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Abstract. — A variety $X$ over a field $K$ is of Hilbert type if $X(K)$ is not thin. We prove that if $f: X \to S$ is a dominant morphism of $K$-varieties and both $S$ and all fibers $f^{-1}(s), s \in S(K)$, are of Hilbert type, then so is $X$. We apply this to answer a question of Serre on products of varieties and to generalize a result of Colliot-Thélène and Sansuc on algebraic groups.

Résumé. — Une variété $X$ sur un corps $K$ a la propriété de Hilbert si $X(K)$ n’est pas mince. Nous montrons que si $f: X \to S$ est un morphisme de $K$-variétés dominant et si $S$ ainsi que toutes les fibres $f^{-1}(s)$ pour $s \in S(K)$ ont la propriété de Hilbert, alors $X$ aussi. Ceci nous permet de répondre à une question de Serre concernant les produits de variétés, et de généraliser un résultat de Colliot-Thélène et Sansuc sur les groupes algébriques.

1. Introduction

In the terminology of thin sets (we recall this notion in Section 2), Hilbert’s irreducibility theorem asserts that $\mathbb{A}^n_K(K)$ is not thin, for any number field $K$ and any $n \geq 1$. As a natural generalization a $K$-variety $X$ is called of Hilbert type if $X(K)$ is not thin. The importance of this definition stems from the observation of Colliot-Thélène and Sansuc [2] that the inverse Galois problem would be settled if every unirational variety over $\mathbb{Q}$ was of Hilbert type.

In this direction, Colliot-Thélène and Sansuc [2, Cor. 7.15] prove that any connected reductive algebraic group over a number field is of Hilbert type. This immediately raises the question whether the same holds for all linear algebraic groups (note that these are unirational). Another question, asked by Serre [19, p. 21], is whether a product of two varieties of Hilbert

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The main result of this paper gives a sufficient condition for a variety to be of Hilbert type:

**Theorem 1.1.** — Let $K$ be a field and $f : X \to S$ a dominant morphism of $K$-varieties. Assume that the set of $s \in S(K)$ for which the fiber $f^{-1}(s)$ is a $K$-variety of Hilbert type is not thin. Then $X$ is of Hilbert type.

As an immediate consequence we get the following result for a family of varieties over a variety of Hilbert type:

**Corollary 1.2.** — Let $K$ be a field and $f : X \to S$ a dominant morphism of $K$-varieties. Assume that $S$ is of Hilbert type and that for every $s \in S(K)$ the fiber $f^{-1}(s)$ is of Hilbert type. Then $X$ is of Hilbert type.

Using this result we resolve both questions discussed above affirmatively, see Corollary 3.4 and Proposition 4.2.

### 2. Background

Let $K$ be a field. A $K$-variety is a separated scheme of finite type over $K$ which is geometrically reduced and geometrically irreducible. Thus, a non-empty open subscheme of a $K$-variety is again a $K$-variety. If $f : X \to S$ is a morphism of $K$-varieties and $s \in S(K)$, then $f^{-1}(s) := X \times_S \text{Spec}(\kappa(s))$, where $\kappa(s)$ is the residue field of $s$, denotes the scheme theoretic fiber of $f$ at $s$. This fiber is a separated scheme of finite type over $K$, which needs not be reduced or connected in general. We identify the set $f^{-1}(s)(K)$ of $K$-rational points of the fiber with the set theoretic fiber $\{x \in X(K) \mid f(x) = s\}$.

Let $X$ be a $K$-variety. A subset $T$ of $X(K)$ is called *thin* if there exists a proper Zariski-closed subset $C$ of $X$, a finite set $I$, and for each $i \in I$ a $K$-variety $Y_i$ with $\dim(Y_i) = \dim(X)$ and a dominant separable morphism $p_i : Y_i \to X$ of degree $\geq 2$ (in particular, $p_i$ is generically étale, cf. Lemma 3.3) such that

$$T \subseteq \bigcup_{i \in I} p_i(Y_i(K)) \cup C(K).$$

A $K$-variety $X$ is of Hilbert type if $X(K)$ is not thin, cf. [19, Def. 3.1.2]. Note that $X$ is of Hilbert type if and only if some (or every) open subscheme of $X$ is of Hilbert type, cf. [19, p. 20]. A field $K$ is Hilbertian if $\mathbb{A}^1_K$ is of Hilbert type. We note that if there exists a $K$-variety $X$ of positive dimension such that $X$ is of Hilbert type, then $K$ is Hilbertian [5, Prop. 13.5.3].
All global fields and, more generally, all infinite fields that are finitely generated over their prime fields are Hilbertian [5, Thm. 13.4.2]. Many more fields are known to be Hilbertian, for example the maximal abelian Galois extension $\mathbb{Q}^{ab}$ of $\mathbb{Q}$, [5, Thm. 16.11.3]. On the other hand, local fields like $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}_p$ and $\mathbb{F}_q((t))$ are not Hilbertian [5, Ex. 15.5.5].

3. Proof of Theorem 1.1

A key tool in the proof of Theorem 1.1 is the following consequence of Stein factorization.

**Lemma 3.1.** — Let $K$ be a field and $\psi: Y \to S$ a dominant morphism of normal $K$-varieties. Then there exists a nonempty open subscheme $U \subset S$, a $K$-variety $T$ and a factorization

$$\psi^{-1}(U) \xrightarrow{g} T \xrightarrow{r} U$$

of $\psi$ such that the fibers of $g$ are geometrically irreducible and $r$ is finite and étale.

**Proof.** — See [13, Lemma 9]. □

**Lemma 3.2.** — Let $K$ be a field and $f: X \to S$ a dominant morphism of normal $K$-varieties. Assume that the set $\Sigma$ of $s \in S(K)$ for which $f^{-1}(s)$ is a $K$-variety of Hilbert type is not thin. Let $I$ be a finite set and let $p_i: Y_i \to X$, $i \in I$, be finite étale morphisms of degree $\geq 2$. Then $X(K) \not\subseteq \bigcup_{i \in I} p_i(Y_i(K))$.

**Proof.** — For $i \in I$ consider the composite morphism $\psi_i := f \circ p_i: Y_i \to S$. By Lemma 3.1 there is a nonempty open subscheme $U_i$ of $S$ and a factorization

$$\psi_i^{-1}(U_i) \xrightarrow{g_i} T_i \xrightarrow{r_i} U_i$$

of $\psi_i$ such that the morphism $g_i$ has geometrically irreducible fibers, $r_i$ is finite and étale, and such that $T_i$ is a $K$-variety. We now replace successively $S$ by $\bigcap_{i \in I} U_i$, $X$ by $f^{-1}(S)$, $T_i$ by $r_i^{-1}(S)$ and $Y_i$ by $p_i^{-1}(X)$, to assume in addition that $r_i: T_i \to S$ is finite étale for every $i \in I$.

For $s \in S(K)$ denote by $X_s := f^{-1}(s)$ the fiber of $f$ over $s$. Then $X_s$ is a $K$-variety of Hilbert type for each $s \in \Sigma$. Furthermore we define $Y_{i,s} := \psi_i^{-1}(s)$ and let $p_{i,s}: Y_{i,s} \to X_s$ be the corresponding projection morphism. Then $p_{i,s}$ is a finite étale morphism of the same degree as $p_i$. In
particular, the $K$-scheme $Y_{i,s}$ is geometrically reduced. For every $s \in S(K)$ and every $i \in I$ we have constructed a commutative diagram

$$
\begin{array}{ccc}
Y_{i,s} & \longrightarrow & Y_i \\
\downarrow p_{i,s} & & \downarrow p_i \\
X_s & \longrightarrow & X \\
& & \downarrow f \\
& & S
\end{array}
$$

in which the left hand rectangle is cartesian. Set $J := \{ i \in I : \deg(r_i) \geq 2 \}$. Then $\bigcup_{i \in J} r_i(T_i(K)) \subseteq S(K)$ is thin, so by assumption there exists $s \in \Sigma \setminus \bigcup_{i \in J} r_i(T_i(K))$.

For $i \in J$ there is no $K$-rational point of $T_i$ over $s$, hence $Y_{i,s}(K) = \emptyset$ for every $i \in J$. For $i \in I \setminus J$, the finite étale morphism $r_i$ is of degree 1, hence an isomorphism, and therefore $Y_{i,s}$ is geometrically irreducible. Thus, $Y_{i,s}$ is a $K$-variety. So since $X_s$ is of Hilbert type, there exists $x \in X_s(K)$ such that $x \notin \bigcup_{i \in I \setminus J} p_{i,s}(Y_{i,s}(K))$. Thus

$$
x \notin \bigcup_{i \in J \setminus I} p_{i,s}(Y_{i,s}(K)) = \bigcup_{i \in I} p_{i,s}(Y_{i,s}(K)),
$$

hence $x \notin \bigcup_{i \in I} p_i(Y_i(K))$, as needed. □

The following fact is well-known, but for the sake of completeness we provide a proof:

**Lemma 3.3.** — Let $K$ be a field, let $X, Y$ be $K$-varieties with $\dim(X) = \dim(Y)$, and let $p : Y \to X$ be a dominant separable morphism. Then there exists a nonempty open subscheme $U$ of $X$ such that the restriction of $p$ to a morphism $p^{-1}(U) \to U$ is finite and étale.

**Proof.** — By the theorem of generic flatness (cf. [11, 6.9.1]) there is a non-empty open subscheme $V$ of $X$ such that the restriction of $p$ to a morphism $p^{-1}(V) \to V$ is flat (and in particular open). This restriction is quasi-finite by [12, 14.2.4], because the generic fiber of $f$ is finite due to our assumption $\dim(X) = \dim(Y)$. By Zariski’s main theorem there exists a $K$-variety $\overline{Y}$, an open immersion $i : p^{-1}(V) \to \overline{Y}$ and a finite morphism $f : \overline{Y} \to V$ such that $f \circ i = p$. The ramification locus $C \subset \overline{Y}$ of $f$ is closed (cf. [8, I.3.3]), and $C \neq \overline{Y}$ because $f$ is separable. Define $U := V \setminus f((\overline{Y} \setminus \text{im}(i)) \cup C)$. Then $U$ is open (cf. [6, 6.1.10]) and non-empty, and $f^{-1}(U) \subset \text{im}(i) \setminus C$. Hence the restriction of $f$ to a morphism $f^{-1}(U) \to U$ is finite and étale, and the assertion follows from that. □
Proof of Theorem 1.1. — Let $K$ be a field, and $f: X \to S$ a dominant morphism of $K$-varieties. Assume that the set $\Sigma$ of those $s \in S(K)$ for which $f^{-1}(s)$ is of Hilbert type is not thin. Let $C \subseteq X$ be a proper Zariski-closed subset. Let $I$ be a finite set and suppose that $Y_i$ is a $K$-variety with $\dim(Y_i) = \dim(X)$ and $p_i: Y_i \to X$ is a dominant separable morphism of degree $\geq 2$, for every $i \in I$. We have to show that $X(K) \not\subseteq C(K) \cup \bigcup_{i \in I} p_i(Y_i(K))$.

By Lemma 3.3 and [11, 6.12.6, 6.13.5] there exists a normal nonempty open subscheme $X' \subset X \setminus C$ such that the restriction of each $p_i$ to a morphism $p_i^{-1}(X') \to X'$ is finite and étale. The image $f(X')$ contains a nonempty open subscheme $S'$ of $S$ (cf. [10, 1.8.4], [7, 9.2.2]). Furthermore, $S'$ contains a nonempty normal open subscheme $S''$. Let us define $X'' := f^{-1}(S'') \cap X'$ and $Y''_i := p_i^{-1}(X'')$. Then the restriction of $f$ to a morphism $f'': X'' \to S''$ is not thin, and $f''(s)$ is of Hilbert type for every $s \in \Sigma \cap S''(K)$ because it is an open subscheme of $f^{-1}(s)$. The restriction $p''_i$ of $p_i$ to a morphism $Y''_i \to X''$ is finite and étale for every $i \in I$. By Lemma 3.2 applied to $f''$ and the $p''_i$ we have

$$\emptyset \neq X''(K) \setminus \bigcup_{i \in I} p''_i(Y''_i(K)) = X''(K) \setminus \bigcup_{i \in I} p_i(Y_i(K)) \subseteq X(K) \setminus \left( C(K) \cup \bigcup_{i \in I} p_i(Y_i(K)) \right),$$

so $X(K) \not\subseteq C(K) \cup \bigcup_{i \in I} p_i(Y_i(K))$, as needed. □

As an immediate consequence we get an affirmative solution of Serre’s question mentioned in the introduction.

Corollary 3.4. — Let $K$ be a field. If $X,Y$ are $K$-varieties of Hilbert type, then $X \times Y$ is of Hilbert type.

Proof. — Denote by $f: X \times Y \to X$ the projection. Then $f^{-1}(x)$ is isomorphic to $Y$ and hence of Hilbert type for every $x \in X(K)$. Thus Corollary 1.2 gives that $X \times Y$ is of Hilbert type. □

4. Algebraic groups of Hilbert type

By an algebraic group over a field $K$ we shall mean a connected smooth group scheme over $K$. Recall that such an algebraic group is a $K$-variety,
see [9, Exp VI A, 0.3, 2.1.2, 2.4]. If \(G\) is an algebraic group over \(K\), then \(G(K_s)\) is a \(\text{Gal}(K)\)-group, where \(K_s\) denotes a separable closure of \(K\) and \(\text{Gal}(K) = \text{Gal}(K_s/K)\) is the absolute Galois group of \(K\), and there is an associated Galois cohomology pointed set \(H^1(K,G) = H^1(\text{Gal}(K),G(K_s))\), which classifies isomorphism classes of \(G(K_s)\)-torsors, cf. [15, Prop. 1.2.3].

**Proposition 4.1.** — Let \(K\) be a field and let

\[1 \to N \to G \xrightarrow{p} Q \to 1\]

be a short exact sequence of algebraic groups over \(K\). If \(H^1(K,N) = 1\) and both \(N\) and \(Q\) are of Hilbert type, then \(G\) is of Hilbert type.

**Proof.** — It suffices to show that \(p^{-1}(x)\) is of Hilbert type for every \(x \in Q(K)\), because then Corollary 1.2 implies the assertion. Let \(x \in Q(K)\) and \(F = p^{-1}(x)\). There is an exact sequence of \(\text{Gal}(K)\)-groups

\[1 \to N(K_s) \to G(K_s) \to Q(K_s) \to 1,\]

where the right hand map is surjective, because for every point \(x \in Q(K_s)\) the fiber over \(x\) is a non-empty \(K_s\)-variety and thus has a \(K_s\)-rational point. Since the \(\text{Gal}(K)\)-set \(F(K_s)\) is a coset of \(N(K_s)\), it is a \(N(K_s)\)-torsor. Our hypothesis \(H^1(K,N) = 1\) implies that \(F(K_s)\) is isomorphic to the trivial \(N(K_s)\)-torsor \(N(K_s)\). It follows that \(F\) is isomorphic to \(N\) as a \(K\)-variety, hence \(F\) is of Hilbert type. \(\Box\)

Using this, we generalize the result of Colliot-Thélène and Sansuc [2, Cor. 7.15] from reductive groups to arbitrary linear groups.

**Theorem 4.2.** — Every linear algebraic group \(G\) over a perfect Hilbertian field \(K\) is of Hilbert type.

**Proof.** — We denote by \(G_u\) the unipotent radical of \(G\) (cf. [14, Prop. XVII.1.2]). We have a short exact sequence of algebraic groups over \(K\)

\[(*) \quad 1 \to G_u \to G \to Q \to 1\]

with \(Q\) reductive, cf. [14, Prop. XVII.2.2]. By [2, Cor. 7.15], \(Q\) is of Hilbert type. Since \(K\) is perfect, \(G_u\) is split, i.e. there exists a series of normal algebraic subgroups

\[1 = U_0 \subseteq \cdots \subseteq U_n = G_u\]

such that \(U_{i+1}/U_i \cong \mathbb{G}_a\) for each \(i\), cf. [1, 15.5(ii)]. The groups \(U_i\) are unipotent, hence \(H^1(K,U_i) = 1\) by [18, Ch. III §2.1, Prop. 6], and \(\mathbb{G}_a\) is of Hilbert type since \(K\) is Hilbertian. Thus, an inductive application of Proposition 4.1 implies that \(G_u\) is of Hilbert type. Finally we apply Proposition 4.1 to the exact sequence \((*)\) to conclude that \(G\) is of Hilbert type. \(\Box\)
Remark 4.3. — In the special case where $K$ is a number field, Sansuc proved a much more precise result: It follows from [17, Cor. 3.5(ii)] that a linear algebraic group $G$ over a number field satisfies the so-called weak weak approximation property [19, Def. 3.5.6], which, by a theorem of Colliot-Thélène and Ekedahl, in particular implies that $G$ is of Hilbert type, cf. [19, Thm. 3.5.7].

Remark 4.4. — The special case of Theorem 4.2 where $G$ is simply connected and $K$ is finitely generated is also a consequence of a result of Corvaja, see [4, Cor. 1.7].

Remark 4.5. — We point out that Theorem 4.2 could be deduced also from Corollary 3.4 (instead of Corollary 1.2) via [16, Cor. 1] and the fact that a unipotent group over a perfect field is rational, cf. [9, XIV, 6.3].

As a consequence of Theorem 4.2, we get a more general statement for homogeneous spaces, which was pointed out to us by Borovoi:

Corollary 4.6. — If $G$ is a linear algebraic group over a perfect Hilbertian field $K$, and $H$ is a connected algebraic subgroup of $G$, then the quotient $G/H$ is of Hilbert type.

Proof. — For the existence of the quotient $Q := G/H$ see for example [1, Ch. II Thm. 6.8]. If $\mathcal{H}$ denotes the generic fiber of $G \to Q$ and $F$ is an algebraic closure of the function field $K(Q)$ of $Q$, then $\mathcal{H}_F \cong H_F$ by translation on $G$. Thus, $\mathcal{H}$ is geometrically irreducible since $H$ is, so [2, Prop. 7.13] implies that $Q$ is of Hilbert type.

We also get a complete classification of the algebraic groups that are of Hilbert type over a number field:

Corollary 4.7. — An algebraic group $G$ over a number field $K$ is of Hilbert type if and only if it is linear.

Proof. — If $G$ is linear, then it is of Hilbert type by Theorem 4.2. Conversely, assume that $G$ is of Hilbert type. Chevalley’s theorem [3, Thm. 1.1] gives a short exact sequence of algebraic groups over $K$,

$$1 \to H \to G \to A \to 1$$

with $H$ linear and $A$ an abelian variety. As in the proof of Corollary 4.6 we conclude that the generic fiber of $G \to A$ is geometrically irreducible, and therefore $A$ is of Hilbert type. Since no nontrivial abelian variety over a number field is of Hilbert type, cf. [5, Remark 13.5.4], $A$ is trivial and $G \cong H$ is linear.
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