Kenichi BANNAI & Shinichi KOBAYASHI

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INTEGRAL STRUCTURES ON $p$-ADIC FOURIER THEORY

by Kenichi BANAI & Shinichi KOBAYASHI (*)

Abstract. — In this article, we give an explicit construction of the $p$-adic Fourier transform by Schneider and Teitelbaum, which allows for the investigation of the integral property. As an application, we give a certain integral basis of the space of $K$-locally analytic functions on the ring of integers $\mathcal{O}_K$ for any finite extension $K$ of $\mathbb{Q}_p$, generalizing the basis constructed by Amice for locally analytic functions on $\mathbb{Z}_p$. We also use our result to prove congruences of Bernoulli-Hurwitz numbers at non-ordinary (i.e. supersingular) primes originally investigated by Katz and Chellali.

Résumé. — Dans cet article, nous donnons une construction explicite de la transformation de Fourier $p$-adique de Schneider et Teitelbaum, qui nous permet d’étudier son integralité. Comme application, pour toute extension finie $K$ de $\mathbb{Q}_p$, nous donnons une certaine base entière de l’espace de fonctions $K$-localement analytiques sur l’anneau des entiers $\mathcal{O}_K$ , en généralisant la base construite par Amice pour les fonctions localement analytiques sur $\mathbb{Z}_p$. Nous utilisons également notre résultat pour démontrer certaines relations de congruence étudiées initialement par Katz et Chellali entre nombres de Bernoulli-Hurwitz aux places non-ordinaires (c’est-à-dire supersingulières).

1. Introduction

One important method in studying the congruences and $p$-adic properties of important invariants in number theory is the use of $p$-adic measures interpolating such values. Such theory was applied to obtain the Kummer congruence between special values of Riemann zeta function as well as the

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construction of the $p$-adic $L$-functions for elliptic curves with ordinary reduction at $p$. When dealing with the non-ordinary case, it is necessary to use the theory of $p$-adic analytic distributions, which is a generalization of the theory of $p$-adic measures. For such $p$-adic distributions on $\mathbb{Z}_p$, the Amice transform gives a one-to-one correspondence between $\mathbb{C}_p$-valued distributions on $\mathbb{Z}_p$ and rigid analytic functions on the open unit disc. The general idea is to study the congruences and $p$-adic properties of the interpolated invariants through the $p$-adic property of the rigid analytic function corresponding to the $p$-adic distribution. However, contrary to the case of $p$-adic measures, the Amice transform is not well-behaved integrally for general $p$-adic distributions, hence it is necessary to investigate in detail the precise integral structure of this transform. Amice [1, §10] investigated the precise integral structure of the Amice transform.

Let $O_K$ be the ring of integers of a finite extension $K$ of $\mathbb{Q}_p$. In [8, §4], Schneider and Teitelbaum constructed the $p$-adic Fourier transform, which is a one-to-one correspondence between $\mathbb{C}_p$-valued distributions on $O_K$ and rigid analytic functions on an open unit disc. The purpose of this article is to give an explicit and elementary construction of the $p$-adic Fourier transform of Schneider-Teitelbaum, which allows investigation of the precise integral structure of this correspondence. We then determine an integral structure on the ring of locally analytic functions on $O_K$. The integrality of the $p$-adic Fourier transform for general $K$ is even less well behaved than for the case of $\mathbb{Q}_p$; even if the rigid analytic function corresponding to a $p$-adic distribution has bounded coefficients, the $p$-adic distribution may not necessarily be a $p$-adic measure. As an application of our result, we obtain the congruences originally proved by Katz [7, Theorem 3.11] and Chellali [4, Théorème 1.1] of Bernoulli-Hurwitz numbers, which are essentially special values of $p$-adic $L$-functions of CM elliptic curves at non-ordinary primes.

We now give the exact statements of our theorems. Let $p$ be a rational prime and let $| \cdot |$ be the absolute value of $\mathbb{C}_p$ such that $|p| = p^{-1}$. Let $\pi$ be an uniformizer of $O_K$, and let $\mathbb{F}_q$ be the residue field of $O_K$. We define $L\mathcal{A}_N(O_K, \mathbb{C}_p)$ to be the space of locally analytic functions on $O_K$ of order $N$ which take values in $\mathbb{C}_p$. That is, $f(x) \in L\mathcal{A}_N(O_K, \mathbb{C}_p)$ if and only if $f(x)$ is defined as a convergent power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ on $a+\pi^N O_K$ for any $a \in O_K$. We let $\|f\|_{a,N} := \max_{a} \{|a_n \pi^{nN}|\}$. The space $L\mathcal{A}_N(O_K, \mathbb{C}_p)$ is a $p$-adic Banach space induced by the norm $\max_{a \in O_K} \{|f(a)\|_{a,N}\}$ and we denote by $L\mathcal{A}_N(O_K, \mathbb{C}_p)_0$ the submodule of elements whose absolute values are less than or equal to 1. We let $\mathcal{G}$ be a Lubin-Tate group of $K$ corresponding to
\[ \pi, \text{ and let } \varpi \in \mathbb{C}_p \text{ be a } p\text{-adic period of } \mathcal{G}. \text{ We let } \]
\[ p(k) = \max_{k \leq m} \{|m!/\varpi^m_p|\}, \quad \rho(k) = \min_{0 \leq m \leq k} \{|m!/\varpi^m_p|\}. \]

See Proposition 3.1 for the properties of these numbers.

Let \( \varphi(t) \) be a rigid analytic function on the open unit disc. In other words, \( \varphi(t) \) is a power series of the form \( \varphi(t) = \sum_{n=0}^{\infty} c_n t^n \) such that \( |c_n| r_0^n \to 0 \) for any \( 0 < r_0 < 1 \). Let \( \mu_\varphi \) be the distribution on \( \mathcal{O}_K \) corresponding to \( \varphi(t) \) given by Schneider-Teitelbaum’s \( p\)-adic Fourier theory [8, Theorem 2.3].

Then we have the following:

**Theorem 1.1.** — Let \( f \in \text{LA}_N(\mathcal{O}_K, \mathbb{C}_p) \). Then we have

\[ \left| \int_{a+\pi^N_\mathcal{O}_K} f(x) \, d\mu_\varphi \right| \leq \frac{\pi|N}{q} \|f\|_{a,N} \|\varphi\|_N \quad (1.1) \]

where

\[ \|\varphi\|_N := \max_k \left\{ |c_k| p \left( \left[ \frac{k}{qN} \right] \right) \right\} \quad (1.2) \]

and \([x]\) is the integral part of \( x \).

The crucial difference from the case when \( K = \mathbb{Q}_p \) is the fact that \(|\pi/q| > 1\) when \( K \neq \mathbb{Q}_p \). A finer version of the above is given as Theorem 4.3. Since \( p \left( \left[ \frac{k}{qN} \right] \right) \sim p^{-kr} \) where \( r = 1/eq^N(q-1) \), the value \( \|\varphi\|_N \) is approximated by

\[ \|\varphi\|_{B(p^{-r})} = \max_{x \in B(p^{-r})} \{ |\varphi(x)| \}, \]

where \( B(p^{-r}) \subset \mathbb{C}_p \) is the closed disc of radius \( p^{-r} \) centered at the origin.

As an application of our main theorem, we obtain an estimate of the Fourier coefficients of Mahler like expansion of functions in \( \text{LA}_N(\mathcal{O}_K, \mathbb{C}_p) \).

Let \( \lambda(t) \) be the formal logarithm of \( \mathcal{G} \), and following [8], we define the polynomial \( P_n(x) \) by

\[ \exp(x\lambda(t)) = \sum_{n=0}^{\infty} P_n(x)t^n. \]

Note that when \( \mathcal{G} \) is the multiplicative formal group \( \mathcal{G} = \widehat{G}_m \), then \( \lambda(t) = \log(1 + t) \) and the above expansion is simply

\[ (1 + t)^x = \sum_{n=0}^{\infty} \binom{x}{n} t^n. \]

Hence the polynomial \( P_n(x) \) is a generalization of the binomial polynomial

\[ \binom{x}{n} = \frac{x(x-1) \cdots (x-n+1)}{n!}. \]
Then we have the following.

**Theorem 1.2** *(Theorem 4.7)*. — The series \( \sum_{n=0}^{\infty} a_n P_n(x^{\varpi_p}) \) converges to an element of \( \mathcal{L}_N(\mathcal{O}_K, \mathbb{C}_p) \) for \( a_n \) satisfying

\[
|a_n| \leq \rho \left( \left\lfloor \frac{n}{q^N} \right\rfloor \right), \quad \lim_{n \to 0} |a_n|/\rho \left( \left\lfloor \frac{n}{q^N} \right\rfloor \right) = 0.
\]

Conversely, if \( f(x) \in \mathcal{L}_N(\mathcal{O}_K, \mathbb{C}_p) \), then it has an expansion

\[
f(x) = \sum_{n=0}^{\infty} a_n P_n(x^{\varpi_p})
\]

of the form

\[
|a_n| \leq c \left| \frac{\pi}{q} \right|^N \rho \left( \left\lfloor \frac{n}{q^N} \right\rfloor \right), \quad \lim_{n \to 0} |a_n|/\rho \left( \left\lfloor \frac{n}{q^N} \right\rfloor \right) = 0,
\]

where \( c = 1 \) if \( e \leq p - 1 \), and \( c = \overline{\rho}(0) \) otherwise.

**Corollary 1.3** *(Corollary 4.8)*. — Suppose

\[
e_{N,n} := \gamma \left( \left\lfloor \frac{n}{q^N} \right\rfloor \right) P_n(x^{\varpi_p}), \quad (n = 0, 1, \ldots),
\]

where \( \gamma(u) \) is an element in \( \mathbb{C}_p \) such that \( \rho(u) = |\gamma(u)| \). If we denote by \( L_N \) the \( \mathcal{O}_{\mathbb{C}_p} \)-module topologically generated by \( e_{N,n} \), then

\[
\overline{\rho}(0)^{-2} \left| \frac{q}{\pi} \right|^N \mathcal{L}_N(\mathcal{O}_K, \mathbb{C}_p) \subset L_N \subset \mathcal{L}_N(\mathcal{O}_K, \mathbb{C}_p).
\]

In particular, \( L_N \otimes \mathbb{Q}_p = \mathcal{L}_N(\mathcal{O}_K, \mathbb{C}_p) \). In other words, the functions \( e_{N,n} \) form a \( p \)-adic Banach basis of \( \mathcal{L}_N(\mathcal{O}_K, \mathbb{C}_p) \). Moreover, if \( e \leq p - 1 \), then

\[
\left| \frac{q}{\pi} \right|^{N+1} \mathcal{L}_N(\mathcal{O}_K, \mathbb{C}_p) \subset L_N \subset \mathcal{L}_N(\mathcal{O}_K, \mathbb{C}_p).
\]

This result for the case \( \mathcal{O}_K = \mathbb{Z}_p \) gives the result of Amice [1, Théorème 3], namely that the functions

\[
\left\lfloor \frac{n}{P^N} \right\rfloor! \left( x \atop n \right) \quad (n = 0, 1, \ldots)
\]

form a topological basis of \( \mathcal{L}_N(\mathbb{Z}_p, \mathbb{C}_p) \) (actually, we can show that it is a basis of \( \mathcal{L}_N(\mathbb{Z}_p, \mathbb{Q}_p) \)).

As another application, in Theorem 5.8, we derive from our estimate of the integral the congruence of Bernoulli-Hurwitz numbers \( BH(n) \) at supersingular primes established by Katz [7, Theorem 3.11] and Chellali [4, Théorème 1.1].
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2. Schneider-Teitelbaum’s $p$-adic Fourier theory.

Let $K$ be a finite extension of $\mathbb{Q}_p$ and $k = \mathbb{F}_q$ the residue field. Let $e$ be the absolute ramification index of $K$. We fix a uniformizer $\pi$ of $K$ and let $\mathcal{G}$ be a Lubin-Tate formal group of $K$ associated to $\pi$. For a natural number $N$ and an element $a$ of $O_K$, we define the space $A(a + \pi^N O_K, \mathbb{C}_p)$ of $K$-analytic functions on $a + \pi^N O_K$ by

$$A(a + \pi^N O_K, \mathbb{C}_p) := \left\{ f : a + \pi^N O_K \to \mathbb{C}_p \mid f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n, a_n \in \mathbb{C}_p, \pi^n a_n \to 0 \right\}.$$

We equip the space $A(a + \pi^N O_K, \mathbb{C}_p)$ with the norm

$$\|f\|_{a,N} := \max_n \{ |\pi^n a_n| \} = \max_{x \in a + \pi^N O_K} \{|f(x)|\}.$$

We also define the space $LA_N(O_K, \mathbb{C}_p)$ of locally $K$-analytic functions on $O_K$ of order $N$ by

$$LA_N(O_K, \mathbb{C}_p) := \left\{ f : O_K \to \mathbb{C}_p \mid f|_{a + \pi^N O_K} \in A(a + \pi^N O_K, \mathbb{C}_p) \text{ for any } a \in O_K \right\},$$

which is a $p$-adic Banach space by the norm $\max_a \{ \|f\|_{a,N} \}$. We denote by $LA_N(O_K, \mathbb{C}_p)_0$ the submodule of elements whose absolute values are less than or equal to 1. We put

$$LA(O_K, \mathbb{C}_p) = \bigcup_N LA_N(O_K, \mathbb{C}_p)$$

and equip it with the inductive limit topology. A continuous $\mathbb{C}_p$-linear function $LA(O_K, \mathbb{C}_p) \to \mathbb{C}_p$ is called a $\mathbb{C}_p$-valued distribution on $O_K$. We denote the space of $\mathbb{C}_p$-valued distributions on $O_K$ by $D(O_K, \mathbb{C}_p)$, i.e.

$$D(O_K, \mathbb{C}_p) = \varprojlim_N \text{Hom}_{\mathbb{C}_p}^{\text{cont}}(LA_N(O_K, \mathbb{C}_p), \mathbb{C}_p).$$
We write an element of $D(O_K, \mathbb{C}_p)$ symbolically as
\[ \int d\mu : LA(O_K, \mathbb{C}_p) \to \mathbb{C}_p, \quad f \mapsto \int d\mu f = \int_{O_K} f(x) d\mu(x). \]

The space $D(O_K, \mathbb{C}_p)$ has a product structure given by the convolution product. For a compact open set $U$ of $O_K$, we let
\[ \int_U f(x) d\mu(x) := \int_{O_K} f(x) \cdot 1_U(x) d\mu(x), \]
where $1_U$ is the characteristic function of $U$.

The structure of $D(O_K, \mathbb{C}_p)$ is well-known for the case $K = \mathbb{Q}_p$ and described through the so-called Amice transform. We denote by $R^{\text{rig}}$ the ring of rigid analytic functions on the open disc of radius 1, that is, the ring of power series of the form $\varphi(T) = \sum_{n=0}^{\infty} c_n T^n$ such that $|c_n| r_0^n \to 0$ for any $0 < r_0 < 1$. Then there exists an isomorphism of topological $\mathbb{C}_p$-algebras
\[ D(\mathbb{Z}_p, \mathbb{C}_p) \cong R^{\text{rig}}, \quad \mu \mapsto \varphi \]
that is characterized by the equation
\[ c_n = \int_{\mathbb{Z}_p} \binom{x}{n} d\mu(x) \]
or equivalently
\[ \varphi(T) = \int_{\mathbb{Z}_p} (1 + T)^x d\mu(x). \]

For the Mahler expansion
\[ f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n} \]
of $f \in LA(\mathbb{Z}_p, \mathbb{C}_p)$, Amice showed that $|a_n| r^n \to 0$ for some $r > 1$ and hence we can compute the integral as
\[ \int_{\mathbb{Z}_p} f(x) d\mu = \sum_{n=0}^{\infty} a_n c_n. \]

Schneider-Teitelbaum [8, Theorem 2.3] constructed an isomorphism analogous to (2.1) for a general local field $K$.

Let $\varpi_p$ be a $p$-adic period of $\mathcal{G}$. By Tate’s theory of $p$-divisible groups and Lubin-Tate theory, we have
\[ \text{Hom}_{O_{\mathbb{C}_p}}(\mathcal{G}, \widehat{\mathbb{G}}_m) \cong \text{Hom}_{\mathbb{Z}_p}(T_p \mathcal{G}, T_p \widehat{\mathbb{G}}_m) \cong O_K. \]
(The last isomorphism is non-canonical.) Hence there exists a generator of the $O_K$-module $\text{Hom}_{O_{\mathbb{C}_p}}(\mathcal{G}, \widehat{\mathbb{G}}_m)$, which is written in the form of the integral power series $\exp(\varpi_p \lambda(t)) \in \mathcal{O}_{\mathbb{C}_p}[[t]]$ where $\lambda(t)$ is the logarithm of
The element $\varpi_p \in \mathcal{O}_C$ is determined uniquely up to an element of $\mathcal{O}_K$. We fix such a $\varpi_p$ and call it the $p$-adic period of $\mathcal{G}$ (if the height of $\mathcal{G}$ is equal to 1, then the inverse of $\varpi_p$ is often called a $p$-adic period of $\mathcal{G}$). For example, see [9]). It is known that $|\varpi_p| = p^{-s}$, where $s = \frac{1}{p-1} - \frac{1}{e(q-1)}$ (see Appendix of [8] or an elementary proof in [3] when $K/\mathbb{Q}_p$ is unramified). We define the polynomials $P_n(X) \in K[X]$ by the formal expansion

$$\text{exp}(X\lambda(t)) = \sum_{n=0}^{\infty} P_n(X) t^n.$$  

Note that in the case $\mathcal{G} = \hat{G}_m$, $\pi = p$ and $\lambda(t) = \log(1 + t)$, the polynomial $P_n(X)$ is simply the binomial polynomial $\binom{X}{n}$. By construction, $P_n(x\varpi_p)$ is in $\mathcal{O}_C$ if $x \in \mathcal{O}_K$.

**Theorem 2.1** (Schneider-Teitelbaum [8, §4]).

i) The series

$$\sum_{n=0}^{\infty} a_n P_n(x\varpi_p)$$

converges to an element of $LA(\mathcal{O}_K, \mathbb{C}_p)$ if $\lim_{n} |a_n|^{\frac{1}{p^n}} < 1$. Conversely, any locally $K$-analytic function $f(x)$ on $\mathcal{O}_K$ has a unique expansion

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x\varpi_p)$$

for some sequence $(a_n)_n$ in $\mathbb{C}_p$ such that $\lim_{n} |a_n|^{\frac{1}{p^n}} < 1$.

ii) There exists an isomorphism of topological $\mathbb{C}_p$-algebras

$$(2.3) \quad D(\mathcal{O}_K, \mathbb{C}_p) \cong \mathbb{R}^{\text{rig}}.$$  

having the following characterization property: if $\varphi(T) = \sum_{n=0}^{\infty} c_n T^n$ corresponds to a distribution $\mu$, then

$$c_n = \int_{\mathcal{O}_K} P_n(x\varpi_p) d\mu(x)$$

or equivalently

$$\varphi(t) = \int_{\mathcal{O}_K} \text{exp}(x\varpi_p \lambda(t)) d\mu(x).$$

Schneider and Teitelbaum called the power series $\varphi(t)$ corresponding to $\mu$ the Fourier transform of $\mu$ and denoted it by $F_\mu(t)$.
3. Power sums

In this section, we give an estimate of the absolute value of the power sum
\[ S_{N,n,k} := \partial_G^n \sum_{t_N \in \mathcal{G}[\pi^N]} (t \oplus t_N)^k |_{t=0}, \]
where \( x \oplus y = \mathcal{G}(x, y) \), \( \partial_G \) is the differential operator \( \lambda'(t)^{-1} (d/dt) \), and \( \mathcal{G}[\pi^N] \) is the kernel of the multiplication \( [\pi^N] \) of \( \mathcal{G} \). This estimate is crucial for everything in this paper. We use Newton’s method to compute this value.

We define \( \overline{p}[l, n] \) and \( \underline{p}[l, n] \) by
\[ \overline{p}[l, n] = \max_{l \leq m \leq n} \{|m!/\varpi_p^m|\}, \quad \underline{p}[l, n] = \min_{l \leq m \leq n} \{|m!/\varpi_p^m|\} \]
for \( l \leq n \). For \( l > n \), we put \( \overline{p}[l, n] = 0 \) and \( \underline{p}[l, n] = \infty \). Then \( \underline{p}(k) = \overline{p}[k, \infty] \) and \( \underline{p}(k) = \overline{p}[0, k] \) are the constants appearing in the introduction.

**Proposition 3.1.**

i) The values \( \overline{p}(k) \) and \( \underline{p}(k) \) are decreasing with \( k \).

ii) We have
\[ \underline{p}(k) \leq \overline{p}(k), \quad \overline{p}(k) \leq \overline{p}(0) \underline{p}(k). \]

iii) We have
\[ \underline{p}(k_1 + \cdots + k_n) \leq \underline{p}(k_1) \cdots \underline{p}(k_n). \]

iv) We have
\[ \underline{p}^{\frac{1}{p-1}} k \lesssim_{(q-1)} \underline{p}(k) \leq 1. \]

**Proof.** — i) is clear. For ii), first we have \( \overline{p}(k) \geq |k!/\varpi_p^k| \geq \underline{p}(k) \). Suppose \( \overline{p}(k) = |k_1!/\varpi_p^{k_1}| \) and \( \underline{p}(k) = |k_2!/\varpi_p^{k_2}| \). Then \( k_1 \geq k \geq k_2 \) and
\[ \left| \frac{k_1!}{\varpi_p^{k_1}}/\frac{k_2!}{\varpi_p^{k_2}} \right| = \left| \frac{k_1!}{k_2!} \frac{(k_1-k_2)!}{\varpi_p^{k_1-k_2}} \right| \leq \overline{p}(0). \]

For iii), suppose that \( \underline{p}(k_i) = |l_i!/\varpi_p^{l_i}| \) for \( l_i \leq k_i \). Then the assertion for \( \underline{p} \) follows from
\[ \underline{p}(k_1 + \cdots + k_n) \leq \left| \frac{(l_1 + \cdots + l_n)!}{\varpi_p^{l_1 + \cdots + l_n}} \right| \leq \left| \frac{(l_1 + \cdots + l_n)!}{l_1! \cdots l_n!} \left| \frac{l_1!}{\varpi_p^{l_1}} \cdots \frac{l_n!}{\varpi_p^{l_n}} \right| . \]

For iv), suppose that \( \underline{p}(k) = |l!/\varpi_p^l| \) for \( l \leq k \). Then
\[ \underline{p}^{\frac{1}{p-1}} k \lesssim_{(q-1)} \underline{p}(k) \leq \left| \frac{l!}{\varpi_p^l} \right| = \underline{p}(k). \]
\[ \square \]
If \( e \leq p - 1 \), then we can determine \( \bar{p}(k) \) and \( \underline{p}(k) \) explicitly.

**Lemma 3.2.** — Let \( k \) be a non-negative integer and let \( q \) be a power of \( p \).

1. For any integer \( 0 \leq r < q \), we have \( \binom{kq+r}{r} \equiv 1 \mod p \).
2. We have \( \binom{k}{q} \in [k/q] \mathbb{Z}_p \).

**Proof.** — i) is clear. For ii), we write \( k = aq + r \) with \( 0 \leq r < q \). We put \( (1 + x)^q = 1 + x^q + pf(x) \) for some integral polynomial \( f(x) \). Then \( (1 + x)^k = (1 + x^q + pf(x))^a(1 + x)^r \equiv (1 + x^q)^a(1 + x)^r \mod ap\mathbb{Z}_p[x] \).

Hence the coefficient of \( x^q \) in the above is in \( a\mathbb{Z}_p \).

**Proposition 3.3.** — Let \( i, e \) and \( h \) be natural numbers. We put \( q = p^h \).

Then we have

\[
v_p(i!) \geq \frac{i}{p-1} - \frac{i}{e(q-1)} - h + \frac{1}{e} + \left[ \frac{i}{q} \right] \left( \frac{1}{e} - \frac{1}{p-1} + \frac{1}{e(q-1)} \right) + v_p \left( \left[ \frac{i}{q} \right] ! \right).
\]

In the above, equality holds if and only if \( i \equiv -1 \mod q \). In particular, if \( e \leq p - 1 \) or \( i < q \), then we have

\[
v_p(i!) \geq \frac{i}{p-1} - \frac{i}{e(q-1)} - h + \frac{1}{e}
\]

and equality holds if and only if \( i = q-1 \). In this case, we have \( \bar{p}(0) = |\pi/q| \).

**Proof.** — First, we assume that \( i < q \). We prove the inequality by induction on \( h \). If \( h = 1 \), then \( i < p \). Hence the left-hand side \( v_p(i!) \) is equal to zero, and the right-hand side takes the maximum value when \( i = p - 1 \), which is also equal to zero. We assume that the inequality holds for natural numbers less than \( h \). Since the right-hand side is strictly increasing for \( i \), and \( v_p(i!) \) strictly increases only when \( p \) divides \( i \), we may assume that \( i \) is of the form \( i = kp - 1 \) for some natural number \( k \leq p^{h-1} \). We have

\[
v_p(i!) = v_p((kp)!) - v_p(kp) = k - 1 + v_p((k-1)!).
\]

On the other hand, we have

\[
\frac{i}{p-1} - \frac{i}{e(q-1)} - h + \frac{1}{e} = (k-1) + \frac{k-1}{p-1} - \frac{k-1}{e(p^{h-1}-1)} - (h-1) + \frac{k-1}{e(p^{h-1}-1)} - \frac{k-1}{e(q-1)} \leq k - 1 + v_p((k-1)!).
\]
In the last inequality, we used the inductive hypothesis and $k \leq p^{h-1}$. Hence we have the desired inequality, and equality holds only when $k = p^{h-1}$, i.e. when $i = q - 1$. For $i \geq q$, by Lemma 3.2 ii) and by induction, we have

\[
v_p(i!) \geq v_p((i - q)!) + v_p(q!) + v_p\left(\left\lfloor \frac{i}{q} \right\rfloor \right) \geq \frac{i}{p - 1} - \frac{i}{e(q - 1)} - h + \frac{1}{e} + \left\lfloor \frac{i}{q} \right\rfloor \left(\frac{1}{e} - \frac{1}{p - 1} + \frac{1}{e(q - 1)}\right) + v_p\left(\left\lfloor \frac{i}{q} \right\rfloor !\right).
\]

From the above argument and by induction, to have the equality, $i$ must be congruent to $-1$ modulo $q$. On the other hand, if $i \equiv -1 \mod q$, then direct calculations give the equality. □

**Proposition 3.4.** — Suppose that $e \leq p - 1$, and that $e > 1$ or $h > 1$.

i) We have $|n!/\varpi_p^n| > 1$ for $0 < n < q$.

ii) For any non-negative integer $n$, $\rho(n) = |n_0!/\varpi_p^{n_0}|$ where $n_0 = \lfloor n/q \rfloor q$.

iii) For $n \equiv -1 \mod q$ and a natural number $i \neq q$, we have

\[
\left|\frac{n!}{\varpi_p^n}\right| > \left|\frac{(n + q)!}{\varpi_p^{n+q}}\right| > \left|\frac{(n + i)!}{\varpi_p^{n+i}}\right|
\]

In particular, for any non-negative integer $n$, we have $\overline{\rho}(n) = |n_1!/\varpi_p^{n_1}|$ where $n_1 = \lfloor n/q \rfloor q + q - 1$.

**Proof.** — We prove i) by induction on $h$ of $q = p^h$. If $h = 1$, then $n!$ is a $p$-adic unit and the assertion is clear. Assume that $h > 1$. We write as $n = kp + r$ with $0 \leq r < p$. Then

\[
\frac{n!}{\varpi_p^n} = \binom{n}{r} \frac{(kp)!}{\varpi_p^{kp}} r! \frac{\varpi_p^r}{\varpi_p^r}.
\]

Hence by Lemma 3.2 i) and the induction on $n$, we may assume that $r = 0$ and $k \geq 1$. Then

\[
v_p\left(\frac{(kp)!}{\varpi_p^{kp}}\right) = v_p((kp)!) - \frac{kp}{p - 1} + \frac{kp}{e(q - 1)} < v_p(k!) - \frac{k}{p - 1} + \frac{k}{e(p^{h-1} - 1)}.
\]

By the inductive hypothesis for $h$, the right-hand side is negative or 0.

Next we prove ii). Suppose that $m < n_0$. Then

\[
\left|\frac{n_0!}{\varpi_p^{n_0}} \cdot \frac{m!}{\varpi_p^m} \right| = \frac{n_0}{\varpi_p} \left(\frac{n_0 - 1}{m}\right) \frac{(n_0 - m - 1)!}{\varpi_p^{n_0 - m - 1}} \leq \frac{n_0}{\varpi_p} \overline{\rho}(0) = \frac{n_0!}{\varpi_p^q} \left|\frac{n_0!}{\varpi_p^q}\right| < 1.
\]
Suppose that \( n \geq m > n_0 \). We write as \( m = \lfloor n/q \rfloor q + r \) with \( 0 < r < q \).

Then i) and Lemma 3.2 i) show that

\[
\left| \frac{n_0!}{\varpi_p^{n_0}} \right| = \left| \frac{m!}{\varpi_p^m} \right| = \left| \frac{(m)^{-1}}{r!} \right| < 1.
\]

Finally, we show iii). Let \( n \) be such that \( n \equiv -1 \mod q \). We have

\[
\left( \frac{n+i}{\varpi_p^{n+i}} \right) = \frac{(n+i)!}{\varpi_p^{n+i}} = q \frac{(i-1)! \pi \varpi_p^{q-1}}{q!},
\]

where \( u = (q-1)^{-1} (n+i) \) is a \( p \)-adic integer by Lemma 3.2 i). By Proposition 3.3, the \( p \)-adic (additive) valuation of the right-hand side is positive.

Since \( v_p(\pi/\varpi_p) > 0 \), the \( p \)-adic (additive) valuation of

\[
\frac{(n+i)!}{\varpi_p^{n+i}} = \frac{(n+i)!}{(n+q)!} \frac{q! \pi}{\varpi_p^{q-1}}
\]

is positive. \( \square \)

Next we investigate the absolute values of the coefficients of a power of the logarithm and the exponential map of the Lubin-Tate group. The case \( k = 1 \) in the proposition below is obtained in [10].

**Proposition 3.5.** — We put \( \partial = d/dt \). Then we have

\[
\left| \frac{\varpi_p^k \partial^n \lambda(t)^k}{k! n!} \right|_{t=0} \leq \rho[k,n]^{-1}, \quad \left| \partial^n \exp G(t) \right|_{t=0} \leq |\varpi_p^n|\bar{\rho}[k,n].
\]

**Proof.** — The case for \( n < k \) or \( k = 0 \) is trivial. Suppose that \( n \geq k \geq 1 \). We first assume that the formal logarithm of \( G \) is given by

\[
\lambda(t) = \sum_{m=0}^{\infty} \frac{t^m}{\pi^m}.
\]

Then it suffices to show inequalities

\[
|\partial^n \lambda(t)^k|_{t=0} \leq |k! \varpi_p^{-k}|, \quad |\partial^n \exp G(t)|_{t=0} \leq |\varpi_p^n|\bar{\rho}[k,n].
\]

When \( k = 1 \), the inequality for \( \lambda(t) \) is proven by direct calculations. We prove the general case by induction on \( k \). We have

\[
\partial^n \lambda(t)^k|_{t=0} = k \partial^{n-1} (\lambda(t)^{k-1} \lambda'(t))|_{t=0}
\]

\[
= k \partial^{n-1} \sum_{m=0}^{\infty} \frac{\lambda(t)^{k-1} q^m t^m}{\pi^m} |_{t=0}
\]

\[
= \sum_{m=0}^{\infty} \binom{n-1}{q^m-1} \frac{q^m!k}{\pi^m} \partial^{n-q^m} \lambda(t)^{k-1} |_{t=0}.
\]
Hence we have \(|\partial^n\lambda(t) \big|_{t=0} \leq |k!\exp_n - k|\).

We put \(\exp_k^G(t) = \sum_{n=k}^\infty a_n t^n\). We prove that \(|n!a_n| \leq |k!\exp_n - k|\) by induction on \(n\). If \(n = k\), then the assertion is true since \(a_k = 1\). We assume that the assertion is true for integers less than \(n\). Since \(\exp_k^G(\lambda(t)) = t^k\), we have

\[t^k = a_k\lambda(t)^k + a_{k+1}\lambda(t)^{k+1} + \cdots + a_n\lambda(t)^n + \cdots.\]

By i) and the inductive hypothesis, we have

\[|a_m\partial^m\lambda(t) \big|_{t=0} \leq |k!\exp_n - k|\]

for \(m < n\). Since \(\partial^n\lambda(t) \big|_{t=0} = n!\) and \(\partial^n\lambda(t) \big|_{t=0} = 0\) for \(n < m\), the assertion is also true for \(n\).

Now we consider a general parameter \(s\). Then the logarithm and the exponential for \(G\) with parameter \(s\) are of the form \(\lambda(\phi(s))\) and \(\psi(\exp_G(s))\) for some \(\phi(s), \psi(s) \in s\mathcal{O}_K[[s]]^\times\). We put \(\lambda(t)^k = \sum_{n=k}^\infty c_n^{(k)} t^n\) and \(\lambda(\phi(s))^k = \sum d_n^{(k)} s^n\). Then we have shown that \(|c_n^{(k)}| \leq |k!\exp_n - k/n|\). Since \(d_n^{(k)}\) is a linear sum of \(c_l^{(k)} (k \leq l \leq n)\) with integral coefficients, we have

\[\frac{|ϖ_p^{k} d_n^{(k)}}{k!} \leq \max_{k \leq l \leq n} \left\{ c_l^{(k)} \frac{ϖ_p^{k}}{k!} \right\} \leq \max_{k \leq l \leq n} \left\{ \frac{a_p^{l}}{l!} \right\} = \rho[k, n]^{-1}.\]

Hence we have the inequality for the logarithm. The inequality for the exponential is straightforward.

\[\square\]

**Lemma 3.6.**

i) Suppose that \(f(t) \in \mathcal{O}_K[[t]]\) satisfies \(f(t \oplus t_N) = f(t)\) for all \(t_N \in G[\pi^N]\). Then there exists a power series \(g(t) \in \mathcal{O}_K[[t]]\) such that \(f(t) = g([\pi^N]t)\).

ii) There exists an integral power series \(g_k(t) \in \mathcal{O}_K[[t]]\) such that

\[\pi^{-N} \sum_{t_N \in G[\pi^N]} (t \oplus t_N)^k = g_k([\pi^N]t)\].

**Proof.** — See [5], Chapter III.

We put

\[F(t, X) = \prod_{t_N \in G[\pi^N]} (1 - (t \oplus t_N)X) = 1 + \alpha_1(t)X + \cdots + \alpha_{q^N}(t)X^{q^N}.\]

For \(\partial_X = \partial/\partial X\), we consider the power series

\begin{equation}
\pi^{-N} \frac{\partial_X F(t, X)}{F(t, X)} = -\sum_{k=0}^\infty \left( \pi^{-N} \sum_{t_N \in G[\pi^N]} (t \oplus t_N)^{k+1} \right) X^k.
\end{equation}

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By Lemma 3.6 and the above formula, \( \pi^{-N} \partial_X F(t, X) \in \mathcal{O}_K[[t]][X]. \)

**Proposition 3.7. —** Let \( k, n \) be non-negative integers and \( N \) a natural number. Then we have

\[
\pi^{-N} \left| \sum_{t_N \in \mathcal{G}[\pi^N]} \partial^n_{\mathcal{G}}(t \oplus t_N)^k \right|_{t=0} \leq \left| \pi^{Nn + k_0(1 - 1/q^n)} \overline{\varphi}_n \right| \left\{ \left[ \frac{k}{q^N} \right] \right\} \varphi(0),
\]

where \( k_0 = \max\{[k/q^N] - n, 0\} \). We also have

\[
\pi^{-N} \left| \sum_{t_N \in \mathcal{G}[\pi^N]} \partial^n_{\mathcal{G}}(t \oplus t_N)^k \right|_{t=0} \leq \left| \pi^{Nn} \overline{\varphi}_n \right| \left\{ \left[ \frac{k}{q^N} \right] \right\}.
\]

Moreover, if \( e \leq p - 1 \), we have

\[
\pi^{-N} \left| \sum_{t_N \in \mathcal{G}[\pi^N]} \partial^n_{\mathcal{G}}(t \oplus t_N)^k \right|_{t=0} \leq \left| \pi^{Nn} \overline{\varphi}_n \right| \left\{ \left[ \frac{k}{q^N} \right] \right\}. 
\]

**Proof. —** We put \( G(t, X) = F(0, X) - F(t, X) \), then \( G(0, X) = G(t, 0) = 0 \). We have

\[
\frac{1}{F(t, X)} = \frac{1}{F(0, X) - G(t, X)} = \sum_{l=0}^{\infty} \frac{G(t, X)^l}{F(0, X)^{l+1}} \in \mathcal{O}_K[[t, X]].
\]

Since \( G(0, X) = 0 \) and \( G(t, X) \) is invariant for the translation \( t \mapsto t_N \), it is of the form

\[
G(t, X) = (|[\pi^n]| t) H([\pi^n] t, X)
\]

for some element \( H \) in \( \mathcal{O}_K[[t]][X] \). Since \( F(0, X) \equiv 1 \mod \pi \), the power series \( F(0, X)^{-l-1} \) is equal to

\[
\sum_{m=0}^{\infty} \binom{-l-1}{m} (F(0, X) - 1)^m = \sum_{m=0}^{\infty} \binom{l+m}{m} \pi^m \left( \frac{1 - F(0, X)}{\pi} \right)^m.
\]

Hence we have

\[
\frac{\pi^{-N} \partial_X F(t, X)}{F(t, X)} = \sum_{l=0}^{\infty} \pi^{-N} \partial_X F(t, X) \cdot G(t, X)^l \cdot F(0, X)^{-l-1}
\]

\[
= \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \binom{l+m}{m} \pi^m (\pi^{-N} \partial_X F(t, X)) \cdot G(t, X)^l \left( \frac{1 - F(0, X)}{\pi} \right)^m.
\]

To show the assertion for \( k + 1 \), we look the coefficient of \( X^k \) in the last term of (3.6). We consider the coefficients of the terms \( X^a \), \( X^b \) and \( X^c \) with \( a + b + c = k \) of \( \pi^{-N} \partial_X F(t, X) \), \( G(t, X)^l \) and \( (1 - F(0, X))^m \pi^{-m} \) respectively. Since \( \deg \partial_X F(t, X) = q^N - 1 \), \( \deg G(t, X) = q^N \) and \( \deg (1 -
\[ F(0, X) = q^N - 1 \text{ as polynomials for } X, \text{ we have } a \leq q^N - 1, b \leq lq^N \text{ and } c \leq m(q^N - 1). \text{ Then by (3.5), the product of these coefficients is an integral linear combination of the terms of the form} \]
\[ (l + m \atop m) \pi^m G_l([\pi^N]t) \]
\[ \text{where } G_l(t) \text{ is a power series in } t^l \mathcal{O}_K[[t]] \text{ and } l, m \text{ satisfies} \]
\[ a + lq^N + m(q^N - 1) \geq a + b + c = k. \]
We estimate the absolute value of
\[ (l + m \atop m) \pi^m \partial^n_G G_l([\pi^N]t) \big|_{t=0}. \]
By Proposition 3.5, we have
\[ \left| \frac{\partial^n_G G_l([\pi^N]t)}{t=0} \right| = \left| \pi^N \frac{d^n}{dz^n} \exp_G(z) \big|_{z=0} \right| \leq \left| \pi^N \varpi^n_p \right| \bar{\varrho}(d, n). \]
Therefore, we have
\[ \left| \frac{\partial^n_G G_l([\pi^N]t)}{t=0} \right| \leq \left| \pi^N \varpi^n_p \right| \bar{\varrho}(l, n). \]
Hence we have (3.3). If \( n < l \), then (3.8) is zero and there is nothing to prove. We assume that \( n \geq l \). We let \( l' \geq l \) be such that \( \rho(l) = \frac{l'!}{\varpi^l}. \)
Then
\[ (l + m \atop m) \pi^m \partial^n_G G_l([\pi^N]t) \big|_{t=0} \leq \left| (l + m \atop m) \pi^{m+N} \varpi^n_p \right| \bar{\varrho}(l, n) \]
\[ \leq \left| \pi^N \varpi^n_p (l + m)! (l' - l)! (l') \frac{\varpi^n_p \pi^m}{m!} \right|. \]
First we consider the case \( a \leq q^N - 2 \) or \( m \neq 0 \). Then by (3.7) we have
\[ l + m \geq \left[ \frac{k+1}{q^N} \right]. \]
In particular, \( m \geq \left[ \frac{(k+1)/q^N} - n \right] \) and the value (3.10) is less than or equal to
\[ \left| \pi^{N+n} \varpi^{n+1} \left[ \frac{k+1}{q^N} \right] \varrho \right| \bar{\varrho}(0) \]
where \( k_0 = \max \{(k+1)/q^N - n, 0\}. \text{ Hence in this case we have (3.2). Suppose that } e \leq p - 1. \text{ If } l' < l + m, \text{ then } |\varpi^m_p| < |\varpi^{l' - l}_p| \text{ and hence the value (3.10) is less than } |\pi^{N+n} \varpi^n_p| \left[ \frac{k+1}{q^N} \right]. \text{ If } l' \geq l + m, \text{ then} \]
\[ \bar{\varrho}(l) = \left| \frac{l'!}{\varpi^{l'}_p} \right| \leq \bar{\varrho}(l + m) \leq \bar{\varrho} \left( \left[ \frac{k+1}{q^N} \right] \right). \]
Hence the value (3.9) is also less than or equal to \(|\pi^{m+Nn}\wedge^n|p\left(\frac{k+1}{q^N}\right)|\).

Hence in this case we have (3.4).

Finally we consider the case when \(a = q^N - 1\) and \(m = 0\). Then the coefficient of \(\pi^{-N}\partial_X F(t, X)\) of degree \(a\) is \((q/\pi)^N\alpha_{q^N}(t)\), which is divisible by \([\pi^N]t\). Hence in this case the product of the coefficient of \(X^a\) in \(\pi^{-N}\partial_X F(t, X)\), the coefficient of \(X^b\) in \(G(t, X)^l\) and the coefficient of \(X^c\) in \((1 - F(0, X))^m\pi^{-m}\) is an integral linear combination of terms in the form \(G_{l+1}(\pi^N t)\) for some \(G_{l+1}(t) \in \mathcal{O}_K[[t]]\). In this case \(l\) satisfies \(l + 1 \geq [(k + 1)/q^N]\). Therefore

\[
|\partial^n G_{l+1}(\pi^N t)|_{t=0} \leq \left|\pi^N \wedge^n p\right|[l + 1, n] \leq \left|\pi^N \wedge^n p\right|\bar{p}\left(\frac{k + 1}{q^N}\right).
\]

If \(n < l + 1\), then (3.8) is zero and there is nothing to prove. We assume that \(n \geq l + 1\). In particular, by (3.7) we have \(n \geq [(k + 1)/q^N]\), and hence \(k_0 = \max\{[(k + 1)/q^N] - n, 0\} = 0\). Therefore we have (3.2) and (3.4). \(\square\)

4. Integral structures on \(p\)-adic Fourier theory

In this section, we give an explicit construction of Schneider-Teitelbaum’s \(p\)-adic distribution associated to a rigid analytic function on the open unit disc.

Let \(\varphi(t)\) be a rigid analytic function on the open unit disc. We will construct a distribution \(\mu_\varphi\) on \(\mathcal{O}_K\) such that

\[
\int_{\mathcal{O}_K} \exp(x\wedge p\lambda(t))d\mu_\varphi = \varphi(t).
\]

If we were able to first prove a Mahler like expansion for \(K\)-analytic functions as in the case of \(K = \mathbb{Q}_p\), then it would be possible to define the integral by (2.2). However, as in [8], we will first define the integral then use this integral to prove the existence of the Mahler like expansion for \(K\)-analytic functions. Our construction of the integral is different from that of [8] in that we investigate directly the explicit power series corresponding to the moments of the integral, instead of formally reducing to the case of \(\mathbb{Z}_p\).

We fix a Lubin-Tate formal group \(G\) associated to \(\pi\), and denote its addition by \(\oplus\). For \(a \in \mathcal{O}_K\) and a natural number \(N\), we let

\[
\int_{a+\pi^N \mathcal{O}_K} (x - a)^n \ d\mu_\varphi := \frac{1}{q^N \wedge^n p}\left|\partial^n G_{t_N \in \mathcal{G}[\pi^N]} \varphi_a(t \oplus t_N)\right|_{t=0}
\]
where 
\[ \varphi_a(t) := \exp(-a\varpi_p\lambda(t))\varphi(t). \]

We put \( \varphi(t) = \sum_{k=0}^{\infty} c_k t^k \) and \( \varphi_a(t) = \sum_{k=0}^{\infty} c_k^{(a)} t^k \). Then by Proposition 3.7, we have

\[
\left| \int_{a+\pi^N\mathcal{O}_K} (x - a)^n \, d\mu_{\varphi} \right| \leq \rho(0) \left| \frac{\pi}{q} \right|^N \left| \frac{k}{q^N} \right| \sup_k \left\{ |c_k^{(a)}| \overline{p}(\left[ \frac{k}{q^N} \right]) \right\}.
\]

Here for the last estimate, we used the facts that \( c_k^{(a)} \) is an integral linear combination of \( c_0, \ldots, c_k \) and the function \( \rho(m) \) for \( m \) is decreasing.

We define the distribution \( \mu_{\varphi} \) on \( \mathcal{L}A_N(O_K, \mathbb{C}_p) \) as follows. For an element \( f \) of \( \mathcal{L}A_N(O_K, \mathbb{C}_p) \), suppose \( f \) is of the form \( \sum_{n=0}^{\infty} a_n (x - a)^n \) such that \( a_n \pi^n \to 0 \) if \( n \to \infty \) on \( a + \pi^N O_K \). Then we define the integral of \( f \) on \( a + \pi^N O_K \) by

\[
\int_{a+\pi^N\mathcal{O}_K} f(x) \, d\mu_{\varphi} := \sum_{n=0}^{\infty} a_n \int_{a+\pi^N\mathcal{O}_K} (x - a)^n \, d\mu_{\varphi}.
\]

We define

\[
\int_{\mathcal{O}_K} f(x) \, d\mu_{\varphi} = \sum_{a \mod \pi^N} \int_{a+\pi^N\mathcal{O}_K} f(x) \, d\mu_{\varphi}.
\]

We have to show the well-definedness of the integral.

**Proposition 4.1.**

i) The integral (4.2) converges and does not depend on the choice of the representative of \( a \mod \pi^N \). The integral (4.3) does not depend on the choice of \( N \). Hence \( \mu_{\varphi} \) gives a well-defined element of \( D(O_K, \mathbb{C}_p) \).

ii) For a polynomial \( f(x) \), we have

\[
\int_{\mathcal{O}_K} f(x) \, d\mu_{\varphi} = f(\varpi_p^{-1} \partial_G)\varphi(t)|_{t=0}.
\]

**Proof.** — Since \( \overline{p}([k/q^N]) \leq C k \frac{1}{e q^N (q-1)} \) for some constant \( C \) which depends only on \( e, q \) and \( N \), the value \( \sup_k \{ |c_k| \overline{p}(\left[ \frac{k}{q^N} \right]) \} \) is finite. Hence the convergence follows from (4.1). We show that the integral (4.2) depends only on the class of \( a \) modulo \( \pi^N \). Since the integral is convergent, we may...
assume that \( f \) is a monomial \( (x - a)^n \). For \( a' \) such that \( a' \equiv a \mod \pi^N \), we put \( b = a' - a \). Since

\[
(x - a)^n|_{a' + \pi^N \mathcal{O}_K} = \sum_{l=0}^{n} \binom{n}{l} b^{n-l} (x - a')^l|_{a' + \pi^N \mathcal{O}_K},
\]

it suffices to show that

\[
\int_{a + \pi^N \mathcal{O}_K} (x - a)^n \, d\mu_\varphi = \sum_{l=0}^{n} \binom{n}{l} b^{n-l} \int_{a' + \pi^N \mathcal{O}_K} (x - a')^l \, d\mu_\varphi.
\]

This follows from

\[
\varpi_p^{-n} \partial_G^n \varphi_a(t \oplus t_m) = \varpi_p^{-n} \partial_G^n (\exp(b \varpi_p \lambda(t)) \varphi_{a'}(t \oplus t_N))
\]

\[
= \exp(b \varpi_p \lambda(t)) \sum_{l=0}^{n} \binom{n}{l} b^{n-l} \varpi_p^{-l} \partial_G^l \varphi_{a'}(t \oplus t_m).
\]

Now we show that the integral (4.3) does not depend on \( N \). It is sufficient to show the distribution relation

\[
(4.4) \quad \int_{a + \pi^N \mathcal{O}_K} f(x) \, d\mu_\varphi = \sum_{b \equiv a \mod \pi^N} \int_{b + \pi^{N+1} \mathcal{O}_K} f(x) \, d\mu_\varphi
\]

where the sum runs over a representative \( b \) of \( \mathcal{O}_K/\pi^{N+1} \) such that \( b \equiv a \mod \pi^N \). To show this, replacing \( \varphi \) by \( \varphi_a \), we may assume that \( a = 0 \) and \( f(x) = x^n \). Then

\[
q^{N+1} \varpi_p^n \sum_{b \equiv 0 \mod \pi^N} \int_{b + \pi^{N+1} \mathcal{O}_K} x^n \, d\mu_\varphi
\]

\[
= \sum_{b \equiv 0 \mod \pi^N} \sum_{i=0}^{k} \binom{n}{k} b^{-k} \left( \varpi_p^{-n-k} \partial_G^k \sum_{t_{N+1} \in \mathcal{G}[\pi^{N+1}]} \varphi_b(t \oplus t_{N+1}) \right)_{t=0}
\]

\[
= \sum_{b \equiv 0 \mod \pi^N} \left( \partial_G^n \sum_{t_{N+1} \in \mathcal{G}[\pi^{N+1}]} \exp(b \varpi_p \lambda(t)) \varphi_b(t \oplus t_{N+1}) \right)_{t=0}
\]

\[
= \sum_{t_{N+1} \in \mathcal{G}[\pi^{N+1}]} \left( \sum_{b \equiv 0 \mod \pi^N} \exp(-b \varpi_p \lambda(t)) \right)_{t=t_{N+1}} \partial_G^n \varphi(t \oplus t_{N+1})_{t=0}
\]

\[
= q^n \left( \partial_G^n \sum_{t_N \in \mathcal{G}[\pi^{N}]} \varphi(t \oplus t_N) \right)_{t=0} = q^{N+1} \varpi_p^n \int_{\pi^N \mathcal{O}_K} x^n \, d\mu_\varphi.
\]

The above calculation is also true when \( a = N = 0 \), and hence we have

\[
\varpi_p^n \sum_{b \in \mathcal{O}_K/\pi} \int_{b + \pi \mathcal{O}_K} x^n \, d\mu_\varphi = \partial_G^n \varphi(t)_{t=0}.
\]
Assertion ii) follows from this equality. 

For $\varphi(t) = \sum_{k=0}^{\infty} c_k t^k \in R^{\text{rig}}$, we define $\|\varphi\|_N$ by

\begin{equation}
\|\varphi\|_N := \max_k \left\{ \left| c_k \right| \bar{p}\left( \left[ \frac{k}{q^N} \right] \right) \right\}.
\end{equation}

Since $\bar{p}\left( \left[ \frac{k}{q^N} \right] \right) \sim p^{-kr}$ where $r = 1/eq^N(q-1)$, the value $\|\varphi\|_N$ is approximately,

$$\|\varphi\|_{B(p^{-r})} = \max_{x \in B(p^{-r})} \{ |\varphi(x)| \}$$

where $B(p^{-r}) \subset \mathbb{C}_p$ is the closed disc with radius $p^{-r}$ at origin.

**Lemma 4.2.** — For an element $a \in O_K$, let $\varphi_a(t) = \exp(-a \pi p \lambda(t)) \varphi(t)$ as before. Then $\|\varphi_a\|_N = \|\varphi\|_N$.

**Proof.** — It suffices to show that $\|\varphi_a\|_N \leq \|\varphi\|_N$. This follows from the same argument showing (4.1). \qed

Then Proposition 3.7 may rewritten as follows, which is a precise version of Theorem 1.1 of the introduction.

**Theorem 4.3.**

i) Suppose that the function $f \in LA_N(O_K, \mathbb{C}_p)$ is given by a polynomial of degree $d$ on $a + \pi^N O_K$ for $a \in O_K$. For $\varphi_k(t) = t^k$, we have

\begin{equation}
\left| \int_{a+\pi^N O_K} f(x) \, d\mu_{\varphi_k} \right| \leq \bar{p}(0) \left[ \frac{\pi^N}{q^N} \right] \| f \|_{a,N}.
\end{equation}

We also have

\begin{equation}
\left| \int_{a+\pi^N O_K} f(x) \, d\mu_{\varphi_k} \right| \leq \bar{p}(0) \left[ \frac{\pi^N}{q^N} \right] \| f \|_{a,N} \bar{p}\left( \left[ \frac{k}{q^N} \right] \right)
\end{equation}

where $k_0 = \max\{[k/q^N] - d, 0\}$. Moreover, if $e \leq p-1$, then we have

\begin{equation}
\left| \int_{a+\pi^N O_K} f(x) \, d\mu_{\varphi_k} \right| \leq \left[ \frac{\pi^N}{q^N} \right] \| f \|_{a,N} \bar{p}\left( \left[ \frac{k}{q^N} \right] \right).
\end{equation}

ii) We have

\begin{equation}
\left| \int_{a+\pi^N O_K} f(x) \, d\mu_{\varphi} \right| \leq \bar{p}(0) \left[ \frac{\pi^N}{q^N} \right] \| f \|_{a,N} \| \varphi \|_N.
\end{equation}

Moreover, if $e \leq p-1$, then

\begin{equation}
\left| \int_{a+\pi^N O_K} f(x) \, d\mu_{\varphi} \right| \leq \left[ \frac{\pi^N}{q^N} \right] \| f \|_{a,N} \| \varphi \|_N.
\end{equation}
Corollary 4.4. — We have

\[ \left| \int_{a+\pi^N\mathcal{O}_K} f(x) \, d\mu_x \right| \leq p^{\frac{p}{p-1} + \frac{1}{e(q-1)}} p(0) |\pi|^N \|f\|_{a,N} \|\varphi\|_{\mathcal{B}^\prime(p^{-r})} \]

where \( r = 1/eq^N(q-1) \) and

\[ \|\varphi\|_{\mathcal{B}^\prime(p^{-r})} := \max_k \{ |c_k| kp^{-kr} \}. \]

Moreover, if \( e \leq p - 1 \), then

\[ \left| \int_{a+\pi^N\mathcal{O}_K} f(x) \, d\mu_x \right| \leq p^{\frac{p}{p-1} + \frac{1}{e(q-1)}} |\pi|^N \|f\|_{a,N} \|\varphi\|_{\mathcal{B}^\prime(p^{-r})}. \]

Proof. — The formula follows from

\[ p \left( \left[ \frac{k}{q^N} \right] \right) \leq kq^{-N} p^{\frac{p}{p-1} + \frac{1}{e(q-1)}} - q^{-N} \left[ \frac{k}{q} \right]. \]

As before, we define polynomials \( P_n \) by

\[ \exp(x\lambda(T)) = \sum_{n=0}^{\infty} P_n(x)T^n. \]

Then by formal computation, we have

\[ P_k(\partial_G)\varphi(t)|_{t=0} = \frac{1}{k!}\partial^k\varphi(t)|_{t=0} \]

where \( \partial = d/dt \) (for example, formula 6 of Lemma 4.2 of [8]). We let \( \varphi_n(t) = t^n \) and \( \mu_{\varphi_n} \) the distribution associated to \( \varphi_n(t) \). Then by Proposition 4.1 ii) we have

\[ \int_{\mathcal{O}_K} P_k(x\varphi_p) \, d\mu_{\varphi_n} = \sum_{n=0}^{\infty} P_k(\partial_G)\varphi_n(t)|_{t=0} = \begin{cases} 1 & (k = n) \\ 0 & (k \neq n) \end{cases}. \]

Hence if \( \varphi(t) = \sum_{k=0}^{\infty} c_k t^k \), then

\[ \int_{\mathcal{O}_K} P_k(x\varphi_p) \, d\mu_{\varphi} = c_k. \]

Equivalently,

\[ \varphi(t) = \int_{\mathcal{O}_K} \exp(x\varphi_p\lambda(t))d\mu_{\varphi}. \]

Proposition 4.5. — For \( N \geq 1 \), we have

\[ \left| \frac{q}{\pi} \right|^N \left[ \frac{n}{q^N} \right]^{-1} c^{-1} \leq \|P_n(x\varphi_p)\|_N \leq \bar{p} \left( \left[ \frac{n}{q^N} \right] \right)^{-1} \]

where \( c = 1 \) if \( e \leq p - 1 \) and \( c = \bar{p}(0) \), otherwise.
Proof. — We have

\[ 1 = \left| \int_{\mathcal{O}_K} P_n(x\overline{\varphi}_p) \, d\mu_{\varphi_n} \right| \leq \max_a \left\{ \left| \int_{a+\pi^N\mathcal{O}_K} P_n(x\overline{\varphi}_p) \, d\mu_{\varphi_n} \right| \right\} \]

\[ \leq \left| \frac{\pi}{q} \right|^N \| P_n(x\overline{\varphi}_p) \|_N \overline{\rho} \left( \left\lfloor \frac{n}{q^N} \right\rfloor \right) \overline{\rho}(0). \]

Similarly, if \( e \leq p - 1 \), then by using (4.8), we obtain the lower estimate.

For the upper estimate, we put \( P_n(x\overline{\varphi}_p) = \sum_{k=1}^{\infty} a_k^{(n)} x^k \) for \( n \geq 1 \). By the definition of \( P_n \), the value \( a_k^{(n)} \) is the coefficient of \( t^n \) of \( \overline{\varphi}_p \lambda([\pi^N]t)^k/k! \).
Since \( \rho(k) \) is decreasing with \( k \), we may assume that \( \lambda(t) = \sum_{i=0}^{\infty} t^{q^i}/\pi^i \).
Since \([\pi^N]t \equiv t^{q^N} \mod \pi\), we have

\[ \lambda([\pi^N]t) \equiv \lambda(t^{q^N}) + \pi t f(t) \]

for some \( f(t) \in \mathcal{O}_{C_p}[[t]] \). Hence we have

\[ \frac{\overline{\varphi}_p \lambda([\pi^N]t)^k}{k!} = \sum_{i=0}^{k} t^i f(t)^i \frac{\overline{\varphi}_p^{q^i} \overline{\varphi}_p^{-i} \lambda(t^{q^N})^{k-i}}{i! (k-i)!}. \]

Therefore by Proposition 3.5 we have

\[ |a_k^{(n)}| \leq \rho \left( \left\lfloor \frac{n}{q^N} \right\rfloor \right)^{-1}. \]

Hence we have \( \| P_n(x\overline{\varphi}_p) \|_{0, N} \leq \rho \left( \left\lfloor \frac{n}{q^N} \right\rfloor \right)^{-1}. \)

Then by the formula before Lemma 4.4 of [8], for \( a \in \mathcal{O}_K \), we have

\[ \| P_n(x\overline{\varphi}_p) \|_{a, N} \leq \max_{0 \leq i \leq n} \| P_i(x\overline{\varphi}_p) \|_{0, N} \leq \rho \left( \left\lfloor \frac{n}{q^N} \right\rfloor \right)^{-1}. \]

\[ \square \]

Now we prove that our definition of the distribution coincides with that of Schneider-Teitelbaum. Namely, we will prove that the distribution has the characterization property (2.3).

Theorem 4.6. — Let \( \mu_{\varphi} \) be the distribution associated to a rigid analytic function \( \varphi(t) \) on the open unit disc. Then

\[ \varphi(t) = \int_{\mathcal{O}_K} \exp(x\overline{\varphi}_p \lambda(t)) \, d\mu_{\varphi}. \]

Conversely, for every distribution \( \mu \), there exists a unique rigid analytic function \( \varphi \) such that \( \mu = \mu_{\varphi} \). Then \( \varphi \) is the Fourier transform of \( \mu \), and we have \( F_{\mu_{\varphi}} = \varphi \). In particular, we have an isomorphism of algebras,

\[ D(\mathcal{O}_K, \mathbb{C}_p) \cong \mathbb{R}^{\text{rig}}. \]
Proof. — We have already shown the first assertion. For a given $\mu$, we put

$$ c_k := \int_{O_K} P_k(x\varpi_p) \, d\mu. $$

Since the distribution is a continuous linear operator on the $p$-adic Banach space $LA_N(O_K, \mathbb{C}_p)$ for every natural number $N$, there exists a positive constant $C$ depending only on $\mu$ and $N$ such that

$$ |c_k| = \left| \int_{O_K} P_k(x\varpi_p) \, d\mu \right| \leq C \|P_k(x\varpi_p)\|_N \leq C p^{-\frac{1}{q^N} + \frac{k}{(q-1)}} $$

where for the last inequality, we used Proposition 3.1 and Proposition 4.5. Hence for any $0 \leq r < 1$, if we choose sufficiently large $N$, we have $|c_k|^k \to 0$ when $k \to \infty$. Hence $\varphi(t) = \sum_{k=0}^{\infty} c_k t^k$ is a rigid analytic function on the open unit disc. Then by construction

$$ \varphi(t) = \int_{O_K} \exp(x\varpi_p \lambda(t)) \, d\mu. $$

Since the function $(x - a)|_{a+\pi^N O_K}$ is given by

$$ \frac{1}{q^N \varpi_p^n} \frac{\partial^n}{\partial t^n} \left( \sum_{t_N \in G[\pi^N]} \exp((x - a)\varpi_p \lambda(t)) \right) |_{t=0}, $$

we have

$$ \int_{a+\pi^N O_K} (x - a)^n \, d\mu = \frac{1}{q^N \varpi_p^n} \frac{\partial^n}{\partial t^n} \sum_{t_N \in G[\pi^N]} \varphi_a(t \oplus t_N) |_{t=0} $$

$$ = \int_{a+\pi^N O_K} (x - a)^n \, d\mu_\varphi. $$

Since $\pi^{-nN}(x - a)^n|_{a+\pi^N O_K}$ for $a \in O_K$ and $n = 0, 1, \ldots$ are topological generators of $LA_n(O_K, \mathbb{C}_p)$, we have

$$ \int_{O_K} f(x) \, d\mu = \int_{O_K} f(x) \, d\mu_\varphi $$

for all $f \in LA_N(O_K, \mathbb{C}_p)$. Hence $\mu = \mu_\varphi$. □

Now we prove Theorem 4.7.

Theorem 4.7.

i) The series $\sum_{n=0}^{\infty} a_n P_n(x\varpi_p)$ converges to an element of $LA_N(O_K, \mathbb{C}_p)_0$ for $a_n$ satisfying

$$ |a_n| \leq \rho \left( \left[ \frac{n}{q^N} \right] \right), \quad \lim_{n \to 0} |a_n|/\rho \left( \left[ \frac{n}{q^N} \right] \right) = 0. $$
ii) If \( f(x) \in LA_N(O_K, \mathbb{C}_p)_{0} \), then it has an expansion
\[
f(x) = \sum_{n=0}^{\infty} a_n P_n(x) \omega_p
\]
of the form
\[
|a_n| \leq c \left\lvert \frac{n}{q} \right\rvert \left( \frac{N}{q^N} \right), \quad \lim_{n \to 0} |a_n|/\bar{p} \left( \frac{n}{q^N} \right) = 0,
\]
where \( c = 1 \) if \( e \leq p - 1 \), and \( c = \bar{p}(0) \), otherwise.

**Proof.** — i) follows from Proposition 4.5. For ii), we proceed as in the proof of Theorem 4.7 of [8] except the estimate of the Mahler coefficients.

We put
\[
a_n := \int_{O_K} f(x) \, d\mu_{\varphi_n}.
\]
Then by Theorem 4.3, we have
\[
|a_n| = \left\lvert \int_{O_K} f(x) \, d\mu_{\varphi_n} \right\rvert \leq c \left\lvert \frac{n}{q} \right\rvert \left( \frac{N}{q^N} \right).
\]
We next prove the limit in ii). We may assume that \( f(x) = \sum_{i=0}^{\infty} c_i (x-a)^i \) on \( a + \pi^N O_K \) and \( f(x) = 0 \) outside of \( a + \pi^N O_K \). For a given \( \epsilon > 0 \), we can take \( N_0 \) so that
\[
\| \sum_{i=N_0}^{\infty} c_i (x-a)^i \|_{a,N} < \epsilon.
\]
Hence by (4.7), we have
\[
\left\lvert \int_{a+\pi^N O_K} \sum_{i=N_0}^{\infty} c_i (x-a)^i \, d\mu_{\varphi_n} \right\rvert \leq \epsilon C_1 \bar{p} \left( \frac{n}{q^N} \right)
\]
where \( C_1 \) is a positive constant independent of \( n \). On the other hand, also by 4.7, we have
\[
\left\lvert \int_{a+\pi^N O_K} \sum_{i=0}^{N_0} c_i (x-a)^i \, d\mu_{\varphi_n} \right\rvert \leq C_2 p^{-\frac{N}{n} \left( \frac{1}{q} - 1 \right)} \bar{p} \left( \frac{n}{q^N} \right)
\]
where \( n_0 = \max\{\lfloor n/q^N \rfloor - N_0, 0\} \) and \( C_2 \) is a positive constant independent of \( n \). Hence we have
\[
\left\lvert \int_{a+\pi^N O_K} f(x) \, d\mu_{\varphi_n} \right\rvert \leq \epsilon C_1 \bar{p} \left( \frac{n}{q^N} \right)
\]
for sufficiently large $n$. Hence we have $|a_n|/\rho\left(\left[\frac{n}{q^N}\right]\right) \to 0$ when $n \to \infty$. Then by i), the series $\sum_{k=0}^{\infty} a_n P_n(x \omega p)$ converges to a function in $L_{A_N}(O_K, \mathbb{C}_p)$. We put

$$g(x) = f(x) - \sum_{k=0}^{\infty} a_n P_n(x \omega p).$$

Then we have $\int_{O_K} g(x) d\mu_{\varphi_n} = 0$ for all $n$, and hence $\int_{O_K} g(x) d\mu = 0$ for all distribution $\mu$. Considering the Dirac distribution $\delta_a : h \mapsto h(a)$, we have $g(a) = 0$ for any $a$. Hence $f(x) = \sum_{n=0}^{\infty} a_n P_n(x \omega p)$. $\square$

**Corollary 4.8.** Suppose

$$e_{N,n} = \gamma\left(\left[\frac{n}{q^N}\right]\right) P_n(x \omega p), \quad (n = 0, 1, \ldots),$$

where $\gamma(u)$ is an element in $\mathbb{C}_p$ satisfying $\rho(u) = |\gamma(u)|$. If $L_N$ is the $O_{C_p}$-module topologically generated by $e_{N,n}$, then

$$\overline{\rho}(0)^{-2} \left| \frac{q}{\pi} \right|^N L_{A_N}(O_K, \mathbb{C}_p)_0 \subset L_N \subset L_{A_N}(O_K, \mathbb{C}_p)_0.$$

In particular, the functions $e_n$ form a topological basis of the $p$-adic Banach space $L_{A_N}(O_K, \mathbb{C}_p)$. Moreover, if $e \leq p - 1$, then

$$\left| \frac{q}{\pi} \right|^{N+1} L_{A_N}(O_K, \mathbb{C}_p)_0 \subset L_N \subset L_{A_N}(O_K, \mathbb{C}_p)_0.$$

In addition, if $O_K = \mathbb{Z}_p$, we recover Amice’s result, namely that

$$\left[ \frac{n}{p^N} \right](x) \binom{x}{n}$$

for $n = 0, 1, \ldots$ form a topological basis of $L_{A_N}(\mathbb{Z}_p, \mathbb{C}_p)_0$.

### 5. Relations to Katz’s and Chellali’s results.

As an application, we reprove Katz’s and Chellali’s results ([4], [7]) by using our results.

First we recall results of Katz [7] and Chellali [4]. Let $E$ be an elliptic curve with complex multiplication by the ring of integer $O_{K}$ of an imaginary quadratic field $K$. For simplicity, we assume that $E$ is defined over $K$, and fix a Weierstrass model

$$y^2 = 4x^3 - g_2 x - g_3, \quad g_2, g_3 \in O_K$$

of $E/K$. Let $p$ be an odd prime. We assume that $p$ is inert in $K$ and does not divide the discriminant of the above Weierstrass model, or equivalently, $E$
has good supersingular reduction at \( p \). Then the Bernoulli-Hurwitz number \( BH(n) \) is defined by

\[
\varphi(z) = \frac{1}{z^2} + \sum_{n \geq 2} \frac{BH(n+2)}{n+2} \frac{z^n}{n!},
\]

where \( \varphi(z) \) is the Weierstrass \( \varphi \)-function for the model. Let \( \epsilon \) be a root of unity in \( \mathcal{O}_K \) such that the multiplication by \( -\epsilon p \) gives the Frobenius \( (x,y) \mapsto (x^{p^2}, y^{p^2}) \) of \( E \mod p \). Let \( \gamma \) be a unit in the Witt ring \( W(\mathbb{F}_p) \) such that

\[
\gamma^{p^2-1} = -\epsilon^{-1} \frac{p^{2!}}{p^{p+1}(p^2-1)}.
\]

For a fixed \( b \in \mathcal{O}_K \) prime to \( p \), we put

\[
L(n) = \frac{(1 - b^{n+2})(1 - p^n) BH(n+2)}{\gamma^{p!} p^{np/(p^2-1)}}.
\]

**Theorem 5.1** (Katz [7]). — The number \( L(n) \) is integral. Let \( l \) and \( n \) be non-negative integers. Then

\[
L(n + p^l(p^2 - 1)) \equiv L(n) \mod p^l.
\]

Later, Chellali [4] refined the congruences as follows.

**Theorem 5.2** (Chellali [4]). — Let \( l \) and \( n \) be non-negative integers. If \( n \not\equiv 0 \mod p^2 - 1 \), we have

\[
L(n + p^l(p^2 - 1)) \equiv L(n) \mod p^{l+1}.
\]

If \( n \equiv 0 \mod p^2 - 1 \) and \( n \not\equiv 0 \), put \( L'(n) = L(n)/n \), then

\[
L'(n + p^l(p^2 - 1)) \equiv L'(n) \mod p^{l+1}.
\]

In the following, let \( K \) be the unramified quadratic extension of \( \mathbb{Q}_p \) and let \( G \) be the Lubin-Tate group of height \( h = 2 \) associated to the uniformizer \( \pi = -ep \). We assume that \( [\pi]T = \pi T + T^q \) for \( q = p^2 \) is an endomorphism of \( G \). It is known that the formal group of \( E \) at \( p \) is isomorphic to \( G \).

**Proposition 5.3.** — Let \( \varphi \) be an integral power series and let \( \mu_\varphi \) be the corresponding distribution associated to \( \varphi \).

i) We have

\[
\left| \int_{\mathcal{O}_K^\times} x^n \, d\mu_\varphi \right| \leq p.
\]

ii) If \( m \equiv n \mod p^l(q - 1) \), then

\[
\left| \int_{\mathcal{O}_K^\times} (x^m - x^n) \, d\mu_\varphi \right| \leq p^{-l+\frac{p}{q-1}}.
\]
iii) If \((q - 1) | n \) and \( m \equiv n \mod p^l(q - 1) \), then
\[
\left| \int_{\mathcal{O}_K^p} \left( \frac{x^m - 1}{m} - \frac{x^n - 1}{n} \right) d\mu_\varphi \right| \leq p^{-l - 1 + \frac{2p}{q - 1}}.
\]

**Proof.** — We have
\[
\int_{a + \pi \mathcal{O}_K} x^n \, d\mu_\varphi = a^n \int_{a + \pi \mathcal{O}_K} d\mu_\varphi + \sum_{k=1}^{n} \int_{a + \pi \mathcal{O}_K} \binom{n}{k} (x - a)^k a^{n-k} \, d\mu_\varphi.
\]
Then by the estimate (4.6) the absolute value of the first integral is less than or equal to \( p \). By the estimate (4.8), the absolute value of the second integral is also less than or equal to \( p \) since \( ||(x - a)||_{a,1} \beta(0) = 1 \). We put \( m - n = k(q - 1) \). Then
\[
x^m - x^n = x^n \sum_{i=1}^{k} \binom{k}{i} (x^{q-1} - 1)^i
\]
\[
= kx^n(x^{q-1} - 1) + x^n \sum_{i=2}^{k} \binom{k}{i-1} (x^{q-1} - 1)^i
\]
\[
= k \left( c_0 + c_1(x - a) + c_2 \frac{(x - a)^2}{2} + c_3 \frac{(x - a)^3}{3} + \cdots \right)
\]
where \( c_i \) are integers satisfying \( p|c_0 \). Since \( ||(x - a)^i/i||_{a,1} \leq p^{-2} \) for \( i \geq 2 \), the assertion ii) follows from the estimates (4.6).

For an integer \( s \), we have
\[
\frac{(x^{q-1})^s - 1}{s} = \sum_{i=1}^{\infty} \frac{(\log_p x^{q-1})^i}{i!} s^{i-1}
\]
\[
= \sum_{i=1}^{\infty} \sum_{n=i}^{\infty} c_{i,n} \frac{(x^{q-1} - 1)^n}{n!}
\]
\[
= \sum_{i=1}^{\infty} \sum_{j+k \geq i} c_{i,j,k} \frac{\pi^k (x - a)^j}{k!} \frac{1}{j!} s^{i-1}
\]
for some integers \( c_{i,n} \) and \( c_{i,j,k} \). If we write \( m = s_1(q - 1) \) and \( n = s_2(q - 1) \), then
\[
\frac{(x^{q-1})^{s_1} - 1}{s_1} - \frac{(x^{q-1})^{s_2} - 1}{s_2} = \sum_{i \geq 2, j+k \geq i} c_{i,j,k} \frac{\pi^k (x - a)^j}{k!} \frac{1}{j!} (s_i^{i-1} - s_i^{i-1})
\]
By the estimate (4.6), the integral of \( \frac{\pi^k (x - a)^j}{k!} \frac{1}{j!} \) is divisible by \( p^{1 - \frac{2p}{q - 1}} \). The assertion iii) follows from this fact. \( \square \)
For $b \in \mathcal{O}_K$ prime to $p$, we put
\[ \wp_b(z) = (1 - b^2[b]^*)\wp(z) \]
and $\phi(t) = \wp_b(z)|_{z=\lambda(t)}$. Then $\wp_b(z)$ has no pole at $z = 0$ and
\[ \wp_b(z) = \sum_{n \geq 2} (1 - b^{n+2}) \frac{BH(n + 2)}{n + 2} z^n n!. \]

It is known that $\phi(t)$ is an integral power series. Similarly, for $c \in \mathcal{O}_K$ prime to $p$, we put
\[ \zeta_c(z) = (c - [c]^*)\zeta(z), \quad \zeta_{b,c}(z) = (1 - b[b]^*)\zeta_c(z), \]
where $\zeta(z)$ is the Weierstrass zeta function and $\psi(t) = \zeta_{b,c}(z)|_{z=\lambda(t)}$. Note that $\zeta_c(z)$ is double periodic and $\zeta_{b,c}(z)$ has no pole at $z = 0$. Then
\[ \zeta_{b,c}(z) = \sum_{n \geq 3} (c - c^n)(1 - b^{n+1}) \frac{BH(n + 1)}{n + 1} z^n n!. \]
and $\psi(t)$ is an integral power series.

**Lemma 5.4.**
\[ \sum_{z_0 \in \frac{1}{p}\Gamma/\Gamma} \wp_b(z + z_0) = p^2\wp_b(pz), \quad \sum_{z_0 \in \frac{1}{p}\Gamma/\Gamma} \zeta_c(z + z_0) = p\zeta_c(pz). \]

**Proof.** — It is known that
\[ \sum_{z_0 \in \frac{1}{p}\Gamma/\Gamma} \wp(z + z_0) = p^2\wp(pz). \]
The first formula follows from this. The above formula also shows that for a set $S$ of representatives of $\frac{1}{p}\Gamma/\Gamma$, there exists a constant $A(S)$ such that
\[ \sum_{z_0 \in S} \zeta(z + z_0) = p\zeta(pz) + A(S). \]
We take $S$ so that $S = -S$. Then since $\zeta(z)$ is an odd function, $A(S)$ should be zero. Therefore,
\[ \sum_{z_0 \in S} \zeta_c(z + z_0) = p\zeta_c(pz). \]
Since $\zeta_c(z)$ is an elliptic function, the left-hand side does not depend on the choice of $S$. \[ \square \]

**Proposition 5.5.** — We put $B(n) = BH(n + 2)/(n + 2)$ if $n \geq 2$ and $0$ if $n = -1, 0, 1$. For $n \geq 0$, we have
\[ \varpi_p^n \int_{K^\times} x^n d\mu_\phi = (1 - p^n)(1 - b^{n+2})B(n), \]
\[ \varpi_p^n \int_{O_K^\times} x^n d\mu_\psi = (1 - p^{n-1})(c - c^n)(1 - b^{n+1})B(n - 1). \]

**Proof.** — Since \( \wp_b(z) \) and \( \zeta_{b,c}(z) \) are double periodic, for \( t_0 \in G[p] \) we have \( \psi(t \oplus t_0) = \zeta_{b,c}(z + z_0)|_{z = \lambda(t)} \) and \( \phi(t \oplus t_0) = \wp_b(z + z_0)|_{z = \lambda(t)} \) where \( z_0 \) is an image of \( t_0 \) by \( G[p] \to E[p] \to \frac{1}{p} \Gamma / \Gamma \) (see for example, [2], Lemma 2.18).

From this fact and the previous lemma, we have
\[
\psi(t) - \frac{1}{q} \sum_{t_0 \in G[p]} \psi(t \oplus t_0) = (\wp_b(z) - \wp_b(pz)) |_{z = \lambda(t)},
\]
\[
\phi(t) - \frac{1}{q} \sum_{t_0 \in G[p]} \phi(t \oplus t_0) = (\zeta_{b,c}(z) - p^{-1}\zeta_{b,c}(pz)) |_{z = \lambda(t)}.
\]

Hence
\[
\varpi_p^n \int_{O_K^\times} x^n d\mu_\phi = \partial_G^n \left. \left( \phi(t) - \frac{1}{q} \sum_{t_0 \in G[p]} \phi(t \oplus t_0) \right) \right|_{t = 0} = \left. \partial_z (\wp_b(z) - \wp_b(pz)) \right|_{z = 0} = (1 - p^n)(1 - b^{n+2})B(n).
\]

The other equality is also shown similarly. \( \square \)

We put
\[ c(n) = (1 - p^n)(1 - b^{n+2}) \frac{BH(n + 2)}{n + 2}. \]

**Corollary 5.6.**

i) We have
\[ \left| \frac{c(n)}{\varpi_p^n} \right| \leq p. \]

Furthermore, if \( n \equiv 0 \mod q - 1 \), then
\[ \left| \frac{c(n)}{\varpi_p^n} \right| \leq p^{\frac{p}{q-1}}. \]

ii) Suppose that \( m \equiv n \mod p^l(q - 1) \). Then
\[ \frac{c(m)}{\varpi_p^m} \equiv \frac{c(n)}{\varpi_p^n} \mod p^{l - \frac{r}{q-1}}O_{C_p}. \]

Furthermore, if \( n \not\equiv 0 \mod q - 1 \), then
\[ \frac{c(m)}{\varpi_p^m} \equiv \frac{c(n)}{\varpi_p^n} \mod p^l O_{C_p}. \]

If \( n \equiv 0 \mod q - 1 \), then
\[ \frac{c(m)}{m \varpi_p^m} \equiv \frac{c(n)}{n\varpi_p^n} \mod p^{l+1 - \frac{2p}{q-1}}. \]
Proof. — For i), the first inequality follows from Proposition 5.3 i) for \(\mu_{\phi}\). The second inequality follows from Proposition 5.3 ii) for \(l = 0\). Note that \(\int_{\mathcal{O}_K^\times} d\mu_{\phi} = 0\). For ii), the first and third congruences follow from Proposition 5.3 for \(\phi\), and the second inequality for \(\psi\). □

Next, we compare \(c(n)\) with \(L(n)\).

**Lemma 5.7.** — We choose \(u \in \mathbb{C}_p\) so that \(\varpi_p^{q-1} = p^\theta u^{q-1}\). Then \(u\) is a unit of \(\mathcal{O}_{\mathbb{C}_p}\) and \(u^{\varpi_p}\{q-1\} \equiv 1 \mod p\).

Proof. — Simple calculation shows the valuation of \(u\) is zero. We have \(\lambda(t) = t + \theta t^q + \cdots\) with \(\theta = 1/\epsilon(p^q - p)\). The \(q\)-th coefficient of the integral power series \(\exp(\varpi_p \lambda(t))\) is

\[
\varpi_p^{q-1} / q! + \varpi_p \theta = \varpi_p \theta \left( \varpi_p^{q-1} / q! + 1 \right).
\]

Since \(\varpi_p \theta\) is not integral, the valuation \(v_p((\varpi_p^{q-1}/\theta q!) + 1) \geq 1\). Thus

\[
\varpi_p^{q-1} / q! + 1 \equiv \left( \frac{u}{\gamma} \right)^{q-1} \frac{(1 - p^{q-1})}{(q - 1)} + 1 \equiv - \left( \frac{u}{\gamma} \right)^{q-1} + 1 \mod p.
\]

must be congruent to zero. □

Let \(n = n'(q-1) + r\) with \(0 \leq r < q - 1\) and put \(c_r = u^{-r} p^{-[p^r/(q-1)]} \varpi_p^r\). Then

\[
\varpi_p^n = c_r p^{[p^n/(q-1) - 1]} u^n.
\]

Hence we have

\[
L(n) = c_r \left( \frac{u}{\gamma} \right)^n \frac{c(n)}{\varpi_p^n}.
\]

Therefore by Corollary 5.6 i), we have \(|L(n)| < p\) (note that if \(n \not\equiv 0 \mod q - 1\), then \(|c_r| < 1\). Since \(L(n)\) is contained in the unramified field \(K\), we have \(L(n) \in \mathcal{O}_K\). Similarly, for \(m \equiv n \mod p^l(q - 1)\), the fact \(L(n) \in \mathcal{O}_K\), Lemma 5.7 and Corollary 5.6 ii) imply the congruence

\[
L(m) \equiv L(n) \left( \frac{u}{\gamma} \right)^{m-n} \equiv L(n) \mod p^l - \frac{p}{q-1}.
\]

Since this is a congruence between elements of \(\mathcal{O}_K\), we have

\[
L(m) \equiv L(n) \mod p^l.
\]

Similarly, from Corollary 5.6, we obtain the congruences originally proved by Katz [7, Theorem 3.1] and Chellali [4, Théorème 1.1].
Theorem 5.8.

i) We have $L(n) \in \mathcal{O}_K$.

ii) Suppose that $m \equiv n \mod p^l(q - 1)$. Then

$$L(m) \equiv L(n) \mod p^l.$$ 

Furthermore, if $n \not\equiv 0 \mod q - 1$, then

$$L(m) \equiv L(n) \mod p^{l+1}.$$ 

If $n \equiv 0 \mod q - 1$, then

$$L'(m) \equiv L'(n) \mod p^{l+1}.$$ 

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Kenichi BANNAI
Department of Mathematics,
Keio University,
3-14-1 Hiyoshi,
Kouhoku-ku, Yokohama
223-8522 (Japan)
banai@math.keio.ac.jp
Shinichi KOBAYASHI
Mathematical Institute,
Tohoku University,
6-3 Aramaki-aza-Aoba,
Aoba-ku, Sendai
980-8578 (Japan)
shinichi@math.tohoku.ac.jp