Annales de l'institut Fourier

B. J. Cole
Theodore W. Gamelin

Weak-star continuous homomorphisms and a decomposition of orthogonal measures

Annales de l'institut Fourier, tome 35, nº 1 (1985), p. 149-189
<http://www.numdam.org/item?id=AIF_1985__35_1_149_0>

Ann. Inst. Fourier, Grenoble
35, 1 (1985), 149-189

WEAK-STAR CONTINUOUS HOMOMORPHISMS AND A DECOMPOSITION OF ORTHOGONAL MEASURES

by B.J. COLE and T.W. GAMELIN

Dedicated to the memory of I. Glicksberg

1. Introduction.

Let A be a uniform algebra on a compact space X, let μ be a (finite regular Borel) measure on X, and let $H^\infty(\mu)$ be the weak-star closure of A in $L^\infty(\mu)$. Let $S(\mu)$ be the set of (nonzero, complex-valued) homomorphisms of $H^\infty(\mu)$ which are weak-star continuous. The unifying theme of this paper is to study the set $S(\mu)$, both from an abstract point of view, and also for certain concrete algebras.

Since any weak-star continuous homomorphism of $H^\infty(\mu)$ is determined uniquely by its action on A, we may regard $S(\mu)$ as the set of those homomorphisms in the spectrum M_A of A which are continuous in the weak-star topology of A determined by $L^\infty(\mu)$. These are precisely the homomorphisms of A which have representing measures absolutely continuous with respect to μ (cf. [8, Theorem II.2.2.]). In Section 7, we will consider the question of whether $S(\mu)$ is a Borel set, regarded as a subset of $M_{H^\infty(\mu)}$ and as a subset of M_A. We observe that it is always an $F_{\sigma\delta}$-subset of $M_{H^\infty(\mu)}$, and under certain special hypotheses (such as A separable) we show that it is a Borel subset of M_A.

In sections 8 through 13 we treat the family of algebras encountered in rational approximation theory, the so-called "T-invariant" subalgebras of C(K), where K is a compact subset of the complex plane **C**, and μ is an arbitrary measure on K. Here we unify and extend the work of D. Sarason [17], who studied the algebra generated by the analytic polynomials, and of

Key-words : Uniform algebra, Maximal ideal space, Weak-star continuous homomorphisms, Rational approximation theory, T-invariant algebras, Infinite-polydisc-algebra.

J. Chaumat [4], who studied the algebras R(K) and A(K). Following Sarason, we describe $H^\infty(\mu)$ in terms of the operation of taking pointwise bounded limits, in Sections 8 through 10. We also show that in this case $S(\mu)$ is the union of the set of nonpeak points of the T-invariant algebra $C(K) \cap H^\infty(\mu)$ and the (at most countable) set at which μ has point masses. In particuiar, for these algebras $S(\mu)$ is always an F_σ-set.

In Section 14 we determine the set $S(\sigma)$ in the case that A is the algebra generated by analytic polynomials on the infinite polydisc and σ is the Haar measure on the infinite torus forming the distinguished boundary. We show that $S(\sigma)$ consists of precisely the set of null sequences in the open polydisc. In this case, $S(\sigma)$ is an $F_{\sigma\delta}$-subset of the polydisc, however it fails to be an F_σ-subset.

In Sections 2 through 6 we consider an abstract uniform algebra and explore the circle of ideas which connect the following theorems:

(i) an abstract version of a theorem of Glicksberg [13], to be discussed below;

(ii) the Cole-König-Seever theorem;

(iii) the abstract F. and M. Riesz theorem ; and

(iv) the Hoffman-Rossi theorem.

The theorem of Glicksberg referred to above pertains to the algebra R(K) of uniform limits on K of rational functions with poles off K, where K is a compact subset of **C**. Glicksberg proved that if the interior K^0 of K is connected and dense in K, then every nonzero measure on the topological boundary ∂K orthogonal to R(K) is mutually absolutely continuous with respect to a representing measure for some point of K^0. The idea of Glicksberg's proof is to apply the Cole-König-Seever theorem to $L^1(\mu)$, where μ is the given orthogonal measure. Let us describe the development in Sections 2 through 6 in more detail.

In Section 2 we present a proof (Cole's original proof) of the L^1-version of the Cole-König-Seever theorem. The proof is based on the Hoffman-Rossi theorem (L^∞-methods), as opposed to the L^2-methods utilized by König and Seever [5].

In Section 3, we define the μ-parts of $S(\mu)$, and use these together with the Cole-König-Seever theorem to obtain a decomposition theorem (Theorem 3.6), which has the following abstract version of Glicksberg's theorem as an immediate consequence.

1.1. THEOREM. – *Let* A *be a uniform algebra on a compact space* X. *Then any measure* $\mu \in A^{\perp}$ *can be decomposed in the form*

$$\mu = \mu_s + \sum_{j \geqslant 1} \mu_j$$

where

(i) *the measures* μ_s *and* μ_j, $j \geqslant 1$, *are pairwise mutually singular, so that in particular the series converges in the total variation norm ;*

(ii) $\mu_s \in A^{\perp}$ *and* $\mu_j \in A^{\perp}$, $j \geqslant 1$;

(iii) *for each* $j \geqslant 1$, μ_j *is mutually absolutely continuous with respect to a representing measure for some point of* M_A.

(iv) *there are no representing measures for points of* M_A *which are absolutely continuous with respect to* μ_s.

Theorem 1.1 should be compared with the version of the abstract F. and M. Riesz theorem which decomposes $\eta \in A^{\perp}$ as a sum of pairwise mutually singular measures $\eta = \eta_s + \sum_{j \geqslant 1} \eta_j$, where each η_j is absolutely continuous with respect to a representing measure for some point of M_A, and η_s is singular to all representing measures for points of M_A. We will show in Section 4 that the decomposition of Theorem 1.1 coincides with the decomposition obtained by applying the abstract F. and M. Riesz theorem to the algebra $H^{\infty}(\mu)$, the weak-star closure of A in $L^{\infty}(\mu)$, regarded as an algebra of continuous functions on the spectrum of $L^{\infty}(\mu)$.

In the case of the algebra $R(K)$, Wilken's theorem shows that the singular summand μ_s is always zero, and we obtain simply $\mu = \Sigma \mu_j$, where each μ_j is mutually absolutely continuous with respect to a representing measure for some point $z_j \in K$. Under the hypotheses of Glicksberg's theorem, only one summand is required. However, infinitely many summands may be required in the decomposition, even in the case of the disc algebra. Such an example is given at the end of Section 3.

Section 4 contains a second proof of the L^1-version of the Cole-König-Seever theorem, also based on the Hoffman-Rossi theorem, which clarifies its relationship with the abstract F. and M. Riesz theorem. In Section 5 we show how the band-version of the Cole-König-Seever theorem is a direct consequence of the L^1-version. In Section 6, the Hoffman-Rossi theorem is extended to bands of measures. This leads immediately to another proof of the band-version of the Cole-König-Seever theorem.

We will use the standard terminology of uniform algebras, as in [3] or [8]. All measures are finite complex-valued regular Borel measures. The characteristic function of a set E will be denoted by χ_E. If ν is a measure, then the measure $\chi_E \nu$ will be denoted by ν_E. Our homomorphisms are always nonzero complex-valued homomorphisms.

2. The L^1-version of the Cole-König-Seever theorem.

The statement (iv) of the next theorem is the L^1-version of the Cole-König-Seever theorem. In Section 5 we will see that the band-version of the theorem follows virtually immediately from the L^1-version.

2.1. THEOREM. – *Let* $\varphi \in S(\mu)$, *that is,* φ *has a representing measure which is absolutely continuous with respect to* μ. *Then there is a Borel set* E *with the following properties:*

(i) φ *has a representing measure which is mutually absolutely continuous with respect to* μ_E;

(ii) *every representing measure for* φ *absolutely continuous with respect to* μ *is absolutely continuous with respect to* μ_E;

(iii) $\chi_E \in H^\infty(\mu)$;

(iv) *if* $\lambda \in A^\perp$ *and* $\lambda \ll \mu$, *then* $\lambda_E \in A^\perp$. *Moreover,* E *is determined uniquely up to sets of zero* $|\mu|$*-measure by properties* (i) *and* (ii).

Proof. – It is easy to see that there exists a Borel set E, unique up to sets of zero $|\mu|$-measure, with properties (i) and (ii). Indeed, for ν a representing measure for φ absolutely continuous with respect to μ, let $E(\nu)$ be the set on which $\frac{d\nu}{d\mu} \neq 0$. Take a sequence $\{\nu_j\}$ of such ν's such that $|\mu|(E(\nu_j))$ tends to a maximum, and set $\nu = \sum_{j \geq 1} \nu_j/2^j$, $E = \cup E(\nu_j)$. Then ν is a representing measure for φ which is mutually absolutely continuous with respect to μ_E, and evidently (ii) is also valid.

The proof of (iii) will be based on the Hoffman-Rossi theorem, applied to the function $u = \chi_E - 1$. First observe that $\int u\, d\tau \geq 0$ for all representing measures τ for φ such that $\tau \ll \mu$. Indeed, any such τ is carried by E, and $u = 0$ on E. By the Hoffman-Rossi theorem ([14], [8, Section IV.2]), these exists $h \in H^\infty(\mu)$ such that $|h| \leq e^u$ and $\varphi(h) = 1$. Then $|h| \leq 1$, and $\int h\, d\nu = 1$, so that $h = 1$ a.e. $(d\nu)$, i.e., $h = 1$ a.e. $(d\mu)$ on E. Since $|h| \leq 1/e$ on the complement of E, the powers h^m converge uniformly to χ_E as $m \longrightarrow \infty$, and $\chi_E \in H^\infty(\mu)$.

To prove (iv), note that if $\lambda \in A^\perp$ satisfies $\lambda \ll \mu$, then $f\lambda \in A^\perp$ for all $f \in H^\infty(\mu)$. In particular, $\lambda_E = \chi_E \lambda \in A^\perp$. □

Note that the statement (iv) of the preceding theorem is essentially a dual version of the statement (iii). We have seen that (iii) implies (iv), and one deduces (iii) readily from (iv) as follows. By duality, one must show that if $h \in L^1(\mu)$ satisfies $\int hf\, d\mu = 0$ for all $f \in A$, then $\int h\, \chi_E\, d\mu = 0$. The former condition is equivalent to $h\mu \in A^\perp$. By (iv) this implies $h\mu_E \in A^\perp$. In particular $0 = \int h\, d\mu_E = \int h\chi_E\, d\mu$, which establishes (iii).

3. The decomposition theorem.

It is easy to deduce an infinite decomposition theorem from Theorem 2.1. We show how this can be done utilizing the notion of a minimal idempotent.

Again let μ be a fixed measure, and let $S(\mu)$ be the set of weak-star continuous homomorphisms of $H^\infty(\mu)$. A function $\chi \in H^\infty(\mu)$ is an *idempotent* if $\chi^2 = \chi$. The idempotents in

$H^\infty(\mu)$ are of the form χ_E, where E is a Borel set with property (iv) of Theorem 2.1, that if $\lambda \in A^\perp$ and $\lambda \ll \mu$, then $\lambda_E \in A^\perp$. An idempotent $\chi \in H^\infty(\mu)$ is *minimal* if χ is not identically zero, while the only idempotents h satisfying $0 \leqslant h \leqslant \chi$ are $h = 0$ and $h = \chi$.

3.1. LEMMA. – *If* χ_0 *and* χ_1 *are two minimal idempotents in* $H^\infty(\mu)$, *then either* $\chi_0 = \chi_1$ *or* $\chi_0\chi_1 = 0$.

Proof. – $\chi_0\chi_1$ is an idempotent that satisfies $0 \leqslant \chi_0\chi_1 \leqslant \chi_0$, so either $\chi_0\chi_1 = 0$ or $\chi_0\chi_1 = \chi_0$. In the latter case, by symmetry, also $\chi_0\chi_1 = \chi_1$, so $\chi_0 = \chi_1$. □

3.2. LEMMA. – *If* χ *is an idempotent in* $H^\infty(\mu)$, *and if* $\nu \ll \mu$ *represents some* $\varphi \in S(\mu)$, *then either* $\chi = 0$ a.e. $(d\nu)$ *or* $\chi = 1$ a.e. $(d\nu)$.

Proof. – Since $\varphi(\chi)^2 = \varphi(\chi^2) = \varphi(\chi)$, either $\varphi(\chi) = 0$ or $\varphi(\chi) = 1$, that is, $\int \chi \, d\nu = 0$ or $\int \chi \, d\nu = 1$. These two cases correspond to the alternatives of the lemma. □

3.3. LEMMA. – *Let* $\varphi \in S(\mu)$, *and let* E *denote the set corresponding to* μ *as in Theorem* 2.1. *Then* χ_E *is a minimal idempotent in* $H^\infty(\mu)$.

Proof. – Let $h \in H^\infty(\mu)$ be a nonzero idempotent such that $0 \leqslant h \leqslant \chi_E$. Choose a representing measure ν for φ such that $\nu \sim \mu_E$. Since h does not vanish a.e. $(d\nu)$, we deduce from the preceding lemma that $h = 1$ a.e. $(d\nu)$. Hence $h = \chi_E$, and χ_E is minimal. □

3.4. LEMMA. – *Let* $\varphi, \psi \in S(\mu)$ *and let* $E(\varphi)$ *and* $E(\psi)$ *be the corresponding sets from Theorem* 2.1. *Then either* $E(\varphi)$ *and* $E(\psi)$ *are essentially disjoint* (i.e. $|\mu|(E(\varphi) \cap E(\psi)) = 0$) *or* $E(\varphi)$ *and* $E(\psi)$ *essentially coincide.*

Proof. – This follows immediately from Lemmas 3.1 and 3.3. □

We say that φ and ψ in $S(\mu)$ belong to the *same μ-part of* $S(\mu)$ if $E(\varphi)$ and $E(\psi)$ essentially coincide, that is, $\chi_{E(\varphi)} = \chi_{E(\psi)}$ a.e. $(d\mu)$. Thus if φ and ψ belong to the same μ-part, then there are mutually absolutely continuous representing measures ν_φ and ν_ψ for φ and ψ respectively, each of which is comparable to $\mu_{E(\varphi)}$. In particular, φ and ψ belong to the same Gleason part of M_A. On the other hand, if φ and ψ belong to different μ-parts, then every representing measure for φ absolutely continuous with respect to μ is singular to every such representing measure for ψ. This does not guarantee, however, that φ and ψ belong to different Gleason parts of M_A ; see the example at the end of this section.

The preceding lemma yields the following corollary, which was pointed out to us by H. König.

3.5. Corollary. – *If μ is a representing measure for $\varphi \in M_A$, and if $\psi \in M_A$ has a representing measure absolutely continuous with respect to μ, then ψ has a representing measure mutually absolutely continuous with respect to μ.*

Proof. – In this case, there is only one μ-part, the μ-part of φ, and all points in this μ-part have representing measures mutually absolutely continuous with respect to μ. □

The following theorem can be regarded as an extended version of Theorem 1.1:

3.6. Theorem. – *Let A be a uniform algebra on X, and let μ be a measure on X. Then there is a partition of X into disjoint Borel sets E_s and $E_1, E_2, \ldots$, and there exist $\varphi_j \in M_A$, $j \geqslant 1$, such that*

(i) *φ_j has a representing measure ν_j mutually absolutely continuous with respect to μ_{E_j}, $j \geqslant 1$;*

(ii) *if λ is a representing measure for φ_j such that $\lambda \ll \mu$, then $\lambda \ll \nu_j$;*

(iii) *if $\varphi \in M_A$, and λ is a representing measure for φ satisfying $\lambda \ll \mu$, then $\lambda(E_s) = 0$;*

(iv) *χ_{E_j} is a minimal idempotent in $H^\infty(\mu)$, $j \geqslant 1$;*

(v) *if $\lambda \in A^\perp$ satisfies $\lambda \ll \mu$, and we denote $\lambda_j = \lambda_{E_j}$, $\lambda_s = \lambda_{E_s}$, then λ_s and the λ_j's belong to $A^\perp$, and*

$$\lambda = \lambda_s + \sum_{j \geqslant 1} \lambda_j ,$$

where the series converges in norm.

Proof. – Choose $\varphi_1, \varphi_2, \dots$ in $S(\mu)$ such that the sets $E(\varphi_j) = E_j$ are pairwise disjoint, and such that $|\mu|(\cup E_j)$ is a maximum. Then set $E_s = X \backslash (\cup_{j \geq 1} E_j)$. □

Theorem 1.1 is now an immediate consequence of Theorem 3.6, obtained by applying Theorem 3.6 in the case where $\lambda = \mu \in A^{\perp}$.

To see that infinitely many summands may be required, consider the following simple example:

Let $K = \overline{\Delta}$ be the closed unit disc in the complex plane, and let $\{\Delta_j\}_{j=1}^{\infty}$ be a sequence of open subdiscs of the open unit disc $\Delta = \{|z| < 1\}$, with pairwise disjoint closures, such that the radii of the Δ_j's are summable, and such that the Δ_j's accumulate towards some fixed point of Δ. Let μ_j be the measure dz on $\partial \Delta_j$, and let $\mu = \Sigma \mu_j$. Then $\mu \perp R(\overline{\Delta})$, and each μ_j is mutually absolutely continuous with respect to harmonic measure (a representing measure) on $\partial \Delta_j$ for the center of Δ_j. Since also $\mu \perp R(\overline{\Delta})$, the decomposition $\mu = \Sigma \mu_j$ has the properties of Theorem 1.1. On the one hand, the closed support of a representing measure for any point of Δ has a connected polynomial hull. On the other hand, the closed support of a measure absolutely continuous with respect to μ does not have a connected polynomial hull unless that measure is carried by one of the $\partial \Delta_j$'s. It follows easily that the decomposition $\mu = \Sigma \mu_j$ above is the *unique* decomposition of μ with the properties of Theorem 1.1. In particular, infinitely many summands are required.

4. Relation to the F. and M. Riesz Theorem.

In this section we will provide another proof of the L^1-version of the Cole-König-Seever theorem. This proof will show that the decomposition given by Theorem 2.1 corresponds to the usual F. and M. Riesz decomposition for the algebra $H^{\infty}(\mu)$.

Let $\Sigma(\mu)$ denote the spectrum of $L^{\infty}(\mu)$, so that $L^{\infty}(\mu) \cong C(\Sigma(\mu))$. We will regard $H^{\infty}(\mu)$ as an algebra of continuous functions on the compact space $\Sigma(\mu)$. While $H^{\infty}(\mu)$ does not necessarily separate the points of $\Sigma(\mu)$, neither the Hoffman-Rossi theorem nor the abstract F. and M. Riesz theorem requires point separation, so that we may appeal to these theorems when appropriate.

Let $\hat{\mu}$ denote the canonical lift of the measure μ to $\Sigma(\mu)$. Every measure absolutely continuous with respect to μ lifts canonically to a measure absolutely continuous with respect to $\hat{\mu}$, and $L^1(\mu) \cong L^1(\hat{\mu})$.

The characteristic function χ_E of a Borel subset E of X is an idempotent in $L^\infty(\mu)$. It corresponds to an idempotent in $C(\Sigma(\mu))$ which is the characteristic function of a closed and open subset of $\Sigma(\mu)$, denoted by $\widetilde{E}$. The lift of μ_E to $\Sigma(\mu)$ is the restriction of $\hat{\mu}$ to $\widetilde{E}$: $\hat{\mu}_E = \hat{\mu}_{\widetilde{E}}$.

4.1. THEOREM. – *Let* $\varphi \in S(\mu)$, *and let* E *be a Borel set satisfying* (i) *and* (ii) *of Theorem* 2.1. *Let* λ *be a measure on* X *such that* $\lambda \ll \mu$, *and set* $\lambda_s = \lambda - \lambda_E$. *Then on* $\Sigma(\mu)$,

$$\hat{\lambda} = \hat{\lambda}_E + \hat{\lambda}_s,$$

where $\hat{\lambda}_E$ *is absolutely continuous with respect to a representing measure for* φ *on* $\Sigma(\mu)$, *and* $\hat{\lambda}_s$ *is singular to all representing measures for* φ *on* $\Sigma(\mu)$.

Proof. – Note that $\hat{\lambda}_E$ is carried by $\widetilde{E}$, while $\hat{\lambda}_s$ is carried by $\Sigma(\mu)\backslash\widetilde{E}$.

Let ν be any representing measure for φ that satisfies $\nu \sim \mu_E$. Then the canonical lift $\hat{\nu}$ of ν to $\Sigma(\mu)$ is a representing measure for φ that satisfies $\hat{\nu} \sim \hat{\mu}_E$. In particular, $\hat{\lambda}_E \ll \hat{\nu}$.

Any representing measure for φ on $\Sigma(\mu)$ which is absolutely continuous with respect to $\hat{\mu}$ is the lift of a representing measure which is absolutely continuous with respect to μ_E, and hence is supported on $\widetilde{E}$. A corollary to the Hoffman-Rossi Theorem [8, Theorem IV.2.3] asserts that the set of representing measures for φ on $\Sigma(\mu)$ is the weak-star closure of the set of representing measures for φ that are absolutely continuous with respect to $\hat{\mu}$. (See Theorem 6.2.) Hence any representing measure for φ on $\Sigma(\mu)$ is supported on $\widetilde{E}$. Since $\hat{\lambda}_s$ is carried by the closed and open set $\Sigma(\mu)\backslash\widetilde{E}$, $\hat{\lambda}_s$ is singular to all representing measures on $\Sigma(\mu)$ for φ. □

Now it is easy to prove statement (iv) of Theorem 2.1. If

$\lambda \in A^\perp$, then $\hat{\lambda} \perp H^\infty(\mu)$. By the abstract F. and M. Riesz Theorem [8, Theorem II.7.6], $\hat{\lambda}_E$ and $\hat{\lambda}_s$ are separately orthogonal to $H^\infty(\mu)$. Hence λ_E and λ_s are in $A^\perp$. As we saw earlier, the statement (iii) of Theorem 2.1, that $\chi_E \in H^\infty(\mu)$, follows directly from statement (iv).

Thus we see also that the μ-parts of $S(\mu)$ are precisely the intersections of $S(\mu)$ with the Gleason parts of $H^\infty(\mu)$. The decomposition of Theorem 3.6 is precisely the abstract F. and M. Riesz decomposition corresponding to the (at most countably many) Gleason parts of homomorphisms in $S(\mu)$. In particular, the lift $\hat{\mu}_s$ to $\Sigma(\mu)$ of the measure μ_s appearing in Theorem 3.6 is singular to all representing measures on $\Sigma(\mu)$ for all $\varphi \in S(\mu)$.

5. Extension to bands of measures.

Recall that a *band* of measures on X is a closed subspace $\mathcal{B}$ of measures on X such that if $\eta \in \mathcal{B}$ then any measure absolutely continuous with respect to η belongs to $\mathcal{B}$. Each measure λ on X can be expressed in the form $\lambda = \lambda_a + \lambda_s$, where $\lambda_a \in \mathcal{B}$ and λ_s is singular to each measure in $\mathcal{B}$. The component λ_a is the *projection* of λ into $\mathcal{B}$. The band-version of the Cole-König-Seever theorem is as follows (cf. [2], [10], [12]).

5.1. THEOREM. – *Let $\mathcal{B}$ be a band of measures on X, and suppose $\varphi \in M_A$ has a representing measure in $\mathcal{B}$. Let $\mathcal{B}_\varphi$ be the subband of measures η such that there is a representing measure ν for φ such that $\nu \in \mathcal{B}$ and $\eta \ll \nu$. Then if $\lambda \in A^\perp \cap \mathcal{B}$, and λ_a is the projection of λ into $\mathcal{B}_\varphi$, then $\lambda_a \in A^\perp$.*

Proof. – Let ν be a representing measure for φ such that $\nu \in \mathcal{B}$ and $\lambda_a \ll \nu$. Set $\mu = |\lambda| + \nu$. Then $\lambda - \lambda_a$ is singular to all representing measures for φ that are absolutely continuous with respect to μ. It follows that the component λ_E of λ given by Theorem 2.1 coincides with λ_a. By that theorem, $\lambda_a \perp A$. □

The preceding theorem, though formally an extension of the abstract F. and M. Riesz Theorem, turns out to be simply the F. and M. Riesz Theorem applied to an appropriate algebra on an appropriate compact space. In this case, it is the F. and M. Riesz theorem for the weak-star closure $H^\infty(\mathcal{B})$ if A in $L^\infty(\mathcal{B})$. In the next section, we define $L^\infty(\mathcal{B})$ and $H^\infty(\mathcal{B})$, we extend the Hoffman-Rossi theorem to $H^\infty(\mathcal{B})$, and we use this to show how the Cole-König-Seever theorem becomes the F. and M. Riesz theorem for $H^\infty(\mathcal{B})$.

6. The Hoffman-Rossi theorem for a band of measures.

Fix a band $\mathcal{B}$ of measures on X. With $\mathcal{B}$ there is associated the algebra $L^\infty(\mathcal{B})$, which plays a role similar to that played by $L^\infty(\mu)$ in the case that $\mathcal{B}$ is the band of measures absolutely continuous with respect to μ ($\mathcal{B} \cong L^1(\mu)$). For more detailed background information on $L^\infty(\mathcal{B})$, see the appendix (section 20) of [5].

The elements of $L^\infty(\mathcal{B})$ are collections $F = \{F_\nu : \nu \in \mathcal{B}\}$ of functions F_ν in $L^\infty(\nu)$, which satisfy the compatibility property that if $\nu \ll \lambda$ then $F_\nu = F_\lambda$ a.e. $(d\nu)$. It will be convenient at times to drop the subscripts and regard F as an element of $L^\infty(\nu)$ for all $\nu \in \mathcal{B}$, though $F \in L^\infty(\mathcal{B})$ cannot generally be realized as a point function on X.

If $F \in L^\infty(\mathcal{B})$, then the supremum of the norms $\|F_\nu\|_{L^\infty(\nu)}$ over $\nu \in \mathcal{B}$ is finite and defines a norm on $L^\infty(\mathcal{B})$. Under the pairing $\langle \nu, F \rangle = \int F_\nu \, d\nu$, the normed space $L^\infty(\mathcal{B})$ is isometrically isomorphic to the dual space $\mathcal{B}^*$ of $\mathcal{B}$. By the weak-star topology of $L^\infty(\mathcal{B})$, we will mean the $\mathcal{B}$-topology determined by this pairing.

Under the obvious multiplication, $L^\infty(\mathcal{B})$ becomes a commutative Banach algebra. Let $\Sigma(\mathcal{B})$ denote the maximal ideal space of $L^\infty(\mathcal{B})$. It can be shown that the Gelfand transform implements an isometric isomorphism

$$L^\infty(\mathcal{B}) \cong C(\Sigma(\mathcal{B})).$$

Each of the measures $\nu \in \mathcal{B}$ lifts canonically to a measure on $\Sigma(\mathcal{B})$, and we may regard $\mathcal{B}$ as a band of measures on the compact space $\Sigma(\mathcal{B})$.

The band-version of the Hoffman-Rossi theorem is as follows.

6.1. THEOREM. – *Let* H *be a weak-star closed subalgebra of* $L^\infty(\mathcal{B})$ *containing the constants, let* $\varphi \in M_H$ *be weak-star continuous, and let* H_φ *denote the kernel of* φ. *Then the following are equivalent, for a fixed* $u \in L^\infty_R(\mathcal{B})$:

(i) $\int u\, d\nu \geq 0$ *for all probability measures* ν *in* $\mathcal{B}$ *that represent* φ.

(ii) $\int u\, d\lambda \geq 0$ *for all probability measures* λ *on the spectrum* $\Sigma(\mathcal{B})$ *of* $L^\infty(\mathcal{B})$ *that represent* φ.

(iii) u *lies in the norm-closure of the cone* $P + \mathrm{Re}H_\varphi$, *where* P *is the cone of nonnegative function in* $L^\infty_R(\mathcal{B})$.

(iv) *For each* $t > 0$, *there is* $h_t \in H$ *such that* $\varphi(h_t) = 1$ *and* $|h_t| \leq e^{tu}$.

Moreover, the set of functions u *that satisfy these equivalent conditions is a weak-star closed convex cone in* $L^\infty_R(\mathcal{B})$.

Proof. – That (ii), (iii) and (iv) are equivalent can be seen rather directly. In fact, that (ii) and (iii) are equivalent follows immediately by duality. We will show that (iii) is also equivalent to (iv).

Suppose that u lies in the norm-closure of $P + \mathrm{Re}H_\varphi$. Choose a sequence $\{g_m\}$ in H and functions $v_m \geq 0$ such that $\varphi(g_m) = 0$, and $v_m + \mathrm{Re}\, g_m$ tends uniformly to u. For $t > 0$, let h_t be a weak-star adherent point of the (bounded) sequence $\{e^{tg_m}\}$. Then $\varphi(h_t) = 1$, while

$$|h_t| \leq \limsup |e^{tg_m}| = \limsup e^{t\mathrm{Re}\, g_m} \leq e^{tu}.$$

Thus (iv) is valid.

Conversely, suppose (iv) is valid, and for each $t > 0$, choose h_t as in (iv). Since $\mathrm{Re}\, h_t \leq |h_t| \leq e^{tu}$, we have $e^{tu} - \mathrm{Re}\, h_t \in P$. Since $\varphi(h_t - 1) = 0$, $\mathrm{Re}\, h_t - 1 \in \mathrm{Re}\, H_\varphi$. Hence

$$e^{tu} - 1 = (e^{tu} - \mathrm{Re}\, h_t) + (\mathrm{Re}\, h_t - 1) \in \mathrm{Re}\, H_\varphi + P.$$

Dividing by t and sending t to 0, we obtain u as a uniform limit of functions in $P + \mathrm{Re}\, H_\varphi$. Note that this argument is more efficient than the original argument in [14].) Thus (iii) and (iv) are equivalent.

Since every representing measure ν in $\mathcal{B}$ lifts naturally to a representing measure on $\Sigma(\mathcal{B})$, it is clear that (ii) implies (i). The crux of the theorem, then, is to prove that (i) implies one of the equivalent assertions (ii), (iii) or (iv). Here we rely on the original argument from [14]. Rather than modifying the argument, we will simply show how the proof that (i) implies (iv) can by reduced to the corresponding statement in the L^1-version of the theorem in [14].

Fix $t > 0$, choose $M > 0$ so that $e^{tu} \leqslant M$, and fix a representing measure $\nu \in \mathcal{B}$ for φ. Suppose $\mu \in \mathcal{B}$ satisfies $\nu \ll \mu$. By the L^1-version of the Hoffman-Rossi theorem, there exists $h^{(\mu)} \in H^\infty(\mu)$ such that $h^{(\mu)} \leqslant e^{tu}$ a.e. $(d\mu)$ and $\varphi(h^{(\mu)}) = 1$. In particular, $|h^{(\mu)}| \leqslant M$. Thus the set $J^{(\mu)}$ of $h \in L^\infty(\mathcal{B})$ which satisfy the conditions $\|h\| \leqslant M$, $h_\mu \in H^\infty(\mu)$, $|h_\mu| \leqslant e^{tu}$ a.e. $(d\mu)$, and $\int h_\mu \, d\nu = 1$, is nonempty. Furthermore, $J^{(\mu)}$ is a weak-star closed subset of $L^\infty(\mathcal{B})$. One checks that the various sets $J^{(\mu)}$, μ as above, have the finite intersection property. Then by compactness, there exists $h \in L^\infty(\mathcal{B})$ such that h belongs to all of the sets $J^{(\mu)}$ above. In particular, $\|h\| \leqslant M$, and $h_\mu \in H^\infty(\mu)$ for all $\mu \in \mathcal{B}$ satisfying $\nu \ll \mu$, so $h \in H^\infty(\mathcal{B}) = H$. Also,

$$\varphi(h) = \int h \, d\nu = 1 ,$$

and $|h| \leqslant e^{tu}$. Thus (iv) is valid, and the conditions (i) through (iv) are equivalent.

The final assertion of the theorem follows immediately from the description of the set given in (i). □

As we have seen, the crux of the Hoffman-Rossi theorem is the equivalence of the assertions (i) and (ii) of the theorem. There is a simple dual reformulation of the equivalence of these assertions, which is often more convenient for applications. The L^1-version of the reformulation is given in [8, Theorem IV.2.3]. The band version, which can be regarded as an equivalent statement of the Hoffman-Rossi theorem, is as follows.

6.2. THEOREM. – *Let* $\mathcal{B}$, H *and* φ *be as above. Then the* *representing measures in* $\mathcal{B}$ *for* φ *are weak-star dense (in* $L^\infty(\mathcal{B})^*$) *in the set of all representing measures on* $\Sigma(\mathcal{B})$ *for* φ.

Proof. – If the representing measures in $\mathcal{B}$ are not weak-star dense, then there are $u \in L^\infty(\mathcal{B})$ and a real number c such that $\int u\, d\lambda \geq c$ for all $\lambda \in \mathcal{B}$ that represent φ, while $\int u\, d\lambda_0 < c$ for some representing measure λ_0 on $\Sigma(\mathcal{B})$. Replacing u by u-c, we may assume that $c = 0$, and we then have a contradiction to the equivalence of (i) and (ii) of Theorem 6.1. □

Now let $\mathcal{B}$, H and φ be as above, let $\mathcal{B}_\varphi$ denote the subband of $\mathcal{B}$ generated by the representing measures for φ that lie in $\mathcal{B}$, and let $\mathcal{B}'_\varphi$ denote its complementary band consisting of all measures in $\mathcal{B}$ singular to each representing measure for φ in $\mathcal{B}$, so that $\mathcal{B} = \mathcal{B}_\varphi \oplus \mathcal{B}'_\varphi$. Let $\chi \in L^\infty(\mathcal{B})$ be the idempotent of $\mathcal{B}_\varphi$, defined so that $\chi = 1$ a.e. $(d\nu)$ for all $\nu \in \mathcal{B}_\varphi$, and $\chi = 0$ a.e. $(d\lambda)$ for all $\lambda \in \mathcal{B}'_\varphi$. The projection of a measure $\eta \in \mathcal{B}$ into $\mathcal{B}_\varphi$ is then $\chi\eta$. Regarded as a continuous function on $\Sigma(\mathcal{B})$, the idempotent χ is the characteristic function of a closed and open subset of $\Sigma(\mathcal{B})$, and it is easy to see that this subset is homeomorphic to $\Sigma(\mathcal{B}_\varphi)$ while its complement is homeomorphic to $\Sigma(\mathcal{B}'_\varphi)$. Thus the band decomposition above leads to a decomposition

$$\Sigma(\mathcal{B}) = \Sigma(\mathcal{B}_\varphi) \cup \Sigma(\mathcal{B}'_\varphi)$$

of $\Sigma(\mathcal{B})$ as a disjoint union of closed subsets. From Theorem 6.2, we obtain immediately the following:

6.3. COROLLARY. – *Let* $\mathcal{B}$, H *and* φ *be as above. Then each representing measure on* $\Sigma(\mathcal{B})$ *for* φ *is supported on* $\Sigma(\mathcal{B}_\varphi)$:

Now we can prove the following version of the Cole-König-Seever theorem by applying the abstract F. and M. Riesz theorem as in Section 4.

6.4. THEOREM. – *Let* H *be a weak-star closed subalgebra of* $L^\infty(\mathcal{B})$, *let* φ *be a weak-star continuous homomorphism of* H, *and let* $\mathcal{B}_\varphi$ *be the band of measures absolutely continuous with respect to some representing measure in* $\mathcal{B}$ *for* φ. *Then:*

(i) *the idempotent* χ *of the band* $\mathcal{B}_\varphi$ *belongs to* H,

(ii) χ *is a minimal idempotent in* H,

(iii) *if* $\lambda \in \mathcal{B}$ *is orthogonal to* H, *and if* λ_a *is the projection of* λ *into* $\mathcal{B}_\varphi$, *then* $\lambda_a \perp$ H.

Proof. – Let $\lambda_s = \lambda - \lambda_a$, and let $\hat{\lambda}, \hat{\lambda}_a$ and $\hat{\lambda}_s$ to the canonical lifts of λ, λ_a and λ_s respectively to measures on $\Sigma(\mathcal{B})$. Using the corollary to Theorem 6.2, as in the proof of Theorem 4.1, we see that the decomposition $\hat{\lambda} = \hat{\lambda}_a + \hat{\lambda}_s$ is the Lebesgue decomposition of $\hat{\lambda}$ with respect to the set of representing measures on $\Sigma(\mathcal{B})$ for φ. By the abstract F. and M. Riesz theorem, $\hat{\lambda}_a \in H^\perp$, so that also $\lambda_a \perp$ H. This proves (iii), and (i) follows from (iii) and duality. The proof of (ii) is the same as that of the corresponding facts in Section 3. □

7. The topology of S(μ).

In this section we will be required to distinguish carrefully between the set of weak-star continuous homomorphisms S(μ), regarded as a subset of $M_{H^\infty(\mu)}$, and the set of their restrictions to A, regarded as a subset of M_A. The latter set will be denoted (in this section only) by $\pi(S(\mu))$, where

$$\pi : M_{H^\infty(\mu)} \longrightarrow M_A$$

is the natural projection obtained by simply restricting a homomorphism of $H^\infty(\mu)$ to A. Thus $\pi(S(\mu))$ is the set of homomorphisms in M_A which extend weak-star continuously to $H^\infty(\mu)$.

The purpose of this section is to give conditions which ensure that $\pi(S(\mu))$ is a Borel subset of M_A. We do not know whether it is a Borel set in general. We begin with the following :

7.1. LEMMA. – S(μ) *is an* $F_{\sigma\delta}$*-subset of the maximal ideal space of* $H^\infty(\mu)$, *with respect to its Gelfand (weak-star) topology.*

Proof. – For integers $k, m \geqslant 1$. let F_{km} be the set of all φ in the spectrum of $H^\infty(\mu)$ which satisfy $|\varphi(f)| \leqslant \frac{1}{m}$ for all $f \in H^\infty(\mu)$

with $|f| \leqslant 1$ and $\int |f|^2 d\mu \leqslant \frac{1}{k}$. Evidently E_{km} is a closed subset of $H^\infty(\mu)$. Hence

$$E = \bigcap_{m=1}^{\infty} \bigcup_{k=1}^{\infty} E_{km}$$

is an $F_{\sigma\delta}$-set, consisting of precisely those φ with the following properties: If $\{f_n\}$ is a sequence in $H^\infty(\mu)$ such that $|f_n| \leqslant 1$ and $f_n \longrightarrow 0$ in $L^2(\mu)$, then $\varphi(f_n) \longrightarrow 0$. Any weak-star continuous φ has this property, so that $S(\mu) \subseteq E$. On the other hand suppose $\varphi \in E$. Let $\{g_m\}$ be a bounded sequence in $H^\infty(\mu) \cap \varphi^{-1}(0)$ such that $g_m \longrightarrow g$ a.e. Then $g \in H^\infty(\mu)$ and $g - g_m \longrightarrow 0$ in $L^2(\mu)$. By the characterizing property for E, we obtain $\varphi(g - g_m) \longrightarrow 0$, so that $\varphi(g) = 0$, and g also belongs to $H^\infty(\mu) \cap \varphi^{-1}(0)$. By the version of the Krein-Schmulian theorem for convex subsets of $L^\infty(\mu)$ [8, Lemma IV2.1], $\varphi^{-1}(0)$ is a weak-star closed subspace of $H^\infty(\mu)$. Hence φ is weak-star continuous, and $\varphi \in S(\mu)$. We conclude that $S(\mu)$ coincides with the $F_{\sigma\delta}$-set E. □

7.2. LEMMA. – *$S(\mu)$ is a norm-closed subset of $H^\infty(\mu)^*$.*

Proof. – The predual $L^1(\mu)/H^\infty(\mu)^\perp$ of $H^\infty(\mu)$ can be regarded as a closed subspace of $H^\infty(\mu)^*$, as can the maximal ideal space of $H^\infty(\mu)$. The intersection of these closed subsets is precisely the set of multiplicative linear functionals which have representing measures in $L^1(\mu)$, that is, which belong to $S(\mu)$. □

7.3. THEOREM. – *Suppose that A is a separable uniform algebra. Then $\pi(S(\mu))$ is a Borel subset of M_A (with the Gelfand topology).*

Proof. – We use the fact that $L^1(\mu)$ is separable, so that the set W of multiplicative probability measures absolutely continuous with respect to μ is a separable set of measures. Since $S(\mu)$ is isometric to the subset $W/H^\infty(\mu)^\perp$ of $L^\infty(\mu)^*/H^\infty(\mu)^\perp$, $S(\mu)$ is also separable. By Lemma 7.2 $S(\mu)$ is closed in $H^\infty(\mu)^*$, hence complete. Thus the norm topology on $S(\mu)$ makes $S(\mu)$ into a separable complete metric space (Polish space). Now π maps $S(\mu)$ in a one-to-one continuous manner into the separable complete metric space M_A. By Souslin's theorem [1, Theorem 3.2.3], $\pi(S(\mu))$ is a Borel subset of M_A. □

When ball A is weak-star dense in ball $H^\infty(\mu)$, the idea of the proof of the first lemma above can be applied to give information about $\pi(S(\mu))$. We begin with a useful criterion for weak-star continuity.

7.4. LEMMA. – *Suppose that* ball A *is weak-star dense in* ball $H^\infty(\mu)$. *Then* $\varphi \in M_A$ *extends to be weak-star continuous on* $H^\infty(\mu)$ *if and only if whenever* $\{f_n\}$ *is a sequence in* A *such that* $\|f_n\| \leqslant 1$ *and* $f_n \longrightarrow 0$ a.e. $(d\mu)$, *then* $f_n(\varphi) \longrightarrow 0$.

Proof. – The forward implication is trivial. For the reverse implication, suppose the condition is valid. First we claim we can extend φ to $H^\infty(\mu)$ so that whenever $f \in H^\infty(\mu)$, and $\{f_n\}$ is a bounded sequence in A such that $f_n \longrightarrow f$ in μ-measure, then $f_n(\varphi) \longrightarrow \varphi(f)$. Indeed, otherwise there are $f \in H^\infty(\mu)$ and a bounded sequence $\{f_n\}$ in A such that $f_n \longrightarrow f$ in measure, while $\{f_n(\varphi)\}$ does not converge. Taking differences of the form $g_k = f_{k_1} - f_{k_2}$, we obtain a bounded sequence in A which converges to 0 in measure, while $g_k(\varphi)$ does not converge to 0. Passing to a subsequence which converges to 0 a.e. $(d\mu)$, we obtain contradiction to the condition.

The extended φ is evidently multiplicative on $H^\infty(\mu)$. To show that it is weak-star continuous, it suffices to show that its kernel is weak-star closed. By the Krein-Schmulian theorem for $L^\infty(\mu)$, it suffices to show that if $\{h_n\}$ is a bounded sequence in $H^\infty(\mu)$ such that $\varphi(h_n) = 0$ and $h_n \longrightarrow h$ in measure, then $\varphi(h) = 0$. For such h_n, choose g_n in A such that $\|g_n\| \leqslant \|h_n\|$, $g_n(\varphi) = 0$, and g_n is close to h_n in measure. Then g_n converges to h in measure, and hence $\varphi(h) = \lim g_n(\varphi) = 0$. □

One useful consequence of the preceding lemma is as follows :

7.5. COROLLARY. – *Suppose that* ball A *is weak-star dense in* ball $H^\infty(\mu)$. *Let* E *be a closed subset of* M_A *consisting of weak-star continuous homomorphisms. Then each homomorphism in the* A*-convex hull* $\hat{E}$ *of* E *is weak-star continuous.*

Proof. – Each $\psi \in \hat{E}$ is represented by a probability measure

ν on E. Suppose that $\{f_n\}$ is a bounded sequence in A such that $f_n \longrightarrow 0$ a.e. $(d\mu)$. Then $f_n(\varphi) \longrightarrow 0$ for all $\varphi \in E$. By the Lebesgue bounded convergence theorem,

$$f_n(\psi) = \int_E f_n \, d\nu(\varphi) \longrightarrow 0 .$$

Hence by Lemma 7.4, ψ is weak-star continuous. □

From Lemma 7.4, we obtain our second result on the measurability of $\pi(S(\mu))$.

7.6. THEOREM. – *If* ball A *is weak-star dense in* ball $H^\infty(\mu)$, *then* $\pi(S(\mu))$ *is an* $F_{\sigma\delta}$*-subset of* M_A *(in the Gelfand topology).*

Proof. – For integers $k, m \geqslant 1$, let F_{km} be the set of all $\varphi \in M_A$ such that $|\varphi(f)| \leqslant 1/m$ for all $f \in A$ satisfying $|f| \leqslant 1$ and $\int |f|^2 \, d\mu < 1/k$. Evidently F_{km} is a closed subset of M_A. The condition above shows that

$$\pi(S(\mu)) = \bigcap_{m=1}^{\infty} \bigcup_{k=1}^{\infty} F_{km} ,$$

as in the proof of the first lemma. □

For the T-invariant algebras treated in Sections 8 through 13, it turns out that $\pi(S(\mu))$ is an F_σ-set (cf. Corollary 10.2). However, we will give an example in Section 14 for which $\pi(S(\mu))$ is not an F_σ-set, even though the hypothesis of Theorem 7.6 is met. Thus the $F_{\sigma\delta}$ condition of Theorem 7.6 cannot be improved.

In Section 14 we will require the following analogue for M_A of Lemma 7.2.

7.7. THEOREM. – *Suppose that* ball A *is weak-star dense in* ball $H^\infty(\mu)$. *Then* $\pi(S(\mu))$ *is a closed subset of* M_A, *with the Gleason topology (norm topology of* A^**).*

Proof. – The hypothesis ensures that the projection π is an isometry from $S(\mu)$, with the Gleason metric of $M_{H^\infty(\mu)}$, to $\pi(S(\mu))$, with the Gleason metric of M_A. By Lemma 7.2, $S(\mu)$ is Gleason-closed in $H^\infty(\mu)^*$, so it is Gleason-complete. It follows that $\pi(S(\mu))$ is Gleason-complete, and hence Gleason-closed. □

8. T-invariant algebras.

Let K be a compact subset of the complex plane. As usual, R(K) will denote the algebra of uniform limits on K of rational functions with poles off K. The Cauchy transform of a measure ν on K is given by

$$\hat{\nu}(w) = \int \frac{d\nu(z)}{z - w},$$

defined whenever the integral converges absolutely. Since the Cauchy transform is the convolution of a finite measure and the locally integrable $(dx\,dy)$ function $1/z$, the integral converges absolutely a.e. $(dx\,dy)$, and $\hat{\nu}$ is defined a.e. $(dx\,dy)$.

The importance of the Cauchy transform in rational approximation theory stems from the fact that a measure ν on K is orthogonal to R(K) if and only if $\hat{\nu} = 0$ on the complement of K (cf. [8,II.8.1]). Dually, R(K) is generated by functions of the form

$$f(w) = \iint \frac{h(z)}{z - w} dx\,dy \tag{8.1}$$

where h is a bounded Borel function on the complex plane with compact support such that $h = 0$ on K. This characterization provides the basis for generalizing the definition of R(K), as follows:

Let E be a Borel subset of K, and denote by R(E) the closed linear span in C(K) of functions of the form (8.1), where h is a bounded Borel function on the complex plane with compact support such that $h = 0$ a.e. $(dx\,dy)$ on E. It is easy to check that $R(E)^{\perp}$ consists of precisely the measures ν on K such that $\hat{\nu} = 0$ a.e. $(dx\,dy)$ on the complement of E. In particular, if we take E = K, we find that R(E) coincides with the algebra R(K) defined earlier. It turns out (cf. [9], [10]) that R(E) is always a subalgebra of C(K), so that it is a uniform algebra on K. The algebra R(E) was used by J. Chaumat [4], to describe the weak-star closure of R(K) in $L^{\infty}(\mu)$.

A closed subalgebra A of C(K) is *T-invariant* if $A \supseteq R(K)$, and if A is invariant under the T_g-operators used in rational approximation theory, defined by

$$(T_g f)(w) = g(w) f(w) + \frac{1}{\pi} \iint \frac{f(z)}{z - w} \frac{\partial g}{\partial \bar{z}} \, dx \, dy \, ,$$

where g is a smooth function with compact support. Here the area integral is taken over K. (It also may be taken over any compact set containing K to which f is extended to be bounded and measurable, since the added integral over the complement of K belongs to R(K).) For a description of the fundamental properties of T-invariant algebras, see [10], or [5, Section 17].

The algebra R(K) is itself a T-invariant algebra, as are the algebras R(E) defined above. If D is any open subset of K^0, then the algebra of continuous functions on K which are analytic on D is T-invariant. As another example, T. Lyons [16] has proved that the algebra of continuous functions on K which are finely holomorphic on the fine interior of K is a T-invariant algebra.

8.1. Lemma. – *Let* A *be a* T-*invariant algebra on* K. *If* $f \in A$ *and* $\nu \in A^{\perp}$, *then*

$$\int \frac{f(w) - f(z)}{w - z} \, d\nu(w) = 0 \text{ a.e. } (dx \, dy) \, .$$

Equivalently,

$$\widehat{f\nu}(z) = f(z) \, \hat{\nu}(z) \qquad \text{a.e. } (dx \, dy).$$

Proof. – First note that $f\nu \in A^{\perp}$, so that $\widehat{f\nu}(z) = 0$ and $\hat{\nu}(z) = 0$ whenever $z \in \mathbf{C} \backslash K$. Thus we need consider the identity only for $z \in K$; it is automatically valid for $z \notin K$, independent of how f might be extended off K.

Let g be a smooth function with compact support. Extend f to be zero in the complement of K. Using Green's formula, we may write

$$(T_g f)(w) = \frac{1}{\pi} \iint \frac{f(z) - f(w)}{z - w} \frac{\partial g}{\partial \bar{z}} \, dx \, dy \, , w \in K \, .$$

Since $T_g f \in A$, we have $\int T_g f \, d\nu = 0$. From the preceding expression and Fubini's theorem, we obtain

$$\int\int\left[\int\frac{f(z)-f(w)}{z-w}\,d\nu(w)\right]\frac{\partial g}{\partial\bar z}\,dx\,dy = 0\,.$$

If h is any smooth function defined in a neighborhood of K, then the Cauchy transform g of h satisfies $\dfrac{\partial g}{\partial \bar z} = -\pi h$ in a neighborhood of K. Modifying g so that it has compact support, and using the fact that the inner integral above is zero off K, we obtain

$$\int\int\left[\int\frac{f(z)-f(w)}{z-w}\,d\nu(w)\right] h(z)\,dx\,dy = 0$$

for all smooth functions h on K. It follows that the inner integral vanishes a.e. $(dy\,dy)$ on K. □

8.2. THEOREM. – *Let* A *be a* T-*invariant algebra on* K, *and let* E *be a Borel subset of* K. *Then the uniform closure* $[A + R(E)]^-$ *of the linear span of* A *and* R(E) *is a* T-*invariant algebra on* K. *The measures orthogonal to* $[A + R(E)]^-$ *are precisely the measures* $\nu \in A^\perp$ *such that* $\hat\nu = 0$ a.e. $(dx\,dy)$ *off* E.

Proof. – The description of the measures orthogonal to $[A + R(E)]^\perp$ follows immediately from the description of $R(E)^\perp$. It suffices then to show that $[A + R(E)]^-$ is an algebra. For this, it suffices to show that if $f \in A$ and $g \in R(E)$, then $fg \in [A + R(E)]^-$.

Let $\nu \in [A + R(E)]^\perp$. Since $\nu \perp A$ and $f \in A$, we obtain $\widehat{f\nu} = f\hat\nu$ by Lemma 8.1. Since $\nu \perp R(E)$, $\hat\nu = 0$ a.e. $(dx\,dy)$ off E, and also $\widehat{f\nu} = 0$ a.e. off E. Hence $f\nu \perp R(E)$ and $\int fg\,d\nu = 0$. By the Hahn-Banach theorem, $fg \in [A + R(E)]^-$. □

There is one case in which the algebra in the preceding theorem is easy to identify.

8.3. LEMMA. – *If* E *and* F *are Borel subsets of* K, *then* $[R(E) + R(F)]^- = R(E \cap F)$.

Proof. – The measures orthogonal to $[R(E) + R(F)]^-$ are precisely those in $R(E)^\perp \cap R(F)^\perp$. These are precisely the measures ν on K such that $\hat\nu = 0$ a.e. $(dx\,dy)$ of E and $\hat\nu = 0$ a.e. $(dx\,dy)$

off F, i.e. $\hat{\nu} = 0$ a.e. $(dx\,dy)$ off $E \cap F$. Thus

$$R(E)^{\perp} \cap R(F)^{\perp} = R(E \cap F)^{\perp},$$

and by the Hahn-Banach theorem, $[R(E) + R(F)]^{-} = R(E \cap F)$. □

Let Q be the set of nonpeak points for the T-invariant algebra A. The proof of the Browder-Wilken Theorem for R(K) extends to T-invariant algebras, to show that Q is an F_{σ}-set with positive area, in fact, with positive area density at each of its points. Furthermore, a standard argument shows that if $\nu \in A^{\perp}$, then $\hat{\nu} = 0$ a.e. $(dx\,dy)$ off Q. In particular, $\nu \perp R(Q)$, so that $R(Q) \subseteq A$.

Note that Q is the smallest set (modulo sets of zero area) for which $R(Q) \subseteq A$. In fact, suppose $Q_0 \subseteq Q$ has area strictly less than Q. Let $q \in Q \backslash Q_0$. Let μ be any representing measure for q with no mass at q, and set $\nu = (z - q)\mu$. Then $\nu \in A^{\perp}$, $\int \frac{d|\nu|(z)}{|z-q|} < \infty$, and $\hat{\nu}(q) \neq 0$. The proof of the Browder density theorem [3, Theorem 3.3.9] shows that the set $\{\hat{\nu} \neq 0\}$ is a subset of Q with full area density at q. In particular $\hat{\nu} \neq 0$ on a set of positive area disjoint from Q_0, and Q indeed has the minimality property.

8.4. LEMMA. – *Let* A *be a* T-*invariant algebra on* K, *let* E *be a Borel subset of* K, *and let* Q_0 *denote the set of nonpeak points of the* T-*invariant algebra* $[A + R(E)]^{-}$. *Then*

$$[A + R(E)]^{-} = [A + R(Q_0)]^{-}.$$

Proof. – Since each $\nu \in R(E)^{\perp}$ satisfies $\hat{\nu} = 0$ a.e. $(dx\,dy)$ outside E, almost all nonpeak points for R(E) are included in E. Since the points in Q_0 are nonpeak points for R(E), $Q_0 \backslash E$ has zero area. Hence $R(E) \subseteq R(Q_0)$, and

$$[A + R(E)]^{-} \subseteq [A + R(Q_0)]^{-}.$$

On the other hand, since $[A + R(E)]^{-}$ is T-invariant, our earlier remark shows that $R(Q_0) \subseteq [A + R(E)]^{-}$. Hence equality must hold above. □

Davie's theorem is valid for any T-invariant algebra A. One version of Davie's theorem is as follows. Let $\mathcal{B}_{\perp}$ denote the band of measures generated by the measures in $A^{\perp}$, and let $\mathcal{S}$ be the band of measures singular to $A^{\perp}$, so that

$$A^{**} \cong H^{\infty}(\mathcal{B}_{\perp}) \oplus L^{\infty}(\mathcal{S}).$$

Let λ_Q denote the area measure on the set Q of nonpeak points of A. Then $\lambda_Q \in \mathcal{B}_{\perp}$, and Davie's theorem states that the projection $H^{\infty}(\mathcal{B}_{\perp}) \longrightarrow H^{\infty}(\lambda_Q)$ is an isometric isomorphism and a weak-star homeomorphism: $H^{\infty}(\mathcal{B}_{\perp}) \cong H^{\infty}(\lambda_Q)$. Moreover, if $f \in H^{\infty}(\lambda_Q)$, there is a sequence $\{f_n\}$ in A such that $\|f_n\| \leqslant \|f\|$ and $f_n \longrightarrow f$ a.e. $(d\lambda_Q)$. This latter statement is equivalent to ball A being weak-star dense in ball $H^{\infty}(\lambda_Q)$.

A theorem of Gamelin and Garnett [11] also extends to T-invariant algebras, to show that $H^{\infty}(\lambda_Q) \cap C(K) = A$. In fact, these theorems are all valid when λ_Q is replaced by a more general class of measures. Since we will require the result, we state it precisely.

8.5. THEOREM. – *Let* A *be a* T-*invariant algebra on* K, *and let* $\mathcal{B}_{\perp}$ *be the band generated by the measures in* $A^{\perp}$. *Let* λ *be any measure in* $\mathcal{B}_{\perp}$ *such that each nonpeak point of* A *has a representing measure absolutely continuous with respect to* λ, *that is, such that* $S(\lambda) = Q$. *Then the projection* $H^{\infty}(\mathcal{B}_{\perp}) \longrightarrow H^{\infty}(\lambda)$ *is an isometric isomorphism and a weak-star homeomorphism:*

$$H^{\infty}(\mathcal{B}_{\perp}) \cong H^{\infty}(\lambda).$$

Moreover, ball A *is weak-star dense in ball* $H^{\infty}(\lambda)$. *Finally,*

$$A = H^{\infty}(\lambda) \cap C(K).$$

Davie's theorem implies that every nonpeak point $q \in Q$ for A has a representing measure absolutely continuous with respect to λ_Q. Thus λ_Q satisfies the hypotheses of Theorem 8.5.

9. Pointwise bounded limits.

Let A be a T-invariant algebra on K, and let μ be a measure on K. Following Sarason [17], we will approach the weak-star closure $H^{\infty}(\mu)$ of A in $L^{\infty}(\mu)$ by first studying the algebra $B(\mu)$ of pointwise bounded limits in $L^\infty(\mu)$ of functions in A. By definition, $B(\mu)$ consists of the functions $f \in L^\infty(\mu)$ for which there is a bounded sequence $\{f_n\}$ in A satisfying $f_n \longrightarrow f$ a.e. $(d\mu)$. Equivalently, $f \in B(\mu)$ if and only if for some $\epsilon > 0$, ϵf is weak-star adherent to ball A in $L^\infty(\mu)$. Evidently $B(\mu)$ is a subalgebra of $H^\infty(\mu)$. However, $B(\mu)$ need not be norm closed, and we will denote by $B(\mu)^-$ the norm-closure of $B(\mu)$ in $H^\infty(\mu)$.

9.1. LEMMA. – *Let* $\mu = \mu_a + \mu_s$, where $\mu_a \in \mathcal{B}_\perp$ *and* μ_s *is singular to* $\mathcal{B}_\perp$. *Then* $B(\mu) \cong B(\mu_a) \oplus L^\infty(\mu_s)$.

Proof. – Let $h \in L^\infty(\mu_s)$ satisfy $\|h\|_\infty \leqslant 1$. Since ball A is weak-star dense in ball A^{**}, the function in $L^\infty(\mu)$ which is h a.e. $(d\mu_s)$ and 0 a.e. $(d\mu_a)$, is weak-star adherent to ball A. A standard argument shows that there is a sequence $\{h_n\}$ in ball A such that $h_n \longrightarrow h$ a.e. $(d\mu_s)$, and $h_n \longrightarrow 0$ a.e. $(d\mu_a)$. Thus $B(\mu)$ includes $\{0\} \oplus L^\infty(\mu_s)$, and it follows that $B(\mu) \cong B(\mu_a) \oplus L^\infty(\mu_s)$. □

9.2. LEMMA. – $B(\mu_a)$ *coincides with the image of*

$$H^\infty(|\mu_a| + \lambda_Q) \quad \text{in} \quad L^\infty(\mu_a).$$

Proof. – Suppose $f \in B(\mu_a)$. Let $\{f_n\}$ be a bounded sequence in A such that $f_n \longrightarrow f$ a.e. $(d\mu_a)$. If $F \in H^\infty(|\mu_a| + \lambda_Q)$ is any weak-star adherent point of $\{f_n\}$, then $F = f$ a.e. $(d\mu_a)$. Conversely, if $F \in H^\infty(|\mu_a| + \lambda_Q)$, then by Davie's theorem there is a bounded sequence $\{f_n\}$ in A such that

$$f_n \longrightarrow f \text{ a.e. } (d|\mu_a| + d\lambda_Q).$$

If f is the projection of F into $L^\infty(\mu_a)$, then evidently $f \in B(\mu_a)$. □

Now fix $\mu = \mu_a + \mu_s$ as before, and let Q denote the set of nonpeak points of A. Define the *envelope of* μ_a, denoted by $E(\mu_a)$, to be the set of $q \in Q$ such that

$$|f(q)| \leqslant \|f\|_{L^\infty(\mu_a)}, \quad \text{all} \quad f \in H^\infty(|\mu_a| + \lambda_Q).$$

Since $f(q)$ depends continuously on $q \in Q$, in the Gleason topology of Q, it is clear that $E(\mu_a)$ is closed in the Gleason topology. However, it is not immediately clear that $E(\mu_a)$ is a

Borel subset of the complex plane. Indeed it is; this is a consequence of the following general lemma and the fact that the set Q is a Gleason-separable subset of M_A.

9.3. LEMMA. – *If* A *is an arbitrary uniform algebra, then any separable Borel subset of* M_A *with respect to the Gleason topology is a Borel subset with respect to the Gelfand topology.*

Proof. – One checks that any closed ball in A^* meets M_A in a Gelfand-closed set. Thus any closed Gleason-ball in M_A is Gelfand-closed, and any open Gleason-ball in M_A is a Gelfand F_σ-set.

Suppose that E is a separable Gleason-closed subset of M_A, and let $\{q_j\}_{j=1}^{\infty}$ be a Gleason-dense sequence in E. Then the union of the Gleason (= norm) $\frac{1}{n}$ balls centered at the q_j's is a Gelfand F_σ-set, and their intersection E is consequently a Borel set with respect to the Gelfand topology. Because E is separable, any subset of E which is relatively open in the Gleason topology is the intersection of E and a countable union of open Gleason-balls, and hence is also a Borel set with respect to the Gelfand topology. It follows that any Borel subset of E with respect to the Gleason topology is a Borel subset with respect to the Gelfand topology. Appealing to this statement, with E the Gleason-closure of a given separable Borel subset of M_A with respect to the Gleason topology, we obtain the lemma. □

Since now $E(\mu_a)$ is a Borel subset of **C**, we can speak about its area.

9.4. LEMMA. – *Either* $E(\mu_a) = Q$, *or else* $E(\mu_a)$ *has area strictly smaller than* Q.

Proof. – From the definition of $E(\mu_a)$, it is clear that $Q \backslash E(\mu_a)$ is an open subset of Q, in the Gleason part metric of Q. Hence by Browder's theorem, $Q \backslash E(\mu_a)$ has full area density at each of its points. Thus either $Q \backslash E(\mu_a)$ is empty, or else $Q \backslash E(\mu_a)$ has positive area. □

9.5. THEOREM. – *if* $E(\mu_a) = Q$, *then*

$$\mathrm{B}(\mu) = \mathrm{H}^\infty(\mu) = \mathrm{H}^\infty(\mu_a) \oplus \mathrm{L}^\infty(\mu_s).$$

Furthermore, each $q \in \mathrm{Q}$ *has a representing measure absolutely continuous with respect to* μ_a, *so that the conclusions of Theorem* 8.5 *are valid for* μ_a.

Proof. – The natural projection $\mathrm{H}^\infty(|\mu_a| + \lambda_\mathrm{Q}) \longrightarrow \mathrm{H}^\infty(\mu_a)$ is norm-decreasing. The hypothesis that $\mathrm{E}(\mu_a) = \mathrm{Q}$ shows that it is an isometry. Since the map is weak-star continuous, the image of the unit ball of $\mathrm{H}^\infty(|\mu_a| + \lambda_\mathrm{Q})$ is weak-star compact, hence weak-star closed in $\mathrm{L}^\infty(\mu_a)$. The Krein-Schmulian theorem then shows that the image of $\mathrm{H}^\infty(|\mu_a| + \lambda_\mathrm{Q})$ is weak-star closed, hence it coincides with $\mathrm{H}^\infty(\mu_a)$. Thus the natural projection maps $\mathrm{H}^\infty(|\mu_a| + \lambda_\mathrm{Q})$ isometrically onto $\mathrm{H}^\infty(\mu_a)$. The map is thus a weak-star homeomorphism. Since the evaluations at points of Q are weak-star continuous on $\mathrm{H}^\infty(\lambda_\mathrm{Q})$, they are also weak-star continuous on $\mathrm{H}^\infty(\mu_a)$, and they have representing measures $\ll \mu_a$. □

In order to determine what happens when $\mathrm{E}(\mu_a) \neq \mathrm{Q}$, we require two technical lemmas, whose proofs may be found elsewhere.

9.5. LEMMA. – *If* $f \in \mathrm{H}^\infty(|\mu_a| + \lambda_\mathrm{Q})$ *extends to be analytic in a neighborhood of some fixed point* w, *then the function* $[f - f(w)]/(z - w)$ *belongs to* $\mathrm{H}^\infty(|\mu_a| + \lambda_\mathrm{Q})$.

Proof. – The proof is the same as that of [5, Lemma 17.10]. One shows that $\mathrm{H}^\infty(|\mu_a| + \lambda_\mathrm{Q})$ is also invariant under the operators T_g, and one uses this invariance for an appropriate function g. □

9.7. LEMMA. – *Suppose* $f \in \mathrm{H}^\infty(|\mu_a| + \lambda_\mathrm{Q})$, $q \in \mathrm{Q}$, *and* $f(q) = 0$. *Then there is a sequence* $\{f_n\}$ *in* $\mathrm{H}^\infty(|\mu_a| + \lambda_\mathrm{Q})$ *such that* f_n *is analytic at* q, $f_n(q) = 0$, $\{f_n\}$ *is bounded, and* $\{f_n\}$ *converges uniformly to* f *on any subset of* K *at a positive distance from* q.

Proof. – This proof is the same as that in [11, Corollary 2.2] for the algebra $\mathrm{R}(\mathrm{K})$. □

The key lemma for our purposes is the following.

9.8. LEMMA. —*With notation as above, we have*

$$R(E(\mu_a)) \subseteq B(\mu)^-,$$

regarded as subset of $L^\infty(\mu)$.

Proof. — Let Y be the quotient space obtained from the spectrum of $L^\infty(|\mu_a| + \lambda_Q)$ by identifying points which are identified by $H^\infty(|\mu_a| + \lambda_Q)$. We may regard $H^\infty(|\mu_a| + \lambda_Q)$ as a uniform algebra on Y. Let $Z : Y \longrightarrow Q$ be the natural projection which coincides with the Gelfand extension of the coordinate function z, regarded as an element of $H^\infty(|\mu_a| + \lambda_Q)$.

By Lemma 9.2, it suffices to show that the uniform closure of $H^\infty(|\mu_a| + \lambda_Q)$ in $L^\infty(\mu_a)$ includes $R(E(\mu_a))$. Note that the measure μ_a has a natural lift to a mesure $\hat{\mu}_a$ on Y. Denoting by supp $\hat{\mu}_a$ its closed support, we see that the uniform closure of $H^\infty(|\mu_a| + \lambda_Q)$ in $L^\infty(\mu_a)$ coincides with the uniform closure of $H^\infty(|\mu_a| + \lambda_Q)$ in $C(\operatorname{supp} \hat{\mu}_a)$. Let η be a measure on $\operatorname{supp} \hat{\mu}_a$ which is orthogonal to $H^\infty(|\mu_a| + \lambda_Q)$. By the Hahn-Banach theorem, it suffices to prove that $\eta \perp R(E(\mu_a))$.

Let $\nu = Z^*\eta$, a measure in $A^\perp$. It suffices to show that $\nu \perp R(E(\mu_a))$, and for this it suffices to show that $\hat{\nu} = 0$ a.e. off $E(\mu_a)$.

Suppose $q \in K$ satisfies

$$\int \frac{1}{|z - q|} \, d(Z^*(|\eta|))(z) < \infty,$$

or equivalently,

$$\int \frac{d|\eta|(\varphi)}{|Z(\varphi) - q|} < \infty. \tag{9.1}$$

Since $|\nu| \leqslant Z^*(|\eta|)$, we then have

$$\int \frac{d|\nu|(z)}{|z - q|} < \infty.$$

Suppose furthermore that q satisfies $\hat{\nu}(q) \neq 0$. Set

$$\alpha = \hat{\nu}(q) = \int \frac{d\nu(z)}{z - q} = \int \frac{d\eta(\varphi)}{Z(\varphi) - q}.$$

Suppose $f \in H^\infty(|\mu_a| + \lambda_Q)$ is analytic at q. By Lemma 9.5, $[f - f(q)]/(z - q) \in H^\infty(|\mu_a| + \lambda_Q)$. Hence

$$0 = \int \frac{f(\varphi) - f(q)}{Z(\varphi) - q}\, d\eta(\varphi) = \int \frac{f(\varphi)\, d\eta(\varphi)}{Z(\varphi) - q} - \alpha f(q). \qquad (9.2)$$

Now if $f \in H^\infty(|\mu_a| + \lambda_Q)$ is arbitrary, we can by Lemma 9.6 find a bounded sequence $\{f_n\}$ in $H^\infty(|\mu_a| + \lambda_Q)$ such that each f_n is analytic at q, and $\{f_n\}$ converges uniformly to f on each subset of K at a positive distance from q. Regarded as functions on Y, $\{f_n\}$ then converges uniformly to f on each compact subset of Y disjoint from $Z^{-1}(\{q\})$. On account of (9.1), $Z^{-1}(\{q\})$ is a null-set for η. Hence we may apply the dominated convergence theorem to conclude that since each f_n satisfies (9.2), so does f. Thus

$$f(q) = \frac{1}{\alpha} \int \frac{f(\varphi)\, d\eta(\varphi)}{Z(\varphi) - q}$$

for all $f \in H^\infty(|\mu_a| + \lambda_Q)$. Thus the homomorphism "evaluation at q" has a representing measure absolutely continuous with respect to η. In particular $|f(q)| \leqslant \|f\|_{\mathrm{supp}\,\eta} \leqslant \|f\|_{L^\infty(\mu_a)}$. This shows that $\hat{\nu} = 0$ a.e. $(dx\, dy)$ off $E(\mu_a)$, as required. □

From the preceding lemma, together with Lemmas 9.1 and 9.2, we obtain immediately the following theorem.

9.9. THEOREM. – *Let* A *be a* T-*invariant algebra on* K, *and let* μ *be a measure on* K. *Let* $\mu = \mu_a + \mu_s$ *be the Lebesgue decomposition of* μ *with respect to the band* $\mathcal{B}_\perp$ *generated by* $A^\perp$, *and let* $E(\mu_a)$ *be the envelope of* μ_a, *as defined above. Then* $H^\infty(\mu) \cong H^\infty(\mu_a) \oplus L^\infty(\mu_s)$, *and furthermore* $H^\infty(\mu_a)$ *coincides with the weak-star closure of the* T-*invariant algebra* $[A + R(E(\mu_a))]^-$ *in* $L^\infty(\mu_a)$.

10. Weak-star closure of A in $L^\infty(\mu)$.

We are now in a position to describe the weak-star closure $H^\infty(\mu)$ of a T-invariant algebra A in $L^\infty(\mu)$ for an arbitrary measure μ. (While many algebras will float through this section, we will reserve the notation H^∞ for the weak-star closure of A.) Following Sarason [17], we will proceed by induction on the countable ordinals, using the work in the preceding section to pass from an ordinal to its successor. Specifically, we will construct by induction a family of T-invariant algebras A_i on K, i a countable ordinal, with the following properties:

$$A_i \subseteq A_j \quad \text{if} \quad i < j, \tag{10.1}$$

$$A_i \subseteq H^\infty(\mu) \quad \text{for all} \quad i, \tag{10.2}$$

$A_i = [A + R(Q_i)]^-$, where Q_i is the set of nonpeak points of A_i. (10.3)

We begin by defining $A_1 = A$. Suppose $j > 1$ is a countable ordinal, and the A_i' s have been defined and have the properties (10.1) through (10.3) for all $i < j$. There are two cases to consider.

Suppose first that j has an immediate predecessor say $j = i + 1$. Let μ_i be the projection of μ into the band generated by $A_i^\perp$, let $E(\mu_i)$ be the envelope of μ_i, and let Q_i be the set of nonpeak points for A_i. Define $A_j = [A + R(E(\mu_i))]^-$. Since $E(\mu_i) \subseteq Q_i$, we have $A_i = [A + R(Q_i)]^- \subseteq [A + R(E(\mu_i))]^- = A_j$, and (10.1) is valid. That (10.3) holds for j follows from Lemma 8.4. By Theorem 9.9, A_j is included in $H^\infty(\mu_i)$. Since $\mu - \mu_i$ is singular to $A_i^\perp$, $H^\infty(\mu) \cong H^\infty(\mu_i) \oplus L^\infty(\mu - \mu_i)$. Hence also $A_j \subseteq H^\infty(\mu)$, so that (10.2) is valid, and the properties are verified for such an ordinal j.

Next suppose that j is a limit ordinal. Define A_j to be the closed linear span of the algebras A_i, for $i < j$. Evidently A_j is T-invariant. Since $A_i^\perp$ consists of the measures $\nu \in A^\perp$ such that $\hat{\nu} = 0$ a.e. $(dx\, dy)$ off Q_i, evidently $A_j^\perp$ consists of the measures $\nu \in A^\perp$ such that $\hat{\nu} = 0$ a.e. $(dx\, dy)$ off

$$T = \cup \{Q_i : i < j\}.$$

Hence $A_j = [A + R(T)]^-$. By Lemma 8.4, (10.3) is valid. Evidently (10.1) and (10.2) hold, so the properties are verified for any ordinal j.

Now let γ be the first ordinal such that $A_\gamma = A_{\gamma+1}$. This occurs just as soon as $Q_{\gamma+1}$ has the same area as Q_γ, which occurs after at most a countable number of steps. By Lemma 9.4, $E(\mu_\gamma) = Q_\gamma$. By Theorems 9.5 and 8.5, the weak-star closure $H^\infty(A_\gamma, \mu_\gamma)$ of A_γ in $L^\infty(\mu_\gamma)$ is isometrically isomorphic and weak-star homeomorphic to its weak-star closure $H^\infty(A_\gamma, \lambda_{Q_\gamma})$ in

$L^\infty(\lambda_{Q_\gamma})$. Since by property (10.2), A_γ is weak-star adherent to A in $L^\infty(\mu)$, we see that A_γ and A have the same weak-star closures $H^\infty(\mu_\gamma)$ and $H^\infty(\lambda_{Q_\gamma})$ in $L^\infty(\mu_\gamma)$ and $L^\infty(\lambda_{Q_\gamma})$ respectively. Moreover, $H^\infty(\mu) \cong H^\infty(\mu_\gamma) \oplus L^\infty(\mu - \mu_\gamma)$.

Now we relate Q_γ to the set $S(\mu)$ of points in K which extend weak-star continuously to $H^\infty(\mu)$. We will write μ_a for μ_γ and μ_s for $\mu - \mu_\gamma$, so that

$$H^\infty(\mu) \cong H^\infty(\mu_a) \oplus L^\infty(\mu_s). \tag{10.4}$$

Every $q \in Q_\gamma$ is weak-star continuous on $H^\infty(\lambda_{Q_\gamma})$, hence on $H^\infty(\mu)$, so that $Q_\gamma \subseteq S(\mu)$. On the other hand, if $q \in S(\mu)$, then q has a representing measure ν absolutely continuous with respect to μ. In view of the decomposition (10.4), we see that either $\nu \ll \mu_a$, or else ν is a singleton at a point at which μ_s has mass. If $\nu \ll \mu_a$, then q is weak-star continuous on $H^\infty(\lambda_{Q_\gamma})$, so that $q \in Q_\gamma$. Hence $S(\mu)$ is the union of Q_γ and an at most countable set, at each point of which μ has mass.

In particular, $S(\mu)\backslash Q_\gamma$ has zero area, so that

$$R(S(\mu)) = R(Q_\gamma),$$

and $A_\gamma = [A + R(S(\mu))]^-$. Our results are summarized in the following theorem, which was obtained for $R(K)$ by Chaumat [4].

10.1. THEOREM. – *Let* A *be a* T-*invariant algebra on the compact subset* K *of the complex plane, and let* μ *be a measure on* K. *Let* $S(\mu)$ *denote the set of points in* K *which determine weak-star continuous homomorphisms of* $H^\infty(\mu)$. *Let* $\widetilde{A}$ *denote the* T-*invariant algebra* $[A + R(S(\mu))]^-$, *and let* $\widetilde{Q}$ *denote the set of nonpeak point of* $\widetilde{A}$. *Let* μ_a *denote the projection of* μ *into the band generated by* $(\widetilde{A})^\perp$, *and* $\mu_s = \mu - \mu_a$. *Then*

(i) $\widetilde{A} \subseteq H^\infty(\mu)$, *so that* $H^\infty(\mu)$ *coincides with the weak-star closure of* $\widetilde{A}$ *in* $L^\infty(\mu)$.

(ii) $S(\mu)$ *is the union of* $\widetilde{Q}$ *and the points of* $K\backslash\widetilde{Q}$ *at which* μ *has mass. In particular,* $S(\mu)\backslash\widetilde{Q}$ *is at most countable.*

(iii) $H^\infty(\mu) \cong H^\infty(\mu_a) \oplus L^\infty(\mu_s)$.

(iv) *The identity map of* A *extends to an isometric* *isomorphism and a weak-star homeomorphism of* $H^\infty(\mu_a)$ *and* $H^\infty(\lambda_{\tilde{Q}})$. *(Here* $H^\infty(\lambda_{\tilde{Q}})$ *is the weak-star closure of* A *in* $L^\infty(\lambda_{\tilde{Q}})$.)

(v) *If* $f \in H^\infty(\mu)$, *there is a sequence* $\{f_n\}$ *in* $\tilde{A}$ *such that* $\|f_n\| \leqslant \|f\|$ *and* $f_n \longrightarrow f$ a.e. $(d\mu)$.

(vi) $\tilde{A} = H^\infty(\mu) \cap C(K)$.

Proof. – The statements (i) through (iv) follow from the preceding discussion. In particular, the measure μ_a satisfies the hypotheses of Theorem 8.5 with respect to the T-invariant algebra $\tilde{A}$. By Theorem 8.5, ball $\tilde{A}$ is weak-star dense in ball $H^\infty(\mu_a)$. Since the measure μ_s is singular to the measures in $\tilde{A}^\perp$, a standard application of the separation theorem for convex sets shows that ball $\tilde{A}$ is weak-star dense in ball $H^\infty(\mu)$. Hence (v) is valid. If we apply Theorem 8.5 to $\tilde{A}$ and μ_a, and take into account (iii) above, we obtain (vi). □

10.2. COROLLARY. – *The set* $S(\mu)$ *of points* $q \in K$ *which are weak-star continuous on* $H^\infty(\mu)$ *is an* F_σ-set.

Proof. – The set of nonpeak points of any separable uniform algebra is an F_σ-set, so that $\tilde{Q}$ is an F_σ-set. Since $S(\mu)$ is the union of $\tilde{Q}$ and an at most countable set, $S(\mu)$ is also an F_σ-set. □

From the description of $H^\infty(\mu)$, it is possible to describe easily the μ-parts of $S(\mu)$.

10.3. THEOREM. – *Let* A *be a* T-*invariant algebra on* K, *and let* μ *be a measure on* K. *Let* $\tilde{A} = [A + R(S(\mu))]^-$, *as in Theorem* 10.1. *Then the* μ-*parts of* $S(\mu)$ *are the various nontrivial Gleason parts of* $\tilde{A}$, *together with the singletons in* $S(\mu)\backslash\tilde{Q}$. *The decomposition of a measure* $\lambda \in A^\perp, \lambda \ll \mu$, *given by Theorem* 3.6 *is precisely the decomposition obtained by applying the abstract F. and M. Riesz theorem to the algebra* $\tilde{A}$, *decomposing* λ *into the constituents corresponding to the various nontrivial Gleason parts of* $\tilde{A}$.

Proof. – Let Q_j be a nontrivial Gleason part of $\tilde{A}$, so that $Q_j \subseteq \tilde{Q}$. Then the characteristic function of Q_j is a minimal idempotent in $H^\infty(d\lambda_{S(\mu)})$, which corresponds to a minimal idempotent $\chi_j \in H^\infty(\mu)$ via Theorem 10.1. Since $\varphi(\chi_j) = 1$ if $\varphi \in Q_j$, and $\varphi(\chi_j) = 0$ if $\varphi \in S(\mu)\backslash Q_j$, the μ-part corresponding to χ_j is precisely Q_j. This establishes the first assertion of the theorem, and the second assertion follows easily. □

11. The local Dirichlet condition.

Again let A be a T-invariant algebra on K, with set Q of nonpeak points, and let μ be a measure on K. Following Sarason [17], Chaumat [4], and Dudziak [7], it is possible to obtain more precise information about the structure of $S(\mu) \cap Q^0$, where Q^0 is the interior of Q with respect to the complex plane. Rather than give formal proofs, we state the results and refer to Dudziak [7] for proofs along the lines we have in mind.

The key result is contained in the following theorem.

11.1. THEOREM. – *Let* A *be a* T-*invariant algebra on* K, *and let* μ *be a measure on* K. *Let* Q *be the set of nonpeak points of* A, *and let* Q^0 *denote the interior of* Q *with respect to the complex plane* **C**. *Let* $\widetilde{A} = H^\infty(\mu) \cap C(K)$ *be as before, and let* W *denote the set of nonpeak points of* $\widetilde{A}$. *Then* $Q^0 \cap W$ *is an open subset of* Q^0, *and each connected component of the complement of* $Q^0\backslash W$ *reaches out to the boundary of* Q^0. *Furthermore, if* D *is any open disc whose closure* $\overline{D}$ *is contained in* Q^0, *then the uniform closure of* $\widetilde{A}$ *in* $C(\overline{D})$ *is a Dirichlet algebra on* $(D\backslash W) \cup \partial D$, *which coincides with the functions in* $R(\overline{D} \cap \overline{W})$, *extended in all possible continuous ways to* $\overline{D}$.

As a simple example, if $A = R(\overline{\Delta})$ is the standard disc algebra on the closed unit disc $\overline{\Delta}$, and if μ is the area measure on an open disc $\Delta_0 \subset \Delta$, then $W = \Delta_0$, and $\widetilde{A}$ consists of the continuous functions on $\overline{\Delta}$ which are analytic on Δ_0. If D is any open disc, $\overline{D} \subset \Delta$, then the uniform closure of $\widetilde{A}$ in $C(\overline{D})$ consists of the continuous functions on $\overline{D}$ which are analytic on $D \cap \Delta_0$.

Theorem 11.1. can be proved by induction, by showing that each of the algebras A_i has the properties ascribed to $\widetilde{A}$ in the theorem. The details depend on some technical results on Dirichlet algebras in the plane. This line of development yields further information on the component μ_a of the measure μ with respect to the band generated by $A^{\perp}$. It shows that the restriction of μ_a to Q^0 is carried by $\overline{W}$, and the restriction of μ_a to $Q^0 \cap \partial W$ is absolutely continuous with respect to harmonic measure for $Q^0 \cap W$ (that is, a convex sum of harmonic measures for the components of $Q^0 \cap W$).

Following Dudziak [7], we may obtain also various corollaries. We retain the notation above.

11.2. COROLLARY. – *If* $Q \backslash Q^0$ *has zero area, then* $S(\mu)\backslash S(\mu)^0$ *has zero area, and* $H^\infty(\mu_a)$ *is isometric and weak-star homeomorphic to a weak-star closed subalgebra* $H^\infty(\lambda_{S(\mu)^0})$ *of the algebra* $H^\infty(S(\mu)^0)$ *of bounded analytic functions on* $S(\mu)^0$.

11.3. COROLLARY. – *Suppose that for all but at most countably many points* $z \in \partial Q$, *the analytic capacity* γ *satisfies the condition* $\liminf_{\delta \longrightarrow 0} \dfrac{\gamma(\Delta(z\,;\delta)\backslash \overline{Q})}{\delta} > 0$. *Then* $S(\mu)\backslash S(\mu)^0$ *is at most countable, and* $H^\infty(\mu_a)$ *is isometric and weak-star homeomorphic to* $H^\infty(S(\mu)^0)$.

12. The algebras R(K) and A(K).

As mentioned before, the results of the preceding sections were obtained by Chaumat [4] for the algebra R(K). In the case of R(K), Theorem 10.1 (vi) and Lemma 8.3 show that the algebra $\widetilde{A} = H^\infty(\mu) \cap C(K)$ coincides with the algebra $R(S(\mu))$.

In the case of the algebra R(K), one can provide a direct proof of Theorem 10.1, which does not proceed by induction on the ordinals, but which still requires the machinery of T-invariant algebras (specifically, Theorem 8.5). The proof, which is essentially the same as that of Chaumat, runs as follows.

We wish to show directly that

$$R(S(\mu)) \subseteq H^\infty(\mu). \tag{12.1}$$

For this, let ν be a measure such that $\nu \ll \mu$ and $\nu \perp R(K)$. A standard argument shows that if $\hat{\nu}(q)$ exists and is nonzero, then $q \in S(\mu)$. It follows that $\hat{\nu} = 0$ off $S(\mu)$, and this implies that $\nu \perp R(S(\mu))$, so that (12.1) is valid. Now with (12.1) in hand, Theorem 8.5 on T-invariant algebras yields directly the version of Theorem 10.1 for $R(K)$.

Chaumat also obtained Theorem 10.1 for the algebra $A(K)$. This algebra already requires more effort than the algebra $R(K)$. In connection with $A(K)$, there remains an interesting problem. Let us briefly describe the situation, for a slightly more general class of algebras.

Let U be an open subset of the complex plane, $U \subset K$, and let $A = A(K,U)$ be the algebra of continuous functions on K which are analytic on U. In this case, the set Q of nonpeak points for A is the union of U and a set of zero area, so that Corollary 11.2 applies. If μ is a measure on K, then the set W of nonpeak points for $\tilde{A}$ meets U in an open set, and $W \cap U$ has full area in W. Thus $H^\infty(\mu)$ is isomorphic to a weak-star closed subalgebra of $H^\infty(W \cap U)$, though it need not coincide with $H^\infty(W \cap U)$.

The problem is to describe more explicitly the algebra $\tilde{A} = H^\infty(\mu) \cap C(K)$. In particular, it is not known whether $\tilde{A} = A(K,V)$ for $V = W \cap U$.

13. The algebra $R(K)$, for K^0 connected and dense in K.

We consider now the case originally treated by Glicksberg in [13], in which K^0 is connected and dense in K. In this case, we do not know the answer to the following questions :

If μ is a nonzero measure on ∂K orthogonal to $R(K)$, does $S(\mu)$ coincide with the set Q of nonpeak points for $R(K)$? (13.1)

Is the restriction map $H^\infty(\lambda_Q) \longrightarrow H^\infty(K^0)$ an isometry ? (13.2)

Is K^0 dense in Q, in the Gleason topology ? (13.3)

An affirmative answer to (13.3) yields immediately an affirmative answer to (13.2). The next theorem shows that if (13.2) is true, then so is (13.1).

13.1. THEOREM. – *Suppose that* K^0 *is connected and dense in* K. *Let* λ_Q *denote the area measure on the set* Q *of nonpeak points for* R(K), *and let* λ *denote the area measure on* K^0. *Suppose the restriction map* $H^\infty(\lambda_Q) \longrightarrow H^\infty(\lambda)$ *is an isometry, i.e.,*

$$\|f\|_Q = \|f\|_{K^0}, \quad all \quad f \in H^\infty(\lambda_Q). \qquad (13.4)$$

Then :

(i) *The restriction map* $H^\infty(\lambda_Q) \longrightarrow H^\infty(\lambda)$ *is an isometric isomorphism and a weak-star homeomorphism of* $H^\infty(\lambda_Q)$ *and* $H^\infty(\lambda)$.

(ii) *If* $f \in H^\infty(\lambda)$, *there exists a sequence* $\{f_n\}_{n=1}^\infty$ *in* R(K) *that converges normally to* f *on* K^0, *such that* $\|f_n\| \leqslant \|f\|$.

(iii) *If* ν *is any representing measure on* ∂K *for a point of* K^0, *then the natural projection* $H^\infty(\lambda_Q) \longrightarrow H^\infty(\nu)$ *is an isometric isomorphism and a weak-star homeomorphism of* $H^\infty(\lambda_Q)$ *and* $H^\infty(\nu)$.

(iv) *If* μ *is any nonzero measure of* ∂K *orthogonal to* R(K), *then* $S(\mu) = Q$.

Proof. – First we claim that the projection of the unit ball of $H^\infty(\lambda_Q)$ is weak-star closed in $H^\infty(\lambda)$. In fact, let $f \in H^\infty(\lambda)$ be a weak-star limit of a net $\{f_\alpha\}$ in ball $H^\infty(\lambda_Q)$. If $F \in H^\infty(\lambda_Q)$ is any weak-star adherent point of $\{f_\alpha\}$, then $\|f\| \leqslant 1$, and $F = f$ on K^0. Thus f also belongs to the projection of ball $H^\infty(\lambda_Q)$. Now the Krein-Schmulian theorem shows that the projection of $H^\infty(\lambda_Q)$ into $H^\infty(\lambda)$ is weak-star closed, hence it coincides with $H^\infty(\lambda)$. Thus $H^\infty(\lambda_Q)$ and $H^\infty(\lambda)$ are isometrically isomorphic. Since the projection is weak-star continuous, it has a preadjoint, which must also be an isometric isomorphism. It follows that the projection is a weak-star homeomorphism, and (i) is established.

Now (ii) follows immediately from the corresponding property for $H^\infty(\lambda_Q)$, which is Davie's theorem. Davie's theorem provides a natural projection $H^\infty(\lambda_Q) \longrightarrow H^\infty(\nu)$. Since each $z \in K^0$ has a representing measure absolutely continuous with respect to ν, there is also a natural projection $H^\infty(\nu) \longrightarrow H^\infty(\lambda)$. Since the composition of these projections is an isometric isomorphism and a weak-star homeomorphism, each of the projections is, and (iii) is established.

To prove (iv), we invoke Glicksberg's original theorem from [13]. By Glicksberg's theorem, if μ is a nonzero measure on ∂K orthogonal to $R(K)$, then μ is mutually absolutely continuous with respect to a representing measure for a point of K^0. Since $H^\infty(\lambda_Q) \cong H^\infty(\nu)$, each point of Q has a representing measure dominated by ν. Hence $Q \subseteq S(\mu)$, and since μ cannot have any point masses at peak points, $Q = S(\mu)$. □

If $Q \backslash K^0$ has zero area, then the hypothesis (13.4) is met trivially. The hypothesis (13.4) can be met even when $Q \backslash K^0$ has positive area. We devote the remainder of this section to showing that the beady hair set [8, VIII.9] satisfies the hypotheses of Theorem 13.1, even though $Q \backslash K^0$ has positive area.

Let Γ be a compact Jordan arc in the closed unit disc $\bar{\Delta}$ such that Γ has positive area, and such that at most one point of Γ lies on $\partial\Delta$. Let $\{\Delta_j\}_{j=1}^{\infty}$ be a sequence of open discs with pairwise disjoint closures, such that the sum of the radii of the $\Delta_j' s$ is finite, each $\bar{\Delta}_j$ is disjoint from $\partial\Delta$, each $\bar{\Delta}_j$ meets Γ in precisely one point, and these points are dense in Γ. Let $K = \bar{\Delta} \backslash (\cup \Delta_j)$. Then ∂K consists of $\partial\Delta$, Γ and the $\partial\Delta_j' s$, while K^0 is connected and dense in K. Let μ be the measure on ∂K defined so that $\mu = dz$ on $\partial\Delta_j$, $j \geqslant 1$, while $\mu = -dz$ on $\partial\Delta$. Then one computes that $\hat{\mu} = 0$ on $\mathbf{C} \backslash K$, while $\hat{\mu} = 2\pi i$ a.e. $(dx\, dy)$ on K. Thus $\mu \perp R(K)$. Every point at which the integral defining $\hat{\mu}$ converges absolutely and satisfies $\hat{\mu}(z) \neq 0$ has a representing measure absolutely continuous with respect to μ, and hence belongs to Q. Thus $K \backslash Q$ has zero area. Since Γ has positive area, and Q includes almost all points of Γ, $Q \backslash K^0$ has positive area.

Next we establish that (13.4) is valid. Let $f \in H^\infty(\lambda_Q)$, and let $\{f_n\}$ be a bounded sequence in $R(K)$ converging weak-star to f in $H^\infty(\lambda_Q)$. Then $\{f_n\}$ converges normally on K^0 to f, and $\{f_n\}$ also converges weak-star in $L^\infty(\mu)$, say to F. As observed before, almost all $(dx\, dy)$ points of Q have representing measures absolutely continuous with respect to μ. Hence

$$\|f\|_{L^\infty(\lambda_Q)} \leqslant \|F\|_{L^\infty(\mu)},$$

so that in fact the correspondence $f \longleftrightarrow F$ is an isometry of

$H^\infty(\lambda_Q)$ and $H^\infty(\mu)$. Now by considering the approximants $\{f_n\}$, one may see that F is the nontangential boundary value function of f on each $\partial\Delta_j$ and also on $\partial\Delta$, so that

$$\|F\|_{L^\infty(\mu)} \leqslant \|f\|_{K^0}.$$

We conclude that $\|f\|_{H^\infty(\lambda_Q)} = \|F\|_{L^\infty(\mu)} = \|f\|_{K^0}$, so that the restriction map $H^\infty(\lambda_Q) \longrightarrow H^\infty(K^0)$ is an isometry, and (13.4) is valid. In particular, from Theorem 13.1 we obtain $S(\mu) = Q$.

14. The infinite polydisc algebra.

Let $\Delta = \{|z| < 1\}$ be the open unit disc in the complex plane, and let $T = \{|z| = 1\}$ be its boundary circle. Let $\Delta^\infty = \Delta \times \Delta \times \ldots$ be the countable product of Δ with itself. Define $T^\infty = T \times T \times \ldots$ similarly to be the infinite torus, and let $d\sigma$ be the Haar measure on T^∞. In this section, we will consider the uniform algebra A on T^∞ generated by the monomials

$$g(e^{i\theta_1}, e^{i\theta_2}, \ldots) = e^{im_1\theta_1} \ldots e^{im_k\theta_k}, \tag{14.1}$$

where $1 \leqslant k < \infty$ and $m_j \geqslant 0$ for $1 \leqslant j \leqslant k$. The maximal ideal space of A is the countable product of the closed unit disc with itself,

$$M_A = \bar{\Delta} \times \bar{\Delta} \times \ldots .$$

The Gelfand extension of the monomial g of (14.1) is given by

$$g(z_1, z_2, \ldots) = z_1^{m_1} \ldots z_k^{m_k}, \quad z = (z_1, z_2, \ldots) \in M_A.$$

The Haar measure σ represents the homomorphism "evaluation at 0" of A:

$$f(0) = \int f\, d\sigma, \quad f \in A.$$

We wish to determine the set $S(\sigma)$ of weak-star continuous evaluations of $H^\infty(\sigma)$. First we make some preliminary observations concerning the Gleason topology for M_A.

For any $\zeta \in \Delta^\infty$ we have the estimate

$$|f(\zeta)| \leqslant (\sup_{1 \leqslant j < \infty} |\zeta_j|)\, \|f\|, f \in A, \; f(0) = 0. \tag{14.2}$$

Indeed, if $\beta = \sup |\zeta_j|$, then $w \longrightarrow f(\zeta_1 w, \zeta_2 w, \ldots)$ depends analytically on the complex parameter w for $|w| < 1/\beta$, and it is bounded by $\|f\|$. Applying the Schwarz inequality to this function, we obtain (14.2).

From (14.2) it follows directly that the Gleason part of $0 \in M_A$ consists of all $\zeta \in \Delta^\infty$ such that $\sup |\zeta_j| < 1$. It also follows from (14.2) that a sequence $\{\zeta^{(n)}\}_{n=1}^\infty$ in Δ^∞ converges to 0 in the Gleason metric if and only if

$$\lim_{n \to \infty} \sup_{1 < j < \infty} |\zeta_j^{(n)}| = 0. \tag{14.3}$$

More generally, a sequence $\{\zeta^{(n)}\}_{n=1}^\infty$ in Δ^∞ converges to $\zeta \in \Delta^\infty$ in the Gleason metric if and only if

$$\lim_{n \to \infty} \sup_{1 \leq j < \infty} \left| \frac{\zeta_j^{(n)} - \zeta_j}{1 - \bar{\zeta}_j \zeta_j^{(n)}} \right| = 0. \tag{14.4}$$

Indeed, for fixed ζ, the map $\Psi : M_A \longrightarrow M_A$ defined by

$$\Psi(z)_j = \frac{z_j - \zeta_j}{1 - \bar{\zeta}_j z_j}$$

is a homeomorphism which leaves A invariant: $f \in A$ if and only if $f \circ \Psi \in A$. It follows that Ψ is an isometry with respect to the Gleason metric. The condition (14.4) follows from (14.3) upon applying Ψ.

14.1. THEOREM. – *The homomorphisms of* A *which are weak-star continuous with respect to Haar measure* σ *are precisely those* $\zeta \in \Delta^\infty$ *such that* $\zeta_j \longrightarrow 0$ *as* $j \longrightarrow \infty$.

Proof. – Since $S(\sigma)$ is contained in the Gleason part of 0, in particular $S(\sigma) \subset \Delta^\infty$.

Suppose that $\zeta \in \Delta^\infty$, and $\zeta_j \not\longrightarrow 0$. Choose $\delta > 0$ and a subsequence $\zeta_{j_1}, \zeta_{j_2}, \ldots$ such that $|\zeta_{j_i}| \geq \delta$, all i. Consider the analytic polynomial

$$f_n = \frac{1}{n}(\bar{\zeta}_{j_1} z_{j_1} + \ldots + \bar{\zeta}_{j_n} z_{j_n}) \in A.$$

Since the z_i's are orthogonal in $L^2(\sigma)$, we obtain

$$\int |f_n|^2 \, d\sigma = \frac{1}{n^2} \sum_{i=1}^{n} |\zeta_{j_i}|^2 \leqslant \frac{1}{n}.$$

Hence $f_n \longrightarrow 0$ in $L^2(\sigma)$, and since $|f_n| \leqslant 1$ we obtain $f_n \longrightarrow 0$ weak-star in $H^\infty(\sigma)$. However, $f_n(\zeta) \geqslant \delta$ for all n, so that the evaluation at ζ is not weak-star continuous.

Suppose next that $\zeta \in \Delta^\infty$ is such that $\zeta_j = 0$ for all large j, say for $j > N$. Then ζ is represented by the measure

$$d\sigma_\zeta = P_{\zeta_1}(\theta_1) \ldots P_{\zeta_n}(\theta_N)\, d\sigma,$$

where P_w is the Poisson kernel for $w \in \Delta$. In particular ζ is weak-star continuous.

Finally, suppose $\zeta \in \Delta^\infty$ satisfies $\zeta_j \longrightarrow 0$. Let

$$\zeta^{(n)} = (\zeta_1, \zeta_2, \ldots, \zeta_n, 0, 0, \ldots) \in \Delta^\infty.$$

As just observed, $\zeta^{(n)} \in S(\sigma)$. By the condition (14.4), $\{\zeta^{(n)}\}$ converges to ζ in the Gleason metric of M_A. Now a standard argument involving convolution with an approximate identity shows that ball A is weak-star dense in ball $H^\infty(\sigma)$. Hence Theorem 7.7 shows that $S(\sigma)$ is closed in the Gleason metric. It follows that $\zeta \in S(\sigma)$. □

Thus we may identify $S(\sigma)$ with the space of null sequences in Δ^∞. Theorem 7.6 shows that $S(\sigma)$ is an $F_{\sigma\delta}$-subset of M_A. However, the following result shows that we cannot expect any more.

14.2. THEOREM. – *The set* $S(\sigma)$ *of weak-star continuous homomorphisms is not an* F_σ*-subset of* M_A.

Proof. – Suppose E_k, $k \geqslant 1$, are compact subsets of M_A contained in $S(\sigma)$. It suffices to construct a point $\zeta \in S(\sigma)$ such that $\zeta \notin \cup E_k$.

Choose N_1 so large that there is no point in E_1 whose first N_1 entries are all equal to $1/2$. (If this were not possible, we could pass to a limit and obtain $\left(\frac{1}{2}, \frac{1}{2}, \ldots\right) \in E_1$, contradicting $E_1 \subset S(\sigma)$.) Then choose $N_2 > N_1$ so large that there is no ζ in

E_2 which satisfies $\zeta_j = \frac{1}{4}$ for $N_1 < j \leq N_2$. Proceeding in this manner, we obtain a sequence $0 = N_0 < N_1 < N_2 < \ldots$ such that no ζ in E_k satisfies $\zeta_j = 1/2^k$ for $N_{k-1} < j \leq N_k$. Then the point ζ defined so that $\zeta_j = \frac{1}{2^k}$ if $N_{k-1} < j < N_k$, $1 \leq k < \infty$, belongs to $S(\sigma)$, but to no E_k. □

BIBLIOGRAPHY

[1] W. ARVESON, *An Invitation to C*Algebras*, Springer-Verlag, 1976.

[2] K. BARBEY and H. KÖNIG, Abstract analytic function theory and Hardy algebras, *Lecture Notes in Math.*, vol. 593, Springer-Verlag, 1977.

[3] A. BROWDER, *Introduction to Function Algebras*, Benjamin, 1969.

[4] J. CHAUMAT, Adhérence faible étoile d'algèbres de fractions rationnelles, *Ann. Inst. Fourier*, Grenoble, 24,4 (1974), 93-120.

[5] B.J. COLE and T.W. GAMELIN, Tight uniform algebras, *Journal of Functional Analysis*, 46 (1982), 158-220.

[6] J.B. CONWAY, *Subnormal operators*, Research Notes in Mathematics # 51, Pitman, 1981.

[7] J. DUDZIAK, Spectral mapping theorems for subnormal operators, *J. Funct. Anal.*, 56 (1984), 360-387.

[8] T.W. GAMELIN, *Uniform Algebras*, Prentice-Hall, 1969.

[9] T.W. GAMELIN, *Rational Approximation Theory*, course lecture notes, UCLA, 1975.

[10] T.W. GAMELIN, Uniform algebras on plane sets, *in Approximation Theory*, Academic Press, 1973, pp. 100-149.

[11] T.W. GAMELIN and J. GARNETT, Bounded approximation by rational functions, *Pac. J. Math.*, 45 (1973), 129-150.

[12] I. GLICKSBERG, Recent results on function algebras, *CBMS Regional Conference Series in Mathematics*, vol. 11, Am. Math. Society, 1972.

[13] I. GLICKSBERG, Equivalence of certain representing measures, *Proc. A.M.S.*, 82 (1981), 374-376.

[14] K. HOFFMAN and H. ROSSI, Extension of positive weak*-continuous functionals, *Duke Math. J.*, 34 (1967), 453-466.

[15] H. KONIG and G.L. SEEVER, The abstract F. and M Riesz theorem, *Duke Math. J.*, 36 (1969), 791-797.

[16] T. LYONS, Finely holomorphic functions, *J. Funct. Anal.*, 37 (1980), 1-18.

[17] D.E. SARASON, Weak-star density of polynomials, *J. Reine Angew Math.*, 252 (1972), 1-15.

Manuscrit reçu le 17 janvier 1984.

B.J. COLE,
Brown University
Dept. of Mathematics
Providence, RI 02912 (U.S.A.)
&
T.W. GAMELIN,
U.C.L.A.
Dept. of Mathematics
Los Angeles CA 90024 (U.S.A.).