac{1}{\sigma!} \int_{\mathbb{R}^k} g(u-v) \varphi_w(v) w^\sigma dv,
\]
for $u \in \mathbb{R}^k$ and $w \in \mathbb{R}, w > 0$.

Then $G : \mathbb{R}^k \times (0, +\infty) \longrightarrow \mathbb{R}$ is a $\mathcal{C}^p$-function and for every $(\alpha, \beta) \in \mathbb{N}^k \times \mathbb{N}$ such that $|\alpha| + \beta \leq p$

\[
\lim_{w \to 0} D^{(\alpha, \beta)} G(u, w) = \begin{cases} 
0, & \text{if } \beta < \sigma \\
D^\alpha g(u) \varphi, & \text{if } \beta = \sigma \\
\sum_{|\gamma| = \beta - \sigma} \omega_\gamma D^{\alpha + \gamma} g(u), & \text{if } \beta > \sigma
\end{cases}
\]
uniformly on compact subsets with respect to $u$, where $\omega_\gamma$ are some constants depending only on $\gamma$, $\sigma$ and $\varphi$.

To prove Proposition 2.1 one needs the following
Lemma 2.2. — For any \( r \in \mathbb{R} \) and \( \lambda \in \{0, \ldots, p\} \)
\[
\frac{\partial^\lambda}{\partial w^\lambda} \left[ w^r \varphi \left( \frac{v}{w} \right) \right] = w^{r-\lambda} \varphi_{\lambda} \left( \frac{v}{w} \right),
\]
where \( \varphi_{\lambda} \) is a \( C^p-\lambda \)-function on \( \mathbb{R}^k \) with a compact support and \( \int \varphi_{\lambda} = (k + r)(k + r - 1) \cdots (k + r - \lambda + 1) \int \varphi. \)

Proof of Lemma 2.2. —
\[
\frac{\partial}{\partial w} \left[ w^r \varphi \left( \frac{v}{w} \right) \right] = rw^{r-1} \varphi \left( \frac{v}{w} \right) + w^r \sum_{i=1}^{k} \frac{\partial \varphi}{\partial v_i} \left( \frac{v}{w} \right) \left( - \frac{v_i}{w^2} \right) = w^{r-1} \varphi_1 \left( \frac{v}{w} \right),
\]
where \( \varphi_1(v) = r \varphi(v) - \sum_{i=1}^{k} v_i \frac{\partial \varphi}{\partial v_i}(v). \) Moreover, integrating by parts,
\[
\int \varphi_1 = r \int \varphi - \sum_{i=1}^{k} \int v_i \frac{\partial \varphi}{\partial v_i} = (r + k) \int \varphi
\]
and Lemma 2.2 follows by induction. \( \square \)

Proof of Proposition 2.1. — \( G \) is of class \( C^p \) on \( \mathbb{R}^k \times (0, +\infty), \) because
\[
G(u, w) = \int g(v) \frac{1}{w^k} \varphi \left( \frac{u - v}{w} \right) w^\sigma dv.
\]
(I) Assume first that \( \beta \leq \sigma \) and \( |\alpha| \leq p - \sigma. \) Then
\[
D^{(\alpha, \beta)}G(u, w) = \frac{1}{\sigma!} \int Dw^\alpha g(u - v)w^{\sigma - \beta - k} \varphi_{\beta} \left( \frac{v}{w} \right) dv = \frac{1}{\sigma!} w^{\sigma - \beta} \int Dw^\alpha g(u - v) \frac{1}{w^k} \varphi_{\beta} \left( \frac{v}{w} \right) dv \rightarrow \frac{1}{\sigma!} 0^{\sigma - \beta} Dw^\alpha g(u) \int \varphi_{\beta},
\]
when \( w \to 0, \) the convergence being uniform on compact subsets with respect to \( u. \) Consequently, the limit is 0, if \( \beta < \sigma \) and \( Dw^\alpha g \int \varphi, \) if \( \beta = \sigma. \)

(II) Now assume that \( \beta \leq \sigma \) and \( |\alpha| > p - \sigma. \) Then \( \alpha = \gamma + \delta, \) where \( |\gamma| = p - \sigma \) and \( \delta \neq 0. \)
\[
D^{(\gamma, \beta)}G(u, w) = \frac{1}{\sigma!} \int Dw^\gamma g(u - v)w^{\sigma - \beta - k} \varphi_{\beta} \left( \frac{v}{w} \right) dv = \frac{1}{\sigma!} \int Dw^\gamma g(v)w^{\sigma - \beta - k} \varphi_{\beta} \left( \frac{u - v}{w} \right) dv.
\]
\[
D^{(\alpha, \beta)}G(u, w) = \frac{1}{\sigma!} \int Dw^\gamma g(v)w^{\sigma - \beta - k} w^{-|\delta|} Dw^\delta \varphi_{\beta} \left( \frac{u - v}{w} \right) dv = \frac{1}{\sigma!} w^{\sigma - \beta - |\delta|} \int Dw^\gamma g(u - vw) Dw^\delta \varphi_{\beta}(v)dv.
\]
Notice that \( \sigma - \beta - |\delta| = p - |\alpha| - \beta \geq 0 \) and \( \int Dw^\delta \varphi_{\beta}(v)dv = 0, \) since \( \varphi_{\beta} \) has a compact support. Consequently, \( D^{(\alpha, \beta)}G(u, w) \rightarrow 0, \) when \( w \to 0. \)
(III) Finally, let $\beta > \sigma$. Then $|\alpha| \leq p - \beta < p - \sigma$ and $\beta = \sigma + \rho$, where $\rho > 0$. By the case (I),

$$D^{(\alpha, \sigma)} G(u, w) = \frac{1}{\sigma!} \int D^\alpha g(u - vw) \varphi_\sigma(v) dv,$$

$D^\alpha g$ being of class $p - \sigma - |\alpha| \geq \rho$, one obtains

$$D^{(\alpha, \beta)} G(u, w) = D^{(0, \rho)} (D^{(\alpha, \sigma)} G)(u, w)$$

$$= \frac{1}{\sigma!} \sum_{|\mu| = \rho} \int D^{\alpha + \mu} g(u - vw)(-v)^\mu \varphi_\sigma(v) dv,$$

which tends to $\sum_{|\mu| = \rho} \omega_\mu D^{\alpha + \mu} g(u)$ uniformly on compact subsets with respect to $u$, when $w \to 0$, where $\omega_\mu = \frac{1}{\sigma!} \int (-v)^\mu \varphi_\sigma(v) dv$. 

**Proposition 2.3.** — Let $\varphi \in \mathcal{C}^p(\mathbb{R}^k)$ be with compact support and such that $\int \varphi = 1$. Then the formula

$$L(g^{W^\sigma})(u, w) = \left( \frac{w}{|w|} \right)^\sigma \left[ \frac{1}{\sigma!} (g \ast \varphi_{|w|})(u)|w|^\sigma \right. - \left. \sum_{0 < |\gamma| \leq p - \sigma} \frac{1}{(\sigma + |\gamma|)!} \omega_\gamma D^{\gamma} g^{W^{\sigma + |\gamma|}}(u, |w|) \right],$$

for $\sigma \in \{1, \ldots, p\}$, $g \in \mathcal{C}^{p-\sigma}(\mathbb{R}^k)$, $u \in \mathbb{R}^k$ and $w \in \mathbb{R} \setminus \{0\}$, completed by putting

$$L(g^{W^\sigma})(u, 0) = 0,$$

and

$$L(g^{W^0}) = L(g) = g,$$ for $g \in \mathcal{C}^p(\mathbb{R}^k)$,

defines (inductively) a continuous linear extension operator $L = L_p : \mathcal{E}^p(\mathbb{R}^k \times 0) \to \mathcal{C}^p(\mathbb{R}^{k+1})$.

Moreover, there exists a constant $M > 0$ (depending only on $k, p$ and $\varphi$) such that if $\omega$ is a modulus of continuity of a field $F \in \mathcal{E}^p(\mathbb{R}^k \times 0)$, then $M \omega$ is a modulus of continuity of the $\mathcal{C}^p$-function $LF$.

**Proof.** — This follows immediately from Proposition 2.1.

Now we generalize our extension operator to any linear subspace of $\mathbb{R}^n$.

**Proposition 2.4.** — Let $\mathbb{R}^k \times 0 \subset \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^l$, where $l > 1$. Then the formula

$$L_p(g^{W_1^{\alpha_1} \cdots W_l^{\alpha_l}}) = L_p(L_{p-\alpha_1}(g^{W_1^{\alpha_1} \cdots W_{l-1}^{\alpha_{l-1}}})W_{l+1}^{\alpha_1}),$$

defines (inductively) a continuous linear extension operator $L = L_p : \mathcal{E}^p(\mathbb{R}^k \times 0) \to \mathcal{C}^p(\mathbb{R}^{k+1})$. 

**Proof.** — This follows immediately from Proposition 2.1.
where \( \alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbb{N}^l, |\alpha| \leq p \) and \( g \in C^{p-|\alpha|}(\mathbb{R}^k) \), defines by induction on \( l \) a linear continuous extension operator \( L = L_p : \mathcal{E}^p(\mathbb{R}^k \times 0) \rightarrow C^p(\mathbb{R}^n) \).

Moreover, there is a constant \( M > 0 \) such that if \( \omega \) is a modulus of continuity for \( F \in \mathcal{E}^p(\mathbb{R}^k \times 0) \), then \( M\omega \) is a modulus of continuity for \( LF \).

Proof. — This follows easily by induction from Proposition 2.3. \( \square \)

3. A generalization to the ideal of \( C^p \)-Whitney fields on \( \Omega \times 0 \) \( p \)-flat on \( \partial \Omega \times 0 \) (\( \Omega \) - an open \( \Lambda_p \)-regular cell in \( \mathbb{R}^k = \mathbb{R}^k \times 0 \subset \mathbb{R}^k \times \mathbb{R}^l \))

If \( A \) is any locally closed subset of \( \mathbb{R}^n \) and \( B \) any closed subset of \( A \), \( \mathcal{E}^p(A, B) \) will denote the ideal of all \( C^p \)-Whitney fields \( F \) on \( A \) \( p \)-flat on \( B \); i.e. \( F^{\alpha}(u) = 0 \), when \( |\alpha| \leq p \) and \( u \in B \). It is closed in \( \mathcal{E}^p(A) \).

Let first \( \Omega \) be any open subset of \( \mathbb{R}^k \). By the Hestenes Lemma (see [12], Lemma 4.3, p.80)

\[
\mathcal{E}^p(\Omega \times 0, \partial \Omega \times 0) = \{ F = \sum_{|\alpha| \leq p} \frac{1}{\alpha!} F^{\alpha} W^{\alpha} : F^{\alpha} \in C^{p-|\alpha|}(\Omega), \\
\lim_{u \to a} D^\beta F^{\alpha}(u) = 0, \text{ if } a \in \partial \Omega, |eta| \leq p - |\alpha| \},
\]

and putting

\[
\tilde{F}^{\alpha}(u) = \begin{cases} F^{\alpha}(u), & \text{if } u \in \Omega \\ 0, & \text{if } u \in \mathbb{R}^k \setminus \Omega \end{cases} \quad \text{and} \quad \tilde{F} = \sum_{\alpha} \frac{1}{\alpha!} \tilde{F}^{\alpha} W^{\alpha},
\]

one obtains a linear continuous extension operator

\[
\mathcal{E}^p(\Omega \times 0, \partial \Omega \times 0) \ni F \rightarrow \tilde{F} \in \mathcal{E}^p(\mathbb{R}^k \times 0, (\mathbb{R}^k \setminus \Omega) \times 0)
\]

preserving modulus of continuity.

Now we will consider the case when \( \Omega \) is an open \( \Lambda_p \)-regular cell in \( \mathbb{R}^k \) (cf. [6]). We will first recall the notion of \( \Lambda_p \)-regular mapping. Let \( \psi : D \rightarrow \mathbb{R}^m \) be a mapping on an open subset \( D \subset \mathbb{R}^n \). We say that \( \psi \) is \( \Lambda_p \)-regular (on \( D \)) if it is of class \( C^p \) and there is a constant \( C \geq 0 \) such that

\[
|D^\kappa \psi(x)| \leq C/d(x, \partial D)^{|\kappa|-1}, \text{ whenever } 1 \leq |\kappa| \leq p \text{ and } x \in D.
\]

Remark 3.1. — Let \( \psi \) be \( \Lambda_p \)-regular on \( D \). Then

1. it is \( \Lambda_p \)-regular on every open \( D' \subset D \);
(2) if $A \subset \Omega$ is a 1-regular subset, then the restriction $\psi|A$ is Lipschitz and thus it has a continuous extension $\overline{\psi}|A$ to $\overline{A}$.

We shall say (after [6]) that $S$ is an open $\Lambda_p$-regular (definable in a given o-minimal structure) cell in $\mathbb{R}^n$ iff

(1) $S$ is an open interval in $\mathbb{R}$, when $n = 1$;

(2) $S = \{(x', x_n) : x' \in T, \; \psi_1(x') < x_n < \psi_2(x')\}$, where $T$ is an open $\Lambda_p$-regular (definable) cell in $\mathbb{R}^{n-1}$ and each $\psi_i$ ($i = 1, 2$) is a function on $T$ being either real $\Lambda_p$-regular (definable) function on $T$, or identically equal to $-\infty$, or identically equal to $+\infty$, and $\psi_1(x') < \psi_2(x')$, for all $x' \in T$, when $n > 1$.

Remark 3.2. — Such a cell $S$ is 1-regular and if $\psi_i$ is finite it is Lipschitz on $T$, thus it admits a continuous extension $\overline{\psi}_i$ to $\overline{T}$.

For any open (definable) $\Lambda_p$-regular cell in $\mathbb{R}^n$, one defines, by induction on $n$, a sequence $\rho_j : \overline{S} \longrightarrow \mathbb{R} \cup \{+\infty\} (j = 1, \ldots, 2n)$ of the functions associated with the cell $S$:

(1) When $n = 1$ and $S = (a_1, a_2)$, we put

$$\rho_1(x) = \begin{cases} x - a_1, & \text{if } a_1 \in \mathbb{R} \\ +\infty, & \text{if } a_1 = -\infty \end{cases} \quad \text{and} \quad \begin{cases} \rho_2(x) = a_2 - x, & \text{if } a_2 \in \mathbb{R} \\ +\infty, & \text{if } a_2 = +\infty. \end{cases}$$

(2) When $n > 1$ and $S = \{(x', x_n) : x' \in T, \; \psi_1(x') < x_n < \psi_2(x')\}$, let $\sigma_j (j = 1, \ldots, 2n - 2)$ be the functions associated with $T$. We put, for any $x = (x', x_n) \in \overline{S}$, $\rho_j(x) = \sigma_j(x')$ for $j = 1, \ldots, 2n - 2$ and

$$\rho_{2n-1}(x) = \begin{cases} x_n - \overline{\psi}_1(x'), & \text{if } \psi_1 : T \rightarrow \mathbb{R} \\ +\infty, & \text{if } \psi_1 \equiv -\infty \end{cases} \quad \text{and} \quad \rho_{2n}(x) = \begin{cases} \overline{\psi}_2(x') - x_n, & \text{if } \psi_2 : T \rightarrow \mathbb{R} \\ +\infty, & \text{if } \psi_2 \equiv +\infty. \end{cases}$$

Remark 3.3 ([6], Lemma 3). — There exists a constant $\Theta > 0$ such that

$$\Theta \min_j \rho_j(x) \leq d(x, \partial S) \leq \min_j \rho_j(x), \quad \text{for } x \in \overline{S}.$$

(We adopt the convention: $d(x, \emptyset) = +\infty$.)

Remark 3.4 ([6], Lemma 4). — The functions $\rho_j$ which are finite are $\Lambda_p$-regular on $S$, Lipschitz on $\overline{S}$ and definable, if $S$ is so.

Lemma 3.5 (cf. [6], Lemma 5). — Let $\varphi_\nu : \Omega \longrightarrow \mathbb{R} \; \; (\nu = 1, \ldots, m)$ be $\Lambda_p$-regular functions on an open subset $\Omega \subset \mathbb{R}^k$. Assume that $r(u) :=$
\[
\left(\sum_{\nu=1}^{m} \varphi_{\nu}^{2}(u)\right)^{\frac{1}{2}} \neq 0 \text{ for each } u \in \Omega. \text{ Then there exists a constant } \tilde{C} > 0 \text{ such that for each } u \in \Omega
\]
\[
\left| D^{\alpha} \left( \frac{1}{r} \right)(u) \right| \leq \frac{\tilde{C}}{r(u) \min(r(u), d(u, \partial \Omega)) |\alpha|}, \text{ where } 0 \leq |\alpha| \leq p;
\]
consequently
\[
\left| D^{\alpha} \left( \frac{1}{r} \right)(u) \right| \leq \frac{\tilde{C}}{\min(r(u), d(u, \partial \Omega)) |\alpha|+1}.
\]

Proof. — Induction on $|\alpha|$. \hfill \Box

Proposition 3.6 (cf. [6], Lemmas 6-7). — Let $\Omega$ be an open subset of $\mathbb{R}^k$, let $f \in C^{\ast}(\Omega \times \mathbb{R}^l)$ and $r \in C^{\ast}(\Omega)$, and let $t : \Omega \to (0, +\infty)$ be any positive function such that $t(u) \leq d(u, \partial \Omega)$ for any $u \in \Omega$. Let $\varepsilon > 0$ and put
\[
\Delta_{\varepsilon} := \{(u, w) \in \Omega \times \mathbb{R}^l : |w| < \varepsilon t(u)\}.
\]

Assume that there exists a constant $\tilde{C} > 0$ such that $|D^{\alpha} \left( \frac{1}{r} \right)| \leq \frac{\tilde{C}}{t|\alpha|+1}$, when $\alpha \in \mathbb{N}^k$, and for each $c \in \partial \Omega$, $D^{\ast} f(u, w) = o(t(u)^{p-|\kappa|})$, when $\Delta_{\varepsilon} \ni (u, w) \to (c, 0)$ and $\kappa \in \mathbb{N}^k \times \mathbb{N}^l, |\kappa| \leq p$.

Let $\xi : \mathbb{R} \to \mathbb{R}$ be any $C^{\ast}$-function. Fix $i \in \{1, \ldots, l\}$ and put
\[
g(u, w) := \xi\left(\frac{w_i}{r(u)}\right) f(u, w), \text{ for } (u, w) \in \Omega \times \mathbb{R}^l.
\]

Then for each $c \in \partial \Omega$, $D^{\ast} g(u, w) = o(t(u)^{p-|\kappa|})$, when $\Delta_{\varepsilon} \ni (u, w) \to (c, 0)$ and $\kappa \in \mathbb{N}^k \times \mathbb{N}^l, |\kappa| \leq p$.

Proof. — Put $h(u, w) = \xi\left(\frac{w_i}{r(u)}\right)$. By the Leibniz formula
\[
D^{\ast} g = \sum_{\lambda \leq \kappa} \binom{\kappa}{\lambda} D^{\lambda} h D^{\kappa - \lambda} f, \text{ so it suffices to check that there exists a constant } C'_{\varepsilon} > 0 \text{ such that } |D^{\lambda} h(u, w)| \leq C'_{\varepsilon} t(u)^{-|\lambda|}, \text{ when } (u, w) \in \Delta_{\varepsilon} \text{ and } |\lambda| \leq p. \text{ First, it is easy to see this for } h_0(u, w) := \frac{w_i}{r(u)} \text{ using Lemma 3.5.}
\]

Then for $h = \xi \circ h_0$ we have
\[
\frac{\partial h}{\partial x_j} = (\xi' \circ h_0) \frac{\partial h_0}{\partial x_j}, \text{ where } (x_1, \ldots, x_n) = (u_1, \ldots, u_k, w_1, \ldots, w_l)
\]
and
\[
D^{\lambda} \left( \frac{\partial h}{\partial x_j} \right) = \sum_{\mu \leq \lambda} \binom{\lambda}{\mu} D^{\mu} (\xi' \circ h_0) D^{\lambda - \mu} \left( \frac{\partial h_0}{\partial x_j} \right), \text{ if } |\lambda| \leq p - 1, \text{ so we conclude by induction.} \hfill \Box
Remark 3.7. — Suppose that $f$ is a $C^p$-function on the whole space $\mathbb{R}^k \times \mathbb{R}^l$ and such that for each $c \in \partial \Omega$, $D^\kappa f(u, 0) = o(t(u)^{p-|\kappa|})$, when $\Omega \ni u \rightarrow c$ and $\kappa \in \mathbb{N}^k \times \mathbb{N}^l$, $|\kappa| \leq p$.

Then for each $c \in \partial \Omega$, $D^\kappa f(u, w) = o(t(u)^{p-|\kappa|})$, when $\Delta_c \ni (u, w) \rightarrow (c, 0)$ and $\kappa \in \mathbb{N}^k \times \mathbb{N}^l$, $|\kappa| \leq p$. This follows immediately from the Taylor formula

$$D^\kappa f(u, w) = \sum_{|\lambda| \leq p-|\kappa|} \frac{1}{\lambda!} D^{\kappa+(0,\lambda)} f(u, 0) w^\lambda + o(|w|^{p-|\kappa|}),$$

when $u \rightarrow c, w \rightarrow 0$.

Let now $\Omega$ be an open $\Lambda_p$-regular cell in $\mathbb{R}^k$ and $\rho_j$ ($j = 1, \ldots, 2k$) - the functions associated with $\Omega$. We define an extension operator

$$\mathcal{L} : \mathcal{E}^p(\overline{\Omega} \times 0, \partial \Omega \times 0) \longrightarrow C^p(\mathbb{R}^n), \quad \text{where} \quad \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^l,$$

by the following formula

$$\mathcal{L} F(u, w) = \begin{cases} \prod_{i=1}^l \prod_{j=1}^{2k} \xi(Q \frac{w_i}{\rho_j(u)}) (L\tilde{F})(u, w), & \text{if } u \in \Omega \\ 0, & \text{if } u \in \mathbb{R}^k \setminus \Omega, \end{cases}$$

where $Q$ is any real number $> \sqrt{l} \Theta^{-1}$, $\Theta$ is a constant from Remark 3.3 and $\xi : \mathbb{R} \longrightarrow \mathbb{R}$ is a (definable, if we wish) $C^p$-function equal to 1 in a neighborhood of 0, and equal to 0 outside the open interval $(-1, 1)$.

To check that $\mathcal{L} F \in C^p(\mathbb{R}^n)$ we use repeatedly Proposition 3.6 with $r = \rho_j \neq +\infty$ and $t(u) = d(u, \partial \Omega)(\text{at the beginning we take } f = L\tilde{F} \text{ as in Remark 3.7})$ and the Hestenes Lemma. The factors involving $\rho_j \equiv +\infty$ being obviously 1 can be omitted in the above formula.

Observe that if $\varepsilon$ is any constant from $(0, 1)$, we can choose $Q$ in such a way that $\mathcal{L} F$ is $p$-flat outside the set

$$\Delta_c(\Omega \times 0) := \{x \in \mathbb{R}^n : d(x, \overline{\Omega} \times 0) < \varepsilon d(x, \partial \Omega \times 0)\}$$

$$= \{(u, w) \in \Omega \times \mathbb{R}^l : |w| < \frac{\varepsilon}{\sqrt{1-\varepsilon^2}} d(u, \partial \Omega)\}.$$

Remark 3.8. — If $r$ and $t$ are as in Proposition 3.6 and $F \in \mathcal{E}^p(\overline{\Omega} \times 0, \partial \Omega \times 0)$ is such that, for each $c \in \partial \Omega$, $F^\kappa(u, 0) = o(t(u)^{p-|\kappa|})$, when $\Omega \ni u \rightarrow c$ and $|\kappa| \leq p$, the above formula for an extension of $F$ can be modified by putting

$$\mathcal{L}' F(u, w) = \begin{cases} \prod_{i=1}^l \xi\left(\sqrt{l} \frac{w_i}{r(u)}\right) \mathcal{L} F(u, w), & \text{if } u \in \Omega \\ 0, & \text{if } u \in \mathbb{R}^k \setminus \Omega. \end{cases}$$
Then $\mathcal{L}'F$ is $p$-flat, outside the neighborhood \{(u, w) \in \Omega \times \mathbb{R}^l : |w| < r(u)\} of \Omega \times 0$ and outside $\Delta_\varepsilon(\Omega \times 0)$.

In order that $\mathcal{L}F$ (or $\mathcal{L}'F$) and $F$ have the same (up to a multiplicative constant) modulus of continuity we will prove the following

**Proposition 3.9.** — Under the assumptions of Proposition 3.6 assume additionally that $\Omega$ is 1-regular, $r \in C^{p+1}(\Omega)$ such that

\[ |D^\alpha \left( \frac{1}{r} \right) | \leq \frac{\tilde{c}}{t|\alpha|+1}, \text{ when } \alpha \in \mathbb{N}^k, |\alpha| \leq p + 1 \]

and $t$ is Lipschitz. Then there exists a constant $M > 0$ such that if $\omega$ is a modulus of continuity for $f$ on $\Delta_\varepsilon$ satisfying

\[ |D^\kappa f(u, w)| \leq \omega(t(u))t(u)^{p-|\kappa|}, \]

when $(u, w) \in \Delta_\varepsilon$ and $|\kappa| \leq p$, then $M\omega$ is a modulus of continuity for $g$ on $\Delta_\varepsilon$ satisfying

\[ |D^\kappa g(u, w)| \leq M\omega(t(u))t(u)^{p-|\kappa|}, \]

when $(u, w) \in \Delta_\varepsilon$ and $|\kappa| \leq p$.

**Proof.** — In view of the proof of Proposition 3.6, it suffices to check that, for a constant $M > 0$, $M\omega$ is a modulus of continuity for $g$ on $\Delta_\varepsilon$. First observe that $\Delta_\varepsilon$ is 1-regular, because $\Omega$ is so and the function $t$ is Lipschitz. There exists a constant $C \geq 1$ such that $|t(u_1) - t(u_2)| \leq C|u_1 - u_2|$, for any $u_1, u_2 \in \Omega$.

Fix any $\kappa \in \mathbb{N}^{k+l}$ such that $|\kappa| = p$, any $\lambda \leq \kappa$ and any two points $x_i = (u_i, w_i) \in \Delta_\varepsilon$ ($i = 1, 2$). We have to estimate

\[ |D^\lambda h(x_1)D^{\kappa-\lambda}f(x_1) - D^\lambda h(x_2)D^{\kappa-\lambda}f(x_2)|. \]

**Case I:** $t(u_i) \leq 2C|x_1 - x_2|$ ($i = 1, 2$).

Then $|D^\lambda h(x_1)D^{\kappa-\lambda}f(x_i)| \leq C't(u_i)^{|\lambda|}\omega(t(u_i))t(u_i)^{p-|\kappa-\lambda|}$

\[ \leq C''\omega(2C|x_1 - x_2|) \leq 2CC''\omega(|x_1 - x_2|). \]

**Case II:** $t(u_1) > 2C|x_1 - x_2|$.

Then $|u_1 - u_2| \leq C|x_1 - x_2| < \frac{1}{2}t(u_1) \leq \frac{1}{2}d(u_1, \Omega)$; thus $[x_1, x_2] \subset \Omega \times \mathbb{R}^l$.

We have $|D^\lambda h(x_1)[D^{\kappa-\lambda}f(x_1) - D^{\kappa-\lambda}f(x_2)]| \leq |D^\lambda h(x_1)||x_1 - x_2|^{p-|\kappa-\lambda|} [\sum_{1\leq |\mu| \leq p-|\kappa-\lambda|} \frac{1}{|\mu|} |D^{\kappa-\lambda+\mu}f(x_1)||x_1 - x_2||\mu| + \omega(|x_1 - x_2|)|x_1 - x_2|^{p-|\kappa-\lambda|}]$.

\[ M_1\omega(t(u_1))t(u_1)^{-1}|x_1 - x_2| + M_2\omega(|x_1 - x_2|) \leq M'\omega(|x_1 - x_2|), \]
where \( M_1, M_2 \) and \( M' \) are positive constants and we use: \( \omega(s)t \leq \omega(t)s \) if \( t \leq s \).

On the other hand

\[
\sup_{x \in [x_1, x_2]} \left| D^{\lambda}(h(x_1) - D^{\lambda}(h(x_2)))D^{\gamma-\lambda}f(x_2) \right| \leq \sum_{k=0}^{k+1} \left| D^{\lambda+\gamma}(h(x)) \right| \left| x_1 - x_2 \right| \left| D^{\gamma-\lambda}f(x_2) \right|.
\]

For any \( x = (u, w) \in [x_1, x_2] \), \( 2|t(u_1) - t(u)| \leq 2C|u_1 - u| \leq 2C|x_1 - x_2| \)
\(< t(u_1) \) and \( 2|w_1 - w| \leq 2C|x_1 - x_2| < t(u_1) \); thus \( \frac{1}{2}t(u_1) < t(u) < \frac{3}{2}t(u_1) \)
and \( |w| \leq |w_1| + |w_1 - w| < \varepsilon t(u_1) + t(u) \leq (2\varepsilon + 1)t(u) \).

Consequently \( x \in \Delta_{2\varepsilon+1} \) and

\[
|D^{\lambda+\gamma}(h(x))| \leq C'_{2\varepsilon+1}t(u)^{-|\gamma|} \leq 2|\lambda|+1C'(2\varepsilon+1)^{-|\gamma|}
\]
and

\[
|D^{\gamma-\lambda}f(x_2)| \leq \omega(t(u_2))t(u_2)^{|\gamma|} \leq \left( \frac{3}{2} \right)^{|\lambda|+1} \omega(t(u_1))t(u_1)^{|\lambda|}.
\]

The needed inequality follows. \( \square \)

**Remark 3.10.** — Suppose that \( f \) is a \( C^p \)-function on the whole space \( \mathbb{R}^k \times \mathbb{R}^l \) and \( \omega \) is its modulus of continuity such that

\[
|D^{\gamma}f(u, 0)| \leq \omega(t(u))t(u)^{p-|\gamma|},
\]
when \( u \in \Omega \) and \( \gamma \in \mathbb{N}^{k+l}, |\gamma| \leq p \).

Then there exists a constant \( M'' > 0 \) such that

\[
|D^{\gamma}f(u, w)| \leq M'' \omega(t(u))t(u)^{p-|\gamma|},
\]
when \( (u, w) \in \Delta_{\varepsilon}, \gamma \in \mathbb{N}^{k+l}, |\gamma| \leq p \).

Indeed, this follows immediately from

\[
|D^{\gamma}f(u, w) - \sum_{|\lambda| \leq p-|\gamma|} \frac{1}{\lambda!}D^{\gamma+0, \lambda}(f(u, 0))w^{\lambda}| \leq \omega(|w|)|w|^{p-|\gamma|}.
\]

**Remark 3.11.** — If \( \Omega \) is an open \( \Lambda_{p+1} \)-regular cell in \( \mathbb{R}^k \) and \( \xi \) is a \( C^{p+1} \)-function, then there exists a positive constant \( M \), such that, for any \( F \in \mathcal{E}^p(\overline{\Omega} \times 0, \partial \Omega \times 0) \) (respectively, fulfilling additional conditions: \( |F^{\gamma}(u, 0)| \leq \omega(r(u))r(u)^{p-|\gamma|}, \) when \( u \in \Omega, \gamma \in \mathbb{N}^{k+l}, |\gamma| \leq p \)) if \( \omega \) is a modulus of continuity for \( F \), then \( M\omega \) is a modulus of continuity for \( LF \) (respectively, for \( L'F \)).
4. A generalization to the ideal of \( C^p \)-Whitney fields on the closure of a \( \Lambda_p \)-regular leaf \( p \)-flat on its boundary

Now we will transpose the extension operator \( \mathcal{L} \) to the closure of any \( \Lambda_p \)-regular leaf. A subset \( E \subset \mathbb{R}^n \) is called a (definable) \( \Lambda_p \)-regular leaf of dimension \( k \) in \( \mathbb{R}^n \) if it is the graph \( E = \{(u, \varphi(u)) : u \in \Omega \} \) of a (definable) \( \Lambda_p \)-regular mapping \( \varphi : \Omega \rightarrow \mathbb{R}^l \) defined on an open (definable) \( \Lambda_p \)-regular cell \( \Omega \) in \( \mathbb{R}^k \). A reduction of this case to the previous one will be by the following Lipschitz automorphism

\[
\overline{\Omega} \times \mathbb{R}^l \ni (u, w) \mapsto (u, w + \varphi(u)) \in \overline{\Omega} \times \mathbb{R}^l
\]

and the following

**Proposition 4.1** (cf. [6], Proposition 3). — Let \( \varphi : \Omega \rightarrow \mathbb{R}^l \) be a \( \Lambda_p \)-regular mapping defined on an open subset \( \Omega \subset \mathbb{R}^k \). Let \( t : \Omega \rightarrow (0, +\infty) \) be any function such that \( t(u) \leq d(u, \partial \Omega) \), for each \( u \in \Omega \). Let \( E \) be any closed subset of \( \Omega \times \mathbb{R}^l \) and

\[
F(u, w; U, W) = \sum_{|\alpha| + |\beta| \leq p} \frac{1}{\alpha! \beta!} F^{(\alpha, \beta)}(u, w) U^\alpha W^\beta \quad \begin{cases} U = (U_1, \ldots, U_k), \\ W = (W_1, \ldots, W_l) \end{cases}
\]

a \( C^p \)-Whitney field on \( E \) such that, for any \( c \in \partial \Omega \)

\[
F^{(\alpha, \beta)}(u, w) = o(t(u)^{|\alpha| - |\beta|}) \quad \text{when} \quad u \to c \quad \text{and} \quad |\alpha| + |\beta| \leq p.
\]

Let \( F_\varphi(u, v; U, V) \) be a polynomial in \( (U, V) \) of degree \( \leq p \) such that

\[
F_\varphi(u, v; U, V) = \sum_{|\alpha| + |\beta| \leq p} \frac{1}{\alpha! \beta!} F^{(\alpha, \beta)}(u, v + \varphi(u)) U^\alpha \\
\left( V + \sum_{1 \leq |\kappa| \leq p} \frac{1}{\kappa!} D^\kappa \varphi(u) U^\kappa \right)^p \mod (U, V)^{p+1}
\]

defined for \( (u, v) \in E_\varphi \), where \( E_\varphi = \{(u, v) \in \Omega \times \mathbb{R}^l : (u, v + \varphi(u)) \in E \} \).

Then \( F_\varphi \) is a \( C^p \)-Whitney field on \( E_\varphi \) such that, for any \( c \in \partial \Omega \)

\[
F^{(\alpha, \beta)}_\varphi(u, v) = o(t(u)^{|\alpha| - |\beta|}) \quad \text{when} \quad u \to c \quad \text{and} \quad |\alpha| + |\beta| \leq p.
\]

**Proof.** — It is easy to check that \( F_\varphi \) fulfills the condition (***) from Introduction, thus it is a \( C^p \)-Whitney field on \( E_\varphi \). Besides

\[
F_\varphi(u, v; U, V) = \sum_{|\alpha| + |\beta| \leq p} \frac{1}{\alpha! \beta!} F^{(\alpha, \beta)}(u, v + \varphi(u)) U^\alpha \times \\
\sum_{\gamma + \sum_{\delta} \delta = \beta} \frac{\beta!}{\gamma! \prod \delta!} V^\gamma \prod_{\kappa} \left[ \frac{1}{\kappa! \delta^\kappa} U^{|\kappa| \delta^\kappa} D^\kappa \varphi(u)^\delta \right] \mod (U, V)^{p+1},
\]

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thus
\[ F^{(\sigma, \gamma)}(u, v) = \sum_{\alpha + \sum_{\kappa} |\delta_{\kappa}|} [. \] F^{(\alpha, \gamma + \sum_{\kappa} \delta_{\kappa})}(u, v + \varphi(u)) \prod_{\kappa} (D^\kappa \varphi(u))^{\delta_{\kappa}}, \]
where [. ] denotes constants. To conclude notice that
\[ F^{(\alpha, \gamma + \sum_{\kappa} \delta_{\kappa})}(u, v + \varphi(u)) \prod_{\kappa} (D^\kappa \varphi(u))^{\delta_{\kappa}} = \]
o(1)t(u)^{p - |\alpha| - |\gamma| - \sum_{\kappa} |\delta_{\kappa}|} C \prod_{\kappa} d(u, \partial \Omega)^{-|\delta_{\kappa}|}|| \sum_{\kappa} |\delta_{\kappa}| + |\delta_{\kappa}| = o(t(u)^{p - |\sigma| - |\gamma|}).

\[ \square \]

Remark 4.2. — If \( E = \{(u, \varphi(u)) : u \in \Omega\} \) (resp. \( E = \Omega \times \mathbb{R}^l \)), then \( F_\varphi \) extends to a \( C^p \)-Whitney field on \( \overline{E}_\varphi = \overline{\Omega} \times 0 \) (resp. \( \overline{E}_\varphi = \overline{\Omega} \times \mathbb{R}^l \)) \( p \)-flat on \( \partial E_\varphi = \partial \Omega \times 0 \) (resp. \( \partial E_\varphi = \partial \Omega \times \mathbb{R}^l \)).

Proof. — The both cases follow from the Hestenes Lemma. \[ \square \]

Proposition 4.3. — Under the assumptions of Proposition 4.1, assume additionally that the mapping \( \varphi \) is \( \Lambda_{p+1} \)-regular, \( E \) and \( \Omega \) are both \( 1 \)-regular and \( \overline{E} \) and \( \partial \Omega \times \mathbb{R}^l \) are simply separated \( \ast \). Then there exists a constant \( M > 0 \) such that, for each \( F \in \mathcal{E}^p(\overline{E}, \partial E) \), if \( \omega \) is a modulus of continuity of \( F \), then \( M \omega \) is a modulus of continuity of \( F_\varphi \).

Moreover, if \( |F^\kappa(u, w)| \leq \omega(t(u))t(u)^{p - |\kappa|} \), when \( (u, w) \in E \) and \( |\kappa| \leq p \), then \( |F^\kappa(u, v)| \leq M \omega(t(u))t(u)^{p - |\kappa|} \), when \( (u, v) \in E_\varphi \) and \( |\kappa| \leq p \).

Proof. — Observe that \( E_\varphi \) is \( 1 \)-regular. Let \( \sigma \in \mathbb{N}^k \), \( \gamma \in \mathbb{N}^l \) be such that \( |\sigma| + |\gamma| = p \) and let \((u_i, v_i) \in E_\varphi, (i = 1, 2) \). We have to estimate
\[ |F^{(\sigma, \gamma)}(u_1, v_1) - F^{(\sigma, \gamma)}(u_2, v_2)| \leq \]
\[ \sum_{\alpha + \sum_{\kappa} |\delta_{\kappa}|} [. \] F^{(\alpha, \gamma + \sum_{\kappa} \delta_{\kappa})}(u_1, v_1 + \varphi(u_1)) \prod_{\kappa} (D^\kappa \varphi(u_1))^{\delta_{\kappa}} - \]
F^{(\alpha, \gamma + \sum_{\kappa} \delta_{\kappa})}(u_2, v_2 + \varphi(u_2)) \prod_{\kappa} (D^\kappa \varphi(u_2))^{\delta_{\kappa}} |.

Fix \( \lambda = (\alpha, \gamma + \sum_{\kappa} \delta_{\kappa}) \) and put \( x_i = (u_i, v_i + \varphi(u_i)) \) and
\[ \theta(u) = \prod_{\kappa} (D^\kappa \varphi(u))^{\delta_{\kappa}}. \]

(*) See the beginning of Section 5 for the definition of simple separation.
Case I: \(|x_1 - x_2| \geq \frac{1}{2}d(u_i, \partial \Omega)\) for \(i = 1, 2\).
\[
|F^\lambda(x_i)\theta(u_i)| \leq \omega(d(x_i, \partial E))d(x_i, \partial E)^{p-|\lambda|}|\theta(u_i)| \leq \omega(Cd(u_i, \partial \Omega)[Cd(u_i, \partial \Omega)]^{p-|\lambda|}|\theta(u_i)| \leq \omega(2C|x_1 - x_2|)[Cd(u_i, \partial \Omega)]^{p-|\lambda|}\prod_{\kappa}d(u_i, \partial \Omega)^{-|\kappa|+[\delta_\kappa]} \leq M\omega(|x_1 - x_2|).
\]

Case II: \(|x_1 - x_2| \leq \frac{1}{2}d(u_1, \partial \Omega)\).
\[
|F^\lambda(x_1)\theta(u_1) - F^\lambda(x_2)\theta(u_2)| \leq \sum_{1 \leq |\mu| \leq p-|\lambda|} \frac{1}{\mu!}|F^{\lambda+\mu}(x_1)||x_2 - x_1|^{|\mu|} + \omega(|x_1 - x_2|)|x_1 - x_2|^{p-|\lambda|} \prod_{\kappa}d(u_i, \partial \Omega)^{-|\kappa|+[\delta_\kappa]} \leq \! M\omega(|x_1 - x_2|) \cdot \square \!
\]

Now it suffices to observe that \(\omega(d(u_1, \partial \Omega))|x_1 - x_2| \leq \omega(|x_1 - x_2|)d(u_1, \partial \Omega)\) and \(d(z, \partial \Omega) \geq d(u_1, \partial \Omega) - |z - u_1| \geq d(u_1, \partial \Omega) - |x_1 - x_2| \geq \frac{1}{2}d(u_1, \partial \Omega),\)
if \(z \in [u_1, u_2]\).

Assume now that \(E = \{(u, \varphi(u)) : u \in \Omega\}\) is a \(\Lambda_p\)-regular leaf of dimension \(k\) in \(\mathbb{R}^n\). We define an extension operator \(L : E^p(E, \partial E) \to C^p(\mathbb{R}^n)\) by the formula
\[
LF = \begin{cases} 
(LF_\varphi)_{-\varphi}, & \text{on } \Omega \times \mathbb{R}^l \\
0, & \text{on } (\mathbb{R}^k \setminus \Omega) \times \mathbb{R}^l.
\end{cases}
\]
For any constant \( \varepsilon > 0 \), we can specify this operator in such a way that for each \( F \in \mathcal{E}^p(\overline{E}, \partial E) \), \( LF \) is flat outside the neighborhood \( \Delta_\varepsilon(E) := \{ x \in \mathbb{R}^n : d(x, E) < \varepsilon d(x, \partial E) \} \).

5. A generalization to a finite tower of \( \Lambda_p \)-regular leaves

Here we will generalize the extension operator \( L \) to the ideal \( \mathcal{E}^p(\overline{E}, \partial E) \), where \( E \) is a finite disjoint union \( E = E_1 \cup \cdots \cup E_s \) of graphs of \( \Lambda_p \)-regular mappings \( \varphi_\sigma : \Omega \to \mathbb{R}^l \) (\( \sigma = 1, \ldots, s \)) defined on a common open \( \Lambda_p \)-regular cell \( \Omega \subset \mathbb{R}^k \). Put \( r_\sigma(u) := |\varphi_\sigma(u) - \varphi_s(u)| \) for \( \sigma = 1, \ldots, s-1 \) and \( u \in \Omega \).

We first define \( LF \) for any \( F \in \mathcal{E}^p(\overline{E}, \overline{E_1} \cup \cdots \cup \overline{E_{s-1}} \cup \partial E_s) \).

Then we put
\[
LF = \begin{cases}
\prod_{\sigma=1}^{s-1} \prod_{i=1}^l \xi \left( \sqrt{1 - \frac{w_i}{r_\sigma(u)}} \right) \mathcal{L}(F|\overline{E}_s)_{\varphi_s}, & \text{on } \Omega \times \mathbb{R}^l \\
0, & \text{on } (\mathbb{R}^k \setminus \Omega) \times \mathbb{R}^l,
\end{cases}
\]
which gives an extension operator according to Proposition 3.6 (used repeatedly with \( t(u) := \min \{ r_\sigma(u) \}, d(u, \partial \Omega) \) ), Remark 3.8 and Proposition 4.1.

Let now consider a general case where \( F \) is any element of \( \mathcal{E}^p(\overline{E}, \partial E) \). Proceeding by induction, assume that \( \mathcal{L}(F|\overline{E_1} \cup \cdots \cup \overline{E_{s-1}}) \) has already been defined. Then \( H := F - T \mathcal{L}(F|\overline{E_1} \cup \cdots \overline{E_{s-1}}|\overline{E} \in \mathcal{E}^p(\overline{E}, \overline{E_1} \cup \cdots \cup \overline{E_{s-1}} \cup \partial E_s) \) and we put
\[
LF = LH + \mathcal{L}(F|\overline{E_1} \cup \cdots \overline{E_{s-1}}).
\]
For any \( \varepsilon > 0 \), we can specify this operator in such a way that \( LF \) is \( p \)-flat outside the set \( \Delta_\varepsilon(E) := \{ x \in \mathbb{R}^n : d(x, E) < \varepsilon d(x, \partial E) \} \).

6. Extension operator for a closed definable subset of \( \mathbb{R}^n \)

**Definition 6.1** (cf. [10]). — Let \( A, B, Z \subset \mathbb{R}^n \). We say that \( A \) and \( B \) are simply \( Z \)-separated if one of the following equivalent conditions holds

1. \( \exists M > 0 \forall x \in A, \ d(x, B) \geq Md(x, Z) \);
2. \( \exists C > 0 \forall x \in \mathbb{R}^n, \ d(x, A) + d(x, B) \geq Cd(x, Z) \). (If (1) holds, one can take \( C = M/(M+1) \).)
We say that $A$ and $B$ are simply separated if they are simply $A \cap B$-separated.

**Proposition 6.2.** — Let $E_i \supset E'_i$ ($i = 1, \ldots, s$) be closed subsets of $\mathbb{R}^n$ and let $C > 0$ be a constant such that, for any $i, j \in \{1, \ldots, s\}, i \neq j$ and any $x \in \mathbb{R}^n$

$$d(x, E_i) + d(x, E_j) \geq Cd(x, E'_i).$$

Let $\varepsilon \in (0, C/2]$. Put $\Gamma_\varepsilon(E_i, E'_i) := \{x \in \mathbb{R}^n : d(x, E_i) < \varepsilon d(x, E'_i)\}$. Suppose that, for each $i = 1, \ldots, s$

$$\mathcal{L}_i : \mathcal{E}^p(E_i, E'_i) \rightarrow C^p(\mathbb{R}^n)$$

is an extension operator such that $\mathcal{L}_i F$ is $p$-flat outside $\Gamma_\varepsilon(E_i, E'_i)$, for any $F \in \mathcal{E}^p(E_i, E'_i)$.

Then the formula

$$\mathcal{L}F = \sum_{i=1}^{s} \mathcal{L}_i(F|E_i)$$

defines an extension operator $\mathcal{L} : \mathcal{E}^p(\bigcup_i E_i, \bigcup_i E'_i) \rightarrow C^p(\mathbb{R}^n)$. Moreover, if each $\mathcal{L}_i$ preserves (up to a multiplicative constant) a modulus of continuity, then $\mathcal{L}$ has the same property.

**Proof.** — It suffices to check that $\Gamma_\varepsilon(E_i, E'_i) \cap \Gamma_\varepsilon(E_j, E'_j) = \emptyset$, if $i \neq j$. If there were $x \in \Gamma_\varepsilon(E_i, E'_i) \cap \Gamma_\varepsilon(E_j, E'_j)$, then

$$2\varepsilon[d(x, E'_i) + d(x, E'_j)] > 2[d(x, E_i) + d(x, E_j)] \geq C[d(x, E'_i) + d(x, E'_j)],$$

a contradiction. \qed

A proof of the following theorem will be given in the next section.

**$\Lambda_p$-regular Decomposition Theorem 6.3.** — Let $E$ be a closed subset of $\mathbb{R}^n$ definable in some fixed o-minimal structure on the ordered field of the real numbers $\mathbb{R}$. Let $k = \dim E$. Let $Z$ be any definable subset of $E$ of dimension $< k$.

Then there exists a finite decomposition

$$E = M_1 \cup \cdots \cup M_s \cup A$$

such that each $M_i$ is a finite tower of $\Lambda_p$-regular $k$-dimensional definable leaves in an appropriate linear coordinate system, $A$ is a closed definable subset of $\dim < k$ containing $Z$ and, for any $i, j \in \{1, \ldots, s\}$ ($i \neq j$), $M_i$ and $M_j$ are simply $\partial M_i$-separated and, for any $i$, $\overline{M_i}$ and $A$ are simply $\partial M_i$-separated.
In order to define an extension operator for any closed definable subset $E \subset \mathbb{R}^n$ we will use induction on $\dim E$. By the induction hypothesis we have an extension operator
\[ \mathcal{L}_0 : \mathcal{E}^p(\bigcup_{i=1}^s \partial M_i \cup A) \rightarrow \mathcal{C}^p(\mathbb{R}^n), \]
and by Section 5 combined with Proposition 6.2 we have an extension operator
\[ \mathcal{L}_1 : \mathcal{E}^p(E, \bigcup_{i=1}^s \partial M_i \cup A) \rightarrow \mathcal{C}^p(\mathbb{R}^n). \]
Now an extension operator for $E$ is defined by the formula
\[ \mathcal{L}F = \mathcal{L}_1[F - T\mathcal{L}_0(F| \bigcup_i \partial M_i \cup A)|E] + \mathcal{L}_0(F| \bigcup_i \partial M_i \cup A). \]

7. Proof of $\Lambda_p$-regular Decomposition Theorem

Let $P \subset \mathbb{R}^n$ be any definable subset and $V$ - a linear subspace of $\mathbb{R}^n$ of dimension $n - k$. Following [10], we will say that $P$ is perfectly situated relative to $V$ if, for a/any linear complement $W$ of $V$ in $\mathbb{R}^n$, $P$ can be represented as a disjoint union
\[ P = \bigcup \{ \hat{\varphi} : \varphi \in \mathcal{F} \} \]
of a finite family $\mathcal{F}$ of definable $C^1$-mappings $\varphi : \Delta \varphi \rightarrow V$ defined on connected $C^1$-submanifolds $\Delta \varphi \subset W$ and with bounded derivatives ($\hat{\varphi}$ stands here for the graph $\{ u + \varphi(u) : u \in \Delta \varphi \}$ of $\varphi$).

We will use the following

**Theorem 7.1** (cf. [10], Theorem 0). — Let $\Sigma = \{ \sigma \subset \{1, \ldots, n\} : \text{card} \sigma = n - k \} = \{ \sigma_1, \ldots, \sigma_q \}$, where $q = \binom{n}{k}$.

Let $V_i = \bigoplus_{\nu \in \sigma_i} \mathbb{R} e_\nu$ ($i = 1, \ldots, q$), where $e_1, \ldots, e_n$ is the canonical basis in $\mathbb{R}^n$.

Any definable closed subset $E \subset \mathbb{R}^n$ of dimension $k$ is a union $E = \bigcup_{i=1}^q E_i$ of definable closed subsets $E_i$ such that, for each $i$, $E_i$ is perfectly situated relative to $V_i$ and, for each $j \neq i$, $E_i$ and $E_j$ are simply separated and $\dim(E_i \cap E_j) < k$.

From the last theorem and easy properties of simply separated sets (see [10], Proposition 2; (1) and (3)), it follows that it suffices to prove $\Lambda_p$-regular Decomposition Theorem for each $E_i$ and $Z_i = (Z \cap E_i) \cup (\bigcup_{j \neq i} E_i \cap E_j)$ separately, therefore - up to a permutation of variables - it suffices to prove it assuming that $E$ is perfectly situated relative to $0 \times \mathbb{R}^l$, where $l = n - k$. The proof in this case is based on the following two propositions.
Proposition 7.2 ([6], Proposition 2). — If \( \varphi : \Omega \rightarrow \mathbb{R} \) is a definable \( \Lambda_1 \)-regular mapping defined on an open \( \Omega \subset \mathbb{R}^k \), then there exists a closed definable subset \( Z \) of \( \Omega \) such that \( \dim Z < k \) and \( \varphi | \Omega \setminus Z \) is \( \Lambda_p \)-regular mapping on \( \Omega \setminus Z \).

Proposition 7.3 ([6], Proposition 4). — For any definable open subset \( \Omega \subset \mathbb{R}^k \), there exists a finite family \( S \) of disjoint subsets of \( \Omega \) such that \( \dim (\Omega \setminus \bigcup S) < k \) and each \( S \in S \) is an open definable \( \Lambda_p \)-regular cell in an appropriate linear system of coordinates in \( \mathbb{R}^k \).

Proof of Proposition 7.3. — See [6], Proposition 4, where the set is assumed bounded, but this assumption is not essential. Alternatively, first one can apply [10]; Theorem 1, \((B_k)\) to get the case \( p = 1 \) of Proposition 7.3, which is the theorem of Kurdyka [5] and Parusiński [9], and then by induction on \( k \) one gets the case of any \( p \geq 1 \), applying Proposition 7.2. \(\square\)

To finish the proof of the theorem, first represent \( E \) as union of graphs with bounded derivatives:

\[
E = \bigcup \{ \hat{\varphi} : \varphi \in \mathcal{F} \},
\]

as in the beginning of the section. Adding to \( Z \) all the graphs with \( \dim \Delta \varphi < k \), one can assume that

\[
E = Z \cup \bigcup \{ \hat{\varphi} : \varphi \in \mathcal{F}_* \},
\]

where \( \mathcal{F}_* = \{ \varphi \in \mathcal{F} : \Delta \varphi \text{ non-empty open in } \mathbb{R}^k \} \). By Proposition 7.2, for each \( \varphi \in \mathcal{F}_* \) there exists a closed definable subset \( K_\varphi \) of \( \Delta \varphi \) of \( \dim < k \) such that \( \varphi | \Delta \varphi \setminus K_\varphi \) is \( \Lambda_p \)-regular. Let

\[
\Theta := \pi(Z) \cup \bigcup \{ \partial \Delta \varphi \cup K_\varphi : \varphi \in \mathcal{F}_* \},
\]

where \( \pi : \mathbb{R}^k \times \mathbb{R} \rightarrow \mathbb{R}^k \) is the canonical projection. Take a family \( S \) as in Proposition 7.3 for the open subset

\[
\Omega := \bigcup \{ \Delta \varphi : \varphi \in \mathcal{F}_* \} \setminus \Theta.
\]

Now it suffices to define, for each \( S \in S \)

\[
M_S := E \cup \pi^{-1}(S) \quad \text{and} \quad A := E \setminus \bigcup \{ M_S : S \in S \}.
\]
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