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HÖLDER CONTINUITY OF SOLUTIONS TO THE MONGE-AMPÈRE EQUATIONS ON COMPACT KÄHLER MANIFOLDS

by Pham Hoang HIEP

Abstract. — We study Hölder continuity of solutions to the Monge-Ampère equations on compact Kähler manifolds. T. C. Dinh, V.A. Nguyen and N. Sibony have shown that the measure \( \omega^u_n \) is moderate if \( u \) is Hölder continuous. We prove a theorem which is a partial converse to this result.

Résumé. — Nous étudions la continuité de Hölder des solutions des équations de Monge-Ampère sur des variétés Kählériennes compactes. T. C. Dinh, V.A. Nguyen et N. Sibony ont prouvé que \( \omega^u_n \) est modéré si \( u \) est Hölder-continue. Nous démontrons dans quelques cas la réciproque de ce résultat.

1. Introduction

Let \( X \) be a compact \( n \)-dimensional Kähler manifold equipped with a fundamental form \( \omega \) satisfying \( \int_X \omega^n = 1 \). An upper semicontinuous function \( \varphi : X \to [-\infty, +\infty) \) is called \( \omega \)-plurisubharmonic (\( \omega \)-psh) if \( \varphi \in L^1(X) \) and \( \omega \varphi := \omega + dd^c \varphi \geq 0 \). By \( \text{PSH}(X, \omega) \) (resp. \( \text{PSH}^-(X, \omega) \)) we denote the set of \( \omega \)-psh (resp. negative \( \omega \)-psh) functions on \( X \). The complex Monge-Ampère equation \( \omega^u_n = f \omega^n \) was solved for smooth positive \( f \) in the fundamental work of S. T. Yau (see [31]). Later S. Kołodziej showed that there exists a continuous solution if \( f \in L^p(\omega^n) \), \( f \geq 0 \), \( p > 1 \) (see [24]). Recently in [27] he proved that this solution is Hölder continuous in this case (see also [18] for the case \( X = \mathbb{C}P^n \)). In Corollary 1.2 in [16] the authors have shown that the measure \( \omega^u_n \) is moderate if \( u \) is Hölder continuous. The main result is the following theorem which give a partial answer to the converse problem:

Keywords: Hölder continuity, complex Monge-Ampère operator, \( \omega \)-plurisubharmonic functions, compact Kähler manifolds.
**Theorem A.** — Let $\mu$ be a non-negative Radon measure on $X$ such that

$$\mu(B(z, r)) \leq Ar^{2n-2+\alpha},$$

for all $B(z, r) \subset X$ ($A, \alpha > 0$ are constants). Then for every $f \in L^p(d\mu)$ with $p > 1$, $\int_X f d\mu = 1$, there exists a Hölder continuous $\omega$-psh function $u$ such that $\omega_u^n = f d\mu$.

The following results are simple applications of Theorem A:

**Corollary B.** — Let $\varphi \in \text{PSH}(X, \omega)$ be a Hölder continuous function. Then for every $f \in L^p(\omega \varphi \wedge \omega^{n-1})$ with $p > 1$, $\int_X f \omega \varphi \wedge \omega^{n-1} = 1$, there exists a Hölder continuous $\omega$-psh function $u$ such that $\omega_u^n = f \omega \varphi \wedge \omega^{n-1}$.

**Corollary C.** — Let $S$ be a $C^1$ smooth real hypersurface in $X$ and $V_S$ be the volume measure on $S$. Then for every $f \in L^p(dV_S)$ with $p > 1$, $\int_X f dV_S = 1$, there exists a Hölder continuous $\omega$-psh function $u$ such that $\omega_u^n = f dV_S$.

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### 2. Preliminaries

First we recall some elements of pluripotential theory that will be used throughout the paper. Details can be found in [2]–[3], [5]–[6], [4], [7], [9]–[8], [13]–[15], [19]–[20], [21], [23]–[27], [28], [29]–[30], [32]–[33].

**2.1.** In [24] Kołodziej introduced the capacity $C_X$ on $X$ by

$$C_X(E) := \sup \left\{ \int_E \omega_{\varphi}^n : \varphi \in \text{PSH}(X, \omega), -1 \leq \varphi \leq 0 \right\}$$

for all Borel sets $E \subset X$.

**2.2.** In [19] Guedj and Zeriahi introduced the Alexander capacity $T_X$ on $X$ by

$$T_X(E) = e^{-\sup_x V_{E.X}^*}$$

for all Borel sets $E \subset X$. Here $V_{E.X}^*$ is the global extremal $\omega$-psh function for $E$ defined as the smallest upper semicontinuous majorant of $V_{E,X}$ i.e,

$$V_{E,X}(z) = \sup \left\{ \varphi(z) : \varphi \in \text{PSH}(X, \omega), \varphi \leq 0 \text{ on } E \right\}.$$
2.3. The following definition was introduced in [18]: A probability measure \( \mu \) on \( X \) is said to satisfy the condition \( \mathcal{H}(\alpha, A) \) \((\alpha, A > 0)\) if
\[
\mu(K) \leq AC_X(K)^{1+\alpha},
\]
for any Borel subset \( K \) of \( X \).

A probability measure \( \mu \) on \( X \) is said to satisfy the condition \( \mathcal{H}(\infty) \) if for any \( \alpha > 0 \) there exist \( A(\alpha) > 0 \) dependent on \( \alpha \) such that
\[
\mu(K) \leq A(\alpha)C_X(K)^{1+\alpha},
\]
for any Borel subset \( K \) of \( X \).

2.4. The following definition was introduced in [17]: A measure \( \mu \) is said to be moderate if for any open set \( U \subset X \), any compact set \( K \subset U \) and any compact family \( \mathcal{F} \) of plurisubharmonic functions on \( U \), there are constants \( \alpha > 0 \) such that
\[
\sup \left\{ \int_K e^{-\alpha \varphi} d\mu : \varphi \in \mathcal{F} \right\} < +\infty.
\]

2.5. The following class of \( \omega \)-psh functions was investigated by Guedj and Zeriahi in [20]:
\[
\mathcal{E}(X, \omega) = \left\{ \varphi \in \text{PSH}(X, \omega) : \lim_{j \to \infty} \int_{\{\varphi > -j\}} \omega^n_{\max(\varphi, -j)} = \int_X \omega^n = 1 \right\}.
\]
Let us also define
\[
\mathcal{E}^-(X, \omega) = \mathcal{E}(X, \omega) \cap \text{PSH}^-(X, \omega).
\]
We refer to [20] for the properties of the class \( \mathcal{E}(X, \omega) \).

2.6. \( S \) is called a \( C^1 \) smooth real hypersurface in \( X \) if for all \( z \in X \) there exists a neighborhood \( U \) of \( z \) and \( \chi \in C^1(U) \) such that \( S \cap U = \{ z \in U : \chi(z) = 0 \} \) and \( D\chi(z) \neq 0 \) for all \( z \in S \cap U \).

Next we state a well-known result needed for our work.

2.7. **Proposition.** — Let \( \mu \) be a non-negative Radon measure on \( X \) such that \( \mu(B(z, r)) \leq Ar^{2n-2+\alpha} \) for all \( B(z, r) \subset X \) \((A, \alpha > 0 \) are constants). Then \( \mu \in \mathcal{H}(\infty) \).

**Proof.** — By Theorem 7.2 in [33] and Proposition 7.1 in [19] we can find \( \epsilon, C > 0 \) such that
\[
\mu(K) \leq Ah^{2n-2+\alpha}(K) \leq \frac{AC}{\alpha} T_X(K)^{\epsilon \alpha} \leq \frac{AC}{\alpha} e^{-\frac{\epsilon \alpha}{c_X(K)^{\pi}}},
\]
for all Borel subsets $K$ of $X$, where $h^{2n-2+\alpha}$ is the Hausdorff content of dimension $2n - 2 + \alpha$. This implies that $\mu \in \mathcal{H}(\infty)$. □

3. Stability of the solutions

The stability estimate of solutions to the Monge-Ampère equations on compact Kähler manifolds was obtained by Kołodziej ([24]). Recently, in [12] S. Dinew and Z. Zhang proved a stronger version of this estimate. We will show a generalization of the stability theorem by S. Kołodziej. As a first step we have the following proposition. This proof follows ideas of the proof of Theorem 2.5 in [11]. We include a proof for the reader’s convenience.

3.1. Proposition. — Let $\varphi, \psi \in \mathcal{E}^-(X, \omega)$ be such that $\omega^n_\varphi \in \mathcal{H}(\alpha, A)$. Then there exist constants $t \in \mathbb{R}$ and $C(\alpha, A) \geq 0$ such that

$$
\int_{\{|\varphi - \psi - t| > a\}} (\omega^n_\varphi + \omega^n_\psi) \leq C(\alpha, A)a^{n+1},
$$

here $a = \left[ \int_X \|\omega^n_\varphi - \omega^n_\psi\| \right]^{\frac{1}{2n+3+\frac{n+1}{\alpha}}}$. 

Proof. — Since $\int_{\{|\varphi - \psi - t| > a\}} (\omega^n_\varphi + \omega^n_\psi) \leq 2$, it suffices to consider the case when $a$ is small. Set

$$
\epsilon = \frac{1}{2} \inf \left\{ \int_{\{|\varphi - \psi - t| > a\}} \omega^n_\varphi : t \in \mathbb{R} \right\}
$$

Hence

$$
\int_{\{|\varphi - \psi - t| \leq a\}} \omega^n_\varphi \leq 1 - 2\epsilon
$$

for all $t \in \mathbb{R}$. Set

$$
t_0 = \sup \left\{ t \in \mathbb{R} : \int_{\{|\varphi - \psi + t| > a\}} \omega^n_\varphi \leq 1 - \epsilon \right\}
$$

Replacing $\psi$ by $\psi + t_0$ we can assume that $t_0 = 0$. Then $\int_{\{|\varphi + t| > a\}} \omega^n_\varphi \leq 1 - \epsilon$ and $\int_{\{|\psi + t| > a\}} \omega^n_\varphi \geq 1 - \epsilon$. Hence

$$
\int_{\{|\varphi + t| > a\}} \omega^n_\varphi = 1 - \int_{\{|\varphi + a| \leq \psi\}} \omega^n_\varphi = 1 - \int_{\{|\varphi + a| \leq \psi\}} \omega^n_\varphi + \int_{\{|\psi - a < \varphi \leq \psi + a\}} \omega^n_\varphi \leq 1 - \epsilon.
$$

Since $\int_{\{|\varphi - \psi| \leq a\}} \omega^n_\varphi \leq 1$ we can choose $s \in [-a + a^{n+2}, a - a^{n+2}]$ satisfying

$$
\int_{\{|\varphi - \psi - s| < a^{n+2}\}} \omega^n_\varphi \leq 2a^{n+1}.
$$
Replacing $\psi$ by $\psi + s$ we can assume that $s = 0$. One easily obtains the following inequalities

\begin{equation}
\int_{\{\varphi < \psi + a_n\}} \omega_\varphi^n \leq 1 - \epsilon, \quad \int_{\{\psi < \varphi + a_n\}} \omega_\varphi^n \leq 1 - \epsilon,
\end{equation}

\begin{equation}
\int_{\{|\varphi - \psi| < a_n\}} \omega_\varphi^n \leq 2a^{n+1}.
\end{equation}

By [20] we can find $\rho \in E(X, \omega)$, such that

\begin{equation}
\omega_\rho^n = \frac{1}{1 - \epsilon} \int_{\{\psi < \varphi\}} \omega_\varphi^n + c \int_{\{\varphi \geq \psi\}} \omega_\varphi^n \quad \text{and} \quad \sup_X \rho = 0,
\end{equation}

($c \geq 0$ is chosen so that the measure has total mass 1). For simplicity of notation we set $\beta = \frac{n+1}{1+\alpha}$. Set

\begin{equation}
U = \left\{ (1 - a^{n+2+\beta}) \varphi < (1 - a^{n+2+\beta}) \psi + a^{n+2+\beta} \rho \right\} \subset \{ \varphi < \psi \}.
\end{equation}

From Theorem 2.1 in [15] and (2) we get

\begin{equation}
\omega_\varphi^{n-1} \land \omega_{(1 - a^{n+2+\beta}) \psi + a^{n+2+\beta} \rho} \geq (1 - a^{n+2+\beta}) \omega_\varphi^{n-1} \land \omega_\psi + \frac{a^{n+2+\beta}}{(1 - \epsilon)^\frac{1}{n}} \omega_\varphi^n,
\end{equation}

on $U$. From Theorem 2.3 in [15], Lemma 2.6 in [11] and (3) we obtain

\begin{align*}
(1 - a^{n+2+\beta}) \int_U \omega_\varphi^{n-1} \land \omega_\psi + \frac{a^{n+2+\beta}}{(1 - \epsilon)^\frac{1}{n}} \int_U \omega_\varphi^n &
\leq \int_U \omega_{(1 - a^{n+2+\beta}) \psi + a^{n+2+\beta} \rho} \land \omega_\varphi^{n-1}
\leq \int_U \omega_{(1 - a^{n+2+\beta}) \varphi} \land \omega_\varphi^{n-1}
\leq (1 - a^{n+2+\beta}) \int_U \omega_\varphi^n + a^{n+2+\beta} \int_U \omega \land \omega_\varphi^{n-1}
\leq (1 - a^{n+2+\beta}) \left( \int_U \omega_\varphi^{n-1} \land \omega_\psi + 2a^{2n+3+\beta} \right) + a^{n+2+\beta} \int_U \omega \land \omega_\varphi^{n-1}.
\end{align*}

Hence

\begin{equation}
\frac{1}{(1 - \epsilon)^\frac{1}{n}} \int_U \omega_\varphi^n \leq 2a^{n+1} + \int_U \omega \land \omega_\varphi^{n-1}.
\end{equation}

From Proposition 3.6 in [19] and (4) we get
(5) \[
\frac{1}{(1 - \epsilon)^\frac{1}{\pi}} \left[ \int_{\{\varphi \leq -a^{n+2}\}} \omega^n_\varphi - C_1(\alpha, A)a^{n+1} \right] \\
\leq \frac{1}{(1 - \epsilon)^\frac{1}{\pi}} \left[ \int_{\{\varphi \leq -a^{n+2}\}} \omega^n_\varphi - A[CX(\{\rho \leq -\frac{1}{2a}\})]^{1+\alpha} \right] \\
\leq \frac{1}{(1 - \epsilon)^\frac{1}{\pi}} \left[ \int_{\{\varphi \leq -a^{n+2}\}} \omega^n_\varphi - \int_{\{\rho \leq -\frac{1}{2a}\}} \omega^n_\varphi \right] \\
\leq \frac{1}{(1 - \epsilon)^\frac{1}{\pi}} \int_U \omega^n_\varphi \\
\leq 2a^{n+1} + \int_U \omega \wedge \omega^{n-1}_\varphi \\
\leq 2a^{n+1} + \int_{\{\varphi < \psi\}} \omega \wedge \omega^{n-1}_\varphi,
\]

Similarly to \(\rho\) we define \(\vartheta \in \mathcal{E}(X, \omega)\), such that
\[
\omega^n_\vartheta = \frac{1}{1 - \epsilon} 1_{\{\varphi < \psi\}} \omega^n_\varphi + l 1_{\{\psi \geq \varphi\}} \omega^n_\varphi \quad \text{and} \quad \sup_X \vartheta = 0,
\]
(l plays the same role as \(c\) above). Set
\[
V = \left\{(1 - a^{n+2+\beta})\psi < (1 - a^{n+2+\beta})\varphi + a^{n+2+\beta}\varphi \right\} \subset \{\psi < \varphi\}.
\]
We get

(6) \[
\frac{1}{(1 - \epsilon)^\frac{1}{\pi}} \left[ \int_{\{\varphi \leq -a^{n+2}\}} \omega^n_\varphi - C_1(\alpha, A)a^{n+1} \right] \leq 2a^{n+1} + \int_{\{\psi < \varphi\}} \omega \wedge \omega^{n-1}_\varphi.
\]

From (1), (5) and (6) we obtain
\[
\frac{1}{(1 - \epsilon)^\frac{1}{\pi}} \left[ 1 - 2a^{n+1} - 2C_1(\alpha, A)a^{n+1} \right] \\
\leq \frac{1}{(1 - \epsilon)^\frac{1}{\pi}} \left[ \int_{|\varphi - \psi| \geq a^{n+1}} \omega^n_\varphi - 2C_1(\alpha, A)a^{1+\alpha} \right] \\
\leq 4a^{n+1} + 1.
\]

Hence
\[
\epsilon \leq 1 - \left[ \frac{1 - 2(C_1(\alpha, A) + 1)a^{n+1}}{4a^{n+1} + 1} \right]^n \leq C_2(\alpha, A)a^{n+1}.
\]

This implies that there exists \(t \in \mathbb{R}\) satisfying
\[
\int_{\{|\varphi - \psi - t| > a\}} \omega^n_\varphi \leq 2C_2(\alpha, A)a^{n+1}.
\]
Finally we have
\[
\int_{\{\|\varphi - \psi - t\| > a\}} (\omega^n_{\varphi} + \omega^n_{\psi}) = 2 \int_{\{\|\varphi - \psi - t\| > a\}} \omega^m_{\varphi} + \int_{\{\|\varphi - \psi - t\| > a\}} (\omega^m_{\psi} - \omega^m_{\varphi}) \\
\leq 2C_2(\alpha, A)a^{n+1} + a^{2n+3+\beta} \leq C(\alpha, A)a^{n+1}.
\]

The second step in proving our stability theorem is the following

**3.2. Proposition.** — Let $\varphi, \psi \in \mathcal{E}^{-}(X, \omega)$ be such that $\omega_{\varphi}^n, \omega_{\psi}^n \in \mathcal{H}(\alpha, A)$. Then there exist constants $t \in \mathbb{R}$ and $C(\alpha, A) \geq 0$ such that
\[
C_X(\{\|\varphi - \psi - t\| > a\}) \leq C(\alpha, A)a,
\]
here $a = [\int_X \|\omega_{\varphi}^n - \omega_{\psi}^n\|]^{\frac{1}{2n+3+\beta}}$.

**Proof.** — Since $C_X(\{\|\varphi - \psi - t\| > a\}) \leq C_X(X) = 1$, it suffices to consider the case when $a$ is small. Without loss of generality we can assume that $\sup_X \varphi = \sup_X \psi = 0$. By Remark 2.5 in [18] there exists $M(\alpha, A) > 0$ such that $\|\varphi\|_{L^\infty(X)} < M(\alpha, A), \|\psi\|_{L^\infty(X)} < M(\alpha, A)$. By Proposition 3.1 we can find $t > 0$ such that
\[
\int_{\{\|\varphi - \psi - t\| > a\}} (\omega_{\varphi}^n + \omega_{\psi}^n) \leq C_1(\alpha, A)a^{n+1}.
\]
We consider the case $a < \min(1, \frac{1}{C_1(\alpha, A)})$. Since $\int_{\{\|\varphi - \psi - t\| > a\}} (\omega_{\varphi}^n + \omega_{\psi}^n) < 1$ we get $\{\|\varphi - \psi - t\| > a\} \neq X$. This implies that $|t| \leq \sup_X \|\varphi - \psi - t\| + 1 \leq M(\alpha, A)+1$. Replacing $\psi$ by $\psi + t$ we can assume that $t = 0$ and $\|\psi\|_{L^\infty(X)} < 2M(\alpha, A) + 1$. Using Lemma 2.3 in [18] for $s = \frac{a}{2}$, $t = \frac{a}{2(M(\alpha, A)+1)}$ we get
\[
C_X(\{\varphi - \psi < -a\}) \leq C_X \left( \left\{ \frac{\varphi - \psi}{a} < -\frac{a}{2} - \frac{a}{2(M(\alpha, A)+1)} \right\} \right) \\
\leq \frac{2^n(2M(\alpha, A)+1)^n}{a^n} \int_{\{\varphi - \psi < -a\}} \omega_{\varphi}^n \\
\leq 2^n(2M(\alpha, A)+1)^nC_1(\alpha, A)a.
\]
Similarly we get
\[
C_X(\{\psi - \varphi < -a\}) \leq 2^n(2M(\alpha, A)+1)^nC_1(\alpha, A)a.
\]
Combination of these inequalities yields
\[
C_X(\{\|\varphi - \psi\| > a\}) \leq C(\alpha, A)a.
\]
Now we prove the promised generalization of Kołodziej stability theorem (Theorem 1.1 in [27]).
3.3. Theorem. — Let $\varphi, \psi \in \mathcal{E}^-(X, \omega)$ be such that $\sup_X \varphi = \sup_X \psi = 0$ and $\omega^n_\varphi, \omega^n_\psi \in \mathcal{H}(\alpha, A)$. Then there exists $C(\alpha, A) > 0$ such that

$$\sup_X |\varphi - \psi| \leq C(\alpha, A) \left[ \int_X \|\omega^n_\varphi - \omega^n_\psi\| \right]^{\frac{\min(1, \frac{n}{n})}{2n^3 + \frac{n}{n+\alpha}}}.$$

Proof. — Set

$$a = \left[ \int_X \|\omega^n_\varphi - \omega^n_\psi\| \right]^{\frac{1}{2n^3 + \frac{n}{n+\alpha}}}.$$

By Proposition 3.2 there exists $C_1(\alpha, A) > 0$ and $t \in \mathbb{R}$ such that $|t| \leq M(\alpha, A) + 1$ and

$$C_X(\{|\varphi - \psi - t| > a\}) \leq C_1(\alpha, A)a.$$

Moreover, by Proposition 2.6 in [18] there exists $C_2(\alpha, A) > 0$ such that

$$\sup_X |\varphi - \psi - t| \leq 2a + C_2(\alpha, A)[C_X(\{|\varphi - \psi - t| > a\})]^{\frac{n}{2}} \leq 2a + C_2(\alpha, A)[C_1(\alpha, A)a]^{\frac{n}{2}} \leq C_3(\alpha, A)a^{\min(1, \frac{n}{n})}.$$

Moreover, since $\sup_X \varphi = \sup_X \psi = 0$ we obtain $|t| \leq C_3(\alpha, A)a^{\min(1, \frac{n}{n})}$. Combination of these inequalities yields

$$\sup_X |\varphi - \psi| \leq \sup_X |\varphi - \psi - t| + |t| \leq 2C_3(\alpha, A)a^{\min(1, \frac{n}{n})}$$

$$= C(\alpha, A) \left[ \int_X \|\omega^n_\varphi - \omega^n_\psi\| \right]^{\frac{\min(1, \frac{n}{n})}{2n^3 + \frac{n}{n+\alpha}}}.$$

\[\square\]

3.4. Corollary. — Let $\mu$ be a non-negative Radon measure on $X$ such that $\mu(B(z, r)) \leq A_r^{2n - 2 + \alpha}$ for all $B(z, r) \subset X$ ($\alpha > 0$ are constants). Given $p > 1, M > 0, \epsilon > 0$ and $f, g \in L^p(d\mu)$ with $\|f\|_{L^p(d\mu)}, \|g\|_{L^p(d\mu)} \leq M$ and $\int_X f d\mu = \int_X g d\mu = 1$. Assume that $\varphi, \psi \in \mathcal{E}^-(X, \omega)$ satisfy $\omega^n_\varphi = f d\mu, \omega^n_\psi = g d\mu$ and $\sup_X \varphi = \sup_X \psi = 0$. Then there exists $C(\alpha, A, M, \epsilon) > 0$ such that

$$\sup_X |\varphi - \psi| \leq C(\alpha, A, M, \epsilon) \left[ \int_X |f - g| d\mu \right]^{\frac{1}{2n^3 + \frac{n}{n+\alpha}}}.$$

Proof. — By Hölder inequality we have

$$\int_K f d\mu \leq \|f\|_{L^p(d\mu)} \mu(K)^{1 - \frac{1}{p}} \leq M[\mu(K)]^{1 - \frac{1}{p}},$$

$$\int_K g d\mu \leq \|g\|_{L^p(d\mu)} \mu(K)^{1 - \frac{1}{p}} \leq M[\mu(K)]^{1 - \frac{1}{p}}.$$
for any Borel subset $K$ of $X$. By Proposition 2.7 we get $f d\mu, g d\mu \in \mathcal{H}(\infty)$. Using Theorem 3.3 we can find $C(\alpha, A, M, \varepsilon) > 0$ such that

$$\sup_X |\varphi - \psi| \leq C(\alpha, A, M, \varepsilon) \left[ \int_X |f - g| d\mu \right]^{\frac{1}{2n+3+\varepsilon}}.$$

□

4. Local estimates in Potential theory

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ ($n \geq 2$). By $\text{SH}(\Omega)$ (resp. $\text{SH}^-(\Omega)$) we denote the set of subharmonic (resp. negative subharmonic) functions on $\Omega$. For each $u \in \text{SH}(\Omega)$ and $\delta > 0$ we denote

$$\tilde{u}_\delta(x) = \frac{1}{c_n \delta^n} \int_{B_\delta} u(x + y) dV_n(y),$$

$$u_\delta(x) = \sup_{y \in B_\delta} u(x + y),$$

for $x \in \Omega_\delta = \{ x \in \Omega: d(x, \partial \Omega) > \delta \}$. Here $B_\delta = \{ x \in \mathbb{R}^n: |x| = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}} < \delta \}$ and $c_n$ is the volume of the unit ball $B_1$. We state some results which will be used in our main theorems.

4.1. Theorem. — Let $\mu$ be a non-negative Radon measure on $\Omega$ such that $\mu(B(z, r)) \leq A r^{n-2+\alpha}$ for all $B(z, r) \subset D \subset \subset \Omega$ ($A, \alpha > 0$ are constants). Then for $K \subset \subset D$ and $\epsilon > 0$ there exists $C(\alpha, A, K, \epsilon)$ such that

$$\int_K [\hat{u}_\delta - u] d\mu \leq C(\alpha, A, K, \epsilon) \int_D \Delta u \delta^{\frac{n-\alpha}{n+3+\epsilon}},$$

for all $u \in \text{SH}(\Omega)$, where $\Delta$ is the Laplace operator.

Proof. — Since the change of radii of the balls does not affect the statement we can assume that $\Omega = B_4$, $D = B_3$, $K = B_1$ and $u$ is smooth on $B_4$. By [22] we have

$$u(x) = \int_{B_2} G(x, z) \Delta u(z) + h(x),$$

where $G(x, y)$ is the fundamental solution of Laplace equation and $h$ is harmonic in $B_2$. By Fubini theorem we have

$$\int_{B_1} [\hat{u}_\delta(x) - u(x)] d\mu(x)$$

$$= \int_{B_1} \frac{1}{c_n \delta^n} \int_{B_\delta} [u(x + y) - u(x)] dV_n(y) d\mu(x)$$

$$\cdot \frac{1}{c_n \delta^n} \int_{B_1} \int_{B_3} \int_{B_2} [G(x + y, z) - G(x, z)] \Delta u(z) dV_n(y) d\mu(x).$$
\[= \int_{B_2} \Delta u(z) \frac{1}{C_n \delta^n} \int_{B_\delta} dV_n(y) \int_{B_1} [G(x + y, z) - G(x, z)]d\mu(x)\]

Set
\[F(y, z) = \int_{B_1} [G(x + y, z) - G(x, z)]d\mu(x).\]

It is enough to prove that \(F(y, z) \leq C(\alpha, A, \epsilon)\delta^{\frac{\alpha - \epsilon}{1 + \alpha}}\) for all \(y \in B_\delta, z \in B_2\). We consider two cases:

Case 1: \(n = 2\). For \(y \in B_\delta, z \in B_2, \delta < \frac{1}{2}\), we have
\[
F(y, z) = \int_{B_1} [\ln |x + y - z| - \ln |x - z|]d\mu(x)
\leq \int_{B_1 \cap \{|x - z| \geq |y|^{\frac{1}{1 + \alpha}}\}} \ln(1 + |y|^{\frac{1}{1 + \alpha}}) |\ln |x - z| |d\mu(x)
+ \ln 4 \int_{B_1 \cap \{|x - z| < |y|^{\frac{1}{1 + \alpha}}\}} d\mu + \int_{B_1 \cap \{|x - z| < |y|^{\frac{1}{1 + \alpha}}\}} \ln \frac{1}{|x - z|} d\mu(x)
\leq |y|^{\frac{\alpha}{1 + \alpha}} \mu(B_1) + A|y|^{\frac{\alpha}{1 + \alpha}} \ln 4
+ |y|^{\frac{\alpha}{1 + \alpha}} \int_{\{|x - z| < |y|^{\frac{1}{1 + \alpha}}\}} \frac{d\mu(x)}{|x - z|^{\alpha - \epsilon}}
\leq A(1 + \ln 4)|y|^{\frac{\alpha}{1 + \alpha}} + |y|^{\frac{\alpha}{1 + \alpha}} C_1(\alpha, \epsilon) \int_{\{|x - z| < 1\}} \frac{d\mu(x)}{|x - z|^{\alpha - \frac{\epsilon}{2}}}
\leq A(1 + \ln 4)|y|^{\frac{\alpha}{1 + \alpha}}
+ C_1(\alpha, \epsilon)|y|^{\frac{\alpha - \epsilon}{1 + \alpha}} \sum_{j=0}^{\infty} \int_{\{|x - z| \leq 2^{-j-1}\}} \frac{d\mu(x)}{|x - z|^{\alpha - \frac{\epsilon}{2}}}
\leq A(1 + \ln 4)|y|^{\frac{\alpha}{1 + \alpha}} + C_1(\alpha, \epsilon)|y|^{\frac{\alpha - \epsilon}{1 + \alpha}} A \sum_{j=0}^{\infty} 2^{(j+1)(\alpha - \frac{\epsilon}{2}) - j\alpha}
\leq C(\alpha, A, \epsilon)|y|^{\frac{\alpha - \epsilon}{1 + \alpha}} \leq C(\alpha, A, \epsilon)\delta^{\frac{\alpha - \epsilon}{1 + \alpha}}.

Case 2: \(n \geq 3\). Similarly for \(y \in B_\delta, z \in B_2, \delta < \frac{1}{2}\), we have
\[
F(y, z) = \int_{B_1} \left[ -\frac{1}{|x + y - z|^{n-2}} + \frac{1}{|x - z|^{n-2}} \right] d\mu(x)
\]
\[
\int_{B_1 \cap \{|x-z| \geq |y|^{1+\alpha}\}} \frac{|x+y-z|^{n-2} - |x-z|^{n-2}}{|x+y-z|^{n-2}|x-z|^{n-2}} d\mu(x) \\
+ \int_{\{|x-z| < |y|^{1+\alpha}\}} \frac{d\mu(x)}{|x-z|^{n-2}} \\
\leq C_2(\alpha) |y|^{\frac{\alpha}{1+\alpha}} \int_{B_1 \cap \{|x-z| \geq |y|^{1+\alpha}\}} d\mu(x) \\
+ |y|^{\frac{\alpha}{1+\alpha}} \int_{\{|x-z| < |y|^{1+\alpha}\}} \frac{d\mu(x)}{|x-z|^{n-2}} \\
\leq AC_2(\alpha) |y|^{\frac{\alpha}{1+\alpha}} + |y|^{\frac{\alpha}{1+\alpha}} \int_{\{|x-z| < 1\}} \frac{d\mu(x)}{|x-z|^{n-2}} \\
\leq C(\alpha, A, \epsilon) |y|^{\frac{\alpha-\epsilon}{1+\alpha}} \\
\leq C(\alpha, A, \epsilon) \delta^{\frac{\alpha-\epsilon}{1+\alpha}}.
\]

\[\square\]

4.2. THEOREM. — Let \( \mu \) be a non-negative Radon measure on \( \Omega \) such that \( \mu(B(z,r)) \leq Ar^{n-2+\alpha} \) for all \( B(z,r) \subset D \subset \subset \Omega \) (\( A, \alpha > 0 \) are constants). Then for \( K \subset \subset D \) and \( \epsilon > 0 \) there exists \( C(\alpha, A, K, \epsilon) \) such that

\[
\int_K [u_\delta - u] d\mu \leq C(\alpha, A, K, \epsilon) \|u\|_{L^\infty(\Omega)} \delta^{\frac{\alpha-\epsilon}{2(1+\alpha)}},
\]

for all \( u \in SH \cap L^\infty(\Omega) \).

We need a well-known lemma:

4.3. LEMMA. — Let \( u \in SH \cap L^\infty(\Omega) \). Then

\[
|\tilde{u}_\delta(x) - \tilde{u}_\delta(y)| \leq \frac{\|u\|_{L^\infty(\Omega)} |x-y|}{\delta},
\]

for all \( x, y \in \Omega_\delta \).

Proof. — Proof of Theorem 4.2 By Lemma 4.3 we have

\[
u_\delta(x) = \sup_{y \in B_\delta} u(x+y) \leq \sup_{y \in B_\delta} \tilde{u}_\delta^{1/2} (x+y) \leq \tilde{u}_\delta^{1/2} (x) + \delta^{1/2} \|u\|_{L^\infty(\Omega)}.
\]

By Theorem 4.1 we get

\[
\int_K [u_\delta - u] d\mu \leq \int_K [\tilde{u}_\delta^{1/2} - u] d\mu + \|u\|_{L^\infty(\Omega)} \mu(K) \delta^{1/2} \\
\leq C(\alpha, A, K, \epsilon) \|u\|_{L^\infty(\Omega)} \delta^{\frac{\alpha-\epsilon}{2(1+\alpha)}}.
\]

Next we state a well-known result is a direct consequence of the Jensen formula (see [1]) \[\square\]
4.4. Proposition. — Let \( u \in SH(B_2) \) be such that \( |u(x) - u(y)| \leq A|x - y|^{\alpha} \) for all \( x, y \in B_2 \). Then there exists \( C(\alpha, A) > 0 \) such that

\[
\int_{B(x,r)} \Delta u \leq C(\alpha, A)r^{n-2+\alpha},
\]

for all \( B(x, r) \subset B_1 \).

5. Main results

Proof of Theorem A. — We use the same scheme as the proof of Theorem 2.1 in [27]. From Corollary 3.4 and from Theorem 4.2 we can replace \( \omega^n \) by \( d\mu \). This implies that \( u \) is Hölder continuous with the Hölder exponent dependent on \( \alpha, A, p, X \) and \( \|f\|_{L^p(d\mu)} \).

Proof of Corollary B. — It follows from Proposition 4.4 and Theorem A.

Proof of Corollary C. — Direct application of Theorem A.

BIBLIOGRAPHY


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