THE DEHN FUNCTIONS OF Out(F_n) AND Aut(F_n)

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ABSTRACT. — For n at least 3, the Dehn functions of Out(F_n) and Aut(F_n) are exponential. Hatcher and Vogtmann proved that they are at most exponential, and the complementary lower bound in the case n = 3 was established by Bridson and Vogtmann. Handel and Mosher completed the proof by reducing the lower bound for n bigger than 3 to the case n = 3. In this note we give a shorter, more direct proof of this last reduction.

Résumé. — Pour n au moins 3, les fonctions de Dehn de Out(F_n) et Aut(F_n) sont exponentielles. Hatcher et Vogtmann ont montré qu’elles étaient au plus exponentielles, et la borne inférieure a été établie par Bridson et Vogtmann dans le cas n = 3. Handel et Mosher ont complété la démonstration en ramenant la preuve de la borne inférieure pour n au moins 4 au cas n = 3. Dans cet article, nous donnons un argument plus direct permettant de passer du cas n = 3 au cas général.

Dehn functions provide upper bounds on the complexity of the word problem in finitely presented groups. They are examples of filling functions: if a group G acts properly and cocompactly on a simplicial complex X, then the Dehn function of G is asymptotically equivalent to the function that provides the optimal upper bound on the area of least-area discs in X, where the bound is expressed as a function of the length of the boundary of the disc. This article is concerned with the Dehn functions of automorphism groups of finitely-generated free groups.

Much of the contemporary study of Out(F_n) and Aut(F_n) is based on the deep analogy between these groups, mapping class groups, and lattices in semisimple Lie groups, particularly SL(n,Z). The Dehn functions of mapping class groups are quadratic [9], as is the Dehn function of SL(n,Z) if n ≥ 5 (see [10]). In contrast, Epstein et al. [6] proved that the Dehn function of SL(3,Z) is exponential. Building on their result, we proved

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in [3] that Aut($F_3$) and Out($F_3$) also have exponential Dehn functions. Hatcher and Vogtmann [8] established an exponential upper bound on the Dehn function of Aut($F_n$) and Out($F_n$) for all $n \geq 3$. The comparison with SL($n, \mathbb{Z}$) might lead one to suspect that this last result is not optimal for large $n$, but recent work of Handel and Mosher [7] shows that in fact it is: they establish an exponential lower bound by using their general results on quasi-retractions to reduce to the case $n = 3$.

**Theorem.** — For $n \geq 3$, the Dehn functions of Aut($F_n$) and Out($F_n$) are exponential.

This theorem answers Questions 35 and 37 of [4].
We learned the contents of [7] from Lee Mosher at Luminy in June 2010 and realized that one can also reduce the Theorem to the case $n = 3$ using a simple observation about natural maps between different-rank Outer spaces and Auter spaces (Lemma 3). The purpose of this note is record this observation and the resulting proof of the Theorem.

1. Definitions

Let $A$ be a 1-connected simplicial complex. We consider simplicial loops $\ell: S \to A^{(1)}$, where $S$ is a simplicial subdivision of the circle. A simplicial filling of $\ell$ is a simplicial map $L: D \to A^{(2)}$, where $D$ is a triangulation of the 2-disc and $L|_{\partial D} = \ell$. Such fillings always exist, by simplicial approximation. The filling area of $\ell$, denoted Area$_A(\ell)$, is the least number of triangles in the domain of any simplicial filling of $\ell$. The Dehn function$^{(1)}$ of $A$ is the least function $\delta_A: \mathbb{N} \to \mathbb{N}$ such that Area$_A(\ell) \leq \delta_A(n)$ for all loops of length $\leq n$ in $A^{(1)}$. The Dehn function of a finitely presented group $G$ is the Dehn function of any 1-connected 2-complex on which $G$ acts simplicially with finite stabilizers and compact quotient. This is well-defined up to the following equivalence relation: functions $f, g: \mathbb{N} \to \mathbb{N}$ are equivalent if $f \preceq g$ and $g \preceq f$, where $f \preceq g$ means that there is a constant $a > 1$ such that $f(n) \leq a g(an + a) + an + a$. The Dehn function can be interpreted as a measure of the complexity of the word problem for $G$ — see [2].

**Lemma 1.** — If $A$ and $B$ are 1-connected simplicial complexes, $F: A \to B$ is a simplicial map, and $\ell$ is a loop in the 1-skeleton of $A$, then Area$_A(\ell) \geq$ Area$_B(F \circ \ell)$.

$^{(1)}$The standard definition of area and Dehn function are phrased in terms of singular discs, but this version is $\simeq$ equivalent.
Proof. — If $L : D \to A$ is a simplicial filling of $\ell$, then $F \circ L$ is a simplicial filling of $F \circ \ell$, with the same number of triangles in the domain $D$. □

Corollary. — Let $A, B$ and $C$ be 1-connected simplicial complexes with simplicial maps $A \to B \to C$. Let $\ell_n$ be a sequence of simplicial loops in $A$ whose length is bounded above by a linear function of $n$, let $\overline{\ell}_n$ be the image loops in $C$ and let $\alpha(n) = \text{Area}_C(\overline{\ell}_n)$. Then the Dehn function of $B$ satisfies $\delta_B(n) \geq \alpha(n)$.

Proof. — This follows from Lemma 1 together with the observation that a simplicial map does not increase the length of any loop in the 1-skeleton. □

2. Simplicial complexes associated to $\text{Out}(F_n)$ and $\text{Aut}(F_n)$

Let $K_n$ denote the spine of Outer space, as defined in [5], and $L_n$ the spine of Auter space, as defined in [8]. These are contractible simplicial complexes with cocompact proper actions by $\text{Out}(F_n)$ and $\text{Aut}(F_n)$ respectively, so we may use them to compute the Dehn functions for these groups.

Recall from [5] that a marked graph is a finite metric graph $\Gamma$ together with a homotopy equivalence $g : R_n \to \Gamma$, where $R_n$ is a fixed graph with one vertex and $n$ loops. A vertex of $K_n$ can be represented either as a marked graph $(g, \Gamma)$ with all vertices of valence at least three, or as a free minimal action of $F_n$ on a simplicial tree (namely the universal cover of $\Gamma$). A vertex of $L_n$ has the same descriptions except that there is a chosen basepoint in the marked graph (respected by the marking) or in the simplicial tree. Note that we allow marked graphs to have separating edges. Both $K_n$ and $L_n$ are flag complexes, so to define them it suffices to describe what it means for vertices to be adjacent. In the marked-graph description, vertices of $K_n$ (or $L_n$) are adjacent if one can be obtained from the other by a forest collapse (i.e. collapsing each component of a forest to a point).

3. Three natural maps

There is a forgetful map $\phi_n : L_n \to K_n$ which simply forgets the basepoint; this map is simplicial.

Let $m < n$. We fix an ordered basis for $F_n$, identify $F_m$ with the subgroup generated by the first $m$ elements of the basis, and identify $\text{Aut}(F_m)$ with
the subgroup of $\text{Aut}(F_n)$ that leaves $F_m < F_n$ invariant and fixes the last $n-m$ basis elements. We consider two maps associated to this choice of basis.

First, there is an equivariant augmentation map $\iota: L_m \to L_n$ which attaches a bouquet of $n-m$ circles to the basepoint of each marked graph and marks them with the last $n-m$ basis elements of $F_n$. This map is simplicial, since a forest collapse has no effect on the bouquet of circles at the basepoint.

Secondly, there is a restriction map $\rho: K_n \to K_m$ which is easiest to describe using trees. A point in $K_n$ is given by a minimal free simplicial action of $F_n$ on a tree $T$ with no vertices of valence 2. We define $\rho(T)$ to be the minimal invariant subtree for $F_m < F_n$; more explicitly, $\rho(T)$ is the union of the axes in $T$ of all elements of $F_m$. (Vertices of $T$ that have valence 2 in $\rho(T)$ are no longer considered to be vertices.)

One can also describe $\rho$ in terms of marked graphs. The chosen embedding $F_m < F_n$ corresponds to choosing an $m$-petal subrose $R_m \subset R_n$. A vertex in $K_n$ is given by a graph $\Gamma$ marked with a homotopy equivalence $g: R_n \to \Gamma$, and the restriction of $g$ to $R_m$ lifts to a homotopy equivalence $\hat{g}: R_m \to \hat{\Gamma}$, where $\hat{\Gamma}$ is the covering space corresponding to $g_*(F_m)$. There is a canonical retraction $r$ of $\hat{\Gamma}$ onto its compact core, i.e. the smallest connected subgraph containing all nontrivial embedded loops in $\Gamma$. Let $\hat{\Gamma}_0$ be the graph obtained by erasing all vertices of valence 2 from the compact core and define $\rho(g, \Gamma) = (r \circ \hat{g}, \hat{\Gamma}_0)$.

**Lemma 2.** — For $m < n$, the restriction map $\rho: K_n \to K_m$ is simplicial.

**Proof.** — Any forest collapse in $\Gamma$ is covered by a forest collapse in $\hat{\Gamma}$ that preserves the compact core, so $\rho$ preserves adjacency. □

**Lemma 3.** — For $m < n$, the following diagram of simplicial maps commutes:

$$
\begin{array}{ccc}
L_m & \xrightarrow{\iota} & L_n \\
\phi_m & \downarrow & \downarrow \phi_n \\
K_m & \xleftarrow{\rho} & K_n
\end{array}
$$

**Proof.** — Given a marked graph with basepoint $(g, \Gamma; v) \in L_n$, the marked graph $\iota(g, \Gamma; v)$ is obtained by attaching $n-m$ loops at $v$ labelled by the elements $a_{m+1}, \ldots, a_n$ of our fixed basis for $F_n$. Then $(g_n, \Gamma_n) := \phi_n \circ \iota(g, \Gamma; v)$ is obtained by forgetting the basepoint, and the cover of $(g_n, \Gamma_n)$ corresponding to $F_m < F_n$ is obtained from a copy of $(g, \Gamma)$ (with its labels) by attaching $2(n-m)$ trees. (These trees are obtained from the Cayley graph of $F_n$ as follows: one cuts at an edge labelled $a^\varepsilon_i$, with
4. Proof of the Theorem

In the light of the Corollary and Lemma 3, it suffices to exhibit a sequence of loops $\ell_i$ in the 1-skeleton of $L_3$ whose lengths are bounded by a linear function of $i$ and whose filling area when projected to $K_3$ grows exponentially as a function of $i$. Such a sequence of loops is essentially described in [3]. What we actually described there were words in the generators of $\text{Aut}(F_3)$ rather than loops in $L_3$, but standard quasi-isometric arguments show that this is equivalent. More explicitly, the words we considered were $w_i = T^i A T^{-i} B T^i A^{-1} T^{-i} B^{-1}$ where

$$T: \begin{cases} 
    a_1 \mapsto a_1^2 a_2 \\
    a_2 \mapsto a_1 a_2 \\
    a_3 \mapsto a_3 
\end{cases}, \quad A: \begin{cases} 
    a_1 \mapsto a_1 \\
    a_2 \mapsto a_2 \\
    a_3 \mapsto a_1 a_3 
\end{cases}, \quad B: \begin{cases} 
    a_1 \mapsto a_1 \\
    a_2 \mapsto a_2 \\
    a_3 \mapsto a_3 a_2 
\end{cases}.$$

To interpret these as loops in the 1-skeleton of $L_3$ (and $K_3$) we note that $A = \lambda_{31}$ and $B = \rho_{32}$ are elementary transvections and $T$ is the composition of two elementary transvections: $T = \lambda_{21} \circ \rho_{12}$. Thus $w_i$ is the product of $8i + 4$ elementary transvections. There is a (connected) subcomplex of the 1-skeleton of $L_3$ spanned by roses (graphs with a single vertex) and Nielsen graphs (which have $(n - 2)$ loops at the base vertex and a further trivalent vertex). We say roses are adjacent if they have distance 2 in this graph.

Let $I \in L_3$ be the rose marked by the identity map $R_3 \to R_3$. Each elementary transvection $\tau$ moves $I$ to an adjacent rose $\tau I$, which is connected to $I$ by a Nielsen graph $N_\tau$. A composition $\tau_1 \ldots \tau_k$ of elementary transvections gives a path through adjacent roses $I, \tau_1 I, \tau_1 \tau_2 I, \ldots, \tau_1 \tau_2 \ldots \tau_k I$; the Nielsen graph connecting $\sigma I$ to $\sigma \tau I$ is $\sigma N_\tau$. Thus the word $w_i$ corresponds to a loop $\ell_i$ of length $16i + 8$ in the 1-skeleton of $L_3$. Theorem A of [3] provides an exponential lower bound on the filling area of $\phi \circ \ell_i$ in $K_3$.

The square of maps in Lemma 3 ought to have many uses beyond the one in this note (cf. [7]). We mention just one, for illustrative purposes. This is a special case of the fact that every infinite cyclic subgroup of $\text{Out}(F_n)$ is quasi-isometrically embedded [1].

**Proposition.** — The cyclic subgroup of $\text{Out}(F_n)$ generated by any Nielsen transformation (elementary transvection) is quasi-isometrically embedded.
Proof. — Each Nielsen transformation is in the image of the map
\[ \Phi: \text{Aut}(F_2) \to \text{Aut}(F_n) \to \text{Out}(F_n) \]
given by the inclusion of a free factor \( F_2 < F_n \). Thus it suffices to prove that if a cyclic subgroup \( C = \langle c \rangle < \text{Aut}(F_2) \) has infinite image in \( \text{Out}(F_2) \), then \( t \mapsto \Phi(c^t) \) is a quasi-geodesic. This is equivalent to the assertion that some (hence any) \( C \)-orbit in \( K_n \) is quasi-isometrically embedded, where \( C \) acts on \( K_n \) as \( \Phi(C) \) and \( K_n \) is given the piecewise Euclidean metric where all edges have length 1.

\( K_2 \) is a tree and \( C \) acts on \( K_2 \) as a hyperbolic isometry, so the \( C \)-orbits in \( K_2 \) are quasi-isometrically embedded. For each \( x \in L_2 \), the \( C \)-orbit of \( \phi_2(x) \) is the image of the quasi-geodesic \( t \mapsto c^t, \phi_2(x) = \phi_2(c^t, x) \). We factor \( \phi_2 \) as a composition of \( C \)-equivariant simplicial maps \( L_2 \xrightarrow{i} K_n \xrightarrow{\phi_n} K_2 \), as in Lemma 3, to deduce that the \( C \)-orbit of \( \phi_n, t(x) \) in \( K_n \) is quasi-isometrically embedded.

A slight variation on the above argument shows that if one lifts a free group of finite index \( \Lambda < \text{Out}(F_2) \) to \( \text{Aut}(F_2) \) and then maps it to \( \text{Out}(F_n) \) by choosing a free factor \( F_2 < F_n \), then the inclusion \( \Lambda \hookrightarrow \text{Out}(F_n) \) will be a quasi-isometric embedding.

\section*{BIBLIOGRAPHY}


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