



# ANNALES

DE

# L'INSTITUT FOURIER

Carolyn S. GORDON, Dorothee SCHUETH & Craig J. SUTTON

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Tome 60, n° 5 (2010), p. 1617-1628.

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# SPECTRAL ISOLATION OF BI-INVARIANT METRICS ON COMPACT LIE GROUPS

by Carolyn S. GORDON,  
Dorothee SCHUETH & Craig J. SUTTON (\*)

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ABSTRACT. — We show that a bi-invariant metric on a compact connected Lie group  $G$  is spectrally isolated within the class of left-invariant metrics. In fact, we prove that given a bi-invariant metric  $g_0$  on  $G$  there is a positive integer  $N$  such that, within a neighborhood of  $g_0$  in the class of left-invariant metrics of at most the same volume,  $g_0$  is uniquely determined by the first  $N$  distinct non-zero eigenvalues of its Laplacian (ignoring multiplicities). In the case where  $G$  is simple,  $N$  can be chosen to be two.

RÉSUMÉ. — Soit  $G$  un groupe de Lie compact et connexe, et soit  $g_0$  une métrique bi-invariante sur  $G$ . On démontre que  $g_0$  est isolée spectralement dans la classe des métriques invariantes à gauche : plus précisément, il existe un entier positif  $N$  tel que, dans un voisinage de  $g_0$  dans la classe des métriques invariantes à gauche et de volume inférieur ou égal à celui de  $g_0$ , la métrique  $g_0$  est déterminée de manière unique par les  $N$  premières valeurs propres strictement positives de son Laplacien (sans multiplicités). Si  $G$  est simple, on peut choisir  $N = 2$ .

## 1. Introduction

Given a connected closed Riemannian manifold  $(M, g)$  its **spectrum**, denoted  $\text{Spec}(M, g)$ , is defined to be the sequence of eigenvalues, counted with multiplicities, of the associated Laplacian  $\Delta$  acting on smooth functions. Two Riemannian manifolds  $(M_1, g_1)$  and  $(M_2, g_2)$  are said to be

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*Keywords:* Laplacian, eigenvalue spectrum, Lie group, left-invariant metric, bi-invariant metric.

*Math. classification:* 53C20, 58J50.

(\*) The first and the third authors were partially supported by National Science Foundation grant DMS 0605247.

The second author was partially supported by DFG Sonderforschungsbereich 647. This work was started during a visit of the second author to Dartmouth College, which she would like to thank for its great hospitality.

**isospectral** if their spectra (counting multiplicities) agree. Inverse spectral geometry is the study of the extent to which geometric properties of a Riemannian manifold  $(M, g)$  are determined by its spectrum.

A long standing question is whether very special Riemannian manifolds — e.g. manifolds of constant curvature or symmetric spaces — may be spectrally distinguishable from other Riemannian manifolds. The strongest results are for constant curvature: Tanno showed that a round sphere of dimension at most six is uniquely determined by its spectrum among all *orientable* Riemannian manifolds [13], and in arbitrary dimensions round metrics on spheres are at least spectrally isolated among all Riemannian metrics on spheres [14]. In contrast, the first and third author have shown that in dimension 7 and higher there are isospectrally deformable metrics on spheres arbitrarily close to the standard metric [2, 11]. Hence, for geometries that are in some sense extremely close to being “nice” or “ideal”, spectral uniqueness can fail profoundly.

While many examples exist of isospectral flat manifolds, Kuwabara [4] has proven that flat metrics are at least spectrally isolated within the space of all metrics. However, even the question of whether a flat torus may be isospectral to a non-flat manifold remains open! One cannot resolve this question by appealing to the heat invariants of a Riemannian manifold as there are examples of non-flat manifolds all of whose heat invariants vanish [8].

Outside of the setting of constant curvature, we are not aware of any examples of Riemannian metrics that are known to be spectrally isolated among arbitrary Riemannian metrics. Various results show that *within certain classes* of Riemannian metrics, isospectral sets are finite. Even here, many of the results involve constant curvature. For example, isospectral sets of flat tori are finite (see [16] or unpublished work of Kneser) as are isospectral sets of Riemann surfaces [6]. As for the class of symmetric spaces, the first and third author have recently shown that any collection of mutually isospectral compact symmetric spaces is finite [3].

This article is motivated by the question of whether one can tell from the spectrum whether a compact Riemannian manifold is symmetric. Given that this question has resisted solution even in the case of spheres, it does not appear tractable at this time to compare the spectrum of a symmetric space with that of a completely arbitrary Riemannian manifold. Instead, we ask whether symmetric spaces can be spectrally distinguished within a larger class of homogeneous Riemannian manifolds.

The compact symmetric spaces fall into two types; the type we consider are those given by bi-invariant Riemannian metrics on compact (not necessarily semisimple) Lie groups. We compare the spectrum of each such symmetric space with the spectra of arbitrary left-invariant metrics on the Lie group. As a departure point we note that the second author showed that there are no non-trivial continuous isospectral deformations of a bi-invariant metric within the class of left-invariant metrics on a compact Lie group [10]. This prompts one to ask whether a bi-invariant metric on a compact Lie group  $G$  is spectrally isolated within the class of left-invariant metrics. We give an affirmative answer; in fact we obtain a significantly stronger result.

Let  $\mathcal{M}_{\text{left}}(G)$  denote the set of left-invariant metrics on a Lie group  $G$ . This set can be canonically identified with the set of Euclidean inner products on the Lie algebra of  $G$ . The latter set can in turn be identified, after some choice of basis, with the set of positive definite symmetric  $(m \times m)$ -matrices, where  $m$  is the dimension of  $G$ . The canonical topology on this set of matrices gives rise to a topology on  $\mathcal{M}_{\text{left}}(G)$  (independent of the choice of basis), and it is this topology that we consider. We call a left-invariant metric  $g_0$  on  $G$  **spectrally isolated** in  $\mathcal{M}_{\text{left}}(G)$  if it is locally spectrally determined within  $\mathcal{M}_{\text{left}}(G)$ ; that is, there is a neighborhood  $\mathcal{U}$  of  $g_0$  in  $\mathcal{M}_{\text{left}}(G)$  such that no  $g \in \mathcal{U} \setminus \{g_0\}$  is isospectral to  $g_0$ . We prove the following:

RESULT. — *Let  $g_0$  be a bi-invariant metric on a compact Lie group  $G$ .*

- (1) *There is a neighborhood  $\mathcal{U}$  of  $g_0$  in  $\mathcal{M}_{\text{left}}(G)$  and a positive integer  $N$  such that if  $g$  is any metric in  $\mathcal{U}$  with  $\text{vol}(g) \leq \text{vol}(g_0)$  and whose first  $N$  distinct eigenvalues (ignoring multiplicities) agree with those of  $g_0$ , then  $g$  is isometric to  $g_0$ . (See Theorem 2.3.)*
- (2) *The metric  $g_0$  is spectrally isolated in  $\mathcal{M}_{\text{left}}(G)$ . (See Corollary 2.4.)*
- (3) *Let  $\alpha_1 < \alpha_2 < \alpha_3$  be three distinct consecutive eigenvalues (ignoring multiplicities) of the associated Laplacian  $\Delta_0$ . If  $G$  is simple, then there exists a neighborhood  $\mathcal{U}$  of  $g_0$  in  $\mathcal{M}_{\text{left}}(G)$  such that if  $g \in \mathcal{U}$  satisfies  $\text{vol}(g) \leq \text{vol}(g_0)$  and the condition that three consecutive distinct eigenvalues of  $\Delta_g$  agree with  $\alpha_1$ ,  $\alpha_2$  and  $\alpha_3$  (again ignoring multiplicities), then  $g = g_0$ . In particular, letting  $\alpha_1 = 0$ , then the first two distinct non-zero eigenvalues along with the volume bound distinguish  $g_0$  within  $\mathcal{U}$ . (See Theorem 3.3.)*

The second result above is immediate from the first since the spectrum of a compact Riemannian manifold determines its volume. Hence, within the class of left-invariant metrics on a compact Lie group  $G$ , any metric

$g \neq g_0$  that is isospectral to a bi-invariant metric  $g_0$  must be sufficiently far away from  $g_0$ . In contrast, we note that the second author exhibited the first examples of continuous isospectral families of left-invariant metrics on compact simple Lie groups [10]; see also [9].

In light of the fact that most examples of isospectral manifolds in the literature exploit metrics with “large” symmetry groups, the spectral isolation results above lend strong support to the conjecture that a bi-invariant metric on a compact Lie group  $G$  is spectrally isolated within the class of *all* metrics on  $G$ . In fact, these results lead one to speculate on whether a bi-invariant metric on a *semisimple* Lie group is uniquely determined by its spectrum.<sup>(1)</sup> In Section 3 we present strong evidence that the bi-invariant metric on a compact simple Lie group is uniquely determined by its spectrum within the class of left-invariant metrics. In particular, we show the following.

**RESULT.** — *Let  $g_0$  be a bi-invariant metric on a compact simple Lie group  $G$ , and let  $g \neq g_0$  be a left-invariant metric on  $G$ , which is isospectral to  $g_0$ . Then there is a constant  $C \equiv C(g) > 1$ , such that for every subspace  $V \leq L^2(G)$  that is invariant under the right regular action of  $G$ , we have*

$$\frac{\mathrm{Tr}(\Delta_g \upharpoonright V)}{\mathrm{Tr}(\Delta_0 \upharpoonright V)} \equiv C > 1.$$

(See Proposition 3.1 for a more precise statement.)

This implies that if  $g \neq g_0 \in \mathcal{M}_{\mathrm{left}}(G)$  is isospectral to the bi-invariant metric  $g_0$ , then a very special rearrangement of the eigenvalues must occur.

The outline of this paper is as follows. In Section 2 we establish the main results for bi-invariant metrics on arbitrary compact Lie groups. In Section 3 we restrict our attention to compact simple Lie groups to obtain the stronger results in this setting.

## 2. Proof of the main result

Following [5] we introduce the notion of **eigenvalue equivalence**, which is weaker than that of isospectrality. The same notion was introduced earlier by Z.I. Szabo [12, p. 212], who referred to it as isotonicity. We also define a notion of partial eigenvalue equivalence.

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<sup>(1)</sup>We must restrict our attention to semisimple Lie groups due to the existence of nontrivial pairs of isospectral flat tori (e.g. [7] and [1]).

DEFINITION 2.1. — Given a compact Riemannian manifold  $(M, g)$ , we define the **eigenvalue set** of  $(M, g)$  to be the ordered collection of eigenvalues of the associated Laplace operator  $\Delta_g$  on functions on  $M$ , not counting multiplicities. We will say that two compact Riemannian manifolds are **eigenvalue equivalent** if their eigenvalue sets coincide. For  $N$  a positive integer, we will say that two compact Riemannian manifolds are **eigenvalue equivalent up to level  $N$**  if the first  $N$  elements of their eigenvalue sets coincide.

LEMMA 2.2. — Let  $G$  be a compact Lie group, and let  $g_0$  be a bi-invariant metric on  $G$  with associated Laplacian  $\Delta_0$ . Let  $V \leq L^2(G)$  be a finite dimensional subspace which is invariant under the right-regular representation of  $G$  on  $L^2(G)$ . Then there exists a positive integer  $N$  and a neighborhood  $\mathcal{U}$  of  $g_0$  in  $\mathcal{M}_{\text{left}}(G)$  such that if  $g \in \mathcal{U}$  is eigenvalue equivalent to  $g_0$  up to level  $N$ , then

$$\Delta_g \upharpoonright V = \Delta_0 \upharpoonright V.$$

*Proof.* — First note that  $V$ , being a finite dimensional subspace of  $L^2(G)$  which is invariant under the right-regular representation, contains only smooth functions. Moreover,  $V$  is a direct sum of finitely many irreducible representations of  $G$ ; it is therefore enough to prove the result in the case that  $V$  is irreducible. For any  $g \in \mathcal{M}_{\text{left}}(G)$  and any  $g$ -orthonormal basis  $\{Y_1, \dots, Y_n\}$  of the Lie algebra of  $G$ , the associated Laplace operator on smooth functions on  $G$  is given by

$$\Delta_g = - \sum_{j=1}^n (\rho_* Y_j)^2,$$

where  $\rho: G \rightarrow U(L^2(G))$  is the right-regular representation of  $G$ . Thus,  $V$  is invariant under  $\Delta_g$ . Since  $g_0$  is bi-invariant, right translations in  $G$  are  $g_0$ -isometries; hence  $\Delta_0: V \rightarrow V$  commutes with the action of  $G$  on  $V$ . Irreducibility of  $V$  implies by Schur's Lemma that  $\Delta_0 \upharpoonright V$  is a multiple of the identity, say  $\Delta_0 \upharpoonright V = \lambda \text{Id}_V$ . We may choose  $\epsilon > 0$  such that  $(\lambda - \epsilon, \lambda + \epsilon) \cap \text{Spec}(\Delta_0) = \{\lambda\}$ . Choose  $N$  large enough so that the  $N$ th element of the eigenvalue set is greater than  $\lambda$  (and hence greater than  $\lambda + \epsilon$ ). The hermitian operators  $\Delta_g \upharpoonright V$  on the finite dimensional vector space  $V$  depend continuously on  $g$ . Therefore, their eigenvalues also depend continuously on  $g$ . Consequently, there is a neighborhood  $\mathcal{U}$  of  $g_0$  in  $\mathcal{M}_{\text{left}}(G)$  such that for each  $g \in \mathcal{U}$  the eigenvalues of  $\Delta_g \upharpoonright V$  must lie in  $(\lambda - \epsilon, \lambda + \epsilon)$ . If  $g \in \mathcal{U}$  is eigenvalue equivalent to  $g_0$  up to level  $N$ , it follows that  $\Delta \upharpoonright V = \Delta_0 \upharpoonright V = \lambda \text{Id}_V$ .  $\square$

We now establish the spectral isolation of bi-invariant metrics on compact connected Lie groups. In fact, we prove a little more; namely, we replace the isospectrality condition by the much weaker condition of partial eigenvalue equivalence together with an upper volume bound.

**THEOREM 2.3.** — *Let  $g_0$  be a bi-invariant metric on a compact connected Lie group  $G$ . Then there is a positive integer  $N$ , depending only on  $g_0$ , and a neighborhood  $\mathcal{U}$  of  $g_0$  in  $\mathcal{M}_{\text{left}}(G)$  such that if  $g \in \mathcal{U}$  is eigenvalue equivalent to  $g_0$  up to order  $N$  and satisfies  $\text{vol}(g) \leq \text{vol}(g_0)$ , then  $g = g_0$ .*

**COROLLARY 2.4.** — *Let  $g_0$  be a bi-invariant metric on a compact connected Lie group  $G$ . Then  $g_0$  is spectrally isolated in  $\mathcal{M}_{\text{left}}(G)$ .*

The corollary follows from the theorem by the fact that isospectrality implies eigenvalue equivalence and equality of volumes; in fact, the volume is the first of the classical heat invariants.

*Proof of Theorem 2.3.* — We have  $G = G_{ss}T$  where  $G_{ss}$  is semisimple,  $T$  is a torus and  $G_{ss} \cap T$  is finite. The Lie algebra of  $G$  is a direct sum  $\mathfrak{g} = \mathfrak{g}_{ss} \oplus \mathfrak{t}$ , where  $\mathfrak{g}_{ss}$  and  $\mathfrak{t}$  are the Lie algebras of  $G_{ss}$  and  $T$ . In particular,  $G_{ss}$  and  $T$  commute, and  $\mathfrak{g}_{ss}$  is  $g_0$ -orthogonal to  $\mathfrak{t}$ .

We first claim that there exists a positive integer  $N'$  and a neighborhood  $\mathcal{U}'$  of  $g_0$  in  $\mathcal{M}_{\text{left}}(G)$  such that if  $g \in \mathcal{U}'$  is eigenvalue equivalent to  $g_0$  up to level  $N'$ , then  $g$  and  $g_0$ , viewed as inner products on  $\mathfrak{g}$ , induce the same inner product on  $\mathfrak{g}/\mathfrak{g}_{ss}$ . By the inner product induced by  $g$  we mean the one obtained by identifying  $\mathfrak{g}/\mathfrak{g}_{ss}$  with the  $g$ -orthogonal complement of  $\mathfrak{g}_{ss}$  in  $\mathfrak{g}$ . To prove the claim, note that the Lie group  $\bar{T} := G/G_{ss} \cong T/(G_{ss} \cap T)$  is a torus which is finitely covered by  $T$ . In particular, the Lie algebra of  $\bar{T}$  is canonically identified with  $\mathfrak{t}$ . Let  $p: G \rightarrow \bar{T}$  be the homomorphic projection. Given  $g \in \mathcal{M}_{\text{left}}(G)$ , denote by  $\bar{g}$  the induced (flat) metric on  $\bar{T}$  (i.e. the metric for which  $p: (G, g) \rightarrow (\bar{T}, \bar{g})$  becomes a Riemannian submersion). Let  $\mathcal{L}$  be the lattice in  $\mathfrak{t}$  which is the kernel of the Lie group exponential map  $\mathfrak{t} \rightarrow \bar{T}$ , and let  $\mathcal{L}^* \subset \mathfrak{t}^*$  be the dual lattice. For  $\mu \in \mathcal{L}^*$ , denote by  $\|\mu\|_{\bar{g}}$  the norm of  $\mu$  with respect to the dual inner product on  $\mathfrak{t}^*$ . Let  $\nu_1, \dots, \nu_k$  be a basis of  $\mathcal{L}^*$ , where  $k = \dim(T)$ . Write  $L := k + \binom{k}{2}$ , and let  $\{\mu_1, \dots, \mu_L\}$  be the set containing the vectors  $\nu_i$  as well as the  $\nu_i + \nu_j$  for  $i \neq j$ . Note that, by polarization, the norm  $\|\cdot\|_{\bar{g}}$  on  $\mathfrak{t}^*$  — and hence  $\bar{g}$  itself — is uniquely determined by the norms of the vectors  $\mu_1, \dots, \mu_L$ . For each  $s \in \{1, \dots, L\}$  let  $\bar{f}_s: \bar{T} \rightarrow U(1)$  (where  $U(1)$  is the unitary group of unit complex numbers) be the associated character of  $\bar{T}$ . Then  $\Delta_{\bar{g}} \bar{f}_s = 4\pi^2 \|\mu_s\|_{\bar{g}}^2 \bar{f}_s$ . Now  $f_s := \bar{f}_s \circ p$  is a character on  $G$ . Since the Riemannian submersion  $p: G \rightarrow \bar{T}$  has minimal fibers,  $f_s$  is an eigenfunction

of  $\Delta_g$  with eigenvalue  $4\pi^2\|\mu_s\|_{\bar{g}}^2$  for each  $s = 1, \dots, L$ . (One can also verify this fact by direct computation.)

The one-dimensional space  $F_s \leq L^2(G)$  spanned by the character  $f_s$  is invariant under the right-regular representation. Let  $N'$  be a positive integer and  $\mathcal{U}'$  be a neighborhood of  $g_0$  in  $\mathcal{M}_{\text{left}}(G)$  satisfying the property from Lemma 2.2 with respect to  $F_1 \oplus \dots \oplus F_L$ , and let  $g \in \mathcal{U}'$  be eigenvalue equivalent to  $g_0$  up to level  $N'$ . Then we must have  $\|\mu_s\|_{\bar{g}} = \|\mu_s\|_{\bar{g}_0}$  for each  $s = 1, \dots, L$ . As remarked above, this implies  $\bar{g} = \bar{g}_0$ . The claim follows.

In the case of the bi-invariant metric  $g_0$ , the metric  $\bar{g}_0$  on  $\mathfrak{t}$  coincides with the restriction of the inner product  $g_0$  to  $\mathfrak{t}$  since  $\mathfrak{g}_{ss}$  and  $\mathfrak{t}$  are  $g_0$ -orthogonal. However, for more general  $g$ , one has only that the differential  $p_*: \mathfrak{g} \rightarrow \mathfrak{t}$  of the projection  $p: G \rightarrow \bar{T}$  restricts to an inner product space isometry  $p_*: (\mathfrak{g}_{ss}^{\perp g}, g) \rightarrow (\mathfrak{t}, \bar{g})$ . In particular, if  $g \in \mathcal{U}'$ , then it follows from the claim that

(2.1) the projection from  $(\mathfrak{g}_{ss}^{\perp g}, g)$  to  $(\mathfrak{t}, g_0)$  along  $\mathfrak{g}_{ss}$  is an isometry.

For the remaining part of the argument, let  $\mathfrak{g}_{ss} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_r$  be the decomposition of  $\mathfrak{g}_{ss}$  into simple Lie subalgebras. The adjoint representation of  $G$  on  $\mathfrak{g}$ , restricted to the invariant subspace  $\mathfrak{g}_\ell$ , is an irreducible representation of  $G$  for each  $\ell = 1, \dots, r$ . Note that for  $\ell \neq \ell'$  these representations of  $G$  are inequivalent even in the case when  $\mathfrak{g}_\ell$  and  $\mathfrak{g}_{\ell'}$  happen to be isomorphic as Lie algebras. By the Peter-Weyl Theorem, every irreducible representation of  $G$  occurs in the right-regular representation of  $G$  on  $L^2(G)$  (with multiplicity equal to its dimension). Thus, for each  $\ell = 1, \dots, r$  we can choose a corresponding irreducible subspace  $V_\ell \leq L^2(G)$ , and the action of  $G$  on the subspace  $V_1 \oplus \dots \oplus V_r$  of  $L^2(G)$  will then be equivalent to the adjoint representation of  $G$  acting on  $\mathfrak{g}_{ss}$ .

Let  $N''$  be a positive integer and  $\mathcal{U}''$  be a neighborhood of  $g_0$  in  $\mathcal{M}_{\text{left}}(G)$  satisfying the property from Lemma 2.2 with respect to  $V_1 \oplus \dots \oplus V_r$ . We are going to show that  $N := \max\{N', N''\}$  and  $\mathcal{U} := \mathcal{U}' \cap \mathcal{U}''$  satisfy the property stated in the Theorem.

If  $g$  is any left-invariant metric on  $G$  and  $\{U_1, \dots, U_m\}$  is a  $g$ -orthonormal basis of  $\mathfrak{g}$ , then

$$(2.2) \quad \text{Tr}(\Delta_g \upharpoonright V_\ell) = - \sum_{j=1}^m \text{Tr}((\text{ad}_{U_j} \upharpoonright \mathfrak{g}_\ell)^2).$$

Since  $g_0$  is bi-invariant,  $\mathfrak{g}_\ell$  is  $g_0$ -orthogonal to  $\mathfrak{g}_{\ell'}$  for  $\ell \neq \ell'$ . Let  $n_\ell$  denote the dimension of  $\mathfrak{g}_\ell$ , and let  $n = n_1 + \dots + n_r$  be the dimension of  $\mathfrak{g}_{ss}$ . Choose a  $g_0$ -orthonormal basis  $\{X_1, \dots, X_n\}$  of  $\mathfrak{g}_{ss}$  such that the first  $n_1$  elements



lie in  $\mathfrak{g}_1$ , the next  $n_2$  elements lie in  $\mathfrak{g}_2$ , etc. Complete to a  $g_0$ -orthonormal basis  $\mathcal{B}_0 = \{X_1, \dots, X_n, Z_1, \dots, Z_k\}$ , where (necessarily)  $Z_1, \dots, Z_k \in \mathfrak{t}$ .

Let  $g$  be a metric in  $\mathcal{U}$  which is eigenvalue equivalent to  $g_0$  up to level  $N$  and satisfies  $\text{vol}(g) \leq \text{vol}(g_0)$ . Since  $g \in \mathcal{U}'$  and  $N \geq N'$ , statement (2.1) holds and thus there exist elements  $W_i \in \mathfrak{g}_{ss}$  such that  $\{Z_1 + W_1, \dots, Z_k + W_k\}$  is a  $g$ -orthonormal basis of  $\mathfrak{g}_{ss}^{\perp g}$ . Complete to a  $g$ -orthonormal basis  $\mathcal{B} = \{Y_1, \dots, Y_n, Z_1 + W_1, \dots, Z_k + W_k\}$  of  $\mathfrak{g}$  with  $Y_1, \dots, Y_k \in \mathfrak{g}_{ss}$ . The change of basis matrix which expresses the elements of  $\mathcal{B}$  in terms of  $\mathcal{B}_0$  is given by

$$\begin{bmatrix} A & R \\ 0 & I_k \end{bmatrix},$$

where  $A = (a_{ij}) \in \text{Mat}_{n \times n}(\mathbb{R})$ ,  $R = (r_{ij}) \in \text{Mat}_{n \times k}(\mathbb{R})$  and  $I_k$  is the  $k \times k$  identity matrix. Hence,  $Y_j = \sum_{i=1}^n a_{ij} X_i$  for  $j = 1, \dots, n$  and  $W_s = \sum_{i=1}^n r_{is} X_i$  for  $s = 1, \dots, k$ . The condition  $\text{vol}(g) \leq \text{vol}(g_0)$  implies that  $|\det(A)| \geq 1$ . Without loss of generality we assume  $\det(A) > 0$  and hence  $\det(A) \geq 1$ .

Since  $g_0$  is bi-invariant, there exist numbers  $c_\ell > 0$  for  $\ell = 1, \dots, r$  such that the restriction of  $g_0$  to  $\mathfrak{g}_\ell$  coincides with  $-c_\ell B_\ell$ , where  $B_\ell$  is the Killing form of  $\mathfrak{g}_\ell$  (which in turn coincides with the restriction to  $\mathfrak{g}_\ell$  of the Killing form  $B: (X, Y) \mapsto \text{Tr}(\text{ad}_X \circ \text{ad}_Y)$  of  $\mathfrak{g}$ ). In particular, by equation (2.2), we have

$$\text{Tr}(\Delta_0 \upharpoonright V_\ell) = \frac{n_\ell}{c_\ell}.$$

Since  $g \in \mathcal{U} \subset \mathcal{U}''$  and  $N \geq N''$ , we have  $\Delta_g \upharpoonright V_\ell = \Delta_0 \upharpoonright V_\ell$  for each  $\ell = 1, \dots, r$ . In particular, for  $\ell = 1$ :

$$\begin{aligned} \frac{n_1}{c_1} &= \text{Tr}(\Delta_0 \upharpoonright V_1) = \text{Tr}(\Delta_g \upharpoonright V_1) \\ &= - \sum_{j=1}^n \text{Tr}((\text{ad}_{Y_j} \upharpoonright \mathfrak{g}_1)^2) - \sum_{s=1}^k \text{Tr}((\text{ad}_{Z_s+W_s} \upharpoonright \mathfrak{g}_1)^2) \\ &= - \sum_{i=1}^{n_1} \sum_{j=1}^n a_{ij}^2 \text{Tr}((\text{ad}_{X_i} \upharpoonright \mathfrak{g}_1)^2) - \sum_{i=1}^{n_1} \sum_{s=1}^k r_{is}^2 \text{Tr}((\text{ad}_{X_i} \upharpoonright \mathfrak{g}_1)^2) \\ &= \frac{1}{c_1} \sum_{i=1}^{n_1} \sum_{j=1}^n a_{ij}^2 + \frac{1}{c_1} \sum_{i=1}^{n_1} \sum_{s=1}^k r_{is}^2, \end{aligned}$$

where the fourth equality holds because  $\text{ad}_X \upharpoonright \mathfrak{g}_1 = 0$  for  $X \in \mathfrak{g}_\ell$  with  $\ell \neq 1$  and because  $\{X_1, \dots, X_{n_1}\}$  is orthonormal with respect to  $-c_1 B_1$ . We hence obtain  $n_1 = \sum_{i=1}^{n_1} \sum_{j=1}^n a_{ij}^2 + \sum_{i=1}^{n_1} \sum_{s=1}^k r_{is}^2$ . Summing over the

analogous equations for  $\ell = 1, \dots, r$  we conclude that

$$n = \|A\|^2 + \|R\|^2,$$

where  $\|\cdot\|$  denotes the standard Euclidean norm of matrices viewed as points in the appropriate  $\mathbb{R}^N$ . However,  $n$  is the minimal value (in fact, the only critical value) of the function  $\text{SL}(n, \mathbb{R}) \ni C \mapsto \|C\|^2 \in \mathbb{R}$  and is attained precisely on  $\text{SO}(n)$ . It thus follows from  $\det(A) \geq 1$  that  $A \in \text{SO}(n)$  and  $R = 0$ ; hence  $g = g_0$ .  $\square$

*Remark 2.5.* — Some of the techniques used in this section are similar to those used by Urakawa in [15].

### 3. A stronger spectral isolation result for simple groups

In the proof of Lemma 2.2 it was observed that if  $g$  is a left-invariant metric on a compact Lie group  $G$ , with associated Laplacian  $\Delta_g$ , then any subspace  $V \leq L^2(G)$  that is invariant under the right-regular representation of  $G$  is also invariant under  $\Delta_g$ . With this in mind we have the following result concerning the trace of the Laplacian on compact simple Lie groups.

**PROPOSITION 3.1.** — *Let  $g_0$  be a bi-invariant metric on a compact simple Lie group  $G$ , and let  $g \neq g_0$  be a left-invariant metric on  $G$  which satisfies  $\text{vol}(g) \leq \text{vol}(g_0)$ . Then there exists a constant  $C = C(g) > 1$  such that*

$$\text{Tr}(\Delta_g \upharpoonright V) = C \text{Tr}(\Delta_0 \upharpoonright V)$$

for every finite dimensional subspace  $V \leq L^2(G)$  which is invariant under the right-regular representation  $\rho$  of  $G$  and on which  $G$  acts nontrivially.

*Proof.* — By rescaling  $g$  and  $g_0$  we can assume without loss of generality that  $g_0$  coincides with  $-B$  on the Lie algebra  $\mathfrak{g}$  of  $G$ , where  $B$  is the Killing form on  $\mathfrak{g}$ . Define  $h: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$  by

$$h(X, Y) := -\text{Tr}(((\rho_* X) \upharpoonright V) \circ ((\rho_* Y) \upharpoonright V)).$$

Obviously  $h$  is bilinear, symmetric, and  $\text{Ad}_G$ -invariant. The map  $X \mapsto (\rho_* X) \upharpoonright V$  is nonzero because  $V$  is not a trivial representation space of  $G$ . Since  $\mathfrak{g}$  is simple, this map has trivial kernel, which implies that  $h$  is positive-definite; in particular, there exists some  $c > 0$  such that  $h = -cB$ . We now proceed similarly as in the last part of the proof of Theorem 2.3: Let  $\{X_1, \dots, X_n\}$  be a  $g_0$ -orthonormal basis and  $\{Y_1, \dots, Y_n\}$  be a  $g$ -orthonormal basis of  $\mathfrak{g}$ , and define  $A = (a_{ij})$  by  $Y_j = \sum_{i=1}^n a_{ij} X_i$

for  $j = 1, \dots, n$ . By the volume condition,  $|\det(A)| \geq 1$ ; we can assume  $\det(A) \geq 1$ . Moreover,  $g \neq g_0$  implies  $A \notin \text{SO}(n)$  and therefore  $\|A\|^2 > n$ . Thus,

$$\begin{aligned} \text{Tr}(\Delta_g \upharpoonright V) &= \sum_{j=1}^n h(Y_j, Y_j) = -c \sum_{j=1}^n B(Y_j, Y_j) \\ &= -c \sum_{i,j=1}^n a_{ij}^2 B(X_i, X_i) = c\|A\|^2 \\ &> cn = \sum_{i=1}^n h(X_i, X_i) = \text{Tr}(\Delta_0 \upharpoonright V). \end{aligned}$$

The proposition follows with  $C = \|A\|^2/n$ .  $\square$

*Remark 3.2.* — Since volume is a spectral invariant the previous proposition implies the following: Let  $g_0$  be a bi-invariant metric on a compact simple Lie group  $G$ , and suppose there exists a left-invariant metric  $g \neq g_0$  on  $G$  which is isospectral to  $g_0$ . Then  $\text{Tr}(\Delta_g \upharpoonright V) > \text{Tr}(\Delta_0 \upharpoonright V)$  for any finite dimensional invariant subspace  $V$  of  $L^2(G)$  on which  $G$  acts non-trivially. Thus  $\Delta_g$ , although isospectral to  $\Delta_0$ , must have greater trace than  $\Delta_0$  on every  $\Delta_0$ -eigenspace (except for the eigenvalue 0), even on each irreducible subspace. This is not a priori a contradiction because some wild reordering of eigenvalues could occur to produce this situation. Nevertheless, this seems a strong indication in support of the conjecture that a bi-invariant metric on a compact simple Lie group is globally spectrally determined among left-invariant metrics.

**THEOREM 3.3.** — *Let  $g_0$  be a bi-invariant metric on a compact simple Lie group  $G$ . Let  $\alpha_1 < \alpha_2 < \alpha_3$  be three consecutive elements of the eigenvalue set of  $(G, g_0)$ . (See Definition 2.1.) Then there exists a neighborhood  $\mathcal{U}$  of  $g_0$  in  $\mathcal{M}_{\text{left}}(G)$  (depending on  $\alpha_1, \alpha_2, \alpha_3$ ) such that if  $g \in \mathcal{U}$  satisfies  $\text{vol}(g) \leq \text{vol}(g_0)$  and if  $\alpha_1, \alpha_2$  and  $\alpha_3$  are also consecutive elements of the eigenvalue set of  $(G, g)$ , then  $g = g_0$ .*

**COROLLARY 3.4.** — *A bi-invariant metric on a compact simple Lie group is locally determined within the set of left-invariant metrics of at most the same volume by its first two distinct non-zero eigenvalues  $0 < \lambda_1 < \lambda_2$  (ignoring multiplicities).*

The corollary follows from the theorem since  $0, \lambda_1, \lambda_2$  are three consecutive elements of the eigenvalue set.

*Proof of Theorem 3.3.* — Since  $\Delta_0$  commutes with right translations in  $G$ , the  $\alpha_2$ -eigenspace  $V$  of  $\Delta_0$  is invariant under the right-regular representation. Note that  $V$  is finite dimensional. As remarked in the proof of Lemma 2.2, the eigenvalues of  $\Delta_g \upharpoonright V$  depend continuously on  $g$ . Thus there exists a neighborhood  $\mathcal{U}$  of  $g_0$  in  $\mathcal{M}_{\text{left}}(G)$  such that for any  $g \in \mathcal{U}$  the eigenvalues of  $\Delta_g$  on  $V$  are contained in the interval  $(\alpha_1, \alpha_3)$ . Let  $g \in \mathcal{U}$  be a metric which satisfies  $\text{vol}(g) \leq \text{vol}(g_0)$  and the condition that  $\alpha_1, \alpha_2$  and  $\alpha_3$  are also consecutive eigenvalues of  $\Delta_g$ . Then necessarily  $\Delta_g \upharpoonright V = \alpha_2 \text{Id}_V = \Delta_0 \upharpoonright V$ . Finally, note that  $G$  acts nontrivially on  $V$  since  $\alpha_2 \neq 0$ . Proposition 3.1 now implies  $g = g_0$ .  $\square$

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Manuscrit reçu le 29 avril 2009,  
accepté le 8 septembre 2009.

Carolyn S. GORDON  
Dartmouth College  
Department of Mathematics  
Hanover, NH 03755 (USA)  
carolyn.s.gordon@dartmouth.edu

Dorothee SCHUETH  
Humboldt-Universität  
Institut für Mathematik  
Unter den Linden 6  
10099, Berlin (Germany)  
schueth@math.hu-berlin.de

Craig J. SUTTON  
Dartmouth College  
Department of Mathematics  
Hanover, NH 03755 (USA)  
craig.j.sutton@dartmouth.edu