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Alessandro CHIODO & Yongbin RUAN

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A GLOBAL MIRROR SYMMETRY FRAMEWORK FOR THE LANDAU–GINZBURG/CALABI–YAU CORRESPONDENCE

by Alessandro CHIODO & Yongbin RUAN

Abstract. — We show how the Landau–Ginzburg/Calabi–Yau correspondence for the quintic three-fold can be cast into a global mirror symmetry framework. Then we draw inspiration from Berglund–Hübsch mirror duality construction to provide an analogue conjectural picture featuring all Calabi–Yau hypersurfaces within weighted projective spaces and certain quotients by finite abelian group actions.

Résumé. — On montre comment la correspondance Landau–Ginzburg/Calabi–Yau pour la variété quintique dans $\mathbb{P}^4$ s'inscrit naturellement dans un cadre de symétrie miroir globale. On s'inspire de la dualité miroir de Berglund–Hübsch pour fournir un cadre conjectural analogue qui incorpore toutes les hypersurfaces de Calabi–Yau dans les espaces projectifs à poids, ainsi que certains quotients par l'action de groupes abéliens finis.

1. Introduction

We survey FJRW theory introduced by Fan, Jarvis, and the second author for the Landau–Ginzburg model following ideas of Witten. We review its connection to related work and we provide a prospectus on the ideas guiding the long term development of FRJW theory. The paper also contains some new results on foundational aspects of FRJW theory.

The theory can be motivated as a tool for the computation of Gromov–Witten invariants. Almost twenty years ago a correspondence was proposed (see [70] and [74]) in order to connect two areas of physics: the Landau–Ginzburg (LG) model and Calabi–Yau (CY) geometry. In simple

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terms, the geometry of certain CY spaces is expected to be completely encoded by another geometrical object, the LG model, which is in many cases easier to study. The case of the quintic three-fold illustrates this well: a smooth hypersurface defined in $\mathbb{P}^4$ by a homogeneous degree-five polynomial plays a central role in Gromov–Witten theory since its early developments. Whereas in genus zero the theory has been completely elucidated in [34] and [56] matching the mirror symmetry conjecture, for positive genus the theory is largely unknown: it has been determined by Zinger [75] for $g = 1$ and is still wide open for $g > 1$ despite the joint effort of mathematicians and physicists over the last twenty years. From the point of view of theoretical physics, the most advanced effort is Huang, Klemm, and Quackenbush's speculation [40] via a physical argument; it is striking however that, even with these far-reaching techniques, there is no prediction beyond $g = 52$. A natural idea to approach the higher genus cases consists in providing a mathematical statement of the physical LG-CY correspondence and using the computational power of the LG singularity model to determine the higher genus Gromov–Witten invariants of the CY manifold. From the mathematical point of view, this conceptual framework is largely incomplete: whereas Gromov–Witten (GW) theory embodies all the relevant information on the CY side, it is not clear which theory plays the same role on the LG side. This is likely to be interesting in its own right; for instance, in a different context, the LG-CY correspondence led to Orlov's equivalence between the derived category of complexes of coherent sheaves and matrix factorizations (see [63], [38] and [47]).

In [31, 32, 30], Fan, Jarvis and the second author construct such a candidate quantum theory of singularities: FJRW theory. In intuitive terms, GW theory may be regarded as the study of the solutions of the Cauchy–Riemann equation $\overline{\partial} f = 0$ for the map $f\colon C \to X_W$, where C is a compact Riemann surface and X_W is a degree-N hypersurface within a projective space with N homogeneous coordinates. On the other hand, in the LG singularity model, we treat W as a holomorphic function on $\mathbb{C}^N$. From this perspective, FJRW theory is about solving a generalized PDE attached to W rather than classifying holomorphic maps from a compact Riemann surface Σ to $\mathbb{C}^N$. The idea comes from Witten's conjecture [72] stated in the early 90's and soon proven by Kontsevich [48]: the intersection theory of Deligne and Mumford's moduli of curves is governed by the KdV integrable hierarchy—*i.e.* the integrable system corresponding to the A_1-singularity. Witten generalized Deligne and Mumford's spaces to new moduli spaces governed by integrable hierarchies attached to more general singularities.

To this effect, he considers the PDE

$$\overline{\partial} s_j + \overline{\partial_j W(s_1, \cdots, s_N)} = 0, \tag{1.1}$$

where W is the same polynomial defining X_W and $\partial_j W$ is the derivative with respect to the jth variable. Faber, Shadrin and Zvonkine proved this conjecture for A_n-singularities. Fan, Jarvis, and the second author [31, 32, 30] extended Witten approach to any singularity and genealized the proof of Witten's statement to all simple singularities. In this way FJRW theory plays the role of Gromov–Witten theory on the LG side for any isolated singularity defined by a quasihomogeneous polynomial. The Witten equation should be viewed as the counterpart to the Cauchy–Riemann equation: when we pass to the LG singularity model we replace the linear Cauchy–Riemann equation on a nonlinear target with the nonlinear Witten equation on a linear target.

Three years ago, a program was launched by the authors in order to establish the LG-CY correspondence mathematically. Since then, a great deal of progress has been made: the proof of classical mirror symmetry statements via the LG model (by the authors [17] and Krawitz [50]), the modularity of the Gromov–Witten theory of elliptic orbifold $\mathbb{P}^1$ (see Krawitz–Shen [51] and work by the second author in collaboration with Milanov [59]) and the connection to Orlov's equivalence (by the authors in collaboration with Iritani [15]). In this survey article, we report on some of the progress within a common framework and we complement at several points our treatment of the quintic threefold [16].

1.1. LG-CY correspondence and "global" mirror symmetry

So far we have presented the LG-CY correspondence from the point of view of the open problem of computing GW theory. The framework of mirror symmetry, however, allows us to recast this transition from CY geometry to the LG side within a geometric setup involving a wider circle of ideas. This is the main focus of this paper.

Recall that mirror symmetry asserts a duality among CY three-folds exchanging the A model invariants with the B model invariants. Naively, the A model contains information such as the Kähler structure and Gromov–Witten invariants, while the B model contains information such as the complex structure and period integrals. From a global point of view, this picture cannot be entirely satisfactory, because the complex moduli space has a nontrivial topology while the Kähler moduli space does not.

1.1.1. Cohomological mirror symmetry

Let us illustrate this issue by means of the example which inspired the whole phenomenon of mirror symmetry [11]. On the one side of the mirror we have the quintic three-fold

$$X_W = \{x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = 0\} \subset \mathbb{P}^4 \tag{1.2}$$

equipped with a natural holomorphic three-form $\omega = dx_1 \wedge dx_2 \wedge dx_3/x_4^4$ (written here in coordinates with $x_5 = 1$). On the other side we take the quotient of X_W by the group $G \cong (\mathbb{Z}_5)^4$ spanned by $x_i \mapsto \alpha x_i$ with $\alpha^5 = 1$ for all $i = 1, \dots, 5$ subject to the condition that ω is preserved$^{(1)}$. The quotient scheme X_W/G is singular; but there is a natural, canonically defined, resolution $Y = (X_W/G)^{\mathrm{res}}$ which is again a CY variety.

In general the existence of resolutions of CY type is not guaranteed. But we can rephrase things in higher generality in terms of orbifolds: let us mod out G by the kernel of $G \to \mathrm{Aut}(X_W)$, the group spanned by the diagonal symmetry j_W scaling all coordinates by the same primitive fifth root ξ_5. Then, the quotient of X_W by $\widetilde{G} = G/\langle j_W \rangle$ equals X_W/G and the group $\widetilde{G}$ acts faithfully. In this way, the resolution Y may be equivalently replaced by the smooth quotient stack (orbifold)

$$X_W^\vee = [X_W/\widetilde{G}] \tag{1.3}$$

(a cohomological equivalence between Y and $X_W^\vee$ holds under the condition that the stabilizers are nontrivial only in codimension 2).

The odd cohomology (primitive cohomology) of $X_W^\vee$ is four-dimensional and unusually simple: the odd-degree Hodge numbers equal $(1, 1, 1, 1)$ and mirror the four hyperplane sections $\mathbf{1}, H, H^2, H^3$ of the projective hypersurface X_W

$$h^{p,q}(X_W^\vee) = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 101 & & 0 & \\ 1 & & 1 & & 1 & & 1 \\ & 0 & & 101 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \qquad h^{p,q}(X_W) = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}.$$

Indeed, this is part of the cohomological mirror symmetry

$$h^{p,q}(X_W) = h^{\dim - p,q}(X_W^\vee). \tag{1.4}$$

(1) In other words, each diagonal transformation $\mathrm{Diag}(\alpha_1 \in \boldsymbol{\mu}_5, \dots, \alpha_5 \in \boldsymbol{\mu}_5)$ should satisfy $\det = \prod_i \alpha_i = 1$.

1.1.2. Mirror symmetry at the large complex structure point

We further illustrate mirror symmetry for this example with special attention to the difference in global geometry between the two sides. On one side of the mirror, for X_W, we consider the (complexified) Kähler moduli space — a *contractible* one-dimensional complex space which should be regarded as an A side invariant

$$\mathcal{A}(X_W).$$

On the other side of the mirror we consider a B model invariant: the (complex structure) deformations of $[X_W/\widetilde{G}]$. These are actually deformations of X_W preserved by the action of $\widetilde{G} = G/\langle j_W\rangle$. We get the Dwork family

$$X_{W,\psi} = \left\{ x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 + 5\psi \prod_{i=1}^{5} x_i = 0 \right\},$$

on which $\widetilde{G}$ operates by preserving the fibres and the form $\omega_\psi = dx_1 \wedge dx_2 \wedge dx_3/(x_4^4 - \psi x_1x_2x_3)$ yielding a family of CY orbifolds $X^\vee_{W,\psi}$ over an open subset of $\mathbb{P}^1_\psi$ (the complement of the divisor where singularities occur). In fact, for $\alpha^5 = 1$, we can let the diagonal symmetry $x_i \mapsto \alpha x_i$ operate on the family so that the action identifies the fibre $X^\vee_{W,\psi}$ over ψ with the isomorphic fibre $X^\vee_{W,\alpha\psi}$ over $\alpha\psi$. Therefore, the Dwork family is ultimately a family of three-dimensional CY orbifolds over $[\mathbb{P}^1/\mathbb{Z}_5]$. Write $t = \psi^5$; then the new family is regular off $t = \infty$ and $t = 1$. These limit points alongside with the stack-theoretic point $t = 0$ are usually referred to as *special limit points*; more precisely, $0, \infty$, and 1 are referred to as the *Gepner point*, the *large complex structure point*, and the *conifold point*. Unlike the Kähler moduli space, this moduli space of complex structures is *not contractible*. For this reason, mirror symmetry has been studied as an identification between the above contractible Kähler moduli space $\mathcal{A}(X_W)$ and a contractible neighborhood of the large complex structure point $t = \infty$

$$\mathcal{B}(X^\vee_{W,\infty}).$$

This leads to a formulation of mirror symmetry as a local statement matching the A model to the B model restricted to a neighborhood of the large complex structure point. Consider the bundle over $\mathcal{B}(X^\vee_{W,\infty})$ minus the origin with four-dimensional fibre $H^3(X^\vee_{W,t}, \mathbb{C})$ over $t \in \mathcal{B}(X^\vee_{W,\infty})$. There is, of course, a flat connection, the Gauss–Manin connection, given by the local system $H^3(X^\vee_{W,t}, \mathbb{Z}) \subset H^3(X^\vee_{W,t}, \mathbb{C})$. Dubrovin has shown how to use Gromov–Witten invariants to put a flat connection on the four-dimensional bundle with fibre $H^{\mathrm{ev}}(X_W)$ over $\mathcal{A}(X_W)$. Under a suitable identification (mirror map)

$$\begin{array}{c} \mathcal{B}(X^\vee_{W,\infty}) \\ \updownarrow \cong \\ \mathcal{A}(X_W) \end{array} \tag{1.5}$$

the two structures are identified (Givental [33], Lian–Liu–Yau [56]). This local point of view dominated the mathematical study of mirror symmetry for the last twenty years.

1.1.3. Global mirror symmetry

It is natural to extend our study to the entire moduli space $[\mathbb{P}^1/\mathbb{Z}_5]$ and to all the special limits. Such a *global* point of view underlies a large part of the physics literature on the subject and leads naturally to the famous holomorphic anomaly equation [8] and, in turn, to the above mentioned spectacular physical predictions [40] on Gromov–Witten invariants of the quintic three-fold up to genus 52. In the early 90's, a physical solution was proposed to complete the Kähler moduli space by including other *phases* [60, 74]. As we shall illustrate, for the quintic three-fold, two phases arise in the A model: the CY geometry and the LG phase. Whereas the CY geometry of the quintic has already been identified by mirror symmetry to a neighborhood of the large complex structure limit point $\mathcal{B}(X^\vee_{W,\infty})$, the LG phase is expected to be mirror to the neighborhood of the Gepner point at 0

$$\mathcal{B}(X^\vee_{W,0}).$$

Then, the LG-CY correspondence can be interpreted as an analytic continuation from the Gepner point to the large complex structure point. From this point of view, the LG-CY correspondence should be viewed as a step towards global mirror symmetry.

From a purely mathematical point of view it may appear difficult to make sense of such a transition of the CY quintic three-fold into a different "phase". Fortunately, Witten has illustrated this in precise mathematical terms as a variation of stability conditions in geometric invariant theory, [74, §4]. Let us consider the explicit example of the Fermat quintic Calabi–Yau three-fold: let $Y = \mathbb{C}^6$ with coordinates $x_1, \dots, x_5$ and p and let $\mathbb{C}^*$ act as

$$x_i \mapsto \lambda x_i, \ \forall i; \qquad p \mapsto \lambda^{-5} p.$$

The presence of nonclosed orbits prevents us from defining a geometric quotient. In order to obtain a geometric quotient, one should necessarily restrict to open $\mathbb{C}^*$-invariant subsets Ω of $V \cong \mathbb{C}^6$ for which $\Omega/\mathbb{C}^*$ exists. The geometric invariant theory (GIT) yields two maximal possibilities: the sets $\Omega_1 = \{\boldsymbol{x} \neq 0\}$ yielding $\mathcal{O}(-5)$ as a quotient by $\mathbb{C}^*$ and the set $\Omega_2 = \{p \neq 0\}$ yielding the orbifold $[\mathbb{C}^5/\mathbb{Z}_5]$. If one adds to the picture a $\mathbb{C}^*$-invariant holomorphic function such as $\widetilde{W}(p, x_1, \dots, x_5) = pW(x_1, \dots, x_5) = p\sum_i x_i^5$ the two geometric models ultimately reduce to the Fermat quintic X_W and to the singularity at the origin of

$$W = \sum_i x_i^5 \colon [\mathbb{C}^5/\mathbb{Z}_5] \longrightarrow \mathbb{C}. \tag{1.6}$$

On $\mathcal{B}(X^\vee_{W,0})$ consider the bundle with fibre $H^3(X^\vee_{W,t}, \mathbb{C})$ over the point t. There is again the flat Gauss–Manin connection induced by the local system $H^3(X^\vee_{W,t}, \mathbb{Z}) \subset H^3(X^\vee_{W,t}, \mathbb{C})$. The work of Fan, Jarvis, and the second author [31] yields — via Dubrovin connection — a flat connection on a vector bundle on a contractible one-dimensional space

$$\mathcal{A}(W, \mathbb{Z}_5)$$

attached to (1.6). (For the abstract formalism of Dubrovin connection we refer to Iritani [42].) The fibre of this bundle is a four-dimensional state space attached to the singularity $W\colon [\mathbb{C}^5/\mathbb{Z}_5] \to \mathbb{C}$ (see §2.1). Under a suitable identification (mirror map)

$$\begin{tikzcd} \mathcal{B}(X^\vee_{W,0}) \arrow[d, leftrightarrow, "\cong"] \\ \mathcal{A}(W, \mathbb{Z}_5) \end{tikzcd} \tag{1.7}$$

we match the two structures in [16]. The LG-CY correspondence can be now carried out via (1.5) and (1.7) on the B side via the local system induced by the family of CY orbifolds $X^\vee_{W,t}$ with t varying in $(\mathbb{P}^1)^\times = \mathbb{P}^1_t \setminus \{0, 1, \infty\}$. Consider Figure 1.1, where the notation $(\quad)^\times$ stands for $(\quad) \setminus \{$special points 0, 1 and $\infty\}$, the horizontal maps to $(\mathbb{P}^1)^\times$ are the natural inclusions, and V, V_0 and V_∞ are the four-dimensional bundles with fibre $H^3(X^\vee_{W,t}, \mathbb{C})$ equipped with the respective Gauss–Manin connections ∇.

$$\begin{tikzcd}
 & (V_0,\nabla_0) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & (V,\nabla) \arrow[d] & (V_\infty,\nabla_\infty) \arrow[l] \arrow[d] \arrow[dl, phantom, "\square"] \\
B \text{ side} \arrow[d, leftrightarrow, "\text{mirror symmetry}"] & \mathcal{B}(X^\vee_{W,0})^\times \arrow[r] \arrow[d, leftrightarrow, "\cong"] & (\mathbb{P}^1)^\times & \mathcal{B}(X^\vee_{W,\infty})^\times \arrow[l] \arrow[d, leftrightarrow, "\cong"] \\
A \text{ side} & \mathcal{A}(W,\mathbb{Z}_5)^\times & & \mathcal{A}(X_W)^\times \\
 & \text{LG side} & \leftarrow \text{correspondence} \rightarrow & \text{CY side}
\end{tikzcd}$$

Figure 1.1. Casting LG-CY correspondence within the global mirror symmetry framework.

On the CY side, *i.e.* on $\mathcal{A}(X_W)^\times$, the isomorphism (1.5) and the study of the variation of the Hodge structure of $X^\vee_{W,t}$ on $\mathcal{B}(X_{W,\infty})^\times$ allow us to associate to a given basis of $H^{\mathrm{ev}}(X_W)$ a basis of multivalued functions from $\mathcal{A}(X_W)^\times$ to $H^{\mathrm{ev}}(X_W)$ which are flat with respect to Dubrovin connection. This amounts to solving Gromov–Witten theory for X_W in genus zero. The analogous problem holds on $\mathcal{A}(W,\mathbb{Z}_5)^\times$ on the LG side; it is solved via (1.7) by computing FJRW theory for $(W,\mathbb{Z}_5)$ in genus zero. Furthermore, via analytic continuation, we can extend the bases of flat sections globally on $(\mathbb{P}^1)^\times$ and find a change of bases matrix

$$\mathbb{U}_{\text{LG-CY}}. \tag{1.8}$$

This is explicitly computed in [16] for the Fermat quintic and is independent of the base parameter t on $(\mathbb{P}^1)^\times$. In [15] we provide a geometric interpretation in terms of Orlov's equivalence relating the bounded derived category of coherent sheaves on X_W and the category of matrix factorizations attached to $(W,\mathbb{Z}_5)$ (see §4.4). This point of view is interesting in its own right, because it short-circuits mirror symmetry in the diagram and provides a conceptual explanation to the fact that $\mathbb{U}_{\text{LG-CY}}$ is symplectic (a crucial fact for Conjecture 4.2). We refer to §4.4.

This paper focuses on the generalization of Figure 1.1. Indeed, the richness of the mirror symmetry picture calls for generalizations in several directions and makes the whole process of figuring out all sides of the story very entertaining: some corners of Figure 1.1 can be more rapidly understood and shed light on entire the picture. For instance, there are cohomological aspects that can be treated in very high generality; *e.g.* we provide results applying to finite group quotients of CY hypersurfaces in weighted projective spaces (see Section 3). On the other hand, the *quantum* cohomological aspects are intrinsically more difficult; there, the results are almost exclusively limited to certain well behaved ambient spaces (Gorenstein weighted projective spaces); in this way, the purely cohomological results can be used to formulate conjectures.

1.2. Structure of the paper

In Section 2, we will review Fan–Jarvis–Ruan–Witten (FJRW) theory which plays a crucial role in these recent developments. In Section 3 we will generalize point (1.4) above discussing results of Berglund, Hübsch and Krawitz on LG phases and of the authors on LG-CY cohomological correspondence. In Section 4 we focus on the quantum counterpart of these theorems; *i.e.* we provide correspondences involving the enumerative geometry of curves. This section is structured in four parts. We will state the LG-CY conjecture and review recent results §4.1. Then, we will cast it in a global mirror symmetry framework in §4.2. We will present in §4.3 a result going very far in providing evidence for this global mirror symmetry framework in higher genus [51] [59]. Finally, in §4.4 we will provide an independent interpretation of the LG-CY correspondence via Orlov's equivalence (work in collaboration with Iritani).

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2. Fan–Jarvis–Ruan–Witten theory

In this section, we review Fan, Jarvis, and Ruan's construction of "quantum" singularity theory based on Witten's partial differential equation (1.1). The treatment given here is more general than that appearing in [16] and complements [31]. The theory provides us with the LG side of the correspondence. The input for the theory is a pair (W, G) where W is a "nondegenerate" quasihomogeneous polynomial $W\colon \mathbb{C}^N \to \mathbb{C}$ and G is a group of diagonal symmetries of W.

We say that $W\colon \mathbb{C}^N \to \mathbb{C}$ is quasihomogeneous (or weighted homogeneous) polynomial of type $(q_1, \dots, q_N)$ for $q_j \in \mathbb{Q}_{>0}$ if the following condition is satisfied. Let $W = \sum_{i=1}^{s} \gamma_i \prod x_j^{m_{i,j}}$ with $m_{i,j} \in \mathbb{Z}$, $m_{i,j} \geqslant 0$ and $\gamma_i \neq 0$; then $\sum_{j=1}^{N} m_{i,j} q_j = 1$. Equivalently, with a slight abuse of notation, we write

$$W(\lambda^{q_1} x_1, \dots, \lambda^{q_N} x_N) = \lambda W(x_1, \dots, x_N)$$

and we refer to $q_1, \dots, q_N$ as the *charges* of W. The polynomial W is *nondegenerate* if: (1) W defines a unique singularity at zero; (2) the choice of $q_1, \dots, q_N$ is unique.

An element $g \in GL(\mathbb{C}^N)$ is a *diagonal symmetry of* W if g is a diagonal matrix of the form $\mathrm{Diag}(\lambda_1, \dots, \lambda_N)$ such that

$$W(\lambda_1 x_1, \dots, \lambda_N x_N) = W(x_1, \dots, x_N).$$

We will use $\mathrm{Aut}(W)$ to denote the group of all diagonal symmetries and we will refer to it as the maximal group of diagonal symmetries. It is easy to see that this group is finite (see for instance [31]). The group is also nontrivial since it contains the element $j_W = \mathrm{Diag}(e^{2\pi \mathrm{i} q_1}, \dots, e^{2\pi \mathrm{i} q_N})$.

FJRW theory applies to a pair (W, G), where $G \subseteq \mathrm{Aut}(W)$. Two conditions will naturally arise in the rest of the paper; their role is specular in the sense of mirror symmetry. We will say that $G \subseteq \mathrm{Aut}(W)$ is *A-admissible* if j_W is contained in G. We will say that it is *B-admissible* if $G \subseteq SL(\mathbb{C}^N)$; *i.e.* if G is included in $SL_W = SL(\mathbb{C}^N) \cap \mathrm{Aut}(W)$.

2.1. A model state space

The state space was introduced by Fan, Jarvis, and Ruan [31, 32, 30] as part of the moduli theory of the Witten equation. We provide a purely mathematical definition, independent from the mirror symmetry motivation.

2.1.1. Lefschetz thimbles from the classical point of view

Consider $W\colon \mathbb{C}^N \to \mathbb{C}$. Let us recall some important facts on the relative homology of $(\mathbb{C}^N, W^{-1}(S^+_M))$ where S^+_M is the half-plane $\{z \in \mathbb{C} \mid \mathrm{Re} z > M\}$ for $M > 0$. We denote it by

$$H_N(\mathbb{C}^N, W^{+\infty}; \mathbb{Z})$$

with $W^{+\infty} = W^{-1}(S^+_M)$ and, by abusing notation, we refer to it as the *space of Lefschetz thimbles*.

Remark 2.1. — Due to the nondegeneracy condition, the origin is the only critical point of W, and W is a fibre bundle on $\mathbb{C}^\times$; for $N > 1$, since $\mathbb{C}^N$ is contractible, we can regard the above relative cohomology as the homology (with compact support) of rank $N-1$ of the fibre over a point of S^+_M ([64, 1.1]).

Remark 2.2. — Standard arguments ([4, ch. 2] and [57, (5.11)]) show that the space of Lefschetz thimbles is freely generated by as many generators μ as the complex dimension of the local algebra $\mathcal{Q}_W$. Furthermore, due to [62] and [71], for any $G \subseteq \mathrm{Aut}(W)$, its dual $\mathrm{Hom}(H, \mathbb{C})$ is isomorphic *as a G-space* to $dx_1 \wedge \cdots \wedge dx_N \cdot \mathcal{Q}_W$ (where a diagonal symmetry $g = \mathrm{Diag}(\lambda_1, \ldots, \lambda_N)$ acts on $dx_1 \wedge \cdots \wedge dx_N$ by multiplication by $\prod_j \lambda_j$).

Remark 2.3. — A nondegenerate pairing can be defined in the following sense. Following [37, §8, Step 2] and [64], we consider the relative homology

$$H_N(\mathbb{C}^N, W^{-\infty}; \mathbb{Z}),$$

where $W^{-\infty}$ denotes $W^{-1}(S^-_M)$ and S^-_M is the half-plane $\{z \in \mathbb{C} \mid \mathrm{Re} z < -M\}$ for $M > 0$. The intersection form for Lefschetz thimbles with boundaries in $W^{+\infty}$ and in $W^{-\infty}$ gives a well defined nondegenerate pairing

$$P\colon H_N(\mathbb{C}^N, W^{+\infty}; \mathbb{Z}) \times H_N(\mathbb{C}^N, W^{-\infty}; \mathbb{Z}) \longrightarrow \mathbb{Z}. \tag{2.1}$$

2.1.2. The state space of (W, G)

In our setup the above facts can be used to define the state space as the space of Lefschetz thimbles for the *stack-theoretic* map

$$W\colon [\mathbb{C}^N/G] \longrightarrow \mathbb{C}$$

where G is an A-admissible group (*i.e.* a group of diagonal symmetries containing j_W). The quasihomogeneity condition yields a state space naturally equipped with a nondegenerate *inner* pairing.

Let us define the state space first; for the scheme-theoretic morphism $W\colon \mathbb{C}^N \to \mathbb{C}$ it is natural to consider the relative cohomology $H^*(\mathbb{C}^N, W^{+\infty})$ which is concentrated in degree N and dual to the above space of Lefschetz thimbles. Since $[\mathbb{C}^N/G]$ is a stack, and the loci $W^{+\infty}$ and $W^{-\infty}$ (preimages of S_M^+ and S_M^-) are substacks, the suitable cohomology theory for this setup is orbifold cohomology (or Chen–Ruan cohomology). Indeed Chen–Ruan cohomology admits a natural relative version

$$H^{a,b}_{\mathrm{CR}}(U,V) = \bigoplus_g H^{a-\mathrm{age}(g),b-\mathrm{age}(g)}(U_g, V_g; \mathbb{C})^G,$$

where, in complete analogy with the standard definition of Chen–Ruan cohomology, the union runs over the elements of the stabilizers of the stack U (*i.e.*, in our case, the elements of G), the notation U_g and V_g stands for the stacks where the automorphism g persists, and $\mathrm{age}(g)$ denotes the age[(2)] of g acting on the normal bundle of U_g in U.

Definition 2.4 (A model state space). — *For any A-admissible group G, we set*

$$\mathcal{H}^{a,b}_{W,G} := H^{a+q,b+q}_{\mathrm{CR}}([\mathbb{C}^N/G], W^{+\infty}) \qquad\qquad q = \textstyle\sum_j q_j.$$

Remark 2.5. — The above state space is the direct sum of two spaces: the image and the kernel of

$$i_*\colon H^*_{\mathrm{CR}}([\mathbb{C}^N/G], W^{+\infty}) \to H^*_{\mathrm{CR}}([\mathbb{C}^N/G])$$

The image of i_* is isomorphic in $\mathcal{H}_{W,G}$ to classes attached to diagonal symmetries g *fixing only the origin*; these are *narrow* states (in Section 2.2, Remark 2.16, we see how these states arise in the geometry of curves and we motivate the terminology "narrow" from this different viewpoint). A special case of narrow state is the fundamental class attached to j_W: since j_W fixes only the origin this class is narrow, and — by construction — its degree vanishes. Such a state plays the role of the unit of $\mathcal{H}_{W,G}$ once the ring structure is set up (see (3.10)). The complementary space of the space of narrow states, *i.e.* the kernel of i_*, is referred to in [31] as the space of *broad* states. These are classes attached to diagonal symmetries fixing a nontrivial subspace of $\mathbb{C}^N$.

(2) We define $\mathrm{age}(\alpha, V) \in \mathbb{Q}$ for any finite order autormorphism α of a vector space V, or — equivalently — for any representation of $\boldsymbol{\mu}_r$ for some $r \in \mathbb{N}$. Each character $\chi\colon \boldsymbol{\mu}_r \to \mathbb{C}^*$ is of the form $t \to t^k$ for a unique integer k with $0 \leqslant k \leqslant r-1$ and, for these representations, we define the age of χ as k/r. Since these characters form a basis for the representation ring of $\boldsymbol{\mu}_r$, this extends to a unique additive homomorphism which we denote by age: $R\boldsymbol{\mu}_r \to \mathbb{Q}$.

Remark 2.6. — By making the above definition explicit we may regard the state space as the direct sum over the elements $g \in G$ of the G-invariant cohomology classes

$$\mathcal{H}_{W,G} = \bigoplus_{g \in G} H^{N_g}(\mathbb{C}^N_g, W_g^{+\infty}; \mathbb{C})^G, \tag{2.2}$$

where N_g is the number of coordinates $x_1, \ldots, x_N$ fixed by g and $\mathbb{C}^N_g$ and $W_g^{+\infty}$ denote the subspaces of $\mathbb{C}^N$ of $W^{+\infty}$ which are fixed by g. In these terms narrow states are spanned by the summands satisfying $N_g \neq 0$. Recall the subspace of G-invariant classes within $H^{N_g}(\mathbb{C}^N_g, W_g^{+\infty})$ is included in the subspace of j_W-invariant classes; this insures that $H^{N_g}(\mathbb{C}^N_g, W_g^{+\infty}; \mathbb{C})^G$ is equipped with a pure Hodge structure of weight N_g; in this way each class has bidegree $(p, N_g - p)$ in standard cohomology and, within $\mathcal{H}_{W,G}$ has bidegree

$$(\deg_A^+, \deg_A^-) = (p, N_g - p) + (\operatorname{age}(g), \operatorname{age}(g)) - (q, q) \qquad (\text{with } q = \textstyle\sum q_j).$$

We will usually write $\mathcal{H}^{a,b}_{W,G}$ for the terms of bidegree (a, b) and $\deg_A$ for the *total* degree $a + b$.

2.1.3. The inner pairing

We now define the nondegenerate inner pairing. The crucial fact is that the quasihomogeneity of the map W allows us to define an automorphism

$$I \colon [\mathbb{C}^N/G] \to [\mathbb{C}^N/G]$$

exchanging $[W^{+\infty}/G]$ with $[W^{-\infty}/G]$. Indeed we can set

$$I(x_1, \ldots, x_N) = (e^{\pi \mathrm{i} q_1}, \ldots, e^{\pi \mathrm{i} q_N})$$

for which

$$W(I(x_1, \ldots, x_N)) = -W(x_1, \ldots, x_N).$$

Recall that automorphisms of $[\mathbb{C}^N/G]$ are defined up to natural transformation (composition with elements of G). The automorphism I induces the nondegenerate *inner* pairing

$$\begin{aligned} \langle \cdot, \cdot \rangle \colon H^N(\mathbb{C}^N, W^{+\infty}; \mathbb{C})^G \times H^N(\mathbb{C}^N, W^{+\infty}; \mathbb{C})^G &\longrightarrow \mathbb{C} \\ (\alpha, \beta) \qquad\qquad &\mapsto \quad P(\alpha, I^*\beta) \end{aligned}$$

via (2.1) and passage to cohomology. Notice that I is defined up to a natural transformation; since we are working with G-invariant cohomology classes this still yields a well defined pairing.

There is an obvious identification ε between $H^{N_g}(\mathbb{C}^N_g, W_g^{+\infty};\mathbb{C})^G$ and $H^{N_h}(\mathbb{C}^{N_h}, W_h^{+\infty};\mathbb{C})^G$ as soon as $g=h^{-1}$ in G. This allows us to define a nondegenerate pairing between these two spaces via $\langle\cdot,\cdot\rangle_g = \langle\cdot,\varepsilon(\cdot)\rangle$ and, in turn, a nondegenerate pairing *globally* on $\mathcal{H}_{W,G}$.

Definition 2.7 (pairing for $\mathcal{H}_{W,G}$). — *We have a nondegenerate inner product*

$$\langle\cdot,\cdot\rangle\colon \mathcal{H}_{W,G}\times\mathcal{H}_{W,G}\to\mathbb{C}$$

pairing $\mathcal{H}^a_{W,G}$ *and* $\mathcal{H}^{2\widehat{c}_W-a}_{W,G}$ *for*

$$\widehat{c}_W = N-2q = \textstyle\sum_j(1-2q_j) \qquad\qquad (\text{central charge}).$$

The above formula follows from the well known relation $\operatorname{age}(g)+\operatorname{age}(g^{-1}) = N-N_g$ from Chen–Ruan cohomology and the overall shift by q in Definition 2.4; it shows that the state space behaves like the cohomology of a variety of complex dimension $\widehat{c}_W$. Under the CY condition $\sum_j q_j = 1$, this equals $N-2$; *i.e.*, precisely the dimension of a weighted projective hypersurface, see Theorem 3.15.

2.2. The moduli space

The relevant moduli space is also defined starting from the pair (W,G) with G A-admissible.

2.2.1. The moduli stack associated to W

The first step is the definition of a moduli stack $W_{g,n}$ attached to the nondegenerate polynomial

$$W=\sum_{i=1}^s \gamma_i\prod_j x_j^{m_{i,j}}. \tag{2.3}$$

We provide and elementary definition, simplifying that of [31] (see Remark 2.13). The moduli stack $W_{g,n}$ is an étale cover of a compactification of the usual moduli stack of curves $\mathcal{M}_{g,n}$. Set

$$\delta=\exp(\operatorname{Aut}(W));$$

i.e., the exponent of the group $\operatorname{Aut}(W)$ (the smallest integer δ for which $g^\delta=1$ for all $\delta\in\operatorname{Aut}(W)$). A *$\delta$-stable* curve is a proper and geometrically connected orbifold curve (or twisted curve in the sense of Abramovich and Vistoli) with finite automorphism group, stabilizers of order δ only at the nodes and at the markings, and trivial stabilizers elsewhere. The stack $\overline{\mathcal{M}}_{g,n,\delta}$ of δ-stable curves is a smooth and proper Deligne–Mumford stack which differs only slightly from the usual $\overline{\mathcal{M}}_{g,n}$ (see [13]). The advantage of working over $\overline{\mathcal{M}}_{g,n,\delta}$ is that $W_{g,n}$ can be regarded as an étale and proper cover of $\overline{\mathcal{M}}_{g,n,\delta}$.

Definition 2.8. — *On a δ-stable curve C, a W-*structure *is the datum of N δth roots*

$$\big(L_j\ ,\ \varphi_j\colon L_j^{\otimes\delta}\xrightarrow{\ \sim\ }\omega_{\log}^{\otimes\delta q_j}\big)_{j=1}^N$$

(as many as the variables of W) satisfying the following s conditions (as many as the monomials $W_1,\dots,W_s$). For each $i=1,\dots,s$ and for $W_i(L_1,\dots,L_N)=\bigotimes_{j=1}^N L_j^{\otimes m_{i,j}}$, the condition

$$W_i(L_1,\dots,L_N)\cong\omega_{\log} \tag{2.4}$$

*holds. A δ-stable curve equipped with a W-structure is called an n-*pointed genus-g W-curve. *We denote by $W_{g,n}$ their moduli stack.*

Remark 2.9. — If we replace $\omega_{\log}$ by the trivial line bundle $\mathcal{O}$ in (2.4), we obtain a different moduli stack $W^0_{g,n}$ which also deserves special attention (see Theorem 2.12).

Remark 2.10. — Since j_W is in $\operatorname{Aut}(W)$, it is automatic that δq_j is integer. On the other hand, the exponent δ of $\operatorname{Aut}(W)$ is not the order $|j_W|$ of j_W. As a counterexample consider the D_4 singularity x^3+xy^2: the order of j_W is 3 but the exponent δ is 6.

Remark 2.11. — It is straightforward to see that $W_{g,n}$ is a proper and étale cover of the proper moduli stack $\overline{\mathcal{M}}_{g,n,\delta}$. Let us introduce the following notation. Given an m-tuple of line bundles $\vec{E}=(E_1,\dots,E_m)$ and an $n\times m$ matrix $A=(a_{i,j})$ we denote by $A\vec{E}$ the n-tuple of line bundles

$$A\vec{E}=(\otimes_j E_i^{\otimes a_{i,j}})_{i=1}^n. \tag{2.5}$$

A similar notation holds for an m-tuple of isomorphisms of line bundles $\vec{f}=(f_1,\dots,f_m)\colon\vec{E}\to\vec{F}$; we write $A\vec{f}$ for the n-tuple of isomorphisms of line bundles $(\otimes_j f_i^{\otimes a_{i,j}})_{i=1}^n$ from $A\vec{E}$ to $A\vec{F}$.

With this notation we may rephrase the definition of $W_{g,n}$. Consider the N roots $(L_j,\varphi_j)_j$ as a pair of vectors as above $(\vec{L},\vec{\varphi})$ and define E_W as the matrix $(m_{i,j})$ from (2.3). Then, $M\vec{\varphi}$ is an s-tuple of isomorphisms $M\vec{\varphi}\colon E_W\vec{L}^{\otimes\delta}\to E_W(\omega_{\log}^{\otimes\delta q_1},\dots,\omega_{\log}^{\otimes dq_N})^t$ identifying the δth tensor powers $W_i(L_1,\dots,L_N)^{\otimes\delta}$ to $\omega_{\log}^{\otimes\delta}$. Hence, it is automatic that $W_i(L_1,\dots,L_N)\otimes\omega_{\log}^\vee$ is δ-torsion and the stack $W_{g,n}$ is merely the open and closed substack where such a line bundle is actually *trivial.* This is an open and closed condition within a fibred product of categories of δth roots. Since the stacks of δth roots of a given line bundle have been shown in [13] to be proper and étale over $\overline{\mathcal{M}}_{g,n,\delta}$, the following theorem follows.

Theorem 2.12. — *Let W be a nondegenerate quasihomogeneous polynomial of type $(q_1,\dots,q_N)$.*

(1) *The stack $W_{g,n}$ is nonempty if and only if $n>0$ or $|j_W|$ divides $2g-2$. It is a proper, smooth, $3g-3+n$-dimensional Deligne–Mumford stack; more precisely, it is étale over $\overline{\mathcal{M}}_{g,n,\delta}$ which is a proper and smooth stack of dimension $3g-3+n$.*

(2) *The stack $W^0_{g,n}$ (see Remark 2.9) carries a structure of a group over the stack of genus-g n-pointed δ-stable curves $\overline{\mathcal{M}}_{g,n,\delta}$ with composition law*

$$W^0_{g,n}\times_\delta W^0_{g,n}\longrightarrow W^0_{g,n},$$

where $\times_\delta$ denotes the fibred product over $\overline{\mathcal{M}}_{g,n,\delta}$. The degree of $W^0_{g,n}$ over $\overline{\mathcal{M}}_{g,n,\delta}$ is equal to $|\mathrm{Aut}(W)|^{2g-1+n}/\delta^N$ for $n>0$ and $|\mathrm{Aut}(W)|^{2g}/\delta^N$ for $n=0$.

(3) *The stack $W_{g,n}$ is a torsor under the group stack $W^0_{g,n}$ over $\overline{\mathcal{M}}_{g,n,\delta}$. In particular, its degree over $\overline{\mathcal{M}}_{g,n,\delta}$ equals that of $W^0_{g,n}$ over $\overline{\mathcal{M}}_{g,n,\delta}$. We have a surjective étale morphism and an action*

$$\begin{array}{c} W_{g,n} \\ \big\downarrow \\ \overline{\mathcal{M}}_{g,n,\delta} \end{array} \qquad\qquad W^0_{g,n}\times_\delta W_{g,n}\longrightarrow W^c_{g,n}\quad .$$

□

Remark 2.13. — The above moduli stack slightly differs from that used in [31]. In the present paper a point representing a curve with trivial automorphism group is equipped with N automorphisms acting by multiplication by ξ_δ on the fibres of $L_1,\dots,L_N$; therefore, as a stack-theoretic point it should be regarded as $B(\boldsymbol{\mu}_\delta)^N$. In [31], the isomorphisms $W_i(L_1,\dots,L_N)=\bigotimes_{j=1}^N L_j^{\otimes m_{i,j}}$ are included in the data defining an object; this involves some technicalities on the compatibility between these isomorphisms [31, 2.1.4]. Adding these extra data imposes further constraints to the multiplications by δth roots of unity along the fibres; hence, the generic automorphism group in [31] may be smaller than $\boldsymbol{\mu}_\delta^N$ and is actually equal to $\mathrm{Aut}(W)\subseteq(\boldsymbol{\mu}_\delta)^N$. It is easy to see that the moduli functor of [31] is an étale cover of $W_{g,n}$, locally isomorphic to

$$B\,\mathrm{Aut}(W) \to B(\boldsymbol{\mu}_\delta)^N;$$

therefore, since we regard the relevant classes defined in [31] as pushforwards to $W_{g,n}$, this issue does not affect the intersection theory on the stack.

2.2.2. Decomposition of $W_{g,n}$ according to the type of the markings

Consider a δth root L of a line bundle pulled back from the universal stable curve of $\overline{\mathcal{M}}_{g,n}$ (e.g., $\omega_{\log}^c$ for some c). An index

$$\mathrm{mult}_{\sigma_i} L = \Theta_i \in [0,1[\tag{2.6}$$

is determined by the local index of the universal δth root L at the ith marking σ_i. More explicitly, the local picture of L over C at the ith marking σ_i is parametrized by the pairs $(x,\lambda) \in \mathbb{C}^2$, where x varies along the curve and λ varies along the fibres of the line bundle. The stabilizer $\boldsymbol{\mu}_\delta$ at the marking acts as $(x,\lambda) \mapsto (\exp(2\pi\mathtt{i}/\delta)x, \exp(2\pi\mathtt{i}\Theta_i)\lambda)$. In this way, the local picture of L provides an explicit definition of $\Theta_1,\dots,\Theta_n$ for the markings $\sigma_1,\dots,\sigma_n$. As a consequence of Definition 2.8, the stack $W_{g,n}$ decomposed into several connected components defined by specifying the multiplicities of the roots $L_1,\dots,L_N$ at the points $\sigma_1,\dots,\sigma_n$. We organize these data into n multi-indices $h_1,\dots,h_n$ each one with N entries.

Definition 2.14. — *Let us fix n multi-indices with N entries $h_i = (e^{2\pi\mathtt{i}\Theta^i_1},\dots,e^{2\pi\mathtt{i}\Theta^i_N}) \in U(1)^N$ for $i=1,\dots,n$ and $\Theta^i_j \in [0,1[$. Then $W(h_1,\dots,h_n)_{g,n}$ is the stack of n-pointed genus-g W-curves satisfying the relation $\Theta^i_j = \mathrm{mult}_{\sigma_i} L_j$, where Θ^i_j is the jth entry of h_i.*

Proposition 2.15. — *Let $n>0$. The stack $W_{g,n}$ is the disjoint union*

$$W_{g,n} = \bigsqcup_{h_1,\dots,h_n \in U(1)^N} W(h_1,\dots,h_n)_{g,n}.$$

The stack $W(h_1,\dots,h_n)_{g,n}$ is nonempty if and only if

$$\begin{cases} h_i = (e^{2\pi\mathtt{i}\Theta^i_1},\dots,e^{2\pi\mathtt{i}\Theta^i_N}) \in \mathrm{Aut}(W) & i=1,\dots,n; \\ q_j(2g-2+n) - \sum_{i=1}^n \Theta^i_j \in \mathbb{Z} & j=1,\dots,N. \end{cases} \tag{2.7}$$

Remark 2.16. — A marking of a W-curve is therefore attached with a multi-index $h=(h_1,\dots,h_N) \in \mathrm{Aut}(W)$. The case where all coordinates of h are nontrivial is special: the sections of the line bundles $L_1,\dots,L_N$ necessarily vanish at such a marking. In this sense the bundle at that marking is "narrow" (this may provide a geometric explanation for the terminology of §2.1). Similarly, a narrow node is a node whose multiplicities h and $h^{-1} \in \mathrm{Aut}(W)$ on the two branches are narrow in the sense of §2.1. Again sections necessarily vanish at such a node.

2.2.3. The moduli stack associated to W and G

We identify open and closed substacks of $W_{g,n,G}$ where the local indices h only belong to a given subgroup G of $\mathrm{Aut}(W)$. This happens because G can be regarded as the group of diagonal symmetries of a polynomial

$$W(x_1, \dots, x_N) + \text{extra quasihomogeneous terms in the variables } x_1, \dots, x_N.$$

We may allow negative exponents in the extra terms; we only require that the extra monomials are distinct from those of W but involve the same variables $x_1, \dots, x_N$ with charges $q_1, \dots, q_N$. The following lemma is due to Krawitz [50].

Lemma 2.17 (Krawitz [50]). — *For any A-admissible subgroup G of* $\mathrm{Aut}(W)$, *there exists a Laurent power series Z in the same variables $x_1, \dots, x_N$ as W such that $W(x_1, \dots, x_N) + Z(x_1, \dots, x_N)$ is quasihomogeneous in the variables $x_1, \dots, x_N$ with charges $q_1, \dots, q_N$ and we have $G = G_{W+Z}$.* □

In this way to each A-admissible subgroup G of $\mathrm{Aut}(W)$ we can associate a substack $W_{g,n,G}$ of $W_{g,n}$ whose object will be referred to as (W, G)-curves.

Definition 2.18. — *Let $W_{g,n,G}$ be the full subcategory of $W_{g,n}$ whose objects $(L_1, \dots, L_N)$ satisfy $Z_t(L_1, \dots, L_N) \cong \omega_{\log}$, where $Z = \sum_t Z_t$ is the sum of monomials Z_t satisfying $G = G_{W+Z}$.*

Remark 2.19. — The above definition of $W_{g,n,G}$ makes sense. It is immediate that the definition of $W_{g,n}$ extends when W is a quasihomogeneous power series. It is also straightforward that the definition of $W_{g,n,G}$ does not depend on the choice of Z. Assume that there are two polynomials Z' and Z'' satisfying $G = G_{W+Z'} = G_{W+Z''}$. We can define a third polynomial $\widetilde{Z}$ by summing all distinct monomials of Z' and Z''. Then we immediately have $G_{W+\widetilde{Z}} = G$ and

$$(W + Z'\text{-conditions})_{g,n} \supseteq (W + \widetilde{Z}\text{-conditions})_{g,n} \subseteq (W + Z''\text{-conditions})_{g,n}.$$

Notice that these inclusions cannot be strict: the fibres over $\overline{\mathcal{M}}_{g,n,\delta}$ of all the three moduli stacks involved are zero-dimensional stacks all isomorphic to the disjoint union of $|G|^{2g-1+n}$ copies of $B(\boldsymbol{\mu}_\delta)^N$.

Remark 2.20. — As in Proposition 2.15, for $n > 0$, we have

$$W_{g,n,G} = \bigsqcup_{h_1,\dots,h_n \in G} W(h_1,\dots,h_n)_{g,n,G},$$

where $h_i \in G$ is the local index at the ith marked point.

Example 2.21. — The case where $G = \langle j_W \rangle$ is easy to work out. The substack $W_{g,n,\langle j_W\rangle} \subseteq W_{g,n}$ is the image of the stack of roots of ω of order $|j_W|$ via the functor

$$(L,\varphi) \mapsto \big((L^{\otimes \delta q_1}, \varphi^{\otimes d q_1}), \dots, (L^{\otimes \delta q_N}, \varphi^{\otimes d q_N})\big)$$

(recall that δ and $|j_W|$ differ in general).

2.2.4. Tautological classes

The so-called psi classes and kappa classes are defined as

$$\psi_i = \sigma_i^* \omega_\pi \qquad \text{(for } i = 1,\dots,n)$$
$$\kappa_h = \pi_*(c_1(\omega_{\log})^{h+1}) \in H^{2h}(W_{g,n}) \qquad \text{(for } h \geqslant 0),$$

where π is the universal curve $\mathcal{C}_{g,n} \to W_{g,n}$ and σ_i denotes the universal section specifying the ith marking. We can identify each stack $W_{g,n+1}(h_1,\dots,h_n,1)$ to the universal curve $\pi\colon \mathcal{C} \to W_{g,n}(h_1,\dots,h_n)$ and express κ_h as $\pi_*(\psi_{n+1}^{h+1})$.

Let us consider the higher direct images of the universal W-structure $(\mathcal{L}_1,\dots,\mathcal{L}_N)$ on the universal d-stable curve $\pi\colon \mathcal{C}_{g,n} \to W_{g,n}$. We express its Chern character in terms of psi classes. The normalization of the boundary locus parametrizing singular curves in $W_{g,n}$ can be identified to the stack parametrixing pairs (W-curves, nodes) in the universal curve. We consider the étale double cover $\mathcal{D}$ given by the moduli space of triples (W-curves, nodes, a branch of the node). The stack $\mathcal{D}$ is naturally equipped with two line bundles whose fibres are the cotangent lines to the branches; we label the corresponding first Chern classes by $\psi, \psi' \in H^2(\mathcal{D})$ starting from the branch attached to the geometric point in $\mathcal{D}$. Recall that a δth root at a node of a δ-stable curve determines local indices $a, b \in [0,1[$ such that $a+b \in \mathbb{Z}$ corresponding to the branches of the node (apply for each branch the definition (2.6) or see [18, §2.2]). In this way on $\mathcal{D}$, the local index attached to the chosen branch determines a natural decomposition into open and closed substacks and natural restriction morphisms of the map to $W_{g,n}$

$$\mathcal{D} = \bigsqcup_{\Theta \in [0,1[} \mathcal{D}_\Theta, \qquad j_\Theta\colon \mathcal{D}_\Theta \to W_{g,n}.$$

Proposition 2.22. — *Let W be a nondegenerate quasihomogeneous polynomial in N variables whose charges equal $q_1,\dots,q_N$. For any $j=1,\dots,N$, consider the higher direct image $R\pi_*\mathcal{L}_j$ of the jth component of the universal W-structure. Let ch_h be the degree-$2h$ term of the restriction of the Chern character to the stack $W(h_1,\dots,h_n)_{g,n}$, where $h_i=(e^{2\pi\mathrm{i}\Theta^i_1},\dots,e^{2\pi\mathrm{i}\Theta^i_N})$ for $\Theta^i_j\in[0,1[^N$. We have*

$$\mathrm{ch}_h(R\pi_*\mathcal{L}_j)=\frac{B_{h+1}(q_j)}{(h+1)!}\kappa_h-\sum_{i=1}^n\frac{B_{h+1}(\Theta^i_j)}{(h+1)!}\psi_i^h$$
$$+\frac{d}{2}\sum_{0\leqslant\Theta<1}\frac{B_{h+1}(\Theta)}{(h+1)!}(j_\Theta)_*\left(\sum_{a+a'=h-1}\psi^a(-\psi')^{a'}\right).$$

Proof. — This is an immediate consequence of the main result of [14]. □

2.3. The virtual cycle

The FJRW invariants of (W,G) fit in the the formalism of Gromov–Witten theory. Fix the genus g and the number of markings n (with $2g-2+n>0$, stability condition); then, for any choice of nonnegative integers $a_1,\dots,a_n$ (associated to powers of psi classes $\psi_1^{a_1},\dots,\psi_1^{a_n}$) and any choice of elements $\alpha_1,\dots,\alpha_n\in\mathcal{H}_{W,G}$ we can define an invariant (a rational number)

$$\langle\tau_{a_1}(\alpha_1),\dots,\tau_{a_n}(\alpha_n)\rangle^{W,G}_{g,n}. \tag{2.8}$$

Once the "target" (W,G) is fixed, the procedure is similar to Gromov–Witten theory and shares many features with orbifold theory. An intrinsic mathematical object is attached to each genus g and each number of markings n: the so called "virtual cycle". Then the psi classes $\psi_1^{a_1},\dots,\psi_1^{a_n}$ and the state space entries $\alpha_1,\dots,\alpha_n\in\mathcal{H}_{W,G}$ naturally yield FJRW invariants via intersection theory carried out on a moduli space classifying the solutions to the Witten equation. This is a moduli space overlying the moduli space of W curves introduced above. We will not provide a treatment at this level of generality, but we identify a number of cases where one can reduce to the moduli space of W curves.

First, let us recall the formalism of [31]. There, the definition of (2.8) is given by extending linearly the treatment of the special case where the entries $\alpha_i\in\mathcal{H}_{W,G}$ lie within a single summand $H^{N_g}(\mathbb{C}^N_g,W^{+\infty};\mathbb{C})$ of (2.2). We denote by h_i the group element satisfying $\alpha_i\in H^{N_{h_i}}(\mathbb{C}^N_{h_i},W^{+\infty};\mathbb{C})$. When all the markings are *narrow* (in the sense of Remark 2.16), a lemma of Witten shows that his equation has only zero solutions; this provides an heuristic explanation for the existence of a definition of the FJRW invariants in terms of intersection of psi classes against an *algebraic* virtual cycle. In other terms, when α_i is narrow for all i, we have

$$H^{N_{h_i}}(\mathbb{C}_g^N, W^{+\infty}; \mathbb{C}) \cong \mathbf{1}_{h_i} \cdot \mathbb{C}; \tag{2.9}$$

i.e. there is a canonical generator $\mathbf{1}_{h_i}$ and, by abuse of notation, we can write $\langle \tau_{a_1}(h_1), \dots, \tau_{a_n}(h_n) \rangle_{g,n}^{W,G}$ and carry the computation directly in the rational cohomology ring of $W(h_1, \dots, h_n)_{g,n}$.

In general, for instance for D-type singularities (see [29]), nonvanishing invariants attached to *broad* entries should be included in order to define a consistent Gromov–Witten-type theory. The presence of nonvanishing invariants attached to these broad entries can be actually probed indirectly via universal relations such as WDVV equation. This observation is the starting point of the FJRW setup. Even if [31] provides a coherent setup, a direct computation is still an open problem in general. FJRW analytic setup via Witten's PDE indicates that this is due to the lack of an effective method to solve Witten's PDE equation. (The reader may refer to [29] for an example of complete treatment of a D-singularity involving the broad sector.)

In this paper, we illustrate the approach which has allowed a large part of the computations available in the literature. Namely, we restrict to well-behaved cases where we can assume that the markings are all within the narrow sector. There, equation (2.9) allows us to focus essentially only on the enumerative geometry of the moduli space. We will return to the general case in the last part of the section: "Cohomological field theory in the general case".

2.3.1. The case of A singularities. A well-behaved "concave" locus.

In [74], Witten considers the case of the A_{r-1} singularity $W = x^r$. Here, the only A-admissible group is $\mathrm{Aut}(W) = \langle j_W \rangle \cong \boldsymbol{\mu}_r$ and the moduli stack is

$$W_{g,n,G}(h_1, \dots, h_N) = W_{g,n}(h_1, \dots, h_n),$$

for $h_i = \exp(e\pi \mathtt{i} \Theta_i)$ with $\Theta_i \in [0, 1[$. We have a universal curve $\pi\colon \mathcal{C} \to W_{g,n}(h_1, \dots, h_n)$ carrying an rth root $\mathcal{L}$ of the relative sheaf of logarithmic differential $\omega_{\log,\pi}$. Let C be a fibre of the universal curve and let L be the rth root on C; consider the space

$$V = H^1(C, L).$$

Generically, and away from a few low genus cases, this vector space has dimension

$$D = (1-g)\left(1-\frac{2}{r}\right) + \sum_{i=1}^{N}\left(\Theta_i - \frac{1}{r}\right). \tag{2.10}$$

This fails precisely when $H^0(C,L)$ is non-zero. When $H^0(C,L)$ is trivial we say that L is concave, the vector space V vary as a complex vector bundle over $W_{g,n}(h_1,\dots,h_N)$ of rank D, the locally free sheaf $R^1\pi_*\mathcal{L}$. In this case, the virtual cycle is Poincaré dual to the top Chern class of $(R^1\pi_*\mathcal{L})^\vee$

$$[W_{g,n}(h_1,\dots,h_N)]^{\mathrm{vir}} = c_{\mathrm{top}}(R^1\pi_*\mathcal{L})^\vee = (-1)^D c_{\mathrm{top}}(R^1\pi_*\mathcal{L}). \tag{2.11}$$

2.3.2. Witten's analytic construction for A-singularities.

Since $H^0(C,L)$ is not trivial in general, Witten suggested the following approach. At least over the open substack of smooth curves one has bundles of Hilbert spaces $\mathcal{E} = \Omega^{0,0}(\mathcal{L})$ and $\mathcal{F} = \Omega^{0,1}(\mathcal{L})$ (consisting respectively of $\mathcal{L}$-valued $(0,0)$-forms and $(0,1)$-forms along the fibres of π), with a family of operators $\overline{\partial}\colon \mathcal{E} \to \mathcal{F}$. Choosing a Hermitian metric on L defines an isomorphism $\overline{L} \cong L^\vee$. In this way the Serre duality (SD) map

$$s \in H^0(C, L^{\otimes r-1}) \hookrightarrow H^0(C, \omega\otimes L^\vee) \overset{\mathrm{SD}}{=} H^1(C,L)^\vee \ni s^{r-1} \tag{2.12}$$

is regarded as a family of maps on the total space of $\mathcal{E}$

$$\begin{aligned} \overline{\partial W}\colon \mathcal{E} &\to p^*\mathcal{F}, \\ s &\mapsto \overline{s}^{r-1}, \end{aligned} \tag{2.13}$$

where p is the projection $\mathcal{E}$ to the moduli stack $W_{g,n}$. Witten considers the section of $p^*\mathcal{F} \to \mathcal{E}$

$$\mathbb{W}(s) = \overline{\partial}(s) + \overline{\partial W}(s), \tag{2.14}$$

for which

$$\mathbb{W}(s) = 0 \quad \Leftrightarrow \quad s = 0; \tag{2.15}$$

i.e. $\mathbb{W}$ vanishes only on the zero section of the total space of $\mathcal{E}$. Then, the above data defines a topological Euler class $(-1)^D e(\mathbb{W} : \mathcal{E} \to \pi^*\mathcal{F})$ which generalizes (2.11). It was not clear, however, how to extend this approach to singular curves and to the whole stack $W_{g,n}(h_1,\dots,h_n)$.

2.3.3. The algebraic counterpart for A-singularities.

The algebraic counterpart of the above analytic construction has been provided in [67] via bivariant intersection theory using MacPherson's graph construction. In [12], the first author provided a compatible construction directly in the K theory ring of $W_{g,n}$. This can be presented in very simple and explicit terms and may clarify the above discussion. Instead of $\mathcal{E} \to \mathcal{F}$, consider a complex of coherent locally free sheaves

$$0 \to E \xrightarrow{\delta} F \to 0 \tag{2.16}$$

representing the pushforward $R\pi_*\mathcal{L}$ in the derived category; this exists because π is of relative dimension one. We have $\operatorname{rk}(F) - \operatorname{rk}(E) = -\chi(\mathcal{L}) = D$ by Riemann–Roch. The case where $H^0(C, L)$ constantly vanishes is the case where we can choose $E = 0$ and define the virtual cycle as $c_{\mathrm{top}}(F^\vee)$; or, equivalently, in terms of Chern and Todd characters via the well known Grothendieck formula $c_{\mathrm{top}}(F^\vee) = \operatorname{ch}\left(\sum_k (-1)^k \Lambda^k F\right) \operatorname{td}(F^\vee)$. Since the Todd class is invertible it makes sense to define $\operatorname{td}(F^\vee - E^\vee) = \operatorname{td}(F^\vee)/\operatorname{td}(E^\vee)$. In this way, the difficulty lies in generalizing the term $\operatorname{ch}\left(\sum_k (-1)^k \Lambda^k F\right)$. To this effect we need to modify $(\Lambda^k F)_{k=0}^{\operatorname{rk} F}$, which should be regarded as a complex with zero differential and K class $\sum_k (-1)^k \Lambda^k F$. Then, this is generalized by a double graded complex of coherent sheaves with two differentials

$$\left(C^{h,k} = \operatorname{Sym}^h E \otimes \Lambda^k F,\quad \delta\colon C^{h,k} \to C^{h-1,k+1},\quad \partial\colon C^{h,k} \to C^{h-r+1,k-1}\right),$$

where the Koszul differentials δ and ∂ are induced by (2.16) and by (2.12-2.13). The reparametrization $(p, q) = (h+k-rk, h+k)$ transforms them into horizontal and vertical differentials of bidegree $(-r, 0)$ and $(0, -r)$, More important, due to (2.15) the differentials commute and the cohomology with respect to the total differential $H^i_{\delta+\partial}(C^{\bullet,\bullet})$ vanishes except for a finite number of ranks i. In this way, the Chern character of $\sum_i (-1)^i H^i_{\delta+\partial}(C^{\bullet,\bullet})$ is well defined and we can set

$$[W_{g,n}(h_1, \dots, h_n)]^{\mathrm{vir}} = \operatorname{ch}\left(\textstyle\sum_i (-1)^i H^i_{\delta+\partial}(C^{\bullet,\bullet})\right) \operatorname{td}(F^\vee - E^\vee), \tag{2.17}$$

which satisfies cohomological field theory axioms (a key property is due to Polishchuk [65]).

2.3.4. Fan, Jarvis and Ruan's construction for the narrow sector

Fan, Jarvis and Ruan extended Witten approach to the case of a general singularity in full generality. The W-structure $L_1, \dots, L_N$ can be assembled into a single vector bundle

$$E = \oplus_{j=1}^{N} L_j.$$

When the markings are all narrow, the codimension of the cycle $[W(h_1, \dots, h_n)]^{\mathrm{vir}}_{g,n}$ in $W(h_1, \dots, h_n)_{g,n}$ equals $-\chi(R\pi_* E)$ for $E = \oplus_{j=1}^{N} L_j$. By Riemann–Roch for orbifold curves [1, Thm. 7.2.1], for $h_i = (e^{2\pi \mathrm{i}\Theta_1^i}, \dots, e^{2\pi \mathrm{i}\Theta_N^i})$, we can explicitly compute

$$\begin{aligned}
-\chi(R\pi_* E) &= -\operatorname{rk}(E)(1-g) - \deg(E) + \sum_{i,j} \Theta_j^i \\
&= (g-1)N - \sum_{j=1}^{N}(2g-2+n)q_i + \sum_{i,j}\Theta_j^i \\
&= (g-1)\sum_{j=1}^{N}(1-2q_j) + \sum_{i=1}^{n}\sum_{j=1}^{N}(\Theta_j^i - q_j) \\
&= (g-1)\widehat{c}_W + \sum_{i=1}^{n}(\operatorname{age}(h_i) - q), \\
&= (g-1)\widehat{c}_W + \frac{1}{2}\sum_{i=1}^{n}\deg\left(\mathbf{1}_{h_i}\right),
\end{aligned} \tag{2.18}$$

where in the last equality we see the role played in FJRW theory by the central charge $\widehat{c}_W = \sum_j (1-2q_j)$, the age shift the constant $q = \sum_j q_j$, and the grading introduced in §2.1. Note how the above formula specializes to (2.10) for $W = x^r$.

Again, since the universal curve $\pi\colon \mathcal{C} \to W_{g,n}(h_1, \dots, h_n)$ is a flat morphism of relative dimension one the pushforward $R\pi_* E$ in the derived categories can be represented by a two-terms complex of the form (2.16); we get $\delta = \oplus \delta_j \colon E \to p^* F$. Then, in [32], Witten's morphism $\overline{\partial W}\colon s \mapsto \overline{s}^{r-1}$ is replaced by the direct sum of all partial derivatives $\overline{\partial_j W}$ for all $j = 1, \dots, N$; we get $\overline{\partial W} = \oplus_j \overline{\partial W_j}\colon E \to p^* F$. Set $\mathbb{W} = \delta + \overline{\partial W}$ as in (2.14). The nondegeneracy condition for $\mathbb{W}$ (2.15) extends immediately by the nondegeneracy of the polynomial W. Then in [32] the virtual cycle is defined via a topological Euler class construction

$$\begin{aligned}
&[W_{g,n,G}(h_1, \cdots, h_n)]^{vir} \\
&\qquad = (-1)^{-\chi(E)} e(\mathbb{W} : E \to p^* F) \cap [W_{g,n,G}(h_1, \cdots, h_n)]
\end{aligned}$$

We highlight two subcases.

Concavity: Suppose that all markings are narrow and suppose that for every fibre C of the universal curve $H^0(C, L_j) = 0$ for all j.

Then the virtual cycle is given by

$$[W_{g,n,G}(h_1,\cdots,h_n)]^{\mathrm{vir}} = c_{\mathrm{top}}\left((R^1\pi_*E)^\vee\right) \cap [W_{g,n,G}(h_1,\cdots,h_n)]\,.$$

Index zero: Suppose that the dimension of $W_{g,n,G}(h_1,\cdots,h_n)$ is zero and that all markings are narrow. Furthermore, let us assume that the π_*E and $R^1\pi_*E$ are both vector spaces and share the same rank. Then the virtual cycle is just the degree of $\mathbb{W}\colon E \to p^*F$.

Remark 2.23. — When W is of Fermat type paired with the group $\langle j\rangle$, the genus-zero theory falls into the concave case. Here, the expression of the virtual cycle via the top Chern class allows explicit computations via the Grothendiek–Riemann–Roch formula of Proposition 2.22; this happens because $c_{\mathrm{top}} = [\exp(\sum_{k>0} s_k \mathrm{ch}_k)]_{\mathrm{top}}$ for $s_k = (-1)^k(k-1)!$ (see for example [16]). Furthermore, since in this case W is a sum of monomials of the form x^r, even in higher genus we can describe the virtual cycle by intersecting cycles defined as in (2.17). We refer again to [16] for more details.

2.3.5. Cohomological field theory in the general case

For sake of completeness we finish the section by presenting the formalism in the general case, beyond the narrow sector. In general the virtual cycle is defined for

$$[W_{g,n,G}(h_1,\cdots,h_n)]^{vir} \in H_*(W_{g,n}(h_1,\cdots,h_n),\mathbb{C}) \otimes \prod_{i=1}^n H_{N_{h_i}}(\mathbb{C}^{N_{h_i}}, W_{h_i}^{+\infty},\mathbb{C})^G$$

of degree

$$2\left((\widehat{c}_W-3)(1-g)+n-\sum_i(\mathrm{age}(h_i)-q)\right)$$

(this is the real dimension $6g-6+2n$ of $W_{g,n}$ minus the same degree already discussed in (2.18)). One of the main achievements of [31] is the proof of the fact that this cycle, satisfies the axioms of a Gromov-Witten type theory. These axioms can be summarized in terms of the abstract notion of cohomological field theory [49]. This amounts to show that the virtual cycle has good factorization properties with respect to the decomposition of the boundary of the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$. To this effect, let us introduce the forgetful morphism

$$f\colon W_{g,n} \to \overline{\mathcal{M}}_{g,n}$$

to the standard Deligne–Mumford space. This map factors through the forgetful map to $\overline{\mathcal{M}}_{g,n,\delta}$ but fails to be étale; nevertheless, Theorem 2.12 determines the degree of f because $\overline{\mathcal{M}}_{g,n,\delta}$ has degree 1 over $\overline{\mathcal{M}}_{g,n}$ (see [13]).

Definition 2.24. — *The operators $\Lambda^W_{g,n,G} \in \operatorname{Hom}(\mathcal{H}_{W,G}^{\otimes n}, H^*(\overline{\mathcal{M}}_{g,n}))$ are defined as follows. For each entry $\alpha_1,\dots,\alpha_n \in \mathcal{H}_{W,G}$ assume there is a group element $h_i \in G$ satisfying $\alpha_i \in H^{N_{h_i}}(\mathbb{C}^N_{h_i}, W^{+\infty}_{h_i};\mathbb{C})^G$. Then, we set*

$$\Lambda^W_{g,n,G}(\alpha_1,\cdots,\alpha_k) := \frac{|G|^g}{\deg(f)} f_*\left([W_{g,n,G}(h_1,\cdots,h_n)]^{\mathrm{vir}} \cap \prod_{i=1}^n \alpha_i\right).$$

We extend the definition linearly to the entire space $\mathcal{H}_{W,G}^{\otimes n}$.

We provide the abstract framework of cohomological field theory. Suppose that H is a graded vector space with a nondegenerate pairing $\langle\cdot,\cdot\rangle$ and a degree zero unit 1. To simplify the signs, we assume that H has only even degree elements and the pairing is symmetric. (When this is not the case, there are systematic solutions in terms of cohomological field theories over super-state spaces) Once and for all, we choose a homogeneous basis ϕ_α ($\alpha = 1,\dots,\dim H$) of H with $\phi_1 = 1$. Let $\eta_{\mu\nu} = \langle\phi_\mu,\phi_\nu\rangle$ and $(\eta^{\mu\nu}) = (\eta_{\mu\nu})^{-1}$.

Definition 2.25. — *A cohomological field theory is a collection of homomorphisms*

$$\Lambda_{g,n} : H^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n},\mathbb{C})$$

satisfying the following properties:

- **C1:** *The element $\Lambda_{g,n}$ is invariant under the action of the symmetric group S_n.*
- **C2:** *Let $g = g_1 + g_2$ and $k = n_1 + n_2$ and cosider $\rho_{\mathrm{tree}}\colon \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g,n}$. Then $\Lambda_{g,n}$ satisfy the composition property*

$$\begin{aligned}&\rho^*_{\mathrm{tree}}\Lambda_{g_1+g_2,n}(\alpha_1,\dots,\alpha_n)\\ &\quad= \Lambda_{g_1,n_1+1}(\alpha_{i_1},\dots,\alpha_{i_{n_1}},\mu)\,\eta^{\mu\nu} \otimes \Lambda_{g_2,k_2+1}(\nu,\alpha_{i_{n_1+1}},\dots,\alpha_{i_{n_1+n_2}})\end{aligned}$$

  for all $\alpha_i \in H$.
- **C3:** *Let $\rho_{loop} : \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$ be the loop-type gluing morphism. Then*

$$\rho^*_{\mathrm{loop}}\,\Lambda_{g,n}(\alpha_1,\dots,\alpha_n) \;=\; \Lambda_{g-1,n+2}\,(\alpha_1,\dots,\alpha_n,\mu,\nu)\,\eta^{\mu\nu}, \tag{2.19}$$

  where α_i, μ, ν, and η are as in C2.
- **C4a:** *For all α_i in H we have*

$$\Lambda_{g,n+1}(\alpha_1,\dots,\alpha_n,1) = \pi^*\Lambda_{g,n}(\alpha_1,\dots,\alpha_n), \tag{2.20}$$

where $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the forgetful morphism.

C4b: *We have*

$$\int_{\overline{\mathcal{M}}_{0,3}} \Lambda_{0,3}(\alpha_1, \alpha_2, 1) = \langle \alpha_1, \alpha_2 \rangle. \tag{2.21}$$

For each cohomological field theory, we can generalize the notion of intersection number, the generating function and total descendant potential function. Let

$$\langle \tau_{a_1,\alpha_1}, \cdots, \tau_{a_n,\alpha_n} \rangle_g^\Lambda = \int_{\overline{\mathcal{M}}_{g,n}} \prod_i \psi_i^{a_i} \Lambda_{g,n}(\phi_{\alpha_1}, \dots, \phi_{\alpha_n}).$$

By associating a formal variable t_i^α to $\tau_{i,\alpha}$, we define generating functions

$$\mathcal{F}_\Lambda^g = \sum_{n\geqslant 0} \frac{t_{a_1}^{\alpha_1} \dots t_{a_n}^{\alpha_k}}{n!} \langle \tau_{a_1,\alpha_1}, \dots, \tau_{a_n,\alpha_n} \rangle_g$$

and their total potential function

$$\mathcal{D}_\Lambda = \exp\left(\textstyle\sum_{g\geqslant 0} h^{g-1} \mathcal{F}_\Lambda^g \right).$$

Theorem 2.26 (Fan–Jarvis–Ruan [31]). — *Let* 1 *be the distinguished generator $\mathbf{1}_{j_W}$ attached j_W lying in the A-admissible group G. Let $\langle \cdot, \cdot \rangle^{W,G}$ denote the pairing on the state space $\mathcal{H}_{W,G}$. Then, the collection $(\mathcal{H}_{W,G}, \langle \cdot, \cdot \rangle^{W,G}, \{\Lambda_{g,n,G}^W\}, 1)$ is a cohomological field theory.*

The following properties hold.

(1) Decomposition. If W_1 and W_2 are two singularities in distinct variables, then the cohomological field theory arising from $(W_1 + W_2, G_1 \times G_2)$ is the tensor product of the cohomological field theories arising from (W_1, G_1) and (W_2, G_2).

(2) Deformation invariance. Suppose that $W_t, t \in [0, 1]$ is a one-parameter family of nondegenerate polynomials such that W_t is G-invariant. Then, we have a canonical isomorphism $\mathcal{H}_{W_0,G} \cong \mathcal{H}_{W_1,G}$. Under the above isomorphism,

$$\Lambda_{g,n,G}^{W_0} = \Lambda_{g,n,G}^{W_1}.$$

Namely, $\Lambda_{g,n}^W$ depends only on $(q_1, \dots, q_N)$ and on G. Note also that, when applied to a deformation of a polynomial W along a loop, this property implies monodromy invariance for $\Lambda_{g,n,G}^W$.

(3) $\mathrm{Aut}(W)$-invariance. The group $\mathrm{Aut}(W)$ acts on $\mathcal{H}_{W,G}$ in an obvious way. Then, $\Lambda_{g,n,G}^W$ is invariant with respect to the action of $\mathrm{Aut}(W)$ on each state space entry $\alpha_1, \dots, \alpha_n \in \mathcal{H}_{W,G}$.

3. State spaces: a complete picture

State spaces are the cornerstones of Gromov–Witten theory and of Fan–Jarvis–Ruan–Witten theory. At their level, we can provide an exhaustive picture featuring LG-CY correspondence as well as mirror symmetry. To this effect, since we already went through A model state spaces, let us introduce the B model state space.

3.1. B model state space

The present discussion parallels the above introduction of the A model state space.

3.1.1. Local algebra from the classical point of view

Consider the local algebra (also known as the chiral ring or the Milnor ring) $\mathcal{Q}_W := \mathbb{C}[x_1,\dots,x_N]/\operatorname{Jac}(W)$, with $\operatorname{Jac}(W)$ equal to the Jacobian ideal generated by partial derivatives $\operatorname{Jac}(W) = (\partial_1 W,\dots,\partial_N W)$. We regard each polynomial $\alpha(x_1,\dots,x_N)$ in $\mathcal{Q}_W$ as an N-form $\alpha(x_1,\dots,x_N)dx_1 \wedge \dots \wedge dx_N$. In this way a diagonal symmetry $\operatorname{Diag}(\lambda_1,\dots,\lambda_N)$ operates on $\prod_j x_j^{m_j} dx_1 \wedge \dots \wedge dx_N$ by multiplication by $\prod_j \lambda_j^{m_j+1}$.

The local algebra is graded by assigning $x_j \mapsto q_j$; in this way $\prod_j x_j^{m_j} dx_1 \wedge \dots \wedge dx_N$ has degree $\sum_j (m_j+1)q_j$. There is a unique element

$$\operatorname{hess}(W) = \det(\partial_i \partial_j W)$$

whose degree is maximal. The dimension of the local algebra is given by the formula

$$\mu(W) = \prod_i \left(\frac{1}{q_i} - 1\right).$$

For $f, g \in \mathcal{Q}_W$, the residue pairing $\langle f, g\rangle$ is determined by writing fg in the form

$$fg = \langle f, g\rangle \frac{\operatorname{hess}(W)}{\mu(W)} + \text{terms of lower degree.}$$

This pairing is well defined and nondegenerate. It endows the local algebra with the structure of a Frobenius algebra (*i.e.* $\langle fg, h\rangle = \langle f, gh\rangle$). For more details, see [4].

3.1.2. The state space of (W, G)

From the modern point of view, the local algebra is regarded as a part of the B model theory of singularities. For its application, it is important to orbifold the construction by G. The orbifold B model graded vector space with pairing $\mathcal{O}_{W,G}$ was essentially worked out by the physicists Intriligator and Vafa [41] (see [46] for a mathematical account). The ring structure was constructed later by Kaufmann [45] and Krawitz [50] in the case of the so called "invertible" W and B-admissible group $G \subseteq \mathrm{Aut}(W)$.

For each $g \in G$, we write as usual $\mathbb{C}^N_g$ for the points of $\mathbb{C}^N$ fixed by g. We write W_g for the restriction of W to $\mathbb{C}^N_g$. In this way W_g is a quasi-homogeneous singularity in a subspace of $\mathbb{C}^N$ and admits a local algebra $\mathcal{Q}_{W_g}$ with a natural G-action.

Definition 3.1 (B model state space). — *For any B-admissible group G, we set*

$$\mathcal{Q}_{W,G} = \bigoplus_{g \in G} (\mathcal{Q}_{W_g})^G,$$

where $(\ \ )^G$ denots the G-invariant subspace.

Remark 3.2. — The state space $\mathcal{Q}_{W,G}$ is clearly a module over $(\mathcal{Q}_W)^G$.

Remark 3.3. — Remark 2.2 may be regarded as saying that the B model state space is — by construction — isomorphic to the A state space of Definition 2.4. On the other hand, the space is equipped with a different Hodge bigrading as follows. For a G-invariant form α of degree p in $\mathcal{Q}_{W_g}$, the bidegree $(\deg^+_B(\alpha), \deg^-_B(\alpha))$ is defined as follows

$$(\deg^+_B(\alpha), \deg^-_B(\alpha)) = (p,p) + (\mathrm{age}(g), \mathrm{age}(g^{-1})) - (q,q) \qquad (\text{with } q = \textstyle\sum q_j).$$

We will usually write $\mathcal{Q}^{a,b}_{W,G}$ for the terms of bidegree $(a = \deg^+_B(\alpha), b = \deg^-_B(\alpha))$. We write $\mathcal{Q}^d_{W,G}$ for the terms whose total degree $\deg_B = a + b$ equals d.

3.1.3. The inner pairing

Notice that $\mathcal{Q}_g$ is canonically isomorphic to $\mathcal{Q}_{g^{-1}}$. The pairing of $\mathcal{Q}_{W,G}$ is the direct sum of residue pairings

$$\langle \cdot, \cdot \rangle \colon \mathcal{Q}_g \otimes \mathcal{Q}_{g^{-1}} \to \mathbb{C}$$

via the pairing of the local algebra.

Definition 3.4 (pairing for $\mathcal{Q}_{W,G}$). — *We have a nondegenerate inner product*

$$\langle\cdot,\cdot\rangle\colon \mathcal{Q}_{W,G}\times\mathcal{Q}_{W,G}\to\mathbb{C}.$$

The pairing descends to a nondegenerate pairing $\mathcal{Q}^{a}_{W,G}\times\mathcal{Q}^{2\widehat{c}_W-a}_{W,G}\to\mathbb{C}$ where $\widehat{c}_W$ is the so-called central charge $\sum_j(1-2q_j)$. This quantity, already appearing in Definition 2.7, plays a fundamental role in singularity theory: the singularities whose central charge c_W is less than 1 are called simple singularities and admit an ADE classification.

3.2. Mirror symmetry between LG models

Berglund and Hübsch [6] consider polynomials in N variables having N monomials

$$W(x_1,\dots,x_N)=\sum_{i=1}^{N}\prod_{j=1}^{N}x_j^{m_{i,j}}. \tag{3.1}$$

Note that each of the N monomials has coefficient one; in fact, since the number of variables equals the number of monomials and W is nondegenerate, any polynomial of the form $\sum_{i=1}^{N}\gamma_i\prod_{j=1}^{N}x_j^{m_{i,j}}$ can be reduced to the above expression by conveniently rescaling the N variables. In this way assigning a polynomial W as above amounts to specifying its exponent square matrix

$$E_W=(m_{i,j})_{1\leqslant i,j\leqslant N}.$$

The polynomials studied in [6] are called "invertible" because the matrix E_W is an invertible $N\times N$ matrix as a consequence of the uniqueness of the charges $q_1,\dots,q_N$ (nondegeneracy of W). There is a strikingly simple classification of invertible nondegenerate singularities by Kreuzer and Skarke [53].

An invertible potential W is nondegenerate if and only if it can be written, for a suitable permutation of the variables, as a sum of invertible potentials (with disjoint sets of variables) of one of the following three types:

$$W_{\mathrm{Fermat}}=x^a. \tag{3.2}$$

$$W_{\mathrm{loop}}=x_1^{a_1}x_2+x_2^{a_2}x_3+\dots+x_{N-1}^{a_{N-1}}x_N+x_N^{a_N}x_1. \tag{3.3}$$

$$W_{\mathrm{chain}}=x_1^{a_1}x_2+x_2^{a_2}x_3+\dots+x_{N-1}^{a_{N-1}}x_N+x_N^{a_N}. \tag{3.4}$$

One can compute the charges $q_1,\dots,q_N$ by simply setting

$$q_i=\textstyle\sum_j m^{i,j}, \tag{3.5}$$

the sum of the entries on the ith line of $E_W^{-1} = (m^{i,j})_{1\leqslant i,j\leqslant N}$.

Each column $(m^{1,j}, \dots, m^{N,j})$ of the matrix E_W^{-1} can be used to define the diagonal matrix

$$\rho_j = \mathrm{Diag}(\exp(2\pi \mathtt{i} m^{1,j}), \dots, \exp(2\pi \mathtt{i} m^{N,j})). \tag{3.6}$$

In fact these matrices satisfy the following properties $\rho_j^* W = W$; *i.e.* W is invariant with respect to ρ_j. Furthermore the group $\mathrm{Aut}(W)$ of diagonal matrices α such that $\alpha^* W = W$ is generated by the elements $\rho_1, \dots, \rho_N$:

$$\mathrm{Aut}(W) := \{\alpha = \mathrm{Diag}(\alpha_1, \dots, \alpha_N) \mid \alpha^* W = W\} = \langle \rho_1, \dots, \rho_N\rangle.$$

For instance, the above mentioned matrix j_W whose diagonal entries are $\exp(2\pi\mathtt{i}q_1), \dots,$ and $\exp(2\pi\mathtt{i}q_N)$ lies in $\mathrm{Aut}(W)$ and is indeed the product $\rho_1 \cdots \rho_N$. Recall that

$$SL_W = \mathrm{Aut}(W) \cap SL(\mathbb{C}^N),$$

the matrices with determinant 1; in Berglund and Hübsch's construction we consider groups G containing j_W (A-admissible) and included in SL_W (B-admissible). We write $\widetilde{G}$ for the quotient $G/\langle j_W\rangle$.

The geometric side of the LG-CY correspondence is an orbifold or smooth Deligne–Mumford stack. More precisely, let d be the least common denominator of $q_1 = w_1/d, \dots, q_N = w_N/d$ (*i.e.* $d = |j_W|$). Then $X_W = \{W = 0\}$ is a degree d hypersurface of the weighted projective space $\mathbb{P}(w_1, \dots, w_N)$. Then, W is nondegenerate (*i.e.* W has a single critical point at the origin) if and only if X_W is a smooth Deligne-Mumford stack. Let W be a nondegenerate invertible potential of charges $q_1, \dots, q_N$ satisfying the *Calabi–Yau condition*

$$\textstyle\sum_j q_j = 1. \tag{3.7}$$

The geometrical meaning of this condition is that $X_W = \{W = 0\}$ is of Calabi–Yau type in the sense that the canonical line bundle ω is trivial (adjuction formula: $d = \sum_j w_j$). Under the CY condition, let us point out a special case where several simplifications occur; namely, the case where w_j divides $d = \sum_j w_j$. Then the CY hypersurface defined X_W is embedded within a weighted projective stack whose coarse space is Gorenstein. This is a very special condition which allows direct computations of GW invariants in genus zero.

3.2.1. The polynomial $W^\vee$

Following Berglund–Hübsch, we consider the transposed polynomial $W^\vee$ defined by the property

$$E_{W^\vee} = (E_W)^\vee.$$

Namely, we set

$$W^\vee(x_1,\dots,x_N)=\sum_{i=1}^N\prod_{j=1}^N x_j^{m_{j,i}} \tag{3.8}$$

by transposing the matrix $(m_{i,j})$ encoding the exponents. This construction respects the above classifications (3.2), (3.3) and (3.4). As a consequence, $W^\vee$ is nondegenerate if and only if W is nondegenerate. Recall that q_j is the sum of the jth column of the inverse matrix E_W^{-1}. Hence, the charges $\overline{q}_1,\dots,\overline{q}_N$ of $W^\vee$ are the sums of the rows of E_W^{-1}. Therefore,

$$\textstyle\sum_j q_j=\sum_j \overline{q}_j.$$

In this way, $W^\vee$ is of Calabi–Yau type if and only if W is of Calabi–Yau type.

The striking idea of Berglund and HÃ¼bsch is that W and $W^\vee$ should be related by mirror symmetry. Clearly this is not true in the naive way: the mirror of a Fermat quintic three-fold X_W is not the quintic itself as one would get by transposing the corresponding exponent matrix E_W. Instead, as already discussed in the introduction, the mirror $X_W^\vee$ is the quotient of X_W by the automorphism group $(\mathbb{Z}_5)^3$. It was already understood by Berglund–Hübsch that the correct statement should read

$$(W,G)\text{ mirror to }(W^\vee,G^\vee)$$

for a conveniently defined dual group $G^\vee$. Many examples of dual groups have been constructed in the literature. The general construction was given only recently by Krawitz [50].

3.2.2. The group $G^\vee$

The group $G^\vee$ is contained in $\mathrm{Aut}(W^\vee)$. Recall that $\mathrm{Aut}(W^\vee)$ is spanned by the diagonal symmetries $\rho_1^\vee,\dots,\rho_N^\vee$ determined by the columns of $(E_W^\vee)^{-1}$ as in (3.6):

$$\mathrm{Aut}(W^\vee)=\langle\rho_1^\vee,\dots,\rho_N^\vee\rangle.$$

Then $G^\vee$ is the subgroup defined by

$$G^\vee=\Big\{\textstyle\prod_{j=1}^N(\rho_i^\vee)^{a_i}\ \big|\ \text{if }\prod_{j=1}^N x_i^{a_i}\text{ is }G\text{-invariant}\Big\}. \tag{3.9}$$

More explicitly, we express any $g\in G$ as $g=\rho_1^{k_1}\dots\rho_N^{k_N}$ and $h\in G^\vee$ as $h=(\rho_1^\vee)^{l_1}\dots(\rho_N^\vee)^{l_N}$. Then, $G^\vee$ is determined by imposing within

$\mathrm{Aut}(W^\vee)$ the following conditions for all $g = \rho_1^{k_1} \dots \rho_N^{k_N} \in G$

$$\begin{bmatrix} k_1 & \dots & k_N \end{bmatrix} E_W^{-1} \begin{bmatrix} l_1 \\ \vdots \\ l_N \end{bmatrix} \in \mathbb{Z}.$$

We have the following properties: transposition is an involution $(G^\vee)^\vee = G$, it is inclusion-reversing ($H \subseteq K \Rightarrow H^\vee \supseteq K^\vee$), it sends the trivial subgroup of $\mathrm{Aut}(W^\vee)$ to the total group $\mathrm{Aut}(W)$, and it exchanges $\langle j_W \rangle$ and $SL_{W^\vee}$.

3.2.3. Mirror symmetry conjectures between LG models

Now, we can state two mirror symmetry conjectures. Here, "mirror" means that the A model and the B model are exchanged. The first one is the Berglund–Hübsch–Krawitz mirror symmetry of the form LG|Ə⅃.

Conjecture 3.5 (mirror symmetry LG|Ə⅃). — *Suppose that W is a nondegenerate invertible polynomial. Then the Landau–Ginzburg models (W, G) and $(W^\vee, G^\vee)$ mirror each other.*

Let W be invertible and of Calabi–Yau type. We say $G \subseteq \mathrm{Aut}(W)$ is of *Calabi–Yau type* if $\langle j_W \rangle \subseteq G \subseteq SL_W$ (the fact that j_W is contained in SL_W follows from the Calabi–Yau condition (3.7)). In this case $\widetilde{G} = G/\langle j_W \rangle$ acts on X_W faithfully and the quotient $[X_W/\widetilde{G}]$ is still an orbifold with trivial canonical bundle (Calabi–Yau type). The properties listed above for the construction associating $G^\vee$ to G show that G is of Calabi–Yau type if and only if $G^\vee$ is of Calabi–Yau type. Then, within the Calabi–Yau category, we obtain a mirror symmetry conjecture of type CY|YƆ.

Conjecture 3.6 (mirror symmetry CY|YƆ). — *Suppose that W and G satisfy the Calabi–Yau condition (automatically the same holds for $W^\vee$ and $G^\vee$). Then the stack $[X_W/\widetilde{G}]$ is the mirror of $[X_{W^\vee}/\widetilde{G^\vee}]$.*

Remark 3.7. — Since we have not given a precise meaning to to the notion of mirror, the above conjectures should be viewed as a guideline instead of a mathematical statement. In the next section the above conjectures are turned into precise mathematical statements. One can regard them as relations in terms of state spaces. Then, they may be read as follows: the A model state space of (W, G) is isomorphic to the B model state space to $(W^\vee, G^\vee)$. Although elementary, the claim is nontrivial. For example it does not fit in Borisov–Batyrev duality of Gorenstein cones [5].

This happens systematically when W is not Fermat as was first noted in [19]. It was proven by Krawitz.

Theorem 3.8 (Krawitz [50]). — *Suppose that W is invertible. Then, there is an isomorphism between bigraded vector spaces*

$$\mathcal{H}_{W,G} \cong \mathcal{Q}_{W^\vee,G^\vee}.$$

Remark 3.9. — The isomorphism in the theorem is interesting in its own right. The basic idea is to exchange monomials with group elements. As already mentioned in Remark 2.2, $\mathcal{H}_{W,G}$ and $\mathcal{Q}_{W,G}$ are isomorphic as a consequence of [62] and [71]. This isomorphism, however, does not respect the gradings. Let us express an element of $\mathcal{Q}_{W,G}$ as $\bigwedge x_i^{l_i} dx_i \mid \prod_i \rho_i^{k_i+1}\rangle$ where $\bigwedge x_i^{l_i} dx_i$ is fixed by $\prod_i \rho_i^{k_i+1}$. Here, we use the presentation of an element of $\mathrm{Aut}(W)$ in terms of the generators ρ_i. Then the mirror map in Krawitz's theorem [50] is of the form

$$\bigwedge_i x_i^{l_i} dx_i \mid \prod_i \rho_i^{k_i+1}\Big\rangle \longmapsto \bigwedge_i x_i^{k_i} dx_i \mid \prod_i \rho_i^{l_i+1}\Big\rangle.$$

The proof uses Kreuzer and Skarke's decomposition of invertible polynomials. Note that there is no analogue decomposition on the CY side. This is the main reason why the LG side is easier to work with in this case. On the other hand, as we will discuss in the last part of this section the LG-CY correspondence sets a connection between the two conjectures given above.

Remark 3.10. — Recently, Borisov has found [10] a new proof of the theorem above via vertex algebras. This approach may actually lead to a unified setup including both Berglund–Hübsch and Borisov–Batyrev duality.

Beyond state spaces the situation is as follows. On the A model side, we have rigorous theories, FJRW theory for the LG model and GW theory for the CY model. The counterpart of these theories for the B model side is incomplete. The genus-zero theory should correspond to a Frobenius manifold structure; however, unless G is a trivial group, it appears delicate to define the suitable G-orbifold version extending the state space $\mathcal{Q}_{W,G} = \bigoplus_{g\in G}(\mathcal{Q}_{W_g})^G$. Due to Kaufmann–Krawitz [45, 50], we can provide at least an orbifold Frobenius algebra construction; *i.e.*, a ring structure on the state space.

Suppose that W is invertible. We define a product on $\bigoplus_h \mathcal{Q}_{W_h}$ and then take G-invariants. The product has the properties

$$\mathcal{Q}_{W_{h_1}} \otimes \mathcal{Q}_{W_{h_2}} \to \mathcal{Q}_{W_{h_1 h_2}}.$$

as well as respecting the $\mathcal{Q}_W$-module structure in the sense that

$$\alpha \mathbf{1}_{g_1} * \beta \mathbf{1}_{g_2} = \alpha\beta \mathbf{1}_{g_1} * \mathbf{1}_{g_2},$$

where $\alpha, \beta \in \mathcal{Q}_{W_e}$ and $\mathbf{1}_g$ is the unit in the algebra $\mathcal{Q}_{W_g}$.

Let

$$\mathbf{1}_{g_1} * \mathbf{1}_{g_2} = \gamma_{g_1,g_2} \mathbf{1}_{g_1 g_2},$$

where

$$\gamma_{g,h} \frac{\operatorname{hess}(W \mid \mathbb{C}_g^N \cap \mathbb{C}_h^N)}{\mu(W \mid \mathbb{C}_g^N \cap \mathbb{C}_h^N)} = \begin{cases} \frac{\operatorname{hess}(W|\mathbb{C}_{gh}^N)}{\mu(W|\mathbb{C}_{gh}^N)} & \text{if } \mathbb{C}_g^N \cup \mathbb{C}_h^N \cup \mathbb{C}_{gh}^N = \mathbb{C}^N \\ 0 & \text{otherwise.} \end{cases} \tag{3.10}$$

(We use the convention that $\operatorname{hess}(W|\{0\}) = 1$.)

Theorem 3.11 (Kaufmann [45], Krawitz [50]). — *When W is invertible and G is B-admissible, the operation $*$ is associative, $\mathbf{1}_e *$ operates as the identity, and $*$ respects the G-action and the double grading. Therefore, the space of G-invariants $\mathcal{Q}_{W,G}$ is equipped with a Frobenius algebra structure.*

Since the cohomological field theory attached to FJRW theory in the previous section automatically yields a Frobenius algebra structure for $\mathcal{H}_{W,G}$, it is natural to further interpret Conjecture 3.5 as a statement relating the Frobenius algebra structure $\mathcal{H}_{W,G}$ on the A side to the Frobenius algebra structure $\mathcal{Q}_{W^\vee,G^\vee}$ on the B side. Krawitz's checked that his vector space isomorphism for the case $G = \operatorname{Aut}(W)$

$$\mathcal{H}_{W,\operatorname{Aut}(W)} \cong \mathcal{Q}_{W^\vee,(e)}$$

respects the Frobenius algebra structure. He also provided evidence for the same statement for $G \subseteq SL_W$ and W of loop type and for other special cases related to Arnold's strange duality. We refer to [50] for precise statements.

Remark 3.12. — We can regard these isomorphisms of Frobenius algebra structures as evidence for an isomorphism between Frobenius manifolds attached to (W, G) on the A side and to $(W^\vee, G^\vee)$ on the B-side. On the other hand we point out again, that — unless the group is trivial — the notion of Frobenius manifold for pairs of the form (W, G) still lacks a rigorous definition. The problem consists in orbifolding the Frobenius manifold structure that can be already defined over the deformation spaces $\operatorname{Def}(W)$. As far as we know, the same issue arises for the B model of Calabi–Yau varieties as soon as they are equipped with a nontrivial orbifold structure.

Problem 3.13. — *Orbifold the Frobenius manifold* $\mathrm{Def}(W)$ *as well as the Calabi–Yau B model for* X_W *and prove a higher genus version of the LG-CY correspondence between them.*

This may well lead to a B model version of the LG-CY correspondence.

3.3. LG-CY correspondence

Let us consider both A model state spaces of FJRW theory and of GW theory. The simplest conjecture from the LG-CY correspondence is the following *cohomological LG-CY correspondence conjecture.*

Conjecture 3.14. — *Suppose that the pair* (W, G) *is of Calabi–Yau type;* i.e. W *is nondegenerate (not necessarily invertible) with* $\sum_j q_j = 1$ *and* G *contains* $\langle j_W \rangle$ *and lies in* SL_W. *Then, there is a bigraded vector space isomorphism*

$$\mathcal{H}^{*,*}_{W,G} \cong H^{*,*}_{\mathrm{CR}}\left([X_W/\widetilde{G}];\mathbb{C}\right), \tag{3.11}$$

where the right-hand side is Chen–Ruan orbifold cohomology of the stack $[X_W/\widetilde{G}]$ *with* $\widetilde{G} = G/\langle j_W \rangle$.

This conjecture is certainly not true without assuming that W is of Calabi–Yau type. For instance a quartic polynomial in five variables provides an immediate counterexample. The Calabi–Yau condition plays a crucial role in the proof of the correspondence. In physics, it reflects a to a supersymmetry condition which is the source of the physical LG-CY correspondence. Even if the formula in the statement above makes sense even for $G \not\subseteq SL_W$, this indicates that the isomorphism may fail without imposing a Calabi–Yau condition to G. Surprisingly the authors found that the above statement still holds when G is not contained in SL_W. We will return to this observation in the end of the paper where we present a higher genus correspondence holding precisely for $G \not\subseteq SL_W$.

Theorem 3.15. — *Suppose that* W *is of Calabi–Yau type and that* G *contains* j_W *(no upper bound for* G*). Then the above cohomological LG-CY correspondence holds.*

The main application is the folllowing *classical mirror symmetry*, which is a direct consequence of the cohomological LG-CY correspondence and Krawitz's mirror symmetry theorem of type LG|⅁˥.

Corollary 3.16. — *Suppose that W is invertible and that the pair (W, G) is of Calabi–Yau type as in Conjecture 3.14. Automatically, also the pair $(W^\vee, G^\vee)$ is of Calabi–Yau type. Furthermore, the Calabi–Yau orbifolds $[X_W/\widetilde{G}]$ and $[X_{W^\vee}/\widetilde{G}^\vee]$ form a mirror pair in the classical sense;* i.e. *we have the following isomorphism between Chen–Ruan cohomology groups*

$$H^{p,q}_{\mathrm{CR}}\left([X_W/\widetilde{G}];\mathbb{C}\right) \cong H^{N-2-p,q}_{\mathrm{CR}}\left([X_{W^\vee}/\widetilde{G}^\vee];\mathbb{C}\right).$$

Corollary 3.17. — *Assume that the quotient schemes $X_W/\widetilde{G}$ and $X_{W^\vee}/\widetilde{G^\vee}$ admit crepant resolutions Z and $Z^\vee$. Then the above statement yields a statement in ordinary cohomology:*

$$h^{p,q}(Z;\mathbb{C}) = h^{N-2-p,q}(Z^\vee;\mathbb{C}).$$

In the case where w_j divides d, Corollary 3.16 can be deduced from Borisov and Batyrev's construction of mirror pairs in toric geometry [5]. As already mentioned, the general case does not fit into polar duality because the associated toric variety is not reflexive. The following example illustrates this well.

Example 3.18. — We consider the quintic hypersurface in $\mathbb{P}^4$ defined as the vanishing locus of

$$W = x_1^4x_2 + x_2^4x_3 + x_3^4x_4 + x_4^4x_5 + x_5^5.$$

This is a chain-type Calabi–Yau variety X whose Hodge diamond is clearly equal to that of the Fermat quintic and is well known: $h^{1,1} = 1$, $h^{0,3} = 1$, $h^{1,2} = 101$. The mirror Calabi–Yau is given by the vanishing of the polynomial

$$W^\vee(x_1, x_2, x_3, x_4, x_5) = x_1^4 + x_1x_2^4 + x_2x_3^4 + x_3x_4^4 + x_4x_5^5,$$

which may be regarded as defining a degree-256 hypersurface $X^\vee$ inside $\mathbb{P}(64, 48, 52, 51, 41)$. This is a degree-256 hypersurface of Calabi–Yau type (256 is indeed the sum of the weights). In this case, the ambient weighted projective stack is no longer Gorenstein (all weights but 64 do not divide the total weight 256). Note that the group SL coincides with $\langle j\rangle$ on both sides; therefore, Corollary 3.16 reads

$$h^{p,q}_{\mathrm{CR}}(X;\mathbb{C}) = h^{3-p,q}_{\mathrm{CR}}(X^\vee;\mathbb{C}).$$

Indeed, the Hodge diamond of $X_{W^\vee}$ satisfies $h^{1,1} = 101$, $h^{0,3} = 1$, $h^{1,2} = 1$ matching (1.4).

Let us explain the role of the Gorenstein condition. Let us call the hypersurface $X_W \subset \mathbb{P}(w_1, \dots, w_N)$ *transverse* if the intersection of X_W with every coordinate subspace of the form $\mathbb{P}(w_{i_1}, \dots, w_{i_k})$ is either empty or a hypersurface. The transversality of X_W amounts essentially to the ambient space being Gorenstein. In another words, if $\mathbb{P}(w_1, \dots, w_N)$ is not Gorenstein, X_W will contain some coordinate subspace. The presence of these coordinate subspaces makes it more difficult to study X_W and its quotients. For instance, it is well known that the enumerative geometry of rational stable maps for these coordinate subspaces is an open problem in Gromov–Witten theory (this is due to the behavior of the virtual fundamental cycle). Initially, we thought that nonGorenstein cases such as loop and chain polynomials may provide counterexamples for the classical mirror symmetry conjecture. We actually found out that the cohomological LG-CY correspondence as well as the classical mirror symmetry conjecture hold in full generality. Similar issues arise in the enumerative geometry of curves; we will discuss them in §4.4.

3.3.1. The proof of the cohomological LG-CY correspondence

To illustrate the idea of the proof, it is instructive to work out the case of the quintic three-fold.

Example 3.19. — Consider $W = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5$ and the cyclic group $G = \langle j \rangle$ of order 5. For each element $j^m = (e^{2\pi \mathrm{i} m/5}, \dots, e^{2\pi \mathrm{i} m/5}) \in G$ with $m = 0, \dots, 4$ we compute $\mathcal{H}_{W,G} = \bigoplus_{g \in G} H^{N_g}(\mathbb{C}_g^N, W_g^{+\infty}; \mathbb{C})^G$ and the total degree of its elements.

Let $m \neq 0$ and consider the elements of the summands corresponding to j^m. These are the narrow states where $H^{N_g}(\mathbb{C}_g^N, W_g^{+\infty}; \mathbb{C})^G$ is isomorphic to $\mathbf{1}_g\mathbb{C}$. The total degree of $\mathbf{1}$ is $2m-2$. We obtain four elements of degree $0, 2, 4$ and 6; they correspond to the generators of $H^0(X_W, \mathbb{C})$, $H^2(X_W, \mathbb{C})$, $H^4(X_W, \mathbb{C})$ and $H^6(X_W, \mathbb{C})$.

Finally consider the remaining states which are not narrow and lie in $H^N(\mathbb{C}_1^N, W_1^{+\infty}, \mathbb{C})^G$. This space is isomorphic to the degree-3 cohomology group of X_W. This holds in full generality as a consequence of the isomorphism between the G-invariant part of the local algebra and the primitive cohomology. The total degree of these elements is 3. Therefore, we recover the desired degree-preserving vector space isomorphism.

We learn from this example that the dichotomy determined by narrow and broad states within the Landau–Ginzburg state space corresponds to the well known dichotomy on the Calabi–Yau side between fixed classes and variable (or primitive) classes. In the orbifold setting, each sector of X_W lies in some subweighted projective coordinate space of the form $\mathbb{P}(w_{i_1}, \dots, w_{i_k})$. Therefore, this dichotomy applies to each sector. We say that an orbifold cohomological class is variable (or primitive) if it comes from a variable (or primitive) cohomology class of some sector. It is straightforward to match the broad sector with variable classes. But it is far trickier to do so for narrow group elements versus fixed classes. We match these classes via a combinatorial construction based on an earlier model for Chen–Ruan orbifold cohomology of weighted projective spaces due to Boissière, Mann, and Perroni [9].

4. LG-CY correspondence: towards a global picture

We state the LG-CY correspondence conjecture at the quantum level; then, we cast it within a mirror symmetry framework. In the last part of this section we review recent results.

4.1. LG-CY correspondence: GW and FJRW theories

The state space isomorphism stated above for any nondegenerate polynomial W and any $G \ni j_W$ allows us to extend the conjecture which we have stated in [16] only for the quintic polynomial W and $G = \langle j_W \rangle$. Let us set up Givental's formalism.

4.1.1. Givental's formalism for GW and FJRW theories

The setup presented here extends the analogue setup for the quintic presented in [16]. The genus-zero invariants of both theories are encoded by two Lagrangian cones, $\mathcal{L}_{\mathrm{GW}}$ and $\mathcal{L}_{\mathrm{FJRW}}$, inside two symplectic vector spaces, $(\mathcal{V}_{\mathrm{GW}}, \Omega_{\mathrm{GW}})$ and $(\mathcal{V}_{\mathrm{FJRW}}, \Omega_{\mathrm{FJRW}})$. The two symplectic vector spaces also allow us to state the conjectural correspondence in higher genera. We recall the two settings simultaneously by using the subscript $\mathbb{W}$, which can be read as GW or FJRW.

We define the vector space $\mathcal{V}_{\mathbb{W}}$ and its symplectic form $\Omega_{\mathbb{W}}$. The elements of $\mathcal{V}_{\mathbb{W}}$ are Laurent series with values in a state space $H_{\mathbb{W}}$;

$$\mathcal{V}_{\mathbb{W}} = H_{\mathbb{W}} \otimes \mathbb{C}((z^{-1})).$$

In FJRW theory the state space is normally the entire space $\mathcal{H}_{W,G}$. In GW theory the state space is $H_{\mathrm{CR}}([X_W/\widetilde{G}];\mathbb{C})$. We choose a basis $\phi_0,\dots,\phi_k$ for the state space of FJRW theory and a basis $\varphi_0,\dots,\varphi_k$ for the state space of GW theory. We label by zero $\mathbf{1}_{j_W}$ and $\mathbf{1}_{X_W}$, respectively (these two classes play a special role at (4.6)). We express the basis of $H_{\mathbb{W}}$ as $\Phi_0,\dots,\Phi_k$ and the dual basis $\Phi^0,\dots,\Phi^k$.

The vector space $\mathcal{V}_{\mathbb{W}}$ is equipped with the symplectic form

$$\Omega_{\mathbb{W}}(f_1,f_2)=\mathrm{Res}_{z=0}\langle f_1(-z),f_2(z)\rangle_{\mathbb{W}},$$

where $\langle\ ,\ \rangle_{\mathbb{W}}$ is the inner pairing discussed above. In this way $\mathcal{V}_{\mathbb{W}}$ is polarized as $\mathcal{V}_{\mathbb{W}}=\mathcal{V}^+_{\mathbb{W}}\oplus\mathcal{V}^-_{\mathbb{W}}$, with $\mathcal{V}^+_{\mathbb{W}}=H_{\mathbb{W}}\otimes\mathbb{C}[z]$ and $\mathcal{V}^-_{\mathbb{W}}=z^{-1}H_{\mathbb{W}}\otimes\mathbb{C}[[z^{-1}]]$, and can be regarded as the total cotangent space of $\mathcal{V}^+_{\mathbb{W}}$. The points of $\mathcal{V}_{\mathbb{W}}$ are parametrized by Darboux coordinates $\{q^h_a,p_{l,j}\}$ and can be written as

$$\sum_{a\geqslant 0}\sum_{h=0}^{k}q^h_a\Phi_h z^a+\sum_{l\geqslant 0}\sum_{j=0}^{k}p_{l,j}\Phi^j(-z)^{-1-l}.$$

We review the definitions of the potentials encoding the invariants of the two theories. In FJRW theory, the invariants are the intersection numbers

$$\langle\tau_{a_1}(\phi_{i_1}),\dots,\tau_{a_n}(\phi_{i_n})\rangle^{\mathrm{FJRW}}_{g,n}=\int_{\overline{\mathcal{M}}_{g,n}}\prod_{i=1}^{n}\psi_i^{a_i}\cap\Lambda^W_{g,n,G}(\phi_{i_1},\dots,\phi_{i_n}),\quad(4.1)$$

with $\Lambda^W_{g,n,G}$ as in §2.3.5. In GW theory, the invariants are the intersection numbers

$$\langle\tau_{a_1}(\varphi_{h_1}),\dots,\tau_{a_n}(\varphi_{h_n})\rangle^{\mathrm{GW}}_{g,n,\delta}=\prod_{i=1}^{n}\mathrm{ev}_i^*(\varphi_{h_i})\psi_i^{a_j}\cap[X_W]^{\mathrm{vir}}_{g,n,\delta}.\qquad(4.2)$$

The generating functions are respectively

$$\mathcal{F}^g_{\mathrm{FJRW}}=\sum_{\substack{a_1,\dots,a_n\\h_1,\dots,h_n}}\langle\tau_{a_1}(\phi_{h_1}),\dots,\tau_{a_n}(\phi_{h_n})\rangle^{\mathrm{FJRW}}_{g,n}\frac{t^{h_1}_{a_1}\dots t^{h_n}_{a_n}}{n!}\qquad(4.3)$$

and

$$\mathcal{F}^g_{\mathrm{GW}}=\sum_{\substack{a_1,\dots,a_n\\h_1,\dots,h_n}}\sum_{\delta\geqslant 0}\langle\tau_{a_1}(\varphi_{h_1}),\dots,\tau_{a_n}(\varphi_{h_n})\rangle^{\mathrm{GW}}_{g,n,\delta}\frac{t^{h_1}_{a_1}\dots t^{h_n}_{a_n}}{n!}.\qquad(4.4)$$

In this way, both theories yield a power series

$$\mathcal{F}^g_{\mathbb{W}}=\sum_{\substack{a_1,\dots,a_n\\h_1,\dots,h_n}}\sum_{\delta\geqslant 0}\langle\tau_{a_1}(\Phi_{h_1}),\dots,\tau_{a_n}(\Phi_{h_n})\rangle^{\mathbb{W}}_{g,n,\delta}\frac{t^{h_1}_{a_1}\dots t^{h_n}_{a_n}}{n!}$$

in the variables t_a^i (for FJRW theory the contribution of the terms $\delta > 0$ is set to zero, whereas $\langle\ \rangle^{\mathbb{W}}_{g,n,0}$ should be read as $\langle\ \rangle^{\mathrm{FJRW}}_{g,n}$).

We can also define the partition function

$$\mathcal{D}_{\mathbb{W}} = \exp\left(\sum\nolimits_{g\geqslant 0}\hbar^{g-1}\mathcal{F}^g_{\mathbb{W}}\right). \tag{4.5}$$

Let us focus on the genus-zero potential $\mathcal{F}^0_{\mathbb{W}}$. The dilaton shift

$$q_a^h = \begin{cases} t_1^0 - 1 & \text{if } (a,h) = (1,0) \\ t_a^h & \text{otherwise.} \end{cases} \tag{4.6}$$

makes $\mathcal{F}^0_{\mathbb{W}}$ into a power series in the Darboux coordinates q_a^h. Now we can define $\mathcal{L}_{\mathbb{W}}$ as the cone

$$\mathcal{L}_{\mathbb{W}} := \{\boldsymbol{p} = d_{\boldsymbol{q}}\mathcal{F}^0_{\mathbb{W}}\} \subset \mathcal{V}_{\mathbb{W}}.$$

With respect to the symplectic form $\Omega_{\mathbb{W}}$, the subvariety $\mathcal{L}_{\mathbb{W}}$ is a Lagrangian cone whose tangent spaces satisfy the geometric condition $zT = \mathcal{L}_{\mathbb{W}} \cap T$ at any point (this happens because both potentials satisfy the equations SE, DE and TRR of [34]; in FJRW theory, this is guaranteed by [31, Thm. 4.2.8]).

Every point of $\mathcal{L}_{\mathbb{W}}$ can be written as follows

$$\begin{aligned} &-z\Phi_0 + \sum_{\substack{0\leqslant h\leqslant k\\ a\geqslant 0}} t_a^h \Phi_h z^a \\ &+\sum_{\substack{n\geqslant 0\\ \delta\geqslant 0}}\ \sum_{\substack{0\leqslant h_1,\dots,h_n\leqslant k\\ a_1,\dots,a_n\geqslant 0}}\ \sum_{\substack{0\leqslant \epsilon\leqslant k\\ l\geqslant 0}} \frac{t_{a_1}^{h_1}\cdots t_{a_n}^{h_n}}{n!(-z)^{l+1}}\langle \tau_{a_1}(\Phi_{h_1}),\dots,\tau_{a_n}(\Phi_{h_n}),\tau_l(\Phi_\epsilon)\rangle^{\mathbb{W}}_{0,n+1,\delta}\Phi^\epsilon, \end{aligned}$$

where the term $-z\Phi_0$ performs the dilaton shift.

Remark 4.1 (J-function). — Setting a and a_i to zero, we obtain the points of the form

$$\begin{aligned} &-z\Phi_0 + \sum_{0\leqslant h\leqslant k} t_0^h \Phi_h \\ &+\sum_{\substack{n\geqslant 0\\ \delta\geqslant 0}}\ \sum_{0\leqslant h_1,\dots,h_n\leqslant k}\ \sum_{\substack{0\leqslant \epsilon\leqslant k\\ l\geqslant 0}} \frac{t_{0}^{h_1}\cdots t_{0}^{h_n}}{n!(-z)^{k+1}}\langle \tau_{0}(\Phi_{h_1}),\dots,\tau_{0}(\Phi_{h_n}),\tau_l(\Phi_\epsilon)\rangle^{\mathbb{W}}_{0,n+1,\delta}\Phi^\epsilon, \end{aligned} \tag{4.7}$$

which uniquely determine the rest of $\mathcal{L}_{\mathbb{W}}$ (via multiplication by $\exp(\alpha/z)$ for any $\alpha \in \mathbb{C}$—*i.e.* via the string equation—and via the divisor equation in GW theory). We define the J-function

$$t=\sum_{h=0}^{k} t_0^h \Phi_h \mapsto J_{\mathbb{W}}(t,z)$$

from the state space $H_{\mathbb{W}}$ to the symplectic vector space $\mathcal{V}_{\mathbb{W}}$ so that $J_{\mathbb{W}}(t,-z)$ equals the expression (4.7).

4.1.2. The conjecture

The following conjecture can be regarded as a geometric version of the physical LG-CY correspondence [70] [74]. A mathematical conjecture was proposed by the second author in [68]. The formalism is analogous to the conjecture of [24, 25] on crepant resolutions of orbifolds and uses Givental's quantization from [34], which is naturally defined in the above symplectic spaces $\mathcal{V}_{\mathrm{FJRW}}$ and $\mathcal{V}_{\mathrm{GW}}$. In [16] we provided a precise mathematical statement for the special case of the quintic three-fold; here, we build upon recent work, and provide a general statement applying to all CY orbifolds that can be written as hypersurfaces X_W in weighted projective spaces and to the finite group quotients $[X_W/\widetilde{G}]$.

Conjecture 4.2 (LG-CY correspondence). — *Consider the Lagrangian cones $\mathcal{L}_{\mathrm{FJRW}}$ and $\mathcal{L}_{\mathrm{GW}}$.*

(1) *There is a degree-preserving $\mathbb{C}[z,z^{-1}]$-valued linear symplectic isomorphism*

$$\mathbb{U}_{\text{LG-CY}}:\mathcal{V}_{\mathrm{FJRW}}\to\mathcal{V}_{\mathrm{GW}}$$

and a choice of analytic continuation of $\mathcal{L}_{\mathrm{FJRW}}$ and $\mathcal{L}_{\mathrm{GW}}$ such that $\mathbb{U}_{\text{LG-CY}}(\mathcal{L}_{\mathrm{FJRW}})=\mathcal{L}_{\mathrm{GW}}$.

(2) *Up to an overall constant and up to a choice of analytic continuation, the total potential functions are related by quantization of $\mathbb{U}_{\text{LG-CY}}$;* i.e.

$$\mathcal{D}_{\mathrm{GW}}=\widehat{\mathbb{U}}_{\text{LG-CY}}(\mathcal{D}_{\mathrm{FJRW}}).$$

Remark 4.3. — For the readers familiar with the crepant resolution conjecture [24, 25], an important difference here is the lack of monodromy condition.

By [25], a direct consequence of the first part of the above conjecture is the following isomorphism between quantum rings.

Corollary 4.4. — *For an explicit specialization of the variable q determined by $\mathbb{U}_{\text{LG-CY}}$, the quantum ring of X_W is isomorphic to the quantum ring of the singularity $\{W=0\}$.*

4.2. Towards global mirror symmetry

Here, we cast the above conjecture into a global mirror symmetry framework. The idea is to extend the results presented in the introduction (in particular Figure 1.1).

There are two difficulties. First, Gromov–Witten theory is largely unknown when entries are taken in the variable (*i.e.* primitive) cohomology part of the state space. Only in some cases, such as the quintic threefold, the whole theory is computable because the invariants associated to the variable cohomology entries are easy to deal with (see [16]). Second, much of the theory on the B side has not yet been figured out. For example, in Problem 3.13 we pointed out that not much is known beyond the untwisted sector; in nontechnical terms, this means that only the case where G is trivial can be treated in a straightforward way. Here, we present a solution allowing us to move beyond this case.

4.2.1. Invariant A-states mirror untwisted B-states

As a first approach to both problems, it is natural to try and single out a subclass of invariants involving only certain state space entries. This requires checking that they really form independent theories (*e.g.* checking that they assemble into a cohomological field theory in the sense of §2.3.5). While doing so, we found out that the two difficulties discussed above are mirror to each other. Namely, we point out that the isomorphism of Theorem 3.16 matches two naturally defined state subspaces. On the A side we consider the untwisted sector; *i.e.* the sector attached to the identity element. On the B side we consider a subspace which contains the fixed cohomology: the space of cohomology classes of $\mathcal{H}_{W,G}$ which are left invariant by $\mathrm{Aut}(W)$. Indeed, Krawitz's Theorem 3.8 yields in particular the identification

$$[\mathcal{H}_{W,G}]^{\mathrm{Aut}(W)} \cong [\mathcal{Q}_{W^\vee,G^\vee}]_{\mathrm{untwisted}} \qquad \begin{array}{l}\text{(the invariant/untwisted}\\ \text{mirror symmetry)}\end{array} \quad . \tag{4.8}$$

The right hand side is the local algebra $\mathcal{Q}$ of $W^\vee$ invariant under $G^\vee$

$$[\mathcal{Q}_{W^\vee,G^\vee}]_{\mathrm{untwisted}} = (\mathcal{Q}_{W^\vee})^{G^\vee}.$$

The left hand side can be regarded via (2.2)

$$[\mathcal{H}_{W,G}]^{\mathrm{Aut}(W)} = \bigoplus_{g\in G} H^{N_g}(\mathbb{C}^N_g, W_g^{+\infty}; \mathbb{C})^{\mathrm{Aut}(W)},$$

where at each summand we have taken invariants with respect of $\mathrm{Aut}(W)$ instead of G.

Remark 4.5. — In the special case where W is of Fermat type, imposing $\mathrm{Aut}(W)$-invariance is the same as restricting to the narrow state space:

$$[\mathcal{H}_{W_{\mathrm{Fermat}},G}]^{\mathrm{Aut}(W_{\mathrm{Fermat}})} = [\mathcal{H}_{W_{\mathrm{Fermat}},G}]^{\mathrm{narrow}} = \bigoplus_{g\in G\ \mid\ \mathbb{C}^N_g=(0)} \mathbf{1}_g\mathbb{C}.$$

Under the LG-CY cohomological correspondence of Theorem 3.15 this is the same as eliminating all variable (primitive) classes. In particular, in the case of the quintic Fermat three-fold discussed in the introductory section, the relation (4.8) shows that fixed cohomology of X_W (in this case H^{ev}) mirrors the variable cohomology of $X_W^\vee$ (in this case H^{odd}). In general, for these Fermat-type cases, we have complete genus-zero computations for Gromov–Witten theory (see [22]) as well as for Fan–Jarvis–Ruan–Witten theory (see [15] and §4.4).

Remark 4.6. — More generally, if W cannot be written as a Fermat polynomial, the charges are not necessarily unitary fractions. In other words the degree d of the corresponding hypersurface is not necessarily a multiple of all weights $w_1, \dots, w_N$. Then, it may happen that $[\mathcal{H}_{W,G}]^{\mathrm{Aut}(W)} \supsetneq [\mathcal{H}_{W,G}]^{\mathrm{narrow}}$. A well known example is $D_4 = x^3 + xy^2$ where $ydx \wedge dy \mid e\rangle$ is $\mathrm{Aut}(D_4)$-invariant and clearly not narrow (see Rem. 3.9 for the notation).

We now prove that the condition of $\mathrm{Aut}(W)$-invariance singles out a self-contained cohomological field theory under the Calabi–Yau condition for W and G.

LEMMA 4.7. — *Assume* $\sum_j q_j = 1$ *and* $\langle j_W\rangle \subseteq G \subseteq \mathrm{Aut}(W)$. *Consider the* $\mathrm{Aut}(W)$*-invariant subspace of either* $\mathcal{H}_{W,G}$ *or* $H_{\mathrm{CR}}([X_W/\widetilde{G}];\mathbb{C})$. *Suppose that* $N \geqslant 5$. *Then the virtual cycle* $\Lambda^W_{g,n,G}(\alpha_1,\dots,\alpha_k)$, *for* $\mathrm{Aut}(W)$*-invariant entries* α_i, *forms a cohomological field theory.*

Proof. — It is enough to check the composition axioms. For tree-type gluing morphisms intervening in Definition 2.25, we encounter cycles of the form

$$\Lambda^W_{g,n,G}(\alpha_1,\dots,\alpha_{k-1},\beta)$$

where α_i denotes an $\mathrm{Aut}(W)$-invariant class for $i = 1,\dots,k-1$. We easily conclude that the condition $\Lambda^W_{g,n,G}(\alpha_1,\dots,\alpha_{k-1},\beta) \neq 0$ holds only if β is $\mathrm{Aut}(W)$-invariant. Next, a dimension argument allows us to verify also the axioms involving loop-type gluing morphism in Definition 2.25. The loop-type gluing situation only appears in the higher genus case: assume $g > 0$. We can use the forgeful morphism and reduce to the cases where no state space entry equals the fundamental class. Then, each entry has degree $\geqslant 2$. This happens because we have $N \geqslant 5$, hence the hypersurface

X_W has dimension $\geqslant 3$ and its lowest degree odd cohomology lies above degree 1. On the other hand the degree of twisted sector entries is bigger than 1 as a consequence of $G \subseteq SL_W$. Hence the state space has no degree 1 class apart from the fundamental class. Therefore, we can assume that $\deg(\alpha_i) \geqslant 2$. Then, a simple computation shows

$$\deg \Lambda^W_{g,n,G}(\alpha_1, \dots, \alpha_k) = (5 - N)(g - 1) + n - \sum_{i=1}^{n} \frac{1}{2} \deg(\alpha_i) \leqslant 0.$$

Therefore, $\rho^*_{loop}\Lambda^W_{g,n,G} = 0$. The right hand side is zero for the same reason. $\square$

It is reasonable to specialize Conjecture 4.2 as follows.

Conjecture 4.8. — *The LG-CY Correspondence Conjecture 4.2 holds for* $\mathrm{Aut}(W)$*-invariant theories on both sides.*

The above conjecture was proved in genus zero for the quintic three-fold by the authors [16] and for Fermat hypersurfaces in general and $G = \langle j \rangle$ by the authors in collaboration with Iritani [15] (this is the same as saying that the correspondence holds for every Calabi–Yau hypersurface within a Gorenstein weighted projective stack). Indeed the case of the quintic three-fold is special, because, there, Conjecture 4.2 in genus zero follows from Conjecture 4.8. Then the proof involves calculating the J-function of FJRW theory and a comparison with the J-function of the GW side obtained by Coates–Corti–Lee–Tseng [22]. We refer to Remark 4.16 for discussion ot the proof of Conjecture 4.8.

The general cases of nonGorenstein hypersurface or larger groups is uncharted territory in GW theory. The starting point of the proof in the Gorenstein case is the observation that the genus-zero theory is concave; *i.e.* the virtual cycle can be phrased as the top Chern class of a bundle. Then, Grothendiek–Riemann–Roch can be applied (see §2.3). Beyond the Gorenstein case, we do not have such a general method to compute invariants. In this sense, the problem is similar in nature to the computation of the higher genus GW theory of the quintic. There, the difficulty is also the lack of concavity. In this genus-zero case it should be noted that the problem may have a chance to be approached via Givental's theory. For this reason it is clear how this problem not only represents an exciting new direction in quantum cohomology, but could also shed new light on GW theory in higher genera.

4.2.2. Global mirror symmetry

Let us set up the B side; *i.e.*, the analogue of the family of Calabi–Yau three-folds $X^\vee_{W,t}$ parametrized by t in $\mathbb{P}^1$ from the introductory section.

It is well known that the genus-zero B model theory corresponds to a period integral vector, a fundamental object in classical complex geometry. Given a nondegenerate quasihomogeneous and invertible polynomial W in N variables of charges $q_1, \dots, q_N$ adding up to 1 (CY condition), we consider the hypersurface $\{W^\vee = 0\}$. It lies naturally in a weighted projective stack $\mathbb{P}(w_1, \dots, w_N)$ for suitable choices of positive integers $w_1, \dots, w_N$. Consider a group of diagonal symmetries G containing j_W (A-admissible) and included in SL_W (B-admissible); then, by Corollary 3.16, the orbifold $[\{W^\vee = 0\}/\widetilde{G}^\vee]$ is the mirror of the hypersurface defined by W. We consider complex deformations of the orbifold $[X_{W^\vee}/\widetilde{G^\vee}]$. Let us focus on the so called *marginal deformations*; *i.e.*, let $M_1, \dots, M_l$ be the monomial generators of the local algebra $\mathcal{Q}_{W^\vee}$ *of degree* 1 and invariant under $G^\vee$. Then, consider the family of hypersurfaces

$$H_{\boldsymbol{a}} = \left\{a_0 W^\vee + \textstyle\sum_{i=1}^l a_i M_i = 0\right\} \subset \mathbb{P}(w_1, \dots, w_N). \tag{4.9}$$

On an open subscheme of $\mathbb{P}^l$ we may regard this as a family of Calabi–Yau orbifolds. The automorphism group $\mathrm{Aut}(W^\vee)$ acts on the family of stacks (4.9) and on the base scheme $\mathbb{P}^l$. Let us mod out the cyclic subgroup $\langle j \rangle$ acting trivially everywhere. Since the morphism is $\mathrm{Aut}(W^\vee)/\langle j \rangle$-equivariant, we obtain a morphism between the corresponding quotient stacks. Furthermore, since $G^\vee$ acts trivially on the base scheme, we get a family of Calabi–Yau orbifolds over an open substack of $[\mathbb{P}^l/Z]$, with $Z = \mathrm{Aut}(W^\vee)/G^\vee$.

In this way, the B side of mirror symmetry is defined. It is a family of quotient stacks of the form

$$X^\vee_{W,\boldsymbol{a}} = \left[H_{\boldsymbol{a}}/\widetilde{G^\vee}\right]$$

parametrized by $\boldsymbol{a} \in [\mathbb{P}^l/Z]$. On an open substack of $[\mathbb{P}^l/Z]$ the family is fibred in Calabi–Yau orbifolds (smooth Deligne–Mumford stacks whose canonical line bundle is trivial)

$$\begin{tikzcd} X^\vee_{W,\boldsymbol{a}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X}^\vee \arrow[d, "\pi"] \\ \boldsymbol{a} \arrow[r] & \Omega \end{tikzcd} \tag{4.10}$$

Remark 4.9. — In the special case wher $G^\vee = SL_{W^\vee}$ the group Z is cyclic. Furthermore, when we start from a Fermat polynomial W of degree d, we have $W^\vee = W$ and $Z = \mathbb{Z}_d$.

Let us define the vector bundle of primitive cohomology with complex coefficients. On the point $\boldsymbol{a}$ of the open substack Ω of $[\mathbb{P}^1/Z]$ consider the vector bundle

$$V \longrightarrow \Omega \subset [\mathbb{P}^l/Z],$$

whose fibre is dual to the $\widetilde{G^\vee}$-invariant part of the kernel of

$$i_* : H_*(H_{\boldsymbol{a}}; \mathbb{C}) \to H_*(\mathbb{P}(\boldsymbol{w}); \mathbb{C}).$$

Indeed i_* is an isomorphism in all degrees up to the middle dimension $N-2$. In degree $N-2$, we have a surjective morphism and the kernel is precisely the so called variable (or primitive) homology of the hypersurface. More systematically we can set $R^{N-2}\pi_*(\mathbb{C}) \otimes \mathcal{O}$, where π is the family (4.10). Then the primitive cohomology sheaf is the kernel of the Lefschetz operator $L\colon R^{N-2}\pi_*(\mathbb{C}) \otimes \mathcal{O} \longrightarrow R^N\pi_*(\mathbb{C}) \otimes \mathcal{O}$.

Remark 4.10 (the relation with the local algebra). — Fibre by fibre, the $\widetilde{G^\vee}$-invariant part may be equivalently regarded as the $G^\vee$-invariant part of the local algebra of $W^\vee_{\boldsymbol{a}} = a_0 W^\vee + \sum_{i=1}^{l} a_i M_i$

$$\mathcal{Q}_{W^\vee_{\boldsymbol{a}}} = \mathbb{C}[x_1, \dots, x_N]/\operatorname{Jac}(W^\vee_{\boldsymbol{a}}).$$

In particular, for $G = \langle j_W \rangle$, we have $G^\vee = SL_{W^\vee}$ and V is a rank-$(N-1)$ vector bundle over a one-dimensional base Ω.

Remark 4.11 (Gauss–Manin connection). — On V there is a flat connection ∇, the Gauss–Manin connection, given by the local system of integer cohomology $H^{N-2}(X^\vee_{W,\boldsymbol{a}}; \mathbb{Z}) \subset H^{N-2}(X^\vee_{W,t}; \mathbb{C})$. This may be regarded as follows. Choose a particular fibre $X^\vee_{W,\boldsymbol{a}}$ and a basis of $(N-2)$-cycles $\Gamma_1, \dots, \Gamma_{\operatorname{rk} V}$ for the primitive homology $\ker(i_*)$. Since the fibration on the scheme Ω is locally trivial, a local trivialization can be used to extend the cycles $\Gamma_1, \dots, \Gamma_{\operatorname{rk} V}$ from the chosen fibre $X^\vee_{W,\boldsymbol{a}}$ to cycles $\Gamma_i(z)$ on nearby fibres $X^\vee_{W,\boldsymbol{a}(z)}$. This may rephrased as saying that V has a connection ∇ and that locally over Ω we can extend a basis of a fibre of V to a basis of flat sections.

Remark 4.12 (monodromy). — Since the base Ω is not contractible, the connection may have nontrivial monodromy. We may phrase this explicitly using the above cycles $\Gamma_i(z)$ extending the basis $\Gamma_1, \dots, \Gamma_{\operatorname{rk} V}$ of $\ker(i_*)$ over $\boldsymbol{a}$. Indeed, they are locally constant in the parameter z. However, transporting Γ_i along each closed path produces a cycle homologous to $T\Gamma_i$ for some linear map T (monodromy operator).

Another approach to this monodromy operator is provided by period integrals. Choose an holomorphic $(n,0)$-form ω on $X^\vee_{W,\boldsymbol{a}}$. We can extend ω

locally on a neighbourhood of $\boldsymbol{a}$ to a holomorphic $(n,0)$-form on the family of CY orbifolds. Then, define the periods integrals of ω as

$$\omega_1(z) = \int_{\Gamma_1(z)} \omega, \quad \dots, \quad \omega_{\mathrm{rk}\, V} = \int_{\Gamma_{\mathrm{rk}\, V}(z)} \omega.$$

They extend by analytic continuation to multiple-valued functions on Ω, transforming according to the same monodromy trasformation T operating on the homology classes of the cycles. We point out that we can always rescale ω by a globally holomorphic function

$$\omega \mapsto f\omega; \tag{4.11}$$

so the period integrals are defined up to rescaling.

Remark 4.13 (the base points 0 *and* ∞*).* — Now we analyse two special fibres of the above family. The prototype case is that of the quintic and more generally that of a homogeneous Fermat polynomial W of degree N in N variables paired with the group $\langle j_W \rangle$. Then, $W = W^\vee$ and $G^\vee = SL_W$, the base is one-dimensional and parametrized by the homogeneous coordinates (a_0, a_1): the monodromy is *maximally unipotent* around $a_0 = 0$ and *diagonalizable* around $a_1 = 0$. This are the Gepner point 0 and the large volume complex structure point ∞.

In the more general set up, we still focus on two points named 0 and ∞. We assume that $\prod_j x_j$ is a nonvanishing element of $\mathcal{Q}_{W^\vee}$. This is the case apart from a few degenerate cases, which are treated for example in [59]. Then let us set $M_1 = \prod_j x_j$ and

$$0 = (1, 0, 0, \dots, 0) \qquad \text{and} \qquad \infty = (0, 1, 0, \dots, 0). \tag{4.12}$$

Consider the overlying fibres $X^\vee_{W,0}$ and $X^\vee_{W,\infty}$. We expect that ∞ is the analogue of the *large complex structure point* in the introductory section, whereas 0 should play the role of the *Gepner point*. More precisely, we propose the following conjectural picture.

We conjecture that the period integral at ∞ encode the Gromov–Witten theory of the CY orbifold $[X_W/\widetilde{G}]$.

Conjecture 4.14 (mirror symmetry CY|YƆ). — *There is a mirror map matching a neighborhood of the origin in* $H^{1,1}_{\mathrm{CR}}([X_W/\widetilde{G}])^{\mathrm{Aut}(W)}$ *with a neighborhood of the large complex structure point* ∞*. The mirror map identifies the* J_{GW}*-function to the* period integrals $\omega_1, \dots, \omega_{\mathrm{rk}\, V}$ *around* ∞ *after a suitable rescaling of the form* (4.11).

It is generally believed that the maximally quasiunipotent monodromy of the periods indicates where the B model can be related to the Gromov–Witten theory of the mirror variety. Since, in our construction, the fibre $X^{\vee}_{W,\infty}$ at ∞ is highly singular, it is delicate to describe the monodromy at infinity. Whereas for $G = \langle j \rangle$, the singularities occur only over isolated points, for general choices of G it may be useful to study birational modifications of $[\mathbb{P}^l/Z]$ and of the overlying Calabi–Yau family. After suitable birational transformations of this base scheme, we still expect some analogue of the condition of maximal unipotency. See Morrison [61] and Deligne [26] for more precise treatments on the relation between mirror symmetry maximal unipotency at the large complex structure point.

At the special point 0, the monodromy is diagonalizable. We expect that the local picture encodes FJRW theory for the potential W with respect to G.

Conjecture 4.15 (mirror symmetry LG|ΥϽ). — *There is a mirror map matching a neighborhood of the origin of* $[\mathcal{H}^{1,1}_{W,G}]^{\mathrm{Aut}(W)}$ *with a neighborhood of the Gepner point* 0*. Via the mirror map, the* J_{FJRW}*-function is matched to the* period basis $\omega_1, \dots, \omega_{\mathrm{rk}\,V}$ *after a suitable rescaling as in* (4.11).

The mirror symmetry CY|ΥϽ conjecture was proved for the smooth case by Givental and Liang–Liu–Yau and for CY hypersurface of Gorenstein weighted projective space by Corti–Coates–Lee–Tseng [22]. For further generalizations we refer to [24] and references therein.

The mirror symmetry conjecture of type LG|ΥϽ was proved for the quintic polynomial in five variables and for the group $\langle j \rangle$ by the authors. For $(W, \langle j \rangle)$ in the Gorenstein case and in the restricted version 4.8, it is proven by the authors with Iritani [15]. We refer to [15] for a precise statement.

Remark 4.16 (LG-CY correspondence via global mirror symmetry). — The proof of these conjectures allows us to perform a parallel transport along ∇ on the B side yielding the identification between the LG model and the CY geometry predicted in the LG-CY correspondence conjecture 4.2. Let us restate the scheme of the argument referring to Figure 1.1 for clarity. The two mirror symmetry conjectures above involve the vertical arrows in Figure 1.1 on the left hand side and on the right hand side respectively. The LG-CY correspondence conjecture 4.2 connects the two A model structures on the bottom of the picture. Mirror symmetry allows us to lift the correspondence to the upper side of the picture, where we can use the vector bundle V and its flat connection ∇.

Indeed, in this way we verify the genus-zero part of Conjecture 4.2 for the quintic; in other words we verify the claim that the parallel transport of a basis of flat sections at 0 is related to a basis of flat sections at ∞ by a constant linear map $\mathbb{U}_{\text{LG-CY}}$. Since $\mathbb{U}_{\text{LG-CY}}$ is symplectic, the genus-zero part of Conjecture 4.2 follows, see [16].

Theorem 4.17. — *The LG-CY correspondence 4.2 holds in genus zero and matches the* GW *theory of the quintic three-folds in* $\mathbb{P}^4$ *to the* FJRW *theory of the isolated singularity of the corresponding affine cone.*

Remark 4.18 (generalizations). — Technically the parallel transport of the cycles $\Gamma_i(z)$ is difficult to perform explicitly. Alternatively, in [16] we study the period integrals ω_i and their Picard–Fuch's equation. Locally, in the case where W is homogeneous and G equals $\langle j_W \rangle$, the period integrals belong to the $(N-1)$-dimensional space of solutions of the Picard–Fuchs equation (this is due to Griffith's transversality). Since, for $G = \langle j_W \rangle$ and $G^\vee = SL_W$, this is precisely the rank of the vector bundle V, we can avoid the parallel transport and carry out analytic continuation of the periods integrals instead. This approach admits generalizations in the quasi-homogeneous setup; indeed, there, the degree of the Picard–Fuchs equation and the dimension of the state space $H^*(X_W)^{\mathrm{Aut}(W)}$ still match.

For the quintic three-fold, this argument is enough to deduce the LG-CY correspondence for genus-zero Gromov–Witten theory entirely. This is not the case in higher dimension. The reason is technical: we can only compute the $J(t,z)$-function for $t \in H^{1,1}$. In dimension three it turns out that all invariants can be deduced from this data. In higher dimension, this may well be not enough to deduce the entire J-function.

The problem is how to recover this information from the B model. In collaboration with Iritani [15] we partly overcome this problem by means of an equivariant theory argument. This requires technical conditions such as assuming that the ambient space is Gorenstein. In both cases, more work is needed in this direction.

4.3. Fulfilling global mirror symmetry in all genera

As we mentioned previously, the physical LG-CY-correspondence requires both W and G to be Calabi–Yau types. While the conjecture is certainly false without W being Calabi–Yau type, our theorem on the state space suggests that it may still be true when G fails to be of Calabi–Yau type. In this section, we pursue this direction for $G = \mathrm{Aut}(W)$. The reason for this choice is simple. By Krawitz's mirror symmetry theorem of type LG|ƆJ, we have $\mathrm{Aut}(W)^\vee = \{1\}$. In this case, we have a well defined B model for all genera. Once Problem 3.13 is solved, the same line of research should hold more generally. Since $[X_W/\widetilde{G}]$ is no longer Calabi–Yau for $G = \mathrm{Aut}(W)$, we cannot expect to have a mirror object of the form of a CY variety. On the other hand the mirror still makes sense. Indeed, we can exploit the Landau–Ginzburg model. Consider the LG side illustrated in Section 3. The state space of FJRW-theory of $(W, \mathrm{Aut}(W))$ is mirror to the state space of $(W^\vee, \{1\})$. This is a unique situation in which we have a B model theory for *all genera*. Here, the genus-zero theory is Saito's Frobenius manifold structure on the tangent bundle of the miniversal deformation of $W^\vee$. The higher genus theory can be obtained via Givental's formalism.

The miniversal deformation is generated by monomials of the local algebra $\mathcal{Q}_{W^\vee}$. Among them, there are the marginal deformation generators of the $j_{W^\vee}$-invariant and degree 1. We label them by $M_{-1}, \dots, M_{-l}$ with $M_{-1} = \prod_i x_i$ and the rest by $M_1, \dots, M_{\mu-l}$ for Milnor number μ. The miniversal deformation is

$$W^\vee_{a_{-l},\dots,a_{-1},a_0,a_1,\dots,a_{\mu-l}} = \sum_{-l}^{\mu-l} a_i M_i,$$

where $M_0 = W^\vee$. Usually in singularity theory one considers only germ of this functions; *i.e.*, one imposes $|a_i| < \epsilon$. Here, we study global singularity theory to allow the marginal deformation parameters $a_{<0}$ to vary to infinity. We still require $|a_i| < \epsilon$ for $i \geqslant 0$. The most interesting aspect of this case is the existence of a rigorous higher genus theory due to Givental. Recall that the Frobenius manifold in this case is generically semisimple (this happens because we can deform any singularity to Morse singularities).

We should mention that the B model Calabi–Yau three-fold has a higher genus potential in physics. It is supposed to have many interesting properties and has been investigated intensively in recent years by Klemm and his collaborators [3, 40]. For example, it is expected to be nonholomorphic and satisfy the so called holomorphic anomaly equation. Furthermore, this anti-holomorphic higher genus generating function is expected to be a modular form. Unfortunately, we cannot access these information due to the lack of a mathematically rigorous definition. On the other hand, for $[X_W/\widetilde{G^\vee}]$, we do have a rigorous theory of on the B side for all genera. This gives us a quite unique opportunity to study higher genus mirror symmetry. The above idea has been put into practice by Krawitz–Shen and Milanov–Ruan in dimension one.

Before describing their work, let us state two mirror conjectures governing the LG-CY correspondence in this case. Again, we refer to $W^\vee_\infty = \prod_i x_i$ as the large complex structure point and $W^\vee_0 = W^\vee$ as the Gepner point (see 4.13).

Conjecture 4.19 (mirror symmetry CY|Ɔ⅃). — *There is a mirror map matching a neighborhood of $H^{1,1}_{\mathrm{CR}}([X_W/\widetilde{G}_W];\mathbb{C})$ with a neighborhood of the large complex structure point $W^\vee_\infty$ such that the genus-g potential $\mathcal{F}^g_{GW}$ is matched to the genus-g formal potential $\mathcal{F}^g_{\mathrm{formal}}$ of Saito–Givental.*

Conjecture 4.20 (mirror symmetry LG|Ɔ⅃). — *Let $G = \mathrm{Aut}(W)$. There is a mirror map matching a neighborhood of $\mathcal{H}^{1,1}_{W,G}$ with a neighborhood of the Gepner point $W^\vee_0$ such that the $\mathcal{F}^g_{\mathrm{FJRW}}$-function is matched to the formal potential $\mathcal{F}^g_{\mathrm{formal}}$ of Saito–Givental.*

Conjecture 4.21. — *Saito–Givental theory at $W^\vee_0$ and $W^\vee_\infty$ are related by analytic continuation and symplectic transformation.*

Remark 4.22. — Naively, one could expect that Saito–Givental theory at $W^\vee_0$ and $W^\vee_\infty$ are related by analytic continuation only; however, notice that the precise statement involves a subtle issue. Indeed, the construction of the Frobenius manifold structure in Saito's theory depends on a choice of primitive form defined by choosing a basis of middle dimension cycles or period integrals. Here, we see the similarity between this case and the Calabi–Yau case.

Now, let us describe the one-dimensional cases. Here, we are concerned with $[X_W/\widetilde{G}_W]$ for an elliptic curve X_W. We obtain three examples which share a common feature. They are one-dimensional stacks of Deligne–Mumford type whose coarse space is $\mathbb{P}^1$ and whose stabilizers are trivial apart from three special points, whose orbifold structure has orders $(k_1, k_2, k_3) = (3,3,3), (2,4,4)$, and $(2,3,6)$. We refer to these orbifolds as Calabi–Yau orbifold $\mathbb{P}^1$, where the terminology "Calabi–Yau" is justified by the fact that, although ω is not trivial, it becomes trivial after taking a suitable tensor power $\omega^{\otimes r}$.

The three cases arise precisely for W equal to the following polynomials

$$\begin{aligned} P_8^\vee &= x_1^3 + x_2^3 + x_3^3 && (\textit{i.e.}\ \ (k_1,k_2,k_3) = (3,3,3)\ ),\\ X_9^\vee &= x_1^2 + x_1x_2^2 + x_3^4 && (\textit{i.e.}\ \ (k_1,k_2,k_3) = (2,4,4)\ ),\\ J_{10}^\vee &= x_1^2x_2 + x_2^3 + x_3^3 && (\textit{i.e.}\ \ (k_1,k_2,k_3) = (2,3,6)\ ). \end{aligned}$$

Clearly, one should not confuse these orbifolds with weighted projective stacks. To this effect we adopt the notation $\mathbb{P}^1[1/k_1, 1/k_2, 1/k_3]$ (instead of $\mathbb{P}(w_1, \ldots, w_N)$). The corresponding LG mirrors are the famous simple elliptic singularities P_8, X_9, and J_{10}.

Remark 4.23. — It is a natural question to classify all CY orbifold $\mathbb{P}^1$, *i.e.* one-dimensional orbifolds with nontrivial stabilizers only over a finite number of points and whose canonical line bundle satisfies $\omega^{\otimes r} \cong \mathcal{O}$ for some integer r. It turns out that there are only four possibilities. The above three cases, alongside with $\mathbb{P}^1[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$. This stack cannot be expressed as $[X_W/\widetilde{G}]$ for $G = \mathrm{Aut}(W)$, but rather as the quotient of an index two subgroup of $\mathrm{Aut}(W)$. Again, this requires solving Problem 3.13.

As we mentioned previously, the key observation is that the primitive form and Frobenius structure are determined by a choice of symplectic basis α, β of H_1 of the corresponding elliptic curve. The parameter a together with a symplectic basis determines a point $\tau \in \mathbb{H}_+$ in the upper half-plane. The space of parameter a can be viewed as the quotient of $\mathbb{H}_+$ by the monodromy group Γ.

Saito's Frobenius manifold structure defines the genus-zero potential function $\mathcal{F}_0(\tau)$. In this situation, Givental has defined a higher genus generating function $\mathcal{F}_g(\tau)$. By studying the transformation of $\mathcal{F}_g$ under $\tau \mapsto g\tau$ for $g \in \Gamma$, the second author obtains, in collaboration with Milanov, the following theorem.

Theorem 4.24. — *For the miniversal deformation of simple elliptic singularities, the Saito–Givental function $\mathcal{F}_g$ transforms as a quasimodular form of a finite index subgroup of $SL_2(\mathbb{Z})$.*

Recall that the second part of the LG-CY correspondence conjecture involves the quantization of a symplectic transformation. The above theorem involves quantization, even if it is not obvious from the statement. Moreover, it provides a much stronger statement; namely, the claim that the symplectic transformation is related to the modular transformation.

The second theorem fulfilling mirror symmetry is the following.

Theorem 4.25 (Krawitz–Shen [51]). — *Both the* CY|Ɔ⅃ *and the* LG|Ɔ⅃ *mirror symmetry conjectures hold for all genera for simple elliptic singularities.*

Theorems 4.24 and 4.25 imply the following corollaries.

Corollary 4.26. — *The LG-CY correspondence holds for all genera for the CY orbifold $\mathbb{P}^1$ of weights* $(3,3,3), (2,4,4), (2,3,6)$.

Corollary 4.27. — *The generating functions of GW theory for CY orbifold* $\mathbb{P}^1$ *of weights* $(3,3,3),(2,4,4),(2,3,6)$ *are quasimodular forms for finite index subgroups of* $SL_2(\mathbb{Z})$.

Remark 4.28. — The first corollary provides the first example of the LG-CY correspondence for all genera.

The most interesting application is probably the modularity of the GW theory of orbifold $\mathbb{P}^1$. As we mentioned in the introduction, a major problem in geometry and physics is to compute Gromov–Witten theory. To do so, we often assemble the numerical Gromov–Witten invariants into a generating function $\mathcal{F}_g$, where g represents the genus. In some extremely fortunate situations, $\mathcal{F}_g$ or, more precisely, the total descendant potential $\mathcal{D}=\sum_{g\geqslant 0} h^{g-1}\mathcal{F}_g$ provides a solution to classical integrable systems. This is the case when the target is a point, by Kontsevich–Witten, or weighted $\mathbb{P}^1$ by (work of Okounkov–Pandharipande, Milanov–Tseng, and Johnson). It is also striking that Okounkov-Pandharipande showed that $\mathcal{F}_g$ for the elliptic curve E is a quasimodular form of $SL_2(\mathbb{Z})$. In this way the study of the LG-CY correspondence, yields another class of examples: the CY orbifold $\mathbb{P}^1$ of weights $(3,3,3),(2,4,4),(2,3,6)$. In many ways, this is much harder to prove because the Chen–Ruan orbifold cohomology of these examples has more generators than that of elliptic curves and the Gromov-Witten invariants are, by definition, more complicated. It would be extremely interesting to investigate this phenomenon in higher dimension.

4.4. LG-CY correspondence shortcircuiting mirror symmetry

The genus-zero LG-CY correspondence has not been proven as a tool to further understand the genus-zero GW theory. The latter was completely elucidated by Givental and Lian–Liu–Yau and does not need to be computed. Instead, we want to use genus-zero information to determine the symplectic transformation $\mathbb{U}_{\text{LG-CY}}$. Via quantization, $\mathbb{U}_{\text{LG-CY}}$ is expected to compute GW theory in higher genus once we know the higher genus FJRW theory. For this purpose, we need to carry out analytic continuation via the Mellin–Barnes method: this yields the desired symplectomorphism as illustrated in Remark 4.16. There is an alternative method that allows us to write down the symplectomorphism $\mathbb{U}_{\text{LG-CY}}$ directly without passing through the B model of mirror symmetry and analytic continuation. As a byproduct, this operation provides an explanation of the fact that $\mathbb{U}_{\text{LG-CY}}$ is symplectic.

4.4.1. Witten's GIT geometric setup

The construction follows the purely mathematical description of the LG-CY correspondence, given by Witten in [74]. As we recalled (in the homogeneous case) in the introduction there are two GIT quotients for $\mathbb{C}^*$ operating on $\mathbb{C}^N \times \mathbb{C}$ with weights $(w_1, \dots, w_N, -d)$.

(1) One of them is the quotient of the open subscheme $(\mathbb{C}^N \setminus \{\mathbf{0}\}) \times \mathbb{C}$ yielding the total space of $\mathcal{O}(-N)$ over the weighted projective stack $\mathbb{P}(\boldsymbol{w})$. This is often referred in the literature as a CY construction as soon as the sum of the weights is zero. Indeed, consider the complex function $\overline{W} = p\sum_{j=1}^N x^{d/w_j}$ defined in coordinates $x_1, \dots, x_N$, and p over $\mathbb{C}^N \times \mathbb{C}$ (for simplicity, we are assuming that d is a multiple of w_j). Then, $\overline{W}$ is $\mathbb{C}^*$-invariant and descends to the quotient $\mathcal{O}(-N)$. There, if we consider the map $\overline{W}$ as a fibration on $\mathbb{C}$ we notice that only the special fibre is singular, precisely along the CY hypersurface X_W.

(2) The second GIT quotient is the quotient of the open subscheme $\mathbb{C}^N \times \mathbb{C}^*$ yielding the stack $[\mathbb{C}^N/\langle j_W \rangle]$. If we consider the above map $\overline{W}$ as a fibration over $\mathbb{C}$, we get the LG singularity model $W\colon [\mathbb{C}^N/\langle j_W\rangle] \to \mathbb{C}$ for $W = \sum_{j=1}^N x_j^{d/w_j}$ with its isolated singularity in the special fibre and no critical points elsewhere.

From this perspective both sides of the correspondence arise from the same $\mathbb{C}^*$-invariant morphism $\overline{W}$ and from the same geometric setup

$$\overline{W}\colon [U/\mathbb{C}^*] \longrightarrow \mathbb{C} \qquad\qquad (\text{for } U = \mathbb{C}^N \times \mathbb{C}). \tag{4.13}$$

4.4.2. Matrix factorizations and Orlov's equivalence

As illustrated in [36] we can exploit the above geometry to present the equivalence between the bounded derived category $\mathcal{D}^b(X_W)$ of coherent sheaves on X_W and the triangulated category of graded matrix factorization $MF^{\mathrm{gr}}(W)$ of $W\colon \mathbb{C}^N \to \mathbb{C}$ (this follows from Orlov theorem [63] we refer to Isik [43] for a complete treatment). We recall that a *matrix factorization* of W is a pair

$$(E, \delta_E) = \left(E^0 \underset{\delta_0}{\overset{\delta_1}{\rightleftarrows}} E^1 \right),$$

where $E = E^0 \oplus E^1$ is a $\mathbb{Z}_2$-graded finitely generated free module over $R = \mathbb{C}[x_1, \dots, x_N]$, and $\delta_E \in \mathrm{End}^1_R(E)$ is a degree $1 \in \mathbb{Z}_2$ endomorphism of E, such that $\delta^2 = W \cdot \mathrm{id}_E$. There is a natural $\mathbb{Z}$-graded version, which gives rise to the triangulated category $MF^{\mathrm{gr}}(W)$ of matrix factorizations.

In [66], Polishchuk and Vaintrob have shown how to apply the Chern character formalism for differential graded categories in general to the special case of $\mathbb{Z}_d$-equivariant matrix factorization. Via this construction, and the natural functor mapping $\mathbb{Z}$-graded matrix factorization to $\mathbb{Z}_d$-equivariant ones, we get the Chern character

$$\mathrm{ch}\colon K(MF^{\mathbb{Z}_d}(W)) \to HH(MF^{\mathbb{Z}_d}(W)),$$

where HH stands for the Hochschild cohomology applied to the differential graded category of $\mathbb{Z}_d$-equivariant matrix factorizations. In fact, in [66] a natural isomorphism involving the FJRW state space

$$HH(MF^{\mathbb{Z}_d}(W)) \cong \mathcal{H}_{W,\langle j_W\rangle}$$

is shown. In this way, Orlov's equivalence

$$MF^{\mathrm{gr}}(W) \xrightarrow{\sim} \mathcal{D}^b(X_W)$$

yields, after passage to K theory and via Serre duality, an isomorphism between the state spaces of GW theory of X_W and the state space of FJRW theory of $W, \langle j\rangle$. The cohomological version of Orlov's equivalence preserves the Euler pairings $\chi(E,F) := \sum_{i\in\mathbb{Z}} \dim \mathrm{Hom}(E, F[i])$ after multiplication on both sides by the Gamma class $\widetilde{\Gamma}$. We refer to [42] and [15] for precise definitions of the Gamma class for the LG model and for the CY hypersurface X_W; the compatibility with the Euler pairings is guaranteed by the following relation with the Todd character: $((-1)^{\frac{\deg}{2}}\widetilde{\Gamma}_{X_W}) \cdot \widetilde{\Gamma}_{X_W} = (2\pi i)^{\deg}\, \mathrm{td}_{X_W}$ (a consequence of $\Gamma(1-z)\Gamma(1+z) = \pi z/\sin(\pi z)$). We finally obtain

$$\Phi_{\mathrm{Orlov}}\colon \mathcal{H}_{W,\langle j\rangle} \longrightarrow H^*_{\mathrm{CR}}(X_W) \tag{4.14}$$

respecting the Euler pairings on both sides. We point out that this isomorphism does not respect the bigrading defined in Section 3.

4.4.3. Short-circuiting mirror symmetry

For simplicity, and in order to connect to the discussion of the introduction, let us focus on the case of the quintic three-fold and refer to [15] for the generalizations to weighted homogeneous polynomials.

Recall that solving GW and FJRW theories amounts to writing a basis of flat sections of a certain vector bundle with connection. Namely, once the state space of the theory is specified $H_{\mathbb{W}}$, we consider the trivial vector bundle $D_{\mathbb{W}} = H_{\mathbb{W}} \times \mathcal{A} \longrightarrow \mathcal{A}$, where $\mathcal{A}$ is a contractible neighbourhood of $H^{1,1}_{\mathbb{W}}$. This vector bundle is equipped with Dubrovin's connection $\nabla_{\mathbb{W}}$ and its fibres are the even-degree parts of the state spaces of the relevant theory: $H^*_{\mathrm{CR}}(X_W)$ for GW theory and $\mathcal{H}_{W,\langle j\rangle}$ for FJRW theory. Solving each theory in genus zero amounts to define morphisms

$$\begin{array}{ccc} \Gamma(D_{\mathrm{FJRW}}, \nabla_{\mathrm{FJRW}}) & \qquad\qquad & \Gamma(D_{\mathrm{GW}}, \nabla_{\mathrm{GW}}) \\ \uparrow & & \uparrow \\ H_{\mathrm{FJRW}} & & H_{\mathrm{GW}} \end{array}$$

identifying $H_{\mathbb{W}}$ with the space of flat sections $\Gamma(D_{\mathbb{W}}, \nabla_{\mathbb{W}})$. The proofs of the mirror symmetry conjectures 4.14 (Givental [34], Lian–Liu–Yau [56]) and 4.19 (by the authors [16]) yield an identification of $(D_{\mathbb{W}}, \nabla_{\mathbb{W}})$ with two local pictures of the B model vector bundle (V, ∇_V). Via analytic continuation this yields an identification between $\Gamma(D_{\mathrm{FJRW}}, \nabla_{\mathrm{FJRW}})$ and $\Gamma(D_{\mathrm{GW}}, \nabla_{\mathrm{GW}})$. More precisely, following Figure 1.1, we can identify both spaces of flat sections to germs of flat sections of (V, ∇_V) around 0 and ∞ and carry out a parallel transport there (this is well defined in terms of multivalued functions, or in terms of a single-valued function once branch cut is chosen, see Remark 4.31). In [15] we prove, in collaboration with Iritani, that analytic continuation can be equivalently replaced by Orlov's isomorphism (4.14).

Theorem 4.29. — *The diagram*

$$\begin{tikzcd} \Gamma(D_{\mathrm{FJRW}}, \nabla_{\mathrm{FJRW}}) \arrow[r, "\text{analytic continuation}"] \arrow[dr, phantom, "\square"] & \Gamma(D_{\mathrm{GW}}, \nabla_{\mathrm{GW}}) \\ H_{\mathrm{FJRW}} \arrow[u] \arrow[r, "\Phi_{\mathrm{Orlov}}"'] & H_{\mathrm{GW}} \arrow[u] \end{tikzcd}$$

is commutative. In this way the linear transformation $\mathbb{U}_{\text{LG-CY}}$ matching the bases of flat sections is encoded by a symplectic matrix expressing Φ_{Orlov} for a given choice of bases of the two state spaces.

Remark 4.30. — The above statement may be regarded as saying that Orlov's categorical equivalence *mirrors* on the A-model the analytic continuation carried out by means of the B-model picture (V, ∇_V). This fits in Iritani's framework developped in [42] describing the integral structures mirroring the local systems $H^3(X^\vee_{W,t}, \mathbb{Z}) \subset H^3(X^\vee_{W,t}, \mathbb{C})$ of the B models.

In physics, this counterpart to parallel transport has been widely treated. Hori, Herbst, and Page rephrase Orlov's equivalence in terms of *brane transport*, see [36]. One of the most interesting aspects of their work is the above mentioned reformulation of Orlov's functor passing through the geometric setup (4.13). There, we are led to extend representations of the cyclic group $\langle j\rangle$ to representations of $\mathbb{C}^*$; clearly, there is not a unique way to do so and this is the reason why Orlov's equivalence should be actually regarded as a set of functors

$$MF^{\mathrm{gr}}(W) \xrightarrow{\sim} \mathcal{D}^b(X_W) \qquad \text{yielding} \qquad \Phi_a\colon H_{\mathrm{FJRW}} \longrightarrow H_{\mathrm{GW}}$$

parametrized by $a \in \mathbb{Z}$ (see [63, §2.2]). Any two of these functors match for a suitable autoequivalence of the source category. We treat this aspect completely in [15].

Remark 4.31. — In complete analogy, the analytic continuation should be carried along an open substack of the one dimensional stack $[\mathbb{P}^1/\mathbb{Z}_d]$; this happens because over the conifold point and over the large complex structure point, the fibres $X^\vee_{W,\boldsymbol{a}}$ are singular and over the Gepner point there is a nontrivial stabilizer. In this way, the analytic continuation of period integrals is defined up to the monodromy operator T at infinity (Remark 4.31). The theorem above should be more precisely stated as follows: there is a choice of analytic continuation commuting with Orlov's isomorphism Φ_0. Let us express Orlov's isomorphisms as symplectic matrices $\mathbb{U}_a$ with respect to the chosen bases for H_{FJRW} and H_{GW}. The linear map T operates on the period integrals and changes the symplectomorphism $\mathbb{U}_{\text{LG-CY}}$ by conjugation. Then, we have the identification

$$\Phi_a = T^{-a}\mathbb{U}_{\text{LG-CY}}T^a$$

via the morphism $H_{\mathrm{GW}} \to \Gamma(D_{\mathrm{GW}}, \nabla_{\mathrm{GW}})$.

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Alessandro CHIODO
Institut de Mathématiques de Jussieu
UMR 7586 CNRS
Université Pierre et Marie Curie
Case 247
4 Place Jussieu
75252 Paris cedex 05
France
chiodoa@math.jussieu

Yongbin RUAN
Department of Mathematics,
University of Michigan,
Ann Arbor, MI 48109-1109,
USA
ruan@umich.edu