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Semiclassical resolvent estimates at trapped sets


<http://aif.cedram.org/item?id=AIF_2012__62_6_2379_0>
SEMICLASSICAL RESOLVENT ESTIMATES
AT TRAPPED SETS

by Kiril DATCHEV & András VASY (*)

Abstract. — We extend our recent results on propagation of semiclassical resolvent estimates through trapped sets when a priori polynomial resolvent bounds hold. Previously we obtained non-trapping estimates in trapping situations when the resolvent was sandwiched between cutoffs $\chi$ microlocally supported away from the trapping: $\|\chi R_h(E + i0)\chi\| = O(h^{-1})$, a microlocal version of a result of Burq and Cardoso-Vodev. We now allow one of the two cutoffs, $\tilde{\chi}$, to be supported at the trapped set, giving $\|\chi R_h(E + i0)\tilde{\chi}\| = O(\sqrt{a(h)}h^{-1})$ when the a priori bound is $\|\tilde{\chi} R_h(E + i0)\tilde{\chi}\| = O(a(h)h^{-1})$.

Résumé. — Nous étendons nos résultats récents sur la propagation d’estimations de résolvantes semi-classiques à travers des ensembles captifs sous des bornes a priori de type polynomial. Précédemment, nous obtenions des estimations non-captives dans des situations captives quand la résolvante est contrôlée par au dessus et en dessous par des fonctions cutoff $\chi$ dont le support microlocal est situé loin de l’ensemble captif : $\|\chi R_h(E + i0)\chi\| = O(h^{-1})$ (version microlocale d’un résultat de Burq et Cardoso-Vodev). Nous considérons maintenant le cas où l’une des deux fonctions cutoff, $\tilde{\chi}$, est à support dans l’ensemble captif, obtenant $\|\chi R_h(E + i0)\tilde{\chi}\| = O(\sqrt{a(h)}h^{-1})$ lorsque la borne a priori est $\|\tilde{\chi} R_h(E + i0)\tilde{\chi}\| = O(a(h)h^{-1})$.

This short article is an addendum to the previous paper by K. Datchev and A. Vasy.

Let $(X, g)$ be a Riemannian manifold which is asymptotically conic or asymptotically hyperbolic in the sense of [6], let $V \in C_0^\infty(X)$ be real valued, let $P = h^2 \Delta_g + V(x)$, where $\Delta_g \geq 0$, and fix $E > 0$.

Keywords: Resolvent estimates, trapping, propagation of singularities.
Math. classification: 58J47, 35L05.
(*) The first author is partially supported by a National Science Foundation postdoctoral fellowship, and the second author is partially supported by the National Science Foundation under grants DMS-0801226 and DMS-1068742.
Theorem 1. — [6, Theorem 1.2] Suppose that for any $\chi_0 \in C_0^\infty(X)$ there exist $C_0, k, h_0 > 0$ such that for any $\varepsilon > 0$, $h \in (0, h_0]$ we have
\[
\|\chi_0 (h^2 \Delta_g + V - E - i\varepsilon)^{-1} \chi_0\|_{L^2(X) \to L^2(X)} \leq C_0 h^{-k}. \tag{1}
\]
Let $K_E \subset T^*X$ be the set of trapped bicharacteristics at energy $E$, and suppose that $b \in C_0^\infty(T^*X)$ is identically $1$ near $K_E$. Then there exist $C_1, h_1 > 0$ such that for any $\varepsilon > 0$, $h \in (0, h_1]$ we have the following nontrapping estimate:
\[
\|\langle r \rangle^{-1/2-\delta} (1 - \text{Op}(b))(h^2 \Delta_g + V - E - i\varepsilon)^{-1} (1 - \text{Op}(b))\langle r \rangle^{-1/2-\delta}\|_{L^2(X) \to L^2(X)} \leq C_1 h^{-1}. \tag{2}
\]
Here by bicharacteristics at energy $E$ we mean integral curves in $p^{-1}(E)$ of the Hamiltonian vector field $H_p$ of the Hamiltonian $p = |\xi|^2 + V(x)$, and the trapped ones are those which remain in a compact set for all time. We use the notation $r = r(z) = d_g(z, z_0)$, where $d_g$ is the distance function on $X$ induced by $g$ and $z_0 \in X$ is fixed but arbitrary.

If $K_E = \emptyset$ then (1) holds with $k = 1$. If $K_E \neq \emptyset$ but the trapping is sufficiently ‘mild’, then (1) holds for some $k > 1$: see [6] for details and examples. The point is that the losses in (1) due to trapping are removed when the resolvent is cutoff away from $K_E$. Theorem 1 is a more precise and microlocal version of an earlier result of Burq [1] and Cardoso and Vodev [3], but the assumption (1) is not needed in [1, 3]. See [6] for additional background and references for semiclassical resolvent estimates and trapping.

In this paper we prove that an improvement over the a priori estimate (1) holds even when one of the factors of $(1 - \text{Op}(b))$ is removed:

Theorem 2. — Suppose that there exist $k > 0$ and $a(h) \leq h^{-k}$ such that for any $\chi_0 \in C_0^\infty(X)$ there exists $h_0 > 0$ such that for any $\varepsilon > 0$, $h \in (0, h_0]$ we have
\[
\|\chi_0 (h^2 \Delta_g + V - E - i\varepsilon)^{-1} \chi_0\|_{L^2(X) \to L^2(X)} \leq a(h)/h. \tag{3}
\]
Suppose that $b \in C_0^\infty(T^*X)$ is identically $1$ near $K_E$. Then there exist $C_1, h_1 > 0$ such that for any $\varepsilon > 0$, $h \in (0, h_1]$,
\[
\|\langle r \rangle^{-1/2-\delta} (1 - \text{Op}(b))(h^2 \Delta_g + V - E - i\varepsilon)^{-1} \langle r \rangle^{-1/2-\delta}\|_{L^2(X) \to L^2(X)} \leq C_1 \sqrt{a(h)}/h. \tag{4}
\]
Note that by taking adjoints, analogous estimates follow if $1 - \text{Op}(b)$ is placed to the other side of $(h^2 \Delta_g + V - E - i\varepsilon)^{-1}$.
Such results were proved by Burq and Zworski [2, Theorem A] and Christianson [4, (1.6)] when $K_E$ consists of a single hyperbolic orbit. Theorem 2 implies an optimal semiclassical resolvent estimate for the example operator of [6, §5.3]: it improves [6, (5.5)] to

$$\|\chi_0 (P - \lambda)^{-1} \chi_0\| \leq C \log(1/h)/h.$$  

Further, this improved estimate can be used to extend polynomial resolvent estimates from complex absorbing potentials to analogous estimates for damped wave equations; this is a result of Christianson, Schenk, Wunsch and the second author [5].

Theorems 1 and 2 follow from microlocal propagation estimates in a neighborhood of $K_E$, or more generally in a neighborhood of a suitable compact invariant subset of a bicharacteristic flow.

To state the general results, suppose $X$ is a manifold, $P \in \Psi^m,0(X)$ a self adjoint, order $m > 0$, semiclassical pseudodifferential operator on $X$, with principal symbol $p$. For $I \subset \mathbb{R}$ compact and fixed, denote the characteristic set by $\Sigma = p^{-1}(I)$, and suppose that the projection to the base, $\pi: \Sigma \to X$, is proper (it is sufficient, for example, to have $p$ classically elliptic). Suppose that $\Gamma \subset T^* X$ is invariant under the bicharacteristic flow in $\Sigma$. Define the forward, resp. backward flowout $\Gamma_+$, resp. $\Gamma_-$, of $\Gamma$ as the set of points $\rho \in \Sigma$, from which the backward, resp. forward bicharacteristic segments tend to $\Gamma$, i.e. for any neighborhood $O$ of $\Gamma$ there exists $T > 0$ such that $-t \geq T$, resp. $t \geq T$, implies $\gamma(t) \in O$, where $\gamma$ is the bicharacteristic with $\gamma(0) = \rho$. Here we think of $\Gamma$ as the trapped set or as part of the trapped set, hence points in $\Gamma_-$, resp. $\Gamma_+$ are backward, resp. forward, trapped. Suppose $V, W$ are neighborhoods of $\Gamma$ with $\overline{V} \subset W$, $W$ compact. Suppose also that

$$\text{If } \rho \in W \setminus \Gamma_+, \text{ resp. } \rho \in W \setminus \Gamma_-, \text{ then the backward, resp. forward bicharacteristic from } \rho \text{ intersects } W \setminus \overline{V}. \tag{5}$$

This means that all bicharacteristics in $V$ which stay in $V$ for all time tend to $\Gamma$.

The main result of [6], from which the other results in the paper follow, is the following:

**Theorem 3.** — [6, Theorem 1.3] Suppose that $\|u\|_{H^{-N}_h} \leq h^{-N}$ for some $N \in \mathbb{N}$ and $(P - \lambda)u = f$, $\text{Re } \lambda \in I$ and $\text{Im } \lambda \geq -O(h^\infty)$. Suppose $f$ is $O(1)$ on $W$, $WF_h(f) \cap \overline{V} = \emptyset$, and $u$ is $O(h^{-1})$ on $W \cap \Gamma_- \setminus \overline{V}$. Then $u$ is $O(h^{-1})$ on $W \cap \Gamma_+ \setminus \Gamma$. 

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Here we say that \( u \) is \( O(a(h)) \) at \( \rho \in T^*X \) if there exists \( B \in \Psi^{0,0}(X) \) elliptic at \( \rho \) with \( \| Bu \|_{L^2} = O(a(h)) \). We say \( u \) is \( O(a(h)) \) on a set \( E \subset T^*X \) if it is \( O(a(h)) \) at each \( \rho \in E \).

Note that there is no conclusion on \( u \) at \( \Gamma \); typically it will be merely \( O(h^{-N}) \) there. However, to obtain \( O(h^{-1}) \) bounds for \( u \) on \( \Gamma_+ \) we only needed to assume \( O(h^{-1}) \) bounds for \( u \) on \( \Gamma_- \) and nowhere else. Note also that by the propagation of singularities, if \( u \) is \( O(h^{-1}) \) at one point on any bicharacteristic, then it is such on the whole forward bicharacteristic. If \( |\Im \lambda| = O(h^\infty) \) then the same is true for backward bicharacteristics.

In this paper we show that a (lesser) improvement on the a priori bound holds even when \( f \) is not assumed to vanish microlocally near \( \Gamma \):

**Theorem 4.** — Suppose that \( \|u\|_{H^{-N}_h} \leq h^{-N} \) for some \( N \in \mathbb{N} \) and \( (P-\lambda)u = f \), \( \Re \lambda \in I \) and \( \Im \lambda \geq -O(h^\infty) \). Suppose \( f \) is \( O(1) \) on \( W \), \( u \) is \( O(a(h)h^{-1}) \) on \( W \), and \( u \) is \( O(h^{-1}) \) on \( W \cap \Gamma_- \setminus \overline{V} \). Then \( u \) is \( O(\sqrt{a(h)}h^{-1}) \) on \( W \cap \Gamma_+ \setminus \Gamma \).

In [6] Theorem 1 is deduced from Theorem 3. Theorem 2 follows from Theorem 4 by the same argument.

**Proof of Theorem 4.** — The argument is a simple modification of the argument of [6, End of Section 4, Proof of Theorem 1.3]; we follow the notation of this proof. Recall first from [6, Lemma 4.1] that if \( U_- \) is a neighborhood of \( (\Gamma_- \setminus \Gamma) \cap (\overline{W} \setminus V) \) then there is a neighborhood \( U \subset V \) of \( \Gamma \) such that if \( \alpha \in U \setminus \Gamma_+ \) then the backward bicharacteristic from \( \alpha \) enters \( U_- \). Thus, if one assumes that \( u \) is \( O(h^{-1}) \) on \( \Gamma_- \) and \( f \) is \( O(1) \) on \( V \), it follows that that \( u \) is \( O(h^{-1}) \) on \( U \setminus \Gamma_+ \), provided \( U_- \) is chosen small enough that \( u \) is \( O(h^{-1}) \) on \( U_- \). Note also that, because \( U \subset V \), \( f \) is \( O(1) \) on \( U \). We will show that \( u \) is \( O(\sqrt{a(h)}h^{-1}) \) on \( U \cap \Gamma_+ \setminus \Gamma \); the conclusion on the larger set \( W \cap \Gamma_+ \setminus \Gamma \) follows by propagation of singularities.

Next, [6, Lemma 4.3] states that if \( U_1 \) and \( U_0 \) are open sets with \( \Gamma \subseteq U_1 \subseteq U_0 \subseteq U \) then there exists a nonnegative function \( q \in C^\infty_0(U) \) such that

\[
q = 1 \text{ near } \Gamma, \quad H_\rho q \leq 0 \text{ near } \Gamma_+, \quad H_\rho q < 0 \text{ on } \Gamma^\infty_+ \setminus U_1.
\]

Moreover, we can take \( q \) such that both \( \sqrt{q} \) and \( \sqrt{-H_\rho q} \) are smooth near \( \Gamma_+ \).

**Remark.** — The last paragraph in the proof of [6, Lemma 4.3] should be replaced by the following: To make \( \sqrt{-H_\rho q} \) smooth, let \( \psi(s) = 0 \) for \( s \leq 0 \), \( \psi(s) = e^{-1/s} \) for \( s > 0 \), and assume as we may that \( U_\rho \cap S_\rho \) is a ball with respect to a Euclidean metric (in local coordinates near \( \rho \)) of

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radius \( r_\rho > 0 \) around \( \rho \). We then choose \( \varphi_\rho \) to behave like \( \psi(r_\rho^2 - |.|^2) \) with \( r_\rho'<r_\rho \) for \(|.| \) close to \( r_\rho' \), bounded away from 0 for smaller values of \(|.|\), and choose \( -\chi_\rho' \) to vanish like \( \psi \) at the boundary of its support. That sums of products of such functions have smooth square roots follows from [7, Lemma 24.4.8].

The proof of Theorem 4 proceeds by induction: we show that if \( u \) is \( \mathcal{O}(h^k) \) on a sufficiently large compact subset of \( U \cap \Gamma_+ \), then \( u \) is \( \mathcal{O}(h^{k+1/2}) \) on \( \Gamma^u \setminus U_1 \), provided \( \sqrt{a(h)}h^{-1} \leq C h^{k+1/2} \).

Now let \( U_- \) be an open neighborhood of \( \Gamma_+ \cap \text{supp} \ q \) which is sufficiently small that \( H_pq \leq 0 \) on \( U_- \) and that \( \sqrt{-H_pq} \) is smooth on \( U_- \). Let \( U_+ \) be an open neighborhood of \( \text{supp} \ q \setminus U_- \) whose closure is disjoint from \( \Gamma_+ \) and from \( T^*X \setminus U \). Define \( \phi_\pm \in \psi^{-\infty}(U_+ \cup U_-) \) with \( \text{supp} \ \phi_\pm \subset U_\pm \) and with \( \phi_+^2 + \phi_-^2 = 1 \) near \( \text{supp} \ q \).

Put
\[
b \equiv \phi_- \sqrt{-H_pq^2}, \quad e \equiv \phi_+^2 H_pq^2.
\]

Let \( Q, B, E \in \psi^{-\infty,0}(X) \) have principal symbols \( q, b, e \), and microsupports \( \text{supp} \ q, \text{supp} \ b, \text{supp} \ e \), so that
\[
\frac{i}{h}[P, Q^*Q] = -B^*B + E + hF,
\]
with \( F \in \psi^{-\infty,0}(X) \) such that \( \text{WF}_h'F \subset \text{supp} \ dq \subset U \setminus \Gamma \). But
\[
\frac{i}{h} \langle [P, Q^*Q]u, u \rangle = \frac{2}{h} \text{Im} \langle Q^*Q(P - \lambda)u, u \rangle + \frac{2}{h} \langle Q^*Q \text{Im} \lambda u, u \rangle \\
\geq -2h^{-1}\|Q(P - \lambda)u\| \|Qu\| - \mathcal{O}(h^\infty)\|u\|^2 \\
\geq -Ch^{-2}a(h) - \mathcal{O}(h^\infty),
\]
where we used \( \text{Im} \lambda \geq -\mathcal{O}(h^\infty) \) and that on \( \text{supp} \ q, (P - \lambda)u \) is \( \mathcal{O}(1) \). So
\[
\|Bu\|^2 \leq \langle Eu, u \rangle + h\langle Fu, u \rangle + Ch^{-2}a(h) + \mathcal{O}(h^\infty).
\]
But \( \langle Eu, u \rangle \leq Ch^{-2} \) because \( \text{WF}_h'E \cap \Gamma_+ = \emptyset \) gives that \( u \) is \( \mathcal{O}(h^{-1}) \) on \( \text{WF}_h'E \) by the first paragraph of the proof. Meanwhile \( \langle Fu, u \rangle \leq C(h^{-2} + h2^k) \) because all points of \( \text{WF}_h'F \) are either in \( U \setminus \Gamma_+ \), where we know \( u \) is \( \mathcal{O}(h^{-1}) \) from the first paragraph of the proof, or on a single compact subset of \( U \cap \Gamma_+ \setminus \Gamma \), where we know that \( u \) is \( \mathcal{O}(h^k) \) by inductive hypothesis. Since \( b = \sqrt{-H_pq^2} > 0 \) on \( \Gamma^u \setminus U_1 \), we can use microlocal elliptic regularity to conclude that \( u \) is \( \mathcal{O}(h^{k+1/2}) \) on \( \Gamma^u \setminus U_1 \), as desired.
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Manuscrit reçu le 12 juin 2012, accepté le 10 janvier 2013.

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