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Second cohomology classes of the group of \( C^1 \)-flat diffeomorphisms


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SECOND COHOMOLOGY CLASSES OF THE GROUP OF $C^1$-FLAT DIFFEOMORPHISMS

by Tomohiko ISHIDA

ABSTRACT. — We study the cohomology of the group consisting of all $C^\infty$-diffeomorphisms of the line, which are $C^1$-flat to the identity at the origin. We construct non-trivial two second real cohomology classes and uncountably many second integral homology classes of this group.

RÉSUMÉ. — On étudie la cohomologie du groupe des $C^\infty$-difféomorphismes de la droite, qui sont $C^1$-tangents à l’identité à l’origine. On construit deux classes non-triviales de cohomologie réelle de degré deux et un nombre non-dénombrable de classes d’homologie de dimension deux de ce groupe.

1. Notations and main results

We denote by $\mathfrak{a}_1$ the Lie algebra of all formal vector fields on $\mathbb{R}$ with the Krull topology. For $k \geq 0$, we denote by $\mathfrak{a}^k_1$ the Lie subalgebra of $\mathfrak{a}_1$ consisting of formal vector fields which are $C^k$-flat at the origin. Let $\text{Diff}^\infty_0(\mathbb{R})$ be the group of orientation-preserving $C^\infty$-diffeomorphisms of $\mathbb{R}$ which fix the origin. Let $G(1)$ be the group of germs of local $C^\infty$-diffeomorphisms at the origin of $\mathbb{R}$. Let $G^\infty(1)$ be the group of $\infty$-jets of local $C^\infty$-diffeomorphisms at the origin of $\mathbb{R}$. For $k \geq 1$, we denote by $\text{Diff}^\infty_k(\mathbb{R})$, $G_k(1)$ and $G^\infty_k(1)$ the subgroup of $\text{Diff}^\infty_0(\mathbb{R})$, $G(1)$ and $G^\infty(1)$ respectively, consisting of elements which are $C^k$-flat to the identity at the origin. The groups $G^\infty(1)$ and $G^\infty_k(1)$ can be considered as infinite-dimensional Lie groups, whose Lie algebras are $\mathfrak{a}^0_1$ and $\mathfrak{a}^k_1$, respectively.

We define the Gel’fand-Fuks cohomology [2] of $\mathfrak{a}_1$ in § 2. It is known to be 2-dimensional for each degree [3][7]. Moreover, Millionschikov proved its generators in degree greater than 1 can be described by the Massey

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products \cite{5}. We carried out the calculation of the Massey products on \(\text{Diff}_k^{\infty}(\mathbb{R})\), and we give two 2-cocycles of \(\text{Diff}_k^{\infty}(\mathbb{R})\) in \(\S\ 3\).

For \(l \geq k\), let \(\alpha_l\) and \(\alpha_{l_1} \ldots \alpha_{l_i}\) be the 1-cochains of \(\text{Diff}_k^{\infty}(\mathbb{R})\) defined by

\[
\alpha_l(f) = \frac{d^l}{dx^l} f(0) \quad \text{for} \ f \in \text{Diff}_k^{\infty}(\mathbb{R}),
\]

and

\[
\alpha_{l_1} \ldots \alpha_{l_i}(f) = \alpha_{l_1}(f) \ldots \alpha_{l_i}(f) \quad \text{for} \ f \in \text{Diff}_k^{\infty}(\mathbb{R}),
\]

respectively. Then the following proposition holds.

**Proposition 1.1.** — The following \(\gamma^-_2\) and \(\gamma^+_2\) are 2-cocycles of the group \(\text{Diff}_1^{\infty}(\mathbb{R})\).

\[
\gamma^-_2 = \left(\frac{1}{2} \alpha_4 - 3 \alpha_2 \alpha_3 + 3 \alpha_2^3\right) \right. - \frac{1}{2} \alpha_2 \left(\alpha_3 - \frac{3}{2} \alpha_2^2\right)^2,
\]

\[
\gamma^+_2 = -\alpha_2 \left(\frac{1}{10} \alpha_3 \alpha_5 - \frac{1}{8} \alpha_4^2 - \frac{3}{20} \alpha_2 \alpha_5
\right.
\]

\[
\left. + \frac{1}{2} \alpha_2 \alpha_3 \alpha_4 - \frac{4}{9} \alpha_3^3 + \frac{1}{2} \alpha_2^2 \alpha_3 - \frac{3}{4} \alpha_2 \alpha_3 + \frac{3}{8} \alpha_2^6\right).
\]

Our main theorem is the following.

**Theorem 1.2.** — Let \(\gamma^2 : H_2(\text{Diff}_1^{\infty}(\mathbb{R}); \mathbb{Z}) \rightarrow \mathbb{R}^2\) be the homomorphism defined by

\[
\gamma^2(\xi) = (\gamma^-_2(\xi), \gamma^+_2(\xi)).
\]

Then \(\gamma^2\) is surjective.

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2. First cohomology of $\text{Diff}^\infty_k(\mathbb{R})$

In this section, we review a result of Fukui [1] and compute 1-cocycles of $H^1(\text{Diff}^\infty_k(\mathbb{R}); \mathbb{R})$.

**Definition 2.1.** For a topological Lie algebra $\mathfrak{g}$, we denote by $A^*_C(\mathfrak{g})$ the differential graded algebra of all continuous alternating forms on $\mathfrak{g}$. The Gel’fand-Fuks cohomology of $\mathfrak{g}$ is defined to be the cohomology of the complex $(A^*_C(\mathfrak{g}), d)$. Here, $d$ is the ordinary differential mapping of cochain complexes of Lie algebras. We denote the Gel’fand-Fuks cohomology of $\mathfrak{g}$ by $H^*_{GF}(\mathfrak{g})$.

In the case $\mathfrak{g} = \mathfrak{a}_1^k$, the complex $A^*_C(\mathfrak{a}_1^k)$ is an exterior algebra generated by $\delta^{(k+1)}, \delta^{(k+2)}, \ldots$, where $\delta^{(l)}$'s are the 1-forms on $\mathfrak{a}_1^k$ defined by

$$\delta^{(l)}\left( f(x) \frac{d}{dx} \right) = (-1)^l f^{(l)}(0) \quad \text{for } f(x) \in \mathbb{R}[[x]].$$

Since it is easily seen that $d\delta^{(l)} = 0$ if and only if $k + 1 \leq l \leq 2k + 1$, we obtain the following proposition.

**Proposition 2.2.** For $k \geq 1$,

$$H^1_{GF}(\mathfrak{a}_1^k) \cong \mathbb{R}^{k+1}.$$ 

Moreover, $\delta^{(k+1)}, \delta^{(k+2)}, \ldots, \delta^{(2k+1)}$ generate $H^1_{GF}(\mathfrak{a}_1^k)$.

In particular, $H^1_{GF}(\mathfrak{a}_1^1)$ is generated by $\delta''$ and $\delta'''$.

On the other hand, Fukui proved a proposition about the homology of groups corresponding to $\mathfrak{a}_1^k$.

**Theorem 2.3** (Fukui[1]). For $k \geq 1$,

$$H_1(\text{Diff}^\infty_k(\mathbb{R}); \mathbb{Z}) \cong \mathbb{R}^{k+1}.$$ 

Theorem 2.3 is obtained from the fact that the group homomorphism

$$\Psi_k : \text{Diff}^\infty_k(\mathbb{R}) \to \mathbb{R}^{k+1}$$

defined by

$$\Psi_k(f) := \left( \frac{1}{(k+1)!} f^{(k+1)}(0), \frac{1}{(k+2)!} f^{(k+2)}(0), \ldots, \frac{1}{(2k+1)!} f^{(2k+1)}(0) \right)$$

induces an isomorphism in the first homology. Here, $\mathbb{R}^{k+1}$ means the group which is $\mathbb{R}^{k+1}$ as a set, where the addition is defined by

$$(a_1, a_2, \ldots, a_{k+1}) + (b_1, b_2, \ldots, b_{k+1}) = (a_1 + b_1, a_2 + b_2, \ldots, a_k + b_k, a_{k+1} + b_{k+1} + (k+1)a_1 b_1).$$
Since $\Psi_k$ is a group homomorphism, $\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_{2k}$ are 1-cocycles of $\text{Diff}_k^\infty(\mathbb{R})$ with real coefficients. Moreover, if we denote the cochain $\tilde{\alpha}_{2k+1}$ by

$$\tilde{\alpha}_{2k+1} = \alpha_{2k+1} - \frac{1}{2} \binom{2k+1}{k} \alpha_{k+1},$$

then it is also a 1-cocycle. In particular, $\alpha_2$ and $\tilde{\alpha}_3 = \alpha_3 - \frac{3}{2} \alpha_2^2$ are 1-cocycles of $\text{Diff}_1^\infty(\mathbb{R})$.

Remark 2.4. — The same argument can be applied to the groups $G_1(1)$ and $G_1^\infty(1)$ instead of $\text{Diff}_1^\infty(\mathbb{R})$. Hence Theorem 2.3 also holds for $G_1(1)$ and $G_1^\infty(1)$. By regarding $\alpha_i$’s as the 1-cochains of $G_1(1)$ and $G_1^\infty(1)$, the 1-cocycles $\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_{2k}$ and $\tilde{\alpha}_{2k+1}$ of $\text{Diff}_1^\infty(\mathbb{R})$ can be considered as 1-cocycles of the groups $G_1(1)$ and $G_1^\infty(1)$, respectively.

3. Construction of the 2-cocycles of $\text{Diff}_1^\infty(\mathbb{R})$

In this section, we recall the definition of the Massey products following [4], and construct the 2-cocycles $\gamma_2^\infty$ of the group $\text{Diff}_1^\infty(\mathbb{R})$.

Definition 3.1 ([4]). — Let $\mathcal{A} = (\mathcal{A}^n, d)$ be a differential graded algebra. For $u_i \in H^p_i(\mathcal{A})$, we set $a_i$ a cocycle representative of $u_i$. We define $p(i, j)$ to be $\sum_{r=1}^j (p_r - 1)$. A collection of cochains $A = (a(i, j))$ for $1 \leq i \leq j \leq k$ and $(i, j) \neq (1, k)$ is a defining system of $\{a_1, \ldots, a_k\}$ if

(i) $a(i, i) = a_i \in \mathcal{A}^{p_i},$

(ii) $a(i, j) \in \mathcal{A}^{p(i, j)+1}$, and

(iii) $da(i, j) = \sum_{r=i}^{j-1} (-1)^{\deg a(i, r)} a(i, r) a(r + 1, j)$.

Definition 3.2 ([4]). — When a defining system $A$ of $\{a_1, \ldots, a_k\}$ exists, we define $c(A) \in \mathcal{A}^{p(1,k)+2}$ by setting

$$c(A) = \sum_{r=1}^{k-1} (-1)^{\deg a(i, r)} a(1, r) a(r + 1, k).$$

Then $c(A)$ is a cocycle and the set

$\{\text{a cohomology class of } c(A); A \text{ is a defining system of } \{a_1, \ldots, a_k\}\}$

depends only on the cohomology classes $u_1, \ldots, u_k$. We call the elements of the set the Massey products of $\{u_1, \ldots, u_k\}$.
By Goncharova’s theorem [3][7], which gives \( \dim H^p_{GF}(a^k_1) \) for any \( p, k \geq 1 \), we know

\[ H^p_{GF}(a^1_1) \cong \mathbb{R}^2 \text{ for any } p. \]

Furthermore, the following theorem is known.

**Theorem 3.3 (Millionschikov[5]).** — For any \( p \geq 2 \), there exist generators \( g^p_+, g^p_- \in H^p_{GF}(a^1_1) \) of \( H^p_{GF}(a^1_1) \cong \mathbb{R}^2 \), which are described by the Massey products. In particular, both the triple Massey product \( \{\delta'', \delta'', \delta''\} \) and the 5-fold Massey product \( \{\delta'', \delta'', \delta'', \delta'', \delta''\} \) determine non-trivial cohomology classes in \( H^2_{GF}(a^1_1) \), which are linearly independent.

In fact, the defining systems of \( \{\delta'', \delta'', \delta''\} \) and \( \{\delta'', \delta'', \delta'', \delta'', \delta''\} \) can be written as

\[ \begin{pmatrix}
\delta'' & -\frac{1}{2} \delta^{(4)} & * \\
\delta'' & 0 & \\
\delta'' & & \\
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
\delta'' & -\frac{1}{2} \delta^{(4)} & -\frac{1}{5} \delta^{(5)} & -\frac{1}{30} \delta^{(6)} & * \\
\delta^{(4)} & \frac{1}{5} \delta^{(5)} & 0 & \delta^{(5)} & 0 \\
\delta^{(4)} & \frac{1}{10} \delta^{(5)} & 0 & \delta^{(5)} & 0 \\
\delta^{(4)} & \frac{1}{3} \delta^{(5)} & 0 & \delta^{(5)} & 0 \\
\delta^{(4)} & \frac{1}{10} \delta^{(5)} & 0 & \delta^{(5)} & 0 \\
\end{pmatrix}, \]

respectively.

**Proof of Proposition 1.1.** — For \( \text{Diff}^\infty_1(\mathbb{R}) \) we checked that the defining systems of both of \( \{\alpha_2, \tilde{\alpha}_3, \tilde{\alpha}_3\} \) and \( \{\alpha_2, \tilde{\alpha}_3, \alpha_2, \alpha_2, \tilde{\alpha}_3\} \) also exist. In fact, they can be written as

\[ \begin{pmatrix}
\alpha_2 & \beta_1 & * \\
\tilde{\alpha}_3 & \beta_2 \\
\tilde{\alpha}_3 & \\
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
\alpha_2 & \beta_1 & \beta_5 & \beta_8 & * \\
\tilde{\alpha}_3 & \beta_3 & \beta_6 & \beta_9 \\
\alpha_2 & \beta_4 & \beta_7 \\
\tilde{\alpha}_3 & \beta_1 \\
\end{pmatrix}, \]
respectively. Here,

\[ \beta_1 = -\frac{1}{2} \alpha_4 + 3 \alpha_2 \alpha_3 - 3 \alpha_2^3, \quad \beta_2 = \frac{1}{2} \tilde{\alpha}_3^2, \]
\[ \beta_3 = \frac{1}{2} \alpha_4 - 2 \alpha_2 \alpha_3 + \frac{3}{2} \alpha_2^3, \quad \beta_4 = \frac{1}{3} \alpha_3, \]
\[ \beta_5 = -\frac{1}{5} \alpha_5 + \frac{3}{2} \alpha_2 \alpha_4 + \alpha_3^2 - 6 \alpha_2^2 \alpha_3 + \frac{15}{4} \alpha_4^2, \]
\[ \beta_6 = \frac{1}{10} \alpha_5 - \frac{1}{2} \alpha_2 \alpha_4 - \frac{1}{3} \alpha_3^2 + \frac{3}{2} \alpha_2^2 \alpha_3 - \frac{3}{4} \alpha_4^2, \]
\[ \beta_7 = \frac{1}{10} \alpha_5 - \alpha_2 \alpha_4 - \frac{1}{3} \alpha_3^2 + 4 \alpha_2^2 \alpha_3 - 3 \alpha_4^2, \]
\[ \beta_8 = -\frac{1}{30} \alpha_6 + \frac{3}{10} \alpha_2 \alpha_5 + \frac{1}{2} \alpha_3 \alpha_4 - \frac{3}{2} \alpha_2^2 \alpha_4 - 2 \alpha_2 \alpha_3^2 + 5 \alpha_2^3 \alpha_3 - \frac{9}{4} \alpha_4^2, \]
\[ \beta_9 = \frac{1}{10} \alpha_3 \alpha_5 - \frac{1}{8} \alpha_4^2 - \frac{3}{20} \alpha_2^2 \alpha_5 + \frac{1}{2} \alpha_2 \alpha_3 \alpha_4 - \frac{4}{9} \alpha_3^3 + \frac{1}{2} \alpha_2^2 \alpha_3^2 - \frac{3}{4} \alpha_2^3 \alpha_3 + \frac{3}{8} \alpha_4^2. \]

Following to the definition of the Massey products, we obtain cocycles

\[ \gamma_2^- = -\alpha_2 \beta_2 - \beta_1 \tilde{\alpha}_3, \]

and

\[ \gamma_2^+ = -\alpha_2 \beta_9 - \beta_1 \beta_7 - \beta_5 \beta_1 - \beta_8 \tilde{\alpha}_3, \]

of Proposition 1.1.

\[ \square \]

4. Proof of the main theorem

Throughout this section, for any two diffeomorphisms \( f \) and \( g \), the multiplication \( fg \) means that \( g \) is applied first.

In this section, we prove the non-triviality of \( \gamma_2^\pm \) by constructing uncountably many 2-cycles \( \xi_2^\pm \in \mathbb{Z}[\text{Diff}_1^\infty(\mathbb{R})^2] \) such that \( \gamma_2^- (\xi_2^-) \neq 0 \) and \( \gamma_2^+ (\xi_2^+) \neq 0 \). Then this proves Theorem 1.2. To construct \( \xi_2^\pm \), we use the following lemma.

**Lemma 4.1.** — For any \( k, l \geq 1 (k \neq l) \) and any \( f \in \text{Diff}_k^\infty(\mathbb{R}) \) and \( g \in \text{Diff}_l^\infty(\mathbb{R}) \) there exist \( h \in \text{Diff}_l^\infty(\mathbb{R}) \) such that \( f = [g, h] \). Here, \([g, h]\) means \( ghg^{-1}h^{-1} \).

Moreover, for any \( k, l \geq 1 (k \neq l) \) it is true that \([\text{Diff}_k^\infty(\mathbb{R}), \text{Diff}_l^\infty(\mathbb{R})] = \text{Diff}_{k+l}^\infty(\mathbb{R}) \). In the case \( k = l \), Fukui proved that \([\text{Diff}_k^\infty(\mathbb{R}), \text{Diff}_k^\infty(\mathbb{R})] = \text{Diff}_{2k+1}^\infty(\mathbb{R}) \) for \( k \geq 1 \) in [1].
Proof. — We may assume that $k > l$ and the $\infty$-jet of $f$ is written as

$$f(x) = x + \sum_{n=k+l+1}^{\infty} a_n x^n.$$  

If we take $h \in \text{Diff}_l^\infty(\mathbb{R})$ so that $h$ can be described as

$$h(x) = x + x^{l+1}$$

in a some neighborhood of 0, then the $\infty$-jet of $fh$ at 0 is written as

$$fh(x) = x + x^{l+1} + a_{k+l+1} x^{k+l+1} + \ldots.$$  

Here, we apply the following theorem of the normal forms of diffeomorphisms of $(\mathbb{R}, 0)$.

**Theorem 4.2** (Takens[6]). — *For any $l \geq 1$ and $\psi \in \text{Diff}_l^\infty(\mathbb{R})$, there exists $\varphi \in \text{Diff}_0^\infty(\mathbb{R})$ such that

$$\varphi \psi \varphi^{-1}(x) = x + \delta x^{l+1} + \alpha x^{2l+1},$$

in a some neighborhood of 0 for some $\delta = \pm 1$ and $\alpha \in \mathbb{R}$. Here $\delta$ and $\alpha$ are uniquely determined by the $(2l + 1)$-jet of $\psi$.***

Because of the uniqueness of $\delta$ and $\alpha$, there exists $\varphi$ such that $\varphi^{-1}fh\varphi = h$ in some neighborhood $U$ of 0. By Takens’ construction of $\varphi$ in Theorem 4.2, it is seen that one can choose $\varphi$ to be $C^l$-flat to the identity at 0. We denote the composition $\varphi^{-1}fh\varphi$ by $\Phi$. If we take $h$ so that both of $h$ and $\Phi$ have no fixed points except for 0, then $\Phi$ is conjugate to $h$. In the case $l$ is odd and $x < 0$, there exists an integer $n_x \geq 0$ such that $\Phi^n(x)$ is in $U$ for any $n \geq n_x$ and we define $\tilde{\varphi}(x) = \Phi^{-n_x}h^{n_x}(x)$. Otherwise, for any $x$ there exists an integer $n_x \geq 0$ such that $\Phi^{-n_x}(x)$ is in $U$ for any $n \geq n_x$ and we define $\tilde{\varphi}(x) = \Phi^{-n_x}h^{-n_x}(x)$. Then $\varphi^{-1}f\tilde{\varphi}$ coincides with $h$. If we set $g = \varphi \tilde{\varphi}$, then $g$ is contained in $\text{Diff}_k^\infty(\mathbb{R})$ and Lemma 4.1 is proved.  

**Proof of Theorem 1.2.** — If $f, g \in \text{Diff}_1^\infty(\mathbb{R})$ and the $\infty$-jet of them are written as

$$f(x) = x + \sum_{n=2}^{\infty} a_n x^n, \quad g(x) = x + \sum_{n=2}^{\infty} b_n x^n,$$

then

$$\gamma^2(f, g) = 36(b_3 - b_2^2)(2a_4 - 6a_2 a_3 + 4a_2^3 - a_2 a_3 + a_2 b_3 + a_2 b_2).$$

Thus if $f_i \in \text{Diff}_i^\infty(\mathbb{R})$ for $i = 1, 2, 3, 4$, then $\gamma^2(f_2, f_3) = \gamma^2(f_1, f_4) = \gamma^2(f_4, f_1) = 0$. On the other hand, if both of the coefficient of $x^4$ in the jet of $f_3$ and the coefficient of $x^3$ in the jet of $f_2$ are non-zero, then $\gamma^2(f_3, f_2) \neq 0$. Therefore, we assume $f_3 \in \text{Diff}_3^\infty(\mathbb{R}) \setminus \text{Diff}_4^\infty(\mathbb{R})$ and $f_2 \in \text{Diff}_2^\infty(\mathbb{R}) \setminus \text{Diff}_3^\infty(\mathbb{R})$.  

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Diff_3^\infty(\mathbb{R}). By Lemma 4.1, we can choose \( f_1 \in \text{Diff}_1^\infty(\mathbb{R}) \) and \( f_4 \in \text{Diff}_3^\infty(\mathbb{R}) \) such that \([f_2, f_3] = [f_1, f_4]\). If we set

\[
\xi^- = (f_3, f_2) - (f_2, f_3) + ([f_2, f_3], f_3 f_2)
\]

\[
- (f_4, f_1) + (f_1, f_4) - ([f_1, f_4], f_4 f_1) \in \mathbb{Z}[\text{Diff}_1^\infty(\mathbb{R})^2],
\]

then \( \xi^- \) is a cycle and

\[
\gamma^2(\xi^-) = \gamma_2^-(\xi(f_3, f_2)) = \frac{1}{2} \alpha_4(f_3) \alpha_3(f_2) \neq 0.
\]

Therefore, the non-triviality of \( \gamma^2^- \) is proved.

The non-triviality of \( \gamma^2_+ \) can be proved similarly. Let \( g_3 \in \text{Diff}_4^\infty(\mathbb{R}) \setminus \text{Diff}_2^\infty(\mathbb{R}) \) and \( g_4 \in \text{Diff}_1^\infty(\mathbb{R}) \setminus \text{Diff}_5^\infty(\mathbb{R}) \). If we choose \( g_1 \in \text{Diff}_1^\infty(\mathbb{R}) \) and \( g_6 \in \text{Diff}_6^\infty(\mathbb{R}) \) such that \([g_3, g_4] = [g_1, g_6]\) and set

\[
\xi^+ = (g_4, g_3) - (g_3, g_4) + ([g_3, g_4], g_4 g_3)
\]

\[
- (g_6, g_1) + (g_1, g_6) - ([g_1, g_6], g_6 g_1) \in \mathbb{Z}[\text{Diff}_1^\infty(\mathbb{R})^2],
\]

then \( \xi^+ \) is a cycle and

\[
\gamma^2(\xi^+) = \gamma_2^+(\xi_-(g_4, g_3) - (g_3, g_4)) = -\frac{1}{20} \alpha_5(g_4) \alpha_4(g_3) \neq 0.
\]

Consequently, \( \gamma^2_\pm \) are non-trivial cohomology classes in \( H^2(\text{Diff}_1^\infty(\mathbb{R}); \mathbb{R}) \).

Furthermore, clearly \( \gamma^2_-(\xi^+) = 0 \) and \( \gamma^2_-(\xi^-) \) can take any real value by changing \( f_2 \) or \( f_3 \). Similarly, \( \gamma^2_+(\xi^+) \) also can take any value. This concludes the proof of Theorem 1.2.

Moreover, the following corollary holds.

**Corollary 4.3.** — For any \( g \geq 2 \), there exist uncountably many isomorphism classes of flat \( \mathbb{R} \)-bundles on genus \( g \) surface \( \Sigma_g \), such that the images of their holonomy homomorphisms

\[
\pi_1(\Sigma_g) \to \text{Diff}_1^\infty(\mathbb{R})
\]

are contained in \( \text{Diff}_1^\infty(\mathbb{R}) \).

**Remark 4.4.** — The same argument in \( \S 3 \), and \( \S 4 \) can be applied to the groups \( G_1(1) \) and \( G_1^\infty(1) \) instead of \( \text{Diff}_1^\infty(\mathbb{R}) \). Therefore, we can regard \( \gamma^2_\pm \) as the 2-cochains of \( G_1(1) \) and \( G_1^\infty(1) \), and Theorem 1.2 also holds for \( G_1(1) \) and \( G_1^\infty(1) \), respectively.

On the other hand, for any group \( G \) and commuting \( f, g, \in G \), the chain \((f, g) - (g, f)\) is the simplest 2-cycle of \( G \). However, if we regard \( \gamma^2_\pm \) as the 2-cocycles of \( G_1^\infty(1) \), then it is seen that \( \gamma^2_\pm((f, g) - (g, f)) = 0 \) for any commuting \( f, g, \in G_1^\infty(1) \). Hence the following is true.
Proposition 4.5. — For any group homomorphism $\rho: \pi_1(T^2) \to G_1^\infty(1)$, 
\[ \rho^* \gamma_\pm^2 = 0. \]

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