Kazuya KATO & Henrik RUSSELL

Albanese varieties with modulus and Hodge theory

<http://aif.cedram.org/item?id=AIF_2012__62_2_783_0>
ALBANESE VARIETIES WITH MODULUS
AND HODGE THEORY

by Kazuya KATO & Henrik RUSSELL (*)

Abstract. — Let $X$ be a proper smooth variety over a field $k$ of characteristic 0 and $Y$ an effective divisor on $X$ with multiplicity. We introduce a generalized Albanese variety $\text{Alb}(X, Y)$ of $X$ of modulus $Y$, as higher dimensional analogue of the generalized Jacobian with modulus of Rosenlicht-Serre. Our construction is algebraic. For $k = \mathbb{C}$ we give a Hodge theoretic description.

Résumé. — Soient $X$ une variété propre et lisse sur un corps $k$ de caractéristique 0 et $Y$ un diviseur effectif avec multiplicité sur $X$. Nous introduisons une variété d’Albanese généralisée $\text{Alb}(X, Y)$ de $X$, de module $Y$, comme analogue en dimension supérieure de la jacobienne généralisée avec module de Rosenlicht-Serre. Notre construction est algébrique. Si $k = \mathbb{C}$, nous donnons une description en termes de théorie de Hodge.

1. Introduction

1.1. Let $X$ be a proper smooth variety over a field $k$ of characteristic 0, and let $\text{Alb}(X)$ be the Albanese variety of $X$. In the work [10], the second author constructed generalized Albanese varieties $\text{Alb}_F(X)$, which are commutative connected algebraic groups over $k$ with surjective homomorphisms $\text{Alb}_F(X) \to \text{Alb}(X)$ (see Section 5 for a review). If $Y$ is an effective divisor on $X$, a special case of $\text{Alb}_F(X)$ becomes the generalized Albanese variety $\text{Alb}(X, Y)$ of $X$ of modulus $Y$ (cf., Section 5). This is a higher dimensional analogue of the generalized Jacobian variety with modulus of Rosenlicht-Serre. Note that the divisor $Y$ can have multiplicity, and so the algebraic group $\text{Alb}(X, Y)$ can have an additive part.

Keywords: generalized Albanese variety, modulus of a rational map, generalized mixed Hodge structure.
Math. classification: 14L10, 14C30, 14F42.

(*) The second author was supported by the DFG.
Assume now $k = \mathbb{C}$. The purpose of this paper is to give Hodge theoretic presentations (Theorem 1.1) of $\text{Alb}(X, Y)$.

The case when $Y$ has no multiplicity was studied in the work [3] of Barbieri-Viale and Srinivas. A Hodge theoretic presentation of a generalized Albanese variety in the case without modulus but allowing singularities on $X$ was given in the work [6] of Esnault, Srinivas and Viehweg.

1.2. First we review the curve case. Let $X$ be a proper smooth curve over $\mathbb{C}$ and let $Y$ be an effective divisor on $X$. In this case, the Albanese variety $\text{Alb}(X, Y)$ of $X$ relative to $Y$ coincides with the generalized Jacobian variety $J(X, Y)$ of $X$ relative to $Y$. In the following, we will write the complex analytic space associated to $X$ simply by $X$, and the sheaf of holomorphic functions on it by $\mathcal{O}_X$. Let $I = \text{Ker}(\mathcal{O}_X \to \mathcal{O}_Y)$ be the ideal of $\mathcal{O}_X$ which defines $Y$. The cohomology below is for the topology of the analytic space $X$ (not for Zariski topology).

The generalized Jacobian variety $J(X, Y)$ is the kernel of the degree map $\text{Pic}(X, Y) \to \mathbb{Z}$ where $\text{Pic}(X, Y) = H^1(X, \text{Ker}(\mathcal{O}_X^\times \to \mathcal{O}_Y^\times))$. Let $j : X - Y \to X$ be the inclusion map and let $j_! \mathbb{Z}(1)$ be the $0$-extension of the constant sheaf $\mathbb{Z}(1)$ of $X - Y$ to $X$. (For $r \in \mathbb{Z}$, $\mathbb{Z}(r)$ denotes $\mathbb{Z}(2\pi i)^r$ as usual.) Then we have an exact sequence

$$0 \to j_! \mathbb{Z}(1) \to I \xrightarrow{\exp} \text{Ker}(\mathcal{O}_X^\times \to \mathcal{O}_Y^\times) \to 0$$

and hence we have an isomorphism

$$\text{Pic}(X, Y) \cong H^2(X, [j_! \mathbb{Z}(1) \to I]).$$

Here in the complex $[j_! \mathbb{Z}(1) \to I]$, $j_! \mathbb{Z}(1)$ is put in degree 0.

We have another presentation of $J(X, Y)$ given in (2) below. Let $I_1$ be the ideal of $\mathcal{O}_X$ which defines the reduced part of $Y$ and let $J = II_1^{-1} \subset \mathcal{O}_X$. Note that the composition of the two inclusion maps of complexes

$$[I \xrightarrow{d} J\Omega^1_X] \to [I \xrightarrow{d} \Omega^1_X] \to [I_1 \xrightarrow{d} \Omega^1_X]$$

is a quasi-isomorphism. Hence we have an isomorphism in the derived category

$$[I \xrightarrow{d} \Omega^1_X] \cong [I_1 \xrightarrow{d} \Omega^1_X] \oplus (\Omega^1_X/J\Omega^1_X)[-1].$$

Since $j_! \mathbb{C} \to [I_1 \xrightarrow{d} \Omega^1_X]$ is a quasi-isomorphism, we have an exact sequence

$$H^0(X, \Omega^1_X) \to H^1_c(X - Y, \mathbb{C}/\mathbb{Z}(1)) \oplus H^0(X, \Omega^1_X/J\Omega^1_X) \to J(X, Y) \to 0.$$  

(Here $H^*_c$ is the cohomology with compact supports.)
1.3. Now let $X$ be a proper smooth variety over $\mathbb{C}$ of dimension $n$ and let $Y$ be an effective divisor on $X$.

Again in the following theorem, cohomology groups are for the topology of the complex analytic spaces, and the notation $\mathcal{O}$ and $\Omega$ stand for analytic sheaves.

Let $I$ be the ideal of $\mathcal{O}_X$ which defines $Y$, let $I_1$ be the ideal of $\mathcal{O}_X$ which defines the reduced part of $Y$, and let $J = II^{-1} \subset \mathcal{O}_X$.

**Theorem 1.1.**

(1) We have an exact sequence

$$0 \to \text{Alb}(X,Y) \to H^{2n}(X,\mathcal{D}_X,Y(n))^{\deg} \to \mathbb{Z} \to 0,$$

where for $r \in \mathbb{Z}$, $\mathcal{D}_{X,Y}(r)$ denotes the kernel of the surjective homomorphism of complexes $\mathcal{D}_X(r) \to \mathcal{D}_Y(r)$ with $\mathcal{D}_X(r)$ the Deligne complex

$$[\mathbb{Z}(r) \to \mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \ldots \xrightarrow{d} \Omega^{r-1}_X]$$

and $\mathcal{D}_Y(r)$ the similar complex

$$[\mathbb{Z}(r)_Y \to \mathcal{O}_Y \xrightarrow{d} \Omega^1_Y \xrightarrow{d} \ldots \xrightarrow{d} \Omega^{r-1}_Y].$$

(2) We have an exact sequence

$$H^{n-1}(X,\Omega^n_X) \to H^{2n-1}_C(X - Y,\mathbb{C}/\mathbb{Z}(n)) \oplus H^{n-1}(X,\Omega^n_X/J\Omega^n_X) \to \text{Alb}(X,Y) \to 0.$$

Note that the case $n = 1$ of Theorem 1.1 (1) (resp. (2)) becomes the presentation of $J(X,Y)$ given by (1) (resp. (2)) in No. 1.2.

**Remark 1.2.** — We give some remarks on this theorem.

(a) The case $Y = 0$ of Theorem 1.1 (1) is nothing but the well known exact sequence

$$0 \to \text{Alb}(X) \to H^{2n}(X,\mathcal{D}_X(n))^{\deg} \to \mathbb{Z} \to 0$$

by using the Deligne cohomology $H^{2n}(X,\mathcal{D}_X(n))$. (Usually the Deligne cohomology $H^m(X,\mathcal{D}_X(r))$ is denoted by $H^m_B(X,\mathbb{Z}(r)).$)

The case $Y = 0$ of Theorem 1.1 (2) is nothing but the usual presentation

$$\text{Alb}(X) \cong H_Z^1(\mathcal{H}_C/F^0\mathcal{H}_C)$$

of the Albanese variety $\text{Alb}(X)$ of $X$, where $(H_Z, H_C, F^\bullet)$ is the following Hodge structure of weight $-1$. $H_Z = H^{2n-1}(X,\mathbb{Z}(n))/(\text{torsion part})$, $H_C = \ldots$
\( \mathbb{C} \otimes_{\mathbb{Z}} H_Z = H^{2n-1}(X, \Omega_X^\bullet) \), and \( F^\bullet \) is the Hodge filtration on \( H_C \) defined as

\[
F^{-1} = H_C, \quad F^0 = H^{n-1}(X, \Omega_X^n), \quad F^1 = 0.
\]

(b) Recall that the presentations (3) and (4) of \( \text{Alb}(X) \) are related as follows. Consider the exact sequence of complexes \( 0 \to \Omega_X^{\leq n-1}[-1] \to D_X(n) \to \mathbb{Z}(n) \to 0 \), where \( \Omega_X^{\leq n-1} \) denotes the part of degree \( \leq n-1 \) of the de Rham complex \( \Omega_X^\bullet \), which is actually a quotient complex of \( \Omega_X^\bullet \). By taking the cohomology associated to this exact sequence, we have an exact sequence

\[
H^{2n-1}(X, \mathbb{Z}(n)) \to H^{2n-1}(X, \Omega_X^{\leq n-1}) \to H^2_{D}(X, \mathbb{Z}(n)) \xrightarrow{\text{deg}} \mathbb{Z} \to 0.
\]

Since

\[
H^{2n-1}(X, \Omega_X^{\leq n-1}) \cong H^{2n-1}(X, \Omega_X^\bullet)/H^{n-1}(X, \Omega_X^n)
\]

\[
\cong H^{2n-1}(X, C)/H^{n-1}(X, \Omega_X^n),
\]

the exact sequence (4) is equivalent to (3).

(c) (1) and (2) of Theorem 1.1 are related similarly. Let \( S \) be the subcomplex of the de Rham complex \( \Omega_X^\bullet \) of \( X \) defined by \( S^p = \text{Ker}(\Omega_X^p \to \Omega_Y^p) \) for \( 0 \leq p \leq n-1 \) and \( S^n = \Omega_X^n \). Then Theorem 1.1 (1) is equivalent to

\[
\text{Alb}(X, Y) \cong H_Z \setminus H^{2n-1}(X, S)/H^{n-1}(X, \Omega_X^n)
\]

where \( H_Z = H^{2n-1}_c(X - Y, \mathbb{Z}(n))/(\text{torsion part}) \). As shown in § 6, we have a commutative diagram with an isomorphism in the lower row

\[
\begin{array}{ccc}
H^{n-1}(X, \Omega_X^n) & = & H^{n-1}(X, \Omega_X^n) \\
\downarrow & & \downarrow \\
H^{2n-1}(X, S) & \cong & H^{2n-1}_c(X - Y, \mathbb{C}) \oplus H^{n-1}(X, \Omega_X^n/J\Omega_X^n).
\end{array}
\]

Thus (1) and (2) of Theorem 1.1 are deduced from each other.

1.4. As mentioned above, Theorem 1.1 shows that \( \text{Alb}(X, Y) \) is expressed as \( H_Z \setminus H_V/F^0 \) where:

\[
H_Z = H^{2n-1}_c(X - Y, \mathbb{Z}(n))/(\text{torsion part}),
\]

\[
H_V = H_C \oplus H^{n-1}(X, \Omega_X^n/J\Omega_X^n) \cong H^{2n-1}(X, S)
\]

\[
(H_C = \mathbb{C} \otimes H_Z \text{ and } S \text{ is as in 1.5 (d)}),
\]

\( F^\bullet \) is the decreasing filtration on \( H_V \) given by

\[
F^{-1} = H_V, \quad F^0 = H^{n-1}(X, \Omega_X^n), \quad F^1 = 0.
\]
Note that \( H_V \) can be different from \( H_C \) here, and so \((H_Z, H_V, F^*)\) here need not be a Hodge structure. It is some kind of “mixed Hodge structure with additive part”. This object \((H_Z, H_V, F^*)\) with a weight filtration, which we will denote by \( H^{2n-1}(X, Y_\cdot)(n) \) in Section 6, belongs to a category \( \mathcal{H} \) introduced in Section 2 which contains the category of mixed Hodge structures but is larger than that. In the proof of Theorem 1.1, it is essential to consider such an object. This category \( \mathcal{H} \) is related to the category of “enriched Hodge structures” of Bloch-Srinivas [4] and to the category of “formal Hodge structures” of Barbieri-Viale [1]. However, the relations between these three categories are not trivial, see 4.6 and [2, 4.2]. Our definition of \( \mathcal{H} \) aims to stick close to the classical language of Hodge structures and to express duality in a simplest possible way. In the proof of Theorem 1.1, we use a Hodge theoretic description of the category of “1-motives with additive parts” over \( \mathbb{C} \) in terms of \( \mathcal{H} \). This description is similar to the result of Barbieri-Viale in [1].

1.5. The theory of generalized Albanese varieties in characteristic \( p > 0 \) is given in [11], basing on duality theory of “1-motives with unipotent parts”.

In characteristic \( p > 0 \), syntomic cohomology is an analogue of Deligne cohomology. We expect that we can have presentations of the \( p \)-adic completion of \( \text{Alb}(X, Y)(k) \) \((k \text{ is the base field})\), which is similar to Theorem 1.1, by using crystalline cohomology theory and syntomic cohomology theory.

We are thankful to Professor Hélène Esnault for advice.

2. Mixed Hodge structures with additive parts

2.1. For a proper smooth variety \( X \) over \( \mathbb{C} \) of dimension \( n \) and for an effective divisor \( Y \) on \( X \), we will have in Section 6 certain structures \( H^1(X, Y_\cdot) \) and \( H^{2n-1}(X, Y_\cdot) \) which are kinds of “mixed Hodge structures with additive parts”. (These structures for the case when \( X \) is a curve are explained in Example 2.1 below.) The authors imagine that there is a nice definition of the category of “mixed Hodge structures with additive parts”, which contains these \( H^1(X, Y_\cdot) \) and \( H^{2n-1}(X, Y_\cdot) \) as objects, but can not define it. Instead, we define a category \( \mathcal{H} \) containing these objects, which may be a very simple approximation of such a nice category.

2.2. The category \( \mathcal{H} \). An object of \( \mathcal{H} \) is by definition a tuple \( H = (H_Z, H_V, W_\cdot H_Q, W_\cdot H_V, F^* H_V, a, b) \), where \( H_Z \) is a finitely generated \( \mathbb{Z} \)-module, \( H_V \) is a finite dimensional \( \mathbb{C} \)-vector space, \( W_\cdot H_Q \) is an increasing
filtration on $H_Q := Q \otimes H_Z$ (called weight filtration), $W_*H_V$ is an increasing filtration on $H_V$ (called weight filtration), $F^\bullet$ is a decreasing filtration on $H_V$ (called Hodge filtration), $a$ is a $\mathbb{C}$-linear map $H_C := \mathbb{C} \otimes H_Z \to H_V$ which sends $W_wH_C := \mathbb{C} \otimes Q W_wH_Q$ into $W_wH_V$ for any $w \in \mathbb{Z}$, and $b$ is a $\mathbb{C}$-linear map $H_V \to H_C$ which sends $W_wH_V$ into $W_wH_C$ for any $w \in \mathbb{Z}$ such that $b \circ a$ is the identity map of $H_C$. We sometimes denote an object $H$ of $\mathcal{H}$ simply by $(H_Z, H_V)$.

A morphism $f : H \to H'$ in $\mathcal{H}$ is a pair of homomorphisms $(f_Z, f_V)$, where $f_Z : H_Z \to H'_Z$ is compatible with the weight filtrations and $f_V : H_V \to H'_V$ is compatible with weight filtrations and Hodge filtrations, which is compatible with the maps $a, b$ and $a', b'$.

The category of mixed Hodge structures is naturally embedded into $\mathcal{H}$ as a full subcategory, by putting $H_V = H_C$.

Similarly as for mixed Hodge structures we can give $\text{Hom}(H, H')$ the structure of an object of $\mathcal{H}$ for $H, H' \in \text{Ob}(\mathcal{H})$. We call $\overline{\text{Hom}}(H, Z)$ the object dual to $H$. The full subcategory of $\mathcal{H}$ consisting of all objects $H$ such that $H_Z$ are torsion free is clearly self-dual.

We will say that a sequence $H' \to H \to H''$ in $\mathcal{H}$ is exact, if and only if the following sequences are all exact:

$$
H'_Z \to H_Z \to H''_Z,
\quad
H'_V \to H_V \to H''_V,
\quad
W_wH'_Q \to W_wH_Q \to W_wH''_Q,
\quad
W_wH'_V \to W_wH_V \to W_wH''_V,
\quad
F^pH'_V \to F^pH_V \to F^pH''_V,
$$

for all $w, p \in \mathbb{Z}$.

See No. 4.6 for the relation of this category $\mathcal{H}$ to the category of enriched Hodge structures of Bloch-Srinivas [4] and to the category of formal Hodge structures of Barbieri-Viale [1].

**Example 2.1.** — Let $X$ be a proper smooth curve over $\mathbb{C}$ and let $Y$ be an effective divisor on $X$. Let $I$ be the ideal of $\mathcal{O}_X$ which defines $Y$, let $I_1$ be the ideal of $\mathcal{O}_X$ which defines the reduced part of $Y$, and let $J = II_1^{-1} \subset \mathcal{O}_X$.

We define objects $H^1(X, Y_+)$ and $H^1(X, Y_-)$ of $\mathcal{H}$.

First, we define $H = H^1(X, Y_+)$. Let

$$
H_Z = H^1(X - Y, Z),
H_V = H^1(X, [\mathcal{O}_X \xrightarrow{d} I^{-1}\Omega^1_X]).
$$

The map $a : H_C \to H_V$ is

$$
H^1(X - Y, \mathbb{C}) \cong H^1(X, [\mathcal{O}_X \to I_1^{-1}\Omega^1_X]) \to H^1(X, [\mathcal{O}_X \to I^{-1}\Omega^1_X]).
$$
The weight filtrations and the Hodge filtration are given by
\[ H^1(X, [\mathcal{O}_X \to I^{-1}\Omega^1_X]) \longrightarrow H^1(X, [J^{-1} \to I^{-1}\Omega^1_X]) \]
\[
\simeq H^1(X, [\mathcal{O}_X \to I_1^{-1}\Omega^1_X]) \simeq H^1(X - Y, \mathbb{C}).
\]
The weight filtrations and the Hodge filtration are given by
\[ W_2 H_\mathbb{Q} = H_\mathbb{Q}, \quad W_1 H_\mathbb{Q} = H^1(X, \mathbb{Q}), \quad W_0 H_\mathbb{Q} = 0, \]
\[ W_2 H_V = H_V, \quad W_1 H_V = H^1(X, \mathbb{C}), \quad W_0 H_V = 0, \]
where \( H^1(X, \mathbb{C}) \) is embedded in \( H_V \) via \( a \), and
\[ F^0 H_V = H_V, \quad F^1 H_V = H^1(X, \mathbb{C}), \quad F^2 H_\mathbb{C} = 0. \]

Next, we define \( H = H^1(X, Y) \). Let
\[ H_Z = H^1_c(X - Y, \mathbb{Z}), \quad H_V = H^1(X, [I \to \Omega^1_X]). \]
The map \( b : H_V \to H_\mathbb{C} \) is
\[ H^1(X, [I \to \Omega^1_X]) \longrightarrow H^1(X, [I \to \Omega^1_X]) \simeq H^1_c(X - Y, \mathbb{C}). \]
The weight filtrations and the Hodge filtration are given by
\[ W_1 H_\mathbb{Q} = H_\mathbb{Q}, \quad W_0 H_\mathbb{Q} = \text{Ker}(H_\mathbb{Q} \to H^1(X, \mathbb{Q})), \quad W_1 H_\mathbb{Q} = 0, \]
\[ W_1 H_V = H_V, \quad W_0 H_V = \text{Ker}(H_V \to H^1(X, \mathbb{C})), \quad W_1 H_V = 0, \]
where \( H^1(X, \mathbb{C}) \) is regarded as quotient of \( H_V \) via \( b \), and
\[ F^0 H_V = H_V, \quad F^1 H_V = \text{Ker}(H_V \to H^1(X, \mathcal{O}_X)) \quad F^2 H_\mathbb{C} = 0. \]

Then we have exact sequences in \( \mathcal{H} \)
\[ 0 \longrightarrow H^1(X) \longrightarrow H^1(X, Y_+) \longrightarrow H^0(Y)(-1) \longrightarrow \mathbb{Z}(-1) \longrightarrow 0, \]
\[ 0 \longrightarrow \mathbb{Z} \longrightarrow H^0(Y) \longrightarrow H^1(X, Y_-) \longrightarrow H^1(X) \longrightarrow 0. \]

Here for \( r \in \mathbb{Z}, \mathbb{Z}(r) \) is the usual Hodge structure \( \mathbb{Z}(r) \) regarded as an object of \( \mathcal{H} \). \( H^1(X) \) is also the usual Hodge structure of weight 1 associated to the first cohomology of \( X \), regarded as an object of \( \mathcal{H} \). Finally the object \( H^0(Y) \) of \( \mathcal{H} \) is defined as below, and \( H^0(Y)(-1) \) is the \(-1\) Tate twist.

The definition of \( H = H^0(Y) \) is as follows. \( H_Z = H^0(Y, \mathbb{Z}) = \oplus_{y \in Y} \mathbb{Z} \).
\( H_V = H^0(Y, \mathcal{O}_Y) \). \( a \) is the canonical map \( H^0(Y, \mathbb{C}) \to H^0(Y, \mathcal{O}_Y) \).
\( b \) is the canonical map \( H^0(Y, \mathcal{O}_Y) \to H^0(Y, \mathbb{C}) \) given by \( \mathcal{O}_Y \to \mathbb{C} \) which is
\( \mathcal{O}_{Y,y} \to \mathcal{O}_{Y,y}/m_y = \mathbb{C} \) at each \( y \in Y \) (\( m_y \) denotes the maximal ideal of \( \mathcal{O}_{Y,y} \)). The weight filtration and the Hodge filtration are given by

\[
\begin{align*}
W_0 H &= H, & W_{-1} H &= 0, \\
F^0 H_V &= H_V, & F^1 H_V &= 0.
\end{align*}
\]

Note that \( H_C \to H_V \) can be like \( \mathbb{C} \to \mathbb{C}[T]/(T^n) \), and need not be an isomorphism.

The evident self-duality \( \text{Hom}(\ , \mathbb{Z}) \) for torsion free objects in \( \mathcal{H} \) induces

\[
H^1(X, Y_-) \cong \text{Hom}(H^1(X, Y_+), \mathbb{Z})(-1).\]

3. 1-motives with additive parts

In [9], Laumon formulated the notion of a “1-motive with additive part” over a field of characteristic 0. We give a short review assuming that the base field is algebraically closed for simplicity.

Fix an algebraically closed field \( k \) of characteristic 0.

3.1. Let \( \mathcal{A}b/k \) be the category of sheaves of abelian groups on the fppf-site of the category of affine schemes over \( k \). Let \( \mathcal{C}^{[-1,0]}(\mathcal{A}b/k) \) be the abelian category of complexes in \( \mathcal{A}b/k \) concentrated in degrees \(-1 \) and \( 0 \).

A 1-motive with additive part over \( k \) is an object of \( \mathcal{C}^{[-1,0]}(\mathcal{A}b/k) \) of the form \( [\mathcal{F} \to \mathcal{G}] \), where \( \mathcal{G} \) is a commutative connected algebraic group over \( k \) and \( \mathcal{F} \cong \mathbb{Z}^t \oplus (\hat{\mathbb{G}}_a)^s \) for some \( t \) and \( s \). (cf., [9, Def. (5.1.1)].) Here \( \mathbb{Z} \) is regarded as a constant sheaf and \( \hat{\mathbb{G}}_a \) denotes the formal completion of the additive group \( \mathbb{G}_a \) at 0. Recall that for any commutative ring \( R \), \( \hat{\mathbb{G}}_a(R) \) is the subgroup of the additive group \( R \) consisting of all nilpotent elements. We have \( \mathcal{F} = \mathcal{F}_{\text{ét}} \oplus \mathcal{F}_{\text{inf}} \), where \( \mathcal{F}_{\text{ét}} \) is the étale part of \( \mathcal{F} \) which corresponds to \( \mathbb{Z}^t \) in the above isomorphism and \( \mathcal{F}_{\text{inf}} \) is the infinitesimal part of \( \mathcal{F} \) which corresponds to \( (\hat{\mathbb{G}}_a)^s \).

We denote the category of 1-motives with additive parts over \( k \) by \( \mathcal{M}_1 \).

3.2. The category \( \mathcal{M}_1 \) admits a notion of duality (called “Cartier duality”). Let \( [\mathcal{F} \to \mathcal{G}] \) be a 1-motive with additive part over \( k \). Then we have the “Cartier dual” \( [\mathcal{F}' \to \mathcal{G}'] \) of \( [\mathcal{F} \to \mathcal{G}] \) which is an object of \( \mathcal{M}_1 \) obtained as follows. Let \( 0 \to L \to \mathcal{G} \to A \to 0 \) be the canonical decomposition of \( \mathcal{G} \) as an extension of an abelian variety \( A \) by a commutative connected affine algebraic group \( L \). Note that \( L \cong (\mathbb{G}_m)^t \oplus (\mathbb{G}_a)^s \) for some \( t \) and \( s \). We have

\[
\mathcal{F}' = \text{Hom}_{\mathcal{A}b/k}(L, \mathbb{G}_m), \quad \mathcal{G}' = \text{Ext}^1_{\mathcal{C}^{[-1,0]}(\mathcal{A}b/k)}([\mathcal{F} \to A], \mathbb{G}_m)
\]
and the homomorphism $\mathcal{F}' \to G'$ is the connecting homomorphism

$$\text{Hom}_{\text{Ab}/k}(L, \mathbb{G}_m) \to \text{Ext}^1_{\mathcal{C}[-1,0](\text{Ab}/k)}([\mathcal{F} \to A], \mathbb{G}_m)$$

associated to the short exact sequence $0 \to L \to [\mathcal{F} \to G] \to [\mathcal{F} \to A] \to 0$ in $\mathcal{C}[-1,0](\text{Ab}/k)$. Since

$$\text{Hom}_{\text{Ab}/k}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}, \quad \text{Hom}_{\text{Ab}/k}(\mathbb{G}_a, \mathbb{G}_m) \cong \hat{\mathbb{G}}_a,$$

we have $\mathcal{F}' \cong \mathbb{Z}^t \oplus (\hat{\mathbb{G}}_a)^s$ for some $t$ and $s$. We have an exact sequence

$$0 \to \text{Hom}_{\text{Ab}/k}(\mathcal{F}, \mathbb{G}_m) \to \text{Ext}^1_{\mathcal{C}[-1,0](\text{Ab}/k)}([\mathcal{F} \to A], \mathbb{G}_m) \to \text{Ext}^1_{\text{Ab}/k}(A, \mathbb{G}_m) \to 0,$$

$\text{Ext}^1_{\text{Ab}/k}(A, \mathbb{G}_m)$ is the dual abelian variety of $A$, and since

$$\text{Hom}_{\text{Ab}/k}(\mathbb{Z}, \mathbb{G}_m) \cong \mathbb{G}_m, \quad \text{Hom}_{\text{Ab}/k}(\hat{\mathbb{G}}_a, \mathbb{G}_m) \cong \mathbb{G}_a,$$

$\text{Hom}_{\text{Ab}/k}(\mathcal{F}, \mathbb{G}_m) \cong \mathbb{G}_m^t \oplus (\hat{\mathbb{G}}_a)^s$ for some $t$ and $s$. Hence $G'$ is a commutative connected algebraic group over $k$. Thus $[\mathcal{F}' \to G']$ is a 1-motive with additive part. The Cartier dual of $[\mathcal{F}' \to G']$ is canonically isomorphic to $[\mathcal{F} \to G]$.

See [9, Section 5] for details or [10, Section 1] for another review.

3.3. Let $\mathcal{M}_{1,\{-1,-2\}}$ be the full subcategory of $\mathcal{M}_1$ consisting of all objects $[\mathcal{F} \to G]$ such that $\mathcal{F} = 0$.

Let $\mathcal{M}_{1,\{0,-1\}}$ be the full subcategory of $\mathcal{M}_1$ consisting of all objects $[\mathcal{F} \to G]$ such that $G$ is an abelian variety.

Then the self-duality of $\mathcal{M}_1$ in No. 3.2 induces an anti-equivalence between the categories $\mathcal{M}_{1,\{-1,-2\}}$ and $\mathcal{M}_{1,\{0,-1\}}$.

4. Equivalences of categories

In [1], Barbieri-Viale constructed a Hodge theoretic category and proved that in the case when the base field is $\mathbb{C}$, the category $\mathcal{M}_1$ is equivalent to his Hodge theoretic category. Here we reformulate his equivalence in the style which is convenient for us, by using the category $\mathcal{H}$ from Section 2.

4.1. The category $\mathcal{H}_1$. An object of $\mathcal{H}_1$ is an object $H$ of $\mathcal{H}$ endowed with a splitting of the weight filtration on $\text{Ker}(H_V \to H_C)$ satisfying the following conditions (i)–(iv).

(i) $H_Z$ is torsion free, $F^{-1} H_V = H_V$, $F^1 H_V = 0$, $W_0 H = H$, $W_{-3} H = 0$. 

TOME 62 (2012), FASCICULE 2
(ii) $\text{gr}^W_{-1} H$ is a polarizable Hodge structure of weight $-1$. That is, $\text{gr}^W_{-1} H_C = \text{gr}^W_{-1} H_V$ and $\text{gr}^W_{-1} H_Z$ with the Hodge filtration on $\text{gr}^W_{-1} H_C$ is a polarizable Hodge structure of weight $-1$.

(iii) $F^0 \text{gr}^W_0 H_V = \text{gr}^W_0 H_V$.

(iv) $F^0 W_{-2} H_V = 0$.

Morphisms of $\mathcal{H}_1$ are the evident ones.

The category $\mathcal{H}_1$ is self-dual by the functor $\text{Hom}(\cdot, \mathbb{Z})(1)$.

4.2. For a subset $\Delta$ of $\{0, -1, -2\}$, let $\mathcal{H}_{1, \Delta}$ be the full subcategory of $\mathcal{H}_1$ consisting of all objects $H$ such that $\text{gr}^w_w H = 0$ unless $w \in \Delta$.

The categories $\mathcal{H}_{1, \{-1, -2\}}$ and $\mathcal{H}_{1, \{0, -1\}}$ are important for us. These categories are in fact defined as full subcategories of $\mathcal{H}$ without reference to the splitting of the weight filtration on $\text{Ker}(H_V \to H_C)$, for the weight filtrations on $\text{Ker}(H_V \to H_C)$ of objects of these categories are pure.

Thus $\mathcal{H}_{1, \{-1, -2\}}$ is the full subcategory of $\mathcal{H}$ consisting of all objects $H$ satisfying the following conditions (i)–(iii).

(i) $H_Z$ is torsion free, $F^{-1} H_V = H_V$, $F^1 H_V = 0$, $W_{-1} H = H$, $W_{-3} H = 0$.

(ii) $\text{gr}^W_{-1} H$ is a polarizable Hodge structure of weight $-1$.

(iii) $F^0 W_{-2} H_V = 0$.

For example, the Tate twist $H^1(X, Y_{-})(1)$ of the object $H^1(X, Y_{-})$ of $\mathcal{H}$ in Example 2.1 belongs to $\mathcal{H}_{1, \{-1, -2\}}$.

Similarly, $\mathcal{H}_{1, \{0, -1\}}$ is the full subcategory of $\mathcal{H}$ consisting of all objects $H$ satisfying the following conditions (i)–(iii).

(i) $H_Z$ is torsion free, $F^{-1} H_V = H_V$, $F^1 H_V = 0$, $W_0 H = H$, $W_{-2} H = 0$.

(ii) $\text{gr}^W_{-1} H$ is a polarizable Hodge structure of weight $-1$.

(iii) $F^0 \text{gr}^W_0 H_V = \text{gr}^W_0 H_V$.

For example, the Tate twist $H^1(X, Y_{+})(1)$ of the object $H^1(X, Y_{+})$ of $\mathcal{H}$ in Example 2.1 belongs to $\mathcal{H}_{1, \{0, -1\}}$.

The self-duality $\text{Hom}(\cdot, \mathbb{Z})(1)$ of $\mathcal{H}_1$ induces an anti-equivalence between the categories $\mathcal{H}_{1, \{-1, -2\}}$ and $\mathcal{H}_{1, \{0, -1\}}$.

**Theorem 4.1.** — (This is an analogue of the equivalence of categories proved by Barbieri-Viale in [1].) We have an equivalence of categories $\mathcal{H}_1 \simeq \mathcal{M}_1$ which is compatible with dualities, and which induces the equivalences

$$\mathcal{H}_{1, \{-1, 0\}} \simeq \mathcal{M}_{1, \{-1, 0\}}, \quad \mathcal{H}_{1, \{-2, 1\}} \simeq \mathcal{M}_{1, \{-2, -1\}}.$$ 

The equivalence $\mathcal{H}_1 \simeq \mathcal{M}_1$ is described in No.s 4.3 and 4.4 below.

4.3. First we define the functor $\mathcal{H}_1 \to \mathcal{M}_1$. 

ANNALES DE L'INSTITUT FOURIER
Let $H$ be an object of $\mathcal{H}_1$. The corresponding object $[F \to G]$ of $\mathcal{M}_1$ is as follows.

$$G = W_{-1}H_Z\backslash W_{-1}H_V / F^0W_{-1}H_V,$$

$$\mathcal{F}_{\text{ét}} = \text{gr}_0^W(H_Z),$$

$$\mathcal{F}_{\text{inf}} = \text{the formal completion of } \text{Ker}(\text{gr}_0^W(H_V) \to \text{gr}_0^W(H_C)).$$

Here homomorphism $\mathcal{F} = \mathcal{F}_{\text{ét}} \oplus \mathcal{F}_{\text{inf}} \to G$ is given as follows.

The part $\mathcal{F}_{\text{ét}} \to G$: Let $x \in \mathcal{F}_{\text{ét}} = \text{gr}_0^W H_Z$. Since the sequence $0 \to W_{-1}H_Z \to H_Z \to \text{gr}_0^W H_Z \to 0$ is exact, we can lift $x$ to an element $y$ of $H_Z$ and this lifting is unique modulo $W_{-1}H_Z$. Since the sequence $0 \to F^0W_{-1}H_V \to F^0H_V \to F^0\text{gr}_0^W H_V \to 0$ is exact, we can lift $x$ to an element $z$ of $F^0H_V$ and this lifting is unique modulo $F^0W_{-1}H_V$. Note that $y - z \in W_{-1}H_V$. We have a well-defined homomorphism

$$\mathcal{F}_{\text{ét}} = \text{gr}_0^W H_Z \to W_{-1}H_Z\backslash W_{-1}H_V / F^0W_{-1}H_V = G; \quad x \mapsto y - z.$$

The part $\mathcal{F}_{\text{inf}} \to G$: Identify $\text{Hom}(\mathcal{F}_{\text{inf}}, G)$ with $\text{Hom}_C(\text{Lie}(\mathcal{F}_{\text{inf}}), \text{Lie}(G))$. We give the corresponding homomorphism $\text{Lie}(\mathcal{F}_{\text{inf}}) = \text{Ker}(\text{gr}_0^W(H_V) \to \text{gr}_0^W(H_C)) \to \text{Lie}(G) = W_{-1}H_V / F^0W_{-1}H_V$. Let $x \in \text{Ker}(\text{gr}_0^W(H_V) \to \text{gr}_0^W(H_C))$. The given splitting of the weight filtration on $\text{Ker}(H_V \to H_C)$ sends $x$ to an element $y$ of $\text{Ker}(H_V \to H_C)$. Since the sequence $0 \to F^0W_{-1}H_V \to F^0H_V \to F^0\text{gr}_0^W H_V \to 0$ is exact, we can lift $x$ to an element $z$ of $F^0H_V$ and this lifting is unique modulo $F^0W_{-1}H_V$. Note that $y - z \in W_{-1}H_V$. We have a well-defined homomorphism

$$\text{Ker}(\text{gr}_0^W H_V \to \text{gr}_0^W H_C) \to W_{-1}H_V / F^0W_{-1}H_V = \text{Lie}(G); \quad x \mapsto y - z.$$

4.4. We give the functor $\mathcal{M}_1 \to \mathcal{H}_1$.

Let $[F \to G]$ be an object of $\mathcal{M}_1$. The corresponding object $H$ of $\mathcal{H}_1$ is as follows. Let $0 \to L \to G \to A \to 0$ be the exact sequence of commutative algebraic groups where $A$ is an abelian variety and $L$ is affine. Let $\mathcal{F}_{\text{ét}}$ be the étale part of $F$ and let $\mathcal{F}_{\text{inf}}$ be the infinitesimal part of $\mathcal{F}$.

First, $H_Z$ is the fiber product of $\mathcal{F}_{\text{ét}} \to G \leftarrow \text{Lie}(G)$, where $\text{Lie}(G) \to G$ is the exponential map, so we have a commutative diagram of exact sequences

$$\begin{array}{ccccccc}
0 & \to & H_1(G, Z) & \to & H_Z & \to & \mathcal{F}_{\text{ét}} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H_1(G, Z) & \to & \text{Lie}(G) & \to & G & \to & 0.
\end{array}$$
The weight filtration on $H_Z$ is given as follows.

\[
\begin{align*}
W_0 H_Z &= H_Z, \\
W_{-1} H_Z &= H_1(G, \mathbb{Z}), \\
W_{-2} H_Z &= H_1(L, \mathbb{Z}) = \text{Ker} \left( H_1(G, \mathbb{Z}) \to H_1(A, \mathbb{Z}) \right), \\
W_{-3} H_Z &= 0.
\end{align*}
\]

Next,

\[H_V = H_C \oplus \text{Lie}(L_a) \oplus \text{Lie}(\mathcal{F}_{\inf})\]

where $L_a$ is the additive part of $L$. The weight filtration on $H_V$ is as follows.

\[
\begin{align*}
W_0 H_V &= H_V, \\
W_{-1} H_V &= H_1(G, \mathbb{C}) \oplus \text{Lie}(L_a), \\
W_{-2} H_V &= H_1(L, \mathbb{C}) \oplus \text{Lie}(L_a), \\
W_{-3} H_V &= 0.
\end{align*}
\]

The splitting of the weight filtration on $\text{Ker}(H_V \to H_C) = \text{Lie}(L_a) \oplus \text{Lie}(\mathcal{F}_{\inf})$ is by definition this direct decomposition.

The Hodge filtration on $H_V$ is given as follows.

\[
\begin{align*}
F^{-1} H_V &= H_V, \\
F^1 H_V &= 0, \\
F^0 H_V &= \text{Ker} \left( H_V \to \text{Lie}(G) \right)
\end{align*}
\]

where $H_V \to \text{Lie}(G)$ is defined as follows. The part $H_C \to \text{Lie}(G)$ of it is the $\mathbb{C}$-linear map induced by the canonical map $H_Z \to \text{Lie}(G)$. The part $\text{Lie}(L_a) \to \text{Lie}(G)$ of it is the inclusion map. The part $\text{Lie}(\mathcal{F}_{\inf}) \to \text{Lie}(G)$ of it is the homomorphism induced by $\mathcal{F}_{\inf} \to G$. We have hence $H_V/F^0 H_V \cong \text{Lie}(G)$.

It is easy to see that this functor $\mathcal{M}_1 \to \mathcal{H}_1$ is quasi-inverse to the functor $\mathcal{H}_1 \to \mathcal{M}_1$ in No. 4.3.

4.5. The induced functor $\mathcal{H}_1,\{-1,-2\} \xrightarrow{\sim} \mathcal{M}_1,\{-1,-2\}$ is especially simple. It is given by

\[H \mapsto [0 \to H_Z \setminus H_V/F^0 H_V].\]

4.6. For those who are familiar with formal Hodge structures from [1] we explain the relation between $\mathcal{H}_1$ and the category $\text{FHS}_{\text{fr}}^1$ of torsion free formal Hodge structures of level $\leq 1$, see [1, Def. 1.1.2]. (This No. is not used in the rest of the paper.)
The categories $\mathcal{H}_1$ and $\text{FHS}_1^{\text{fr}}$ are equivalent. The functor $\mathcal{H}_1 \to \text{FHS}_1^{\text{fr}}$ is given by $(H_Z, H_V) \mapsto (\mathcal{F}, V)$, where $(\mathcal{F}, V)$ is the following object of $\text{FHS}_1^{\text{fr}}$.

\[
\mathcal{F} = \mathcal{F}_{\text{ét}} \oplus \mathcal{F}_{\text{inf}}, \\
\mathcal{F}_{\text{ét}} = H_Z, \\
\mathcal{F}_{\text{inf}} = \text{formal completion of } \ker \left( \text{gr} \mathcal{W}_0 (H_V) \to \text{gr} \mathcal{W}_0 (H_{\mathbb{C}}) \right), \\
V = W_{-1} H_V / W_{-1} F^0 H_V \\
\supseteq V^1 = W_{-2} H_V \\
\supseteq V^0 = \ker (W_{-2} H_V \to W_{-2} H_{\mathbb{C}}),
\]

$v: \mathcal{F} \to V$ is defined by \(v|_{\mathcal{F}_{\text{ét}}} = a|_{H_Z} \mod F^0 H_V\) (we have $V = H_V / F^0 H_V$), $v|_{\mathcal{F}_{\text{inf}}}$ is the map $\mathcal{F}_{\text{inf}} \subset \text{Lie}(\mathcal{F}_{\text{inf}}) \to \text{Lie}(G)$ as in No. 4.3,

\[
H_{\mathbb{C}} / F^0 H_{\mathbb{C}} \to V / V^0 \text{ is the map induced by } a.
\]

The functor $\text{FHS}_1^{\text{fr}} \to \mathcal{H}_1$ is given by $(\mathcal{F}, V) \mapsto (H_Z, H_V)$, where $(H_Z, H_V)$ is the following object of $\mathcal{H}_1$.

\[
H_Z = \mathcal{F}_{\text{ét}}, \\
H_V = H_{\mathbb{C}} \oplus \text{Lie}(\mathcal{F}_{\text{inf}}) \oplus V^0, \\
F^{-1} H_V = H_V, \\
F^0 H_V = \ker (H_V \to V), \\
F^1 H_V = 0, \\
W_0 H_V = H_V, \\
W_{-1} H_V = W_{-1} H_{\mathbb{C}} \oplus V^0, \\
W_{-2} H_V = W_{-2} H_{\mathbb{C}} \oplus V^0, \\
W_{-3} H_V = 0,
\]

where $H_V \to V$ is the map given by $(v|_{\mathcal{F}_{\text{ét}}} \otimes \mathbb{C}, \text{Lie}(v|_{\mathcal{F}_{\text{inf}}}), V^0 \hookrightarrow V)$.

These functors are quasi-inverse to each other and yield an equivalence of categories $\mathcal{H}_1 \simeq \text{FHS}_1^{\text{fr}}$. The relation between $\text{FHS}_1$ and the category $\text{EHS}_1$ of enriched Hodge structures of level $\leq 1$ from [4] is given in [2, 4.2] by explicit functors. Composition yields an explicit functor $\text{EHS}_1^{\text{fr}} \to \mathcal{H}_1$ (left to the reader). The category $\text{EHS}_1^{\text{fr}}$ of torsion free enriched Hodge structures of level $\leq 1$ is equivalent to a subcategory of $\text{FHS}_1^{\text{fr}}$ resp. $\mathcal{H}_1$, see [2, Prop. 4.2.3].
5. Generalized Albanese varieties

Let $k$ be an algebraically closed field of characteristic 0 and let $X$ be a proper smooth algebraic variety over $k$ of dimension $n$. We review generalized Albanese varieties $\text{Alb}_\mathcal{F}(X)$ defined in [10] (1). For an effective divisor $Y$ on $X$, the generalized Albanese variety $\text{Alb}(X,Y)$ of modulus $Y$ is a special case of $\text{Alb}_\mathcal{F}(X)$.

The Albanese variety $\text{Alb}(X)$ is defined by a universal mapping property for morphisms from $X$ to abelian varieties. Similarly, the generalized Albanese variety $\text{Alb}(X,Y)$ of modulus $Y$ is characterized by a universal property for morphisms from $X - Y$ into commutative algebraic groups with “modulus” $\leq Y$. See Proposition 5.1.

5.1. Let $\mathcal{Div}_X$ be the sheaf of abelian groups on $Ab/k$ defined as follows. For any commutative ring $R$ over $k$, $\mathcal{Div}_X(R)$ is the group of all Cartier divisors on $X \otimes_k R$ generated locally on $\text{Spec}(R)$ by effective Cartier divisors which are flat over $R$. Let $\text{Pic}_X$ be the Picard functor, and let $\text{Pic}^0_X \subset \text{Pic}_X$ be the Picard variety of $X$. We have the class map $\mathcal{Div}_X \to \text{Pic}_X$. Let $\mathcal{Div}^0_X \subset \mathcal{Div}_X$ be the inverse image of $\text{Pic}^0_X$.

5.2. Let $\Lambda$ be the set of all subgroup sheaves $\mathcal{F}$ of $\mathcal{Div}^0_X$ such that $\mathcal{F} \cong \mathbb{Z}^t \oplus (\hat{\mathbb{G}}_a)^s$ for some $t$ and $s$. For $\mathcal{F} \in \Lambda$, we have an object $[\mathcal{F} \to \text{Pic}^0_X]$ of $\mathcal{M}_{1,\{0,-1\}}$. The generalized Albanese variety $\text{Alb}_\mathcal{F}(X)$ is defined in [10] to be the Cartier dual of $[\mathcal{F} \to \text{Pic}^0_X]$. It is an object of $\mathcal{M}_{1,\{-1,-2\}}$ and hence is a commutative connected algebraic group over $k$.

If $\mathcal{F}, \mathcal{F}' \in \Lambda$ and $\mathcal{F} \subset \mathcal{F}'$, we have a canonical surjective homomorphism $\text{Alb}_{\mathcal{F}'}(X) \to \text{Alb}_\mathcal{F}(X)$. In the case $\mathcal{F} = 0$, $\text{Alb}_{\mathcal{F}}(X) = \text{Alb}(X)$.

5.3. Let $Y$ be an effective divisor of $X$. Then the generalized Albanese variety with modulus $Y$ is defined as $\text{Alb}_\mathcal{F}(X)$ where $\mathcal{F} = \mathcal{F}_{X,Y} \in \Lambda$ is defined as follows. The étale part $\mathcal{F}_{\text{ét}}$ of $\mathcal{F}$ is the subgroup of $\mathcal{Div}^0_X(k)$ consisting of all divisors whose support is contained in the support of $Y$. The infinitesimal part $\mathcal{F}_{\text{inf}}$ of $\mathcal{F}$ is as follows. Let $I$ be the ideal of $\mathcal{O}_X$ (though the notation $\mathcal{O}_X$ is often used in this paper for the sheaf of analytic functions, $\mathcal{O}_X$ here stands for the usual algebraic object on the Zariski site) defining $Y$, let $I_1$ be the ideal of $\mathcal{O}_X$ which defines the reduced part of $Y$, and let $J = II_1^{-1} \subset \mathcal{O}_X$. Then $\mathcal{F}_{\text{inf}}$ is the formal completion $\hat{\mathbb{G}}_a \otimes_k H^0(X,J^{-1}/\mathcal{O}_X)$ of the finite dimensional $k$-vector space $H^0(X,J^{-1}/\mathcal{O}_X)$.

(1) In [10], $X$ was assumed to be projective. This assumption was used only for singular $X$, which is not our concern here. The construction of the $\text{Alb}_{\mathcal{F}}(X)$ is valid in the same way for proper $X$.  

ANNALES DE L’INSTITUT FOURIER
which is embedded in $\text{Div}_X^0$ by the exponential map
\[
\exp : \hat{\mathbb{G}}_a \otimes_k H^0(X, J^{-1}/\mathcal{O}_X) \rightarrow \text{Div}_X^0.
\]
If $Y'$ is an effective divisor on $X$ such that $Y' \geq Y$, then $\mathcal{F}_{X,Y'} \supset \mathcal{F}_{X,Y}$ and hence we have a canonical surjective homomorphism $\text{Alb}(X,Y') \rightarrow \text{Alb}(X,Y)$. In the case $Y = 0$, $\text{Alb}(X,Y) = \text{Alb}(X)$.

In the case when $X$ is a curve, $\text{Alb}(X,Y)$ coincides with the generalized Jacobian variety $J(X,Y)$ of $X$ with modulus $Y$ as is explained in [10, Exm. 2.34].

5.4. As in [10], for $\mathcal{F} \in \Lambda$ we have a rational map
\[
\alpha_{\mathcal{F}} : X \rightarrow \text{Alb}_{\mathcal{F}}(X)
\]
which is canonically defined up to translation by a $k$-rational point of $\text{Alb}_{\mathcal{F}}(X)$. If $\mathcal{F}' \in \Lambda$ and $\mathcal{F} \subset \mathcal{F}'$, then $\alpha_{\mathcal{F}}$ and $\alpha_{\mathcal{F}'}$ are compatible via the canonical surjection $\text{Alb}_{\mathcal{F}'}(X) \rightarrow \text{Alb}_{\mathcal{F}}(X)$.

For an effective divisor $Y$ on $X$, we denote the rational map $\alpha_{\mathcal{F}_{X,Y}}$ simply by $\alpha_{X,Y}$. In Proposition 5.1 (2) below, we give a universal property of $\alpha_{X,Y} : X \rightarrow \text{Alb}_{\mathcal{F}_{X,Y}}(X)$ concerning rational maps from $X$ to commutative algebraic groups. This property follows from a general universal property of $\alpha_{\mathcal{F}} : X \rightarrow \text{Alb}_{\mathcal{F}}(X)$ obtained in [10], as is shown in No. 5.6 below.

5.5. Let $G$ be a commutative connected algebraic group over $k$ and let $\varphi : X \rightarrow G$ be a rational map. We define an effective divisor $\text{mod}(\varphi)$ on $X$ which we call the modulus of $\varphi$.

We treat $X$ as a scheme. This divisor $\text{mod}(\varphi)$ is written in the form
\[
\sum_u \text{mod}_u(\varphi) v,
\]
where $v$ ranges over all points of $X$ of codimension one and $\text{mod}_u(\varphi)$ is a non-negative integer defined as follows.

Let $0 \rightarrow L \rightarrow G \rightarrow A \rightarrow 0$ be the canonical decomposition of $G$ and take an isomorphism
\[
\text{(1)} \quad L_a \cong (\mathbb{G}_a)^s
\]
where $L_a$ is the additive part of $L$.

Let $K$ be the function field of $X$, and regard $\varphi$ as an element of $G(K)$. Since the local ring $\mathcal{O}_{X,v}$ of $X$ at $v$ is a discrete valuation ring and since $A$ is proper, we have $A(\mathcal{O}_{X,v}) = A(K)$. By the commutative diagram with exact rows
\[
\begin{array}{cccccc}
0 & \rightarrow & L(\mathcal{O}_{X,v}) & \rightarrow & G(\mathcal{O}_{X,v}) & \rightarrow & A(\mathcal{O}_{X,v}) & \rightarrow & 0 \\
& & \cap & \cap & || & & \\
0 & \rightarrow & L(K) & \rightarrow & G(K) & \rightarrow & A(K) & \rightarrow & 0,
\end{array}
\]
we have $G(K) = L(K)G(\mathcal{O}_{X,v})$. Write $\varphi \in G(K)$ as

\[\text{TOME 62 (2012), FASCICULE 2}\]
(2) \( \varphi = l g \) with \( l \in L(K) \) and \( g \in G(O_{X,v}) \).

Let \( (l_j)_{1 \leq j \leq s} \) be the image of \( l \) in \( (\mathbb{G}_a)^s(K) \).

If \( \varphi \) belongs to \( G(O_{X,v}) \), we define \( \text{mod}_v(\varphi) = 0 \). Assume that \( \varphi \) does not belong to \( G(O_{X,v}) \). Then we define

\[
\text{mod}_v(\varphi) = 1 + \max\{\text{ord}_v(l_j) \mid 1 \leq j \leq s\} \cup \{0\}.
\]

This integer \( \text{mod}_v(\varphi) \) is independent of the choice of the isomorphism (1) and of the choice of the presentation (2) of \( \varphi \).

For example, if \( G = \mathbb{G}_m \), \( \text{mod}_v(\varphi) \) is 0 if the element \( \varphi \) of \( G(K) = K^\times \) belongs to \( \mathcal{O}_{X,v}^\times \), and is 1 otherwise. If \( G = \mathbb{G}_a \), \( \text{mod}_v(\varphi) \) is 0 if the element \( \varphi \) of \( G(K) = K \) belongs to \( \mathcal{O}_{X,v} \), and is \( m + 1 \) if \( \varphi \) has a pole of order \( m \geq 1 \) at \( v \).

**Proposition 5.1.** — Let \( G \) be a commutative connected algebraic group over \( k \) and let \( \varphi : X \to G \) be a rational map.

1. For a dense open set \( U \) of \( X \), \( \varphi \) induces a morphism \( U \to G \) (not only a rational map) if and only if the support of \( \text{mod}(\varphi) \) does not meet \( U \).

2. Let \( Y \) be an effective divisor on \( X \). Then the following two conditions (i) and (ii) are equivalent.

   (i) There is a homomorphism \( h : \text{Alb}(X,Y) \to G \) such that \( \varphi \) coincides with \( h \circ \alpha_{X,Y} \) modulo a translation by \( G(k) \).

   (ii) \( \text{mod}(\varphi) \leq Y \).

Furthermore, if these equivalent conditions are satisfied, such homomorphism \( h \) is unique.

It is easy to prove (1). The proof of (2) is given in No. 5.7 below after we review results on \( \text{Alb}_F(X) \) from [10].

5.6. We review a general universal property of \( \text{Alb}_F(X) \) proved in [10] concerning rational maps from \( X \) into commutative algebraic groups.

Let \( \varphi : X \to G \) be a rational map into a commutative connected algebraic group \( G \), and let \( L \) be the canonical connected affine subgroup such that the quotient \( G/L \) is an abelian variety. One observes that \( \varphi \) induces a natural transformation \( \tau_\varphi : L^\vee \to \text{Div}^0_X \) (see [10, Section 2.2]), where \( L^\vee = \text{Hom}_{Ab/k}(L, \mathbb{G}_m) \) is the Cartier dual of \( L \). It is shown in [10, Section 2.3] that if \( F \in \Lambda \), there is a rational map \( \alpha_F : X \to \text{Alb}_F(X) \) for which the corresponding homomorphism \( \tau_{\alpha_F} : F \to \text{Div}^0_X \) coincides with the inclusion map, and such rational map \( \alpha_F \) is unique up to translation by a \( k \)-rational point of \( \text{Alb}_F(X) \). For a rational map \( \varphi : X \to G \) into a commutative connected algebraic group \( G \) and for \( F \in \Lambda \), there is a homomorphism
$h : \text{Alb}_F(X) \to G$ such that $f$ coincides with $h \circ \alpha_F$ up to translation by an element of $G(k)$ if and only if the image of the homomorphism $\tau_\varphi : L^Y \to \text{Div}_X^0$ is contained in $F$. Furthermore, if such $h$ exists, it is unique.

Moreover, any rational map $\varphi : X \to G$ into a commutative connected algebraic group $G$ coincides with $h \circ \alpha_F$ up to translation by an element of $G(k)$ if and only if the image of the homomorphism $\tau_\varphi : L^Y \to \text{Div}_X^0$ is contained in $F$. Furthermore, if such $h$ exists, it is unique.

Moreover, any rational map $\varphi : X \to G$ into a commutative connected algebraic group $G$ coincides with $h \circ \alpha_F$ up to translation by an element of $G(k)$ if and only if the image of the homomorphism $\tau_\varphi : L^Y \to \text{Div}_X^0$ is contained in $F$. Furthermore, if such $h$ exists, it is unique.

5.7. We prove Proposition 5.1. By No. 5.6 we find that condition (i) of Proposition 5.1 (2) is equivalent to (i') The image of $\tau_\varphi$ is contained in $F_{X,Y}$.

Write

$$Y = \sum_v e_v v$$

where $v$ ranges over all points of $X$ of codimension one and $e_v \in \mathbb{N}$. Condition (ii) of Proposition 5.1 (2) is expressed as (ii') $\mod_v (\varphi) \leq e_v$ for all points $v$ of codimension one in $X$.

By construction of the transformation $\tau_\varphi$ in [10, Section 2.2], we have the following (b) and (c).

(b) The étale part of $\tau_\varphi$

$$\tau_{\varphi, \text{ét}} : \mathbb{Z}^t \longrightarrow \text{Div}_X^0(k)$$

sends the $j$-th base of $\mathbb{Z}^t$ ($1 \leq j \leq t$) to the divisor $\sum_v \text{ord}_v(l'_{v,j}) v$.

(c) The infinitesimal part of $\tau_\varphi$

$$\tau_{\varphi, \text{inf}} : (\hat{\mathbb{G}}_a)^s \longrightarrow \text{Div}_X^0$$

has the form

$$(a_j)_{1 \leq j \leq s} \longmapsto \exp \left( \sum_{j=1}^s a_j f_j \right)$$

for some $f_j \in \Gamma(X, K/O_X) = \text{Lie}(\text{Div}_X^0)$ ($1 \leq j \leq s$) such that for any point $v$ of $X$ of codimension one, the stalk of $f_j$ at $v$ coincides with $l_{v,j} \mod O_{X,v}$.
Condition (i') is equivalent to the condition that the following (i'$_{\text{ét}}$) and (i'$_{\text{inf}}$) are satisfied.

(i'$_{\text{ét}}$) The image of $\tau_\varphi,\text{ét}$ is contained in the étale part of $\mathcal{F}_{X,Y}$.

(i'$_{\text{inf}}$) The image of $\tau_\varphi,\text{inf}$ is contained in the infinitesimal part of $\mathcal{F}_{X,Y}$.

By the above (b), (i'$_{\text{ét}}$) is equivalent to the condition that the following (i'$_{\text{ét},v}$) is satisfied for any point $v$ of $X$ of codimension one.

(i'$_{\text{ét},v}$) If $e_v = 0$, then $l_{v,j} \in \mathcal{O}_{X,v}^\times$ for $1 \leq j \leq t$.

On the other hand, by the above (c), (i'$_{\text{inf}}$) is equivalent to

\[ f_j \in \Gamma(X, J^{-1}/\mathcal{O}_X) \text{ for } 1 \leq j \leq s, \]

and hence equivalent to the condition that the following (i'$_{\text{inf},v}$) is satisfied for any point $v$ of $X$ of codimension one.

(i'$_{\text{inf},v}$) If $e_v = 0$, then $l_{v,j} \in \mathcal{O}_{X,v}$ for $1 \leq j \leq s$.

If $e_v \geq 1$, then $\text{ord}_v(l_{v,j}) \geq 1 - e_v$ for $1 \leq j \leq s$.

By (a) above, for each $v$, (i'$_{\text{ét},v}$) and (i'$_{\text{inf},v}$) are satisfied if and only if $\text{mod}_v(\varphi) \leq e_v$. □

**Corollary 5.2.** — For any $\mathcal{F} \in \Lambda$, there exists an effective divisor $Y$ such that $\mathcal{F} \subset \mathcal{F}_{X,Y}$.

**Proof.** — Let $Y = \text{mod}(\alpha_\mathcal{F})$ be the modulus of the rational map $\alpha_\mathcal{F}: X \to \text{Alb}_\mathcal{F}(X)$ associated with $\mathcal{F} \in \Lambda$. Then $\mathcal{F} = \text{Image}(\tau_{\alpha_\mathcal{F}}) \subset \mathcal{F}_{X,Y}$. □

### 6. Proof of Theorem 1.1

We prove Theorem 1.1. Let $X$ be a proper smooth algebraic variety over $\mathbb{C}$ of dimension $n$, and let $Y$ be an effective divisor on $X$. Let $I$ be the ideal of $\mathcal{O}_X$ which defines $Y$, let $I_1$ be the ideal of $\mathcal{O}_X$ which defines the reduced part of $Y$, and let $J = II_1^{-1} \subset \mathcal{O}_X$.

**6.1.** Let $H^1(X, Y_+)(1)$ be the object of $\mathcal{H}_{1,\{0,-1\}}$ corresponding to the object $[\mathcal{F}_{X,Y} \to \text{Pic}^0(X)]$ of $\mathcal{M}_{1,\{0,-1\}}$ in the equivalence of categories of Theorem 4.1. Let $H^{2n-1}(X, Y_-)(n)$ be the object of $\mathcal{H}_{1,\{-1,-2\}}$ corresponding to the object $\text{Alb}(X, Y)$ of $\mathcal{M}_{1,\{-1,-2\}}$.

Since the equivalence of categories in Theorem 4.1 is compatible with dualities, we have

\[
H^{2n-1}(X, Y_-)(n) \cong \text{Hom}(H^1(X, Y_+)(1), \mathbb{Z})(1).
\]

We prove Theorem 1.1 in the following way. First in No. 6.3, we give an explicit description of $H^1(X, Y_+)(1)$. From this, by (6.1), we can obtain an
explicit description of $H^{2n-1}(X, Y_\ast)(n)$ as in No. 6.4. Since $\text{Alb}(X, Y)$ corresponds to $H^{2n-1}(X, Y_\ast)(n)$ in the equivalence of categories $\mathcal{H}_{1,\{-1,-2\}} \simeq \mathcal{M}_{1,\{-1,-2\}}$, we can obtain from No. 6.4 the explicit descriptions of $\text{Alb}(X, Y)$ as stated in Theorem 1.1.

We define objects $H^1(X, Y_+)$ and $H^{2n-1}(X, Y_-)$ of $\mathcal{H}$ as follows: $H^1(X, Y_+)$ is the Tate twist $(H^1(X, Y_+)(1))(-1)$ of $H^1(X, Y_+)(1)$, and $H^{2n-1}(X, Y_-)$ is the Tate twist $(H^{2n-1}(X, Y_-)(n))(-n)$ of $H^{2n-1}(X, Y_+)(n)$. These are natural generalizations of the objects of $\mathcal{H}$ for the curve case considered in Example 2.1.

6.2. We define canonical $\mathbb{C}$-linear maps

\begin{equation}
H^1(X - Y, \mathbb{C}) \rightarrow H^1(X, \mathcal{O}_X),
\end{equation}

\begin{equation}
H^{n-1}(X, \Omega^n_X) \rightarrow H^{2n-1}_c(X - Y, \mathbb{C})
\end{equation}

First assume that $Y$ is with normal crossings. Then by [5], we have canonical isomorphisms

\[
H^m(X - Y, \mathbb{C}) \cong H^m(X, \Omega^\bullet_X(\log(Y))),
\]

\[
H^m_c(X - Y, \mathbb{C}) \cong H^m(X, \Omega^\bullet_X(-\log(Y)))
\]

for $m \in \mathbb{Z}$, where $\Omega^p_X(\log(Y))$ is the sheaf of differential $p$-forms with log poles along $Y$, and $\Omega^p_X(-\log(Y)) = I_1 \Omega^p_X(\log(Y))$. Since $\mathcal{O}_X = \Omega^1_X(\log(Y))$ and $\Omega^p_X = \Omega^p_X(-\log(Y))$, we have canonical maps of complexes $\Omega^\bullet_X(\log(Y)) \rightarrow \mathcal{O}_X$ and $\Omega^m_X[-n] \rightarrow \Omega^\bullet_X(-\log(Y))$. These maps induce the maps (6.2) and (6.3) in the case $Y$ is with normal crossings, respectively.

In general, take a birational morphism $X' \rightarrow X$ of proper smooth algebraic varieties over $\mathbb{C}$ such that the inverse image $Y'$ of $Y$ on $X'$ is with normal crossings. Then we have maps

\[
H^{n-1}(X, \Omega^n_X) \rightarrow H^{n-1}(X', \Omega^n_{X'}) \rightarrow H^{2n-1}_c(X' - Y', \mathbb{C}) = H^{2n-1}_c(X - Y, \mathbb{C})
\]

where the second arrow is the map (6.3) for $X'$, and the composition $H^{n-1}(X, \Omega^n_X) \rightarrow H^{2n-1}_c(X - Y, \mathbb{C})$ is independent of the choice of $X' \rightarrow X$. The $\mathbb{C}$-linear dual of (6.3) with respect to the Poincaré duality and Serre duality gives the map (6.2). The map (6.2) is also obtained as the composition

\[
H^1(X - Y, \mathbb{C}) = H^1(X' - Y', \mathbb{C}) \rightarrow H^1(X', \mathcal{O}_{X'}) \leftarrow H^1(X, \mathcal{O}_X).
\]

6.3. Let $H = H^1(X, Y_+)(1)$, the object of $\mathcal{H}_{1,\{0,-1\}}$ corresponding to the object $[\mathcal{F}_{X, Y} \rightarrow \text{Pic}^0(X)]$ of $\mathcal{M}_{1,\{0,-1\}}$. We describe $H$. By [3, Thm. 4.7] which treats the case when $Y$ has no multiplicity, we can identify $H_\mathbb{Z}$ with $H^1(X - Y, \mathbb{Z}(1))$ and identify the map $H_\mathbb{C} \rightarrow \text{Lie}(\text{Pic}^0(X)) = H^1(X, \mathcal{O}_X)$.
with the map (6.2) in No. 6.2. We have \( H_V = H_C \oplus H^0(X, J^{-1}/\mathcal{O}_X) \), the maps \( a : H_C \to H_V \) and \( b : H_V \to H_C \) are the evident ones, the weight filtration is given by \( W_0H = H, W_{-2}H = 0, \)
\[
W_{-1}H_Q = H^1(X, \mathbb{Q}(1)),
\]
\[
W_{-1}H_V = H^1(X, \mathbb{C}),
\]
and the Hodge filtration is given by \( F^{-1}H_V = H_V, F^1H_V = 0, \) and
\[
F^0H_V = \text{Ker} \left( H^1(X - Y, \mathbb{C}) \oplus H^0(J^{-1}/\mathcal{O}_X) \to H^1(X, \mathcal{O}_X) \right)
\]
where the map \( H^0(J^{-1}/\mathcal{O}_X) \to H^1(X, \mathcal{O}_X) \) is the connecting map of the exact sequence \( 0 \to \mathcal{O}_X \to J^{-1} \to J^{-1}/\mathcal{O}_X \to 0. \)

6.4. Let \( H = H^{2n-1}(X, Y_-)(n) \), the object of \( \mathcal{H}_{1,\{−1,−2\}} \) corresponding to the object \( \text{Alb}(X, Y) \) of \( \mathcal{M}_{1,\{−1,−2\}} \). By (6.1) in No. 6.1, we obtain the following description of \( H \) from the description of \( H^1(X, Y_+)(1) \) in No. 6.3.
\[
H_Z = H^{2n-1}_c(X - Y, \mathbb{Z})/(\text{torsion}),
\]
\[
H_V = H_C \oplus H^{n-1}(X, \Omega^n_X/J\Omega^n_X),
\]
the maps \( a : H_C \to H_V \) and \( b : H_V \to H_C \) are the evident ones, the weight filtration is given by \( W_{-1}H = H, W_{-3}H = 0, \)
\[
W_{-2}H_Q = \text{Ker} \left( H_Q \to H^{2n-1}(X, \mathbb{Q}(n)) \right),
\]
\[
W_{-2}H_V = \text{Ker} \left( H_V \to H^{2n-1}(X, \mathbb{C}) \right),
\]
and the Hodge filtration is given by \( F^{-1}H_V = H_V, F^1H_V = 0, \) and
\[
F^0H_V = \text{Image} \left( H^{n-1}(X, \Omega^n_X) \to H^{2n-1}_c(X - Y, \mathbb{C}) \oplus H^{n-1}(X, \Omega^n_X/J\Omega^n_X) \right)
\]
where the map \( H^{n-1}(X, \Omega^n_X) \to H^{2n-1}_c(X - Y, \mathbb{C}) \) is (6.3) in No. 6.2 and the map \( H^{n-1}(X, \Omega^n_X) \to H^{n-1}(X, \Omega^n_X/J\Omega^n_X) \) is the evident one.

6.5. We prove Theorem 1.1 (2). Let \( H = H^{2n-1}(X, Y_-)(n) \). Then
\[
\text{Alb}(X, Y) = H_Z \backslash H_V / F^0H_V
\]
by No. 4.5. Hence the description of \( H^{2n-1}(X, Y_-)(n) \) in No. 6.4 proves Theorem 1.1 (2).

6.6. As a preparation for the proof of Theorem 1.1 (1), we review a kind of Serre-duality obtained in the appendix by Deligne of the book [8].

Let \( S \) be a proper scheme over a field \( k \), let \( C \) be a closed subscheme of \( S \), let \( U = S - C \), and let \( I_C \) be the ideal of \( \mathcal{O}_S \) which defines \( C \). Assume \( U \) is
smooth over \(k\) and purely of dimension \(n\). Let \(\mathcal{F}\) be a coherent \(O_S\)-module. Then for any \(p \in \mathbb{Z}\), we have a canonical isomorphism
\[
H^p(U, R\text{Hom}_{O_U}(\mathcal{F}|_U, \Omega^p_U)) \cong \lim_{\to m} \text{Hom}_k \left( H^{n-p}(X, I^n_{O_U} \mathcal{F}), k \right).
\]

In the case when \(C\) is empty and \(\mathcal{F}\) is locally free, this is the usual Serre duality
\[
H^p(X, \text{Hom}_{O_X}(\mathcal{F}, \Omega^p_X)) \cong \text{Hom}_k \left( H^{n-p}(X, \mathcal{F}), k \right).
\]

6.7. We start the proof of Theorem 1.1 (1).
Let \(C_Y\) be the subcomplex of \(\Omega_X^\bullet\) defined as
\[
C_Y^p = \ker(\Omega^p_X \to \Omega^p_Y) \quad \text{for} \quad 0 \leq p \leq n-1, \quad C^n_Y = J\Omega_X^n.
\]

**Proposition 6.1.** — For \(p = 2n, 2n-1\), the maps \(H^p(X - Y, \mathbb{C}) \to H^p(X, C_Y)\) induced by the homomorphism \(j^! C \to C_Y\) are isomorphisms.

6.8. We prove Proposition 6.1 in the case \(Y = Y_1\). We have an exact sequence of complexes
\[
0 \to C_{Y_1} \to \Omega^\bullet_X \to \Omega^{\leq n-1}_{Y_1} \to 0.
\]
Since the support of \(\Omega^{\leq n-1}_{Y_1}\) is of dimension \(\leq n-1\) and since \(\Omega^{\leq n-1}_{Y_1}\) has only terms of degree \(\leq n-1\), we have \(H^p(X, \Omega^{\leq n-1}_{Y_1}) = 0\) for \(p \geq 2n - 1\). Hence
\[
H^{2n}(X, C_{Y_1}) \cong H^{2n}(X, \Omega^\bullet_X) \cong H^{2n}(X, \mathbb{C}) \cong H^{2n}_c(X - Y, \mathbb{C}).
\]
The above exact sequence of complexes induces the lower row of the commutative diagram with exact rows
\[
\begin{array}{cccccc}
H^{2n-2}(X, \mathbb{C}) & \to & H^{2n-2}(Y_1, \mathbb{C}) & \to & H^{2n-1}_c(X \setminus Y_1, \mathbb{C}) & \to & H^{2n-1}(X, \mathbb{C}) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
H^{2n-2}(X, \Omega^\bullet_X) & \to & H^{2n-2}(Y_1, \Omega^{\leq n-1}_{Y_1}) & \to & H^{2n-1}(X, C_{Y_1}) & \to & H^{2n-1}(X, \Omega^\bullet_X) & \to & 0.
\end{array}
\]
The vertical arrows except possibly the map \(H^{2n-1}_c(X - Y_1, \mathbb{C}) \to H^{2n-1}(X, C_{Y_1})\) are isomorphisms. Hence the last map is also an isomorphism.

**Lemma 6.2.** — Let \(Y'\) and \(Y''\) be effective divisors on \(X\) whose supports coincide with \(Y_1\) and assume \(Y' \supseteq Y''\). Then the canonical map \(H^{2n-1}(X, C_{Y'}) \to H^{2n-1}(X, C_{Y''})\) is surjective and the canonical map \(H^{2n}(X, C_{Y'}) \to H^{2n}(X, C_{Y''})\) is an isomorphism.

**Proof.** — Let \(N = C_{Y''}/C_{Y'}\). We have
\[
N^p = \ker(\Omega^p_{Y''}, \to \Omega^p_{Y''}) \quad \text{for} \quad 0 \leq p \leq n-1, \quad N^n = J''\Omega^n_X/J'\Omega^n_X.
\]
Here, $J' = I' I_1^{-1}$, $J'' = I'' I_1^{-1}$ with $I'$ (resp. $I''$) the ideal of $\mathcal{O}_X$ which defines $Y'$ (resp. $Y''$). Since the support of $N$ is of dimension $\leq n - 1$ and $N$ has only terms of degree $\leq n$, we have $H^{2n}(X, N) = 0$. Hence it is sufficient to prove $H^{2n-1}(X, N) = 0$.

Let $\Sigma$ be the set of all singular points of $Y_1$. Then $\Sigma$ is of dimension $\leq n - 2$. Let $\Omega^n_Y(\log(Y_1))$ be the de Rham complex on $X - \Sigma$ with log poles along $Y_1 - \Sigma$. Then, as is easily seen, the restriction of $C_Y$ to $X - \Sigma$ coincides with $\Omega^n_X(\log(Y_1))$. Let $I_\Sigma$ be the ideal of $\mathcal{O}_X$ defining $\Sigma$ (here $\Sigma$ is endowed with the reduced structure). For $k \geq 0$, let $N_k$ be the subcomplex of $N$ defined by $N_k^p = I_\Sigma^{\max(p, 0)}N^p$. In particular, $N_0 = N$. Then if $k \geq j \geq 0$, since the support of $N_j/N_k$ is of dimension $\leq n - 2$ and $N_j/N_k$ has only terms of degree $\leq n$, we have $H^{2n-1}(X, N_j/N_k) = 0$. Hence $H^{2n-1}(X, N_k) \to H^{2n-1}(X, N_j)$ is surjective. Applying No. 6.6 for $S = X$ and $C = \Sigma$ yields that $\lim_k H^{2n-1}(X, N_k)$ is the dual vector space of $H^0(X - \Sigma, ([J')^{-1}]^{-1} d (J')^{-1} \Omega_X(\log Y_1)/([J'')^{-1} \Omega_X(\log Y_1))]$. Since $d : (J')^{-1}/(J'')^{-1} \to (J')^{-1} \Omega_X(\log Y_1)/(J'')^{-1} \Omega_X(\log Y_1)$ is injective, the last cohomology group is 0. Hence $H^{2n-1}(X, N_k) = 0$ for all $k \geq 0$. In particular, $H^{2n-1}(X, N) = 0$.  

6.9. We prove Proposition 6.1 in general. By Lemma 6.2, the map $\lim_{Y'} H^{2n-1}(X, C_{Y'}) \to H^{2n-1}(X, C_Y)$ is surjective, where $Y'$ ranges over all effective divisors on $X$ whose supports coincide with $Y_1$. By No. 6.6, which we apply by taking $S = X$ and $C = Y$, we have that $\lim_{Y'} H^{2n-1}(X, C_{Y'})$ is the dual vector space of $H^1((X - Y)_\text{zar}, \Omega^n_{X - Y, \text{alg}})$, where “zar” means Zariski topology and “alg” means the algebraic version. But $H^1((X - Y)_\text{zar}, \Omega^n_{X - Y, \text{alg}}) \simeq H^1(X - Y, \mathbb{C})$ by Grothendieck’s Theorem [7, Thm. 1]. This proves $\lim_{Y'} H^{2n-1}(X, C_{Y'}) \simeq H^{2n-1}_c(X - Y, \mathbb{C})$. Hence the map $H^{2n-1}_c(X - Y, \mathbb{C}) \to H^{2n-1}(X, C_Y)$ is surjective. Since the composition $H^{2n-1}_c(X - Y, \mathbb{C}) \to H^{2n-1}(X, C_Y) \to H^{2n-1}(X, C_{Y'}) \simeq H^{2n-1}_c(X - Y, \mathbb{C})$ is the identity map, the map $H^{2n-1}_c(X - Y, \mathbb{C}) \to H^{2n-1}(X, C_Y)$ is an isomorphism.

6.10. We prove (1) of Theorem 1.1. Let $S_Y = \text{Ker}(\Omega^n_X \to \Omega^n_Y)$. Then $C_Y \subset S_Y \subset C_{Y_1}$. We have an exact sequence of complexes

$$0 \to C_Y \to S_Y \to \Omega^n_X / J \Omega^n_X [-n] \to 0.$$
Hence we have an exact sequence
\[ H^{2n-1}(X, C_Y) \to H^{2n-1}(X, S_Y) \to H^{n-1}(X, \Omega^n_X/J\Omega^n_X) \to H^{2n}(X, C_Y) \]
\[ \to H^{2n}(X, S_Y). \]

Note that for \( p = 2n, 2n - 1 \), the compositions
\[ H^p(X, C_Y) \longrightarrow H^p(X, S_Y) \longrightarrow H^p(X, C_Y_1) \]
are isomorphisms by Proposition 6.1. Hence by Proposition 6.1, we have an isomorphism
\[ H^{2n-1}(X, S_Y) \cong H^{2n-1}_c(X - Y, \mathbb{C}) \oplus H^{n-1}(X, \Omega^n_X/J\Omega^n_X) \]
which is compatible with the maps from \( H^{n-1}(X, \Omega^n_X) \). Hence (1) of Theorem 1.1 follows from (2) of Theorem 1.1.

**BIBLIOGRAPHY**


Manuscrit reçu le 26 avril 2011,
accepté le 8 février 2011.

Kazuya KATO
University of Chicago
Department of Mathematics
Chicago, IL 60637 (USA)
kkato@math.uchicago.edu

Henrik RUSSELL
Universität Duisburg-Essen
FB6 Mathematik, Campus Essen
45117 Essen (Germany)
henrik.russell@uni-due.de