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WALTER SEAMAN

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THE THIRD BETTI NUMBER OF A POSITIVELY PINCHED RIEMANNIAN SIX MANIFOLD

by **Walter SEAMAN**

Introduction.

The Sphere theorem, together with the classification of symmetric spaces, provides a complete classification for compact simply connected Riemannian manifolds whose sectional curvature, K, satisfies $1 \geqslant K \geqslant \frac{1}{4}\cdot$ Namely, each such space is topologically a sphere, or a projective space over the complex, quaternionic or Cayley numbers. These spaces are sometimes called the « model » spaces. Recently, Berger [3] has shown that there is a number, $\varepsilon(n) \in \left(0, \frac{1}{4}\right)$ such that any $2n$-dimensional compact simply connected manifold with $1 \geqslant K \geqslant \varepsilon(n)$ is still topologically a model space. Unfortunately, $\varepsilon(n)$ is « purement idéal », and one would like to have some quantitative information about it. We hasten to add that there are known examples of compact simply connected positively curved manifold topologically distinct from the model spaces, and in fact the even dimensional homogeneous ones have been classified. These include, besides the model spaces, examples in dimensions 6, 7, 12, 13 and 24 c.f. [9]. There are also known examples, in dimension 7, of infinitely many simply connected, homotopically distinct, positively curved compact manifolds [1].

One indirect way to get a feeling for Berger's $\varepsilon(n)$ would be to exhibit actual numbers δ, with the property that any « δ-pinched » compact manifold (i.e. $1 \geqslant K \geqslant \delta$) is at least cohomologically similar to a model space. The procedure here is to use the Weitzenböch – Bochner method,

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which states that the positive definiteness of an operator $\mathcal{R}_K$ on k-forms guarantees that there are no nonzero harmonic k-forms, together with the Hodge-DeRham theorem, which then implies that the k^{th} real Betti number must be zero. This method has an extensive history [5,2,6], and we use it to prove our:

THEOREM. — *If a compact six manifold has sectional curvature* K, *satisfying* $1 \geqslant K > \dfrac{4\sqrt{10}-4}{4\sqrt{10}+23} \cong .2426$, *then* $b_3(M;\mathbf{R}) = 0$.

The proof amounts to showing that the cited restriction on K guarantees that $\mathcal{R}_3$ is positive definite. Our result is motivated by the fact that all the model spaces have zero odd dimensional cohomology, so it is natural to try to show that this property is preserved when the curvature goes slightly below $\frac{1}{4}$. Furthermore, the only known (at least to us) positively curved compact six manifolds, S^6, $\mathbf{CP}^3$, and Wallach's $SU(3)/T^2$, all have $b_3 = 0$.

We would like to thank the referee for suggestions simplifying the form of $\mathcal{R}_3$, and Professor Richard Randell for many helpful topological discussions. Finally, we add that we believe that our current lower bound for K can be improved to $\frac{8}{35}$.

1. Preliminaries.

Let M be an n-dimensional Riemannian manifold, and ω a k-form on M. Define a k-tensor field on M, as follows, at each $p \in M$:

$$(1) \quad \mathcal{R}_k\omega_p(v_1,\dots,v_k) = \sum_{i=1}^{n}\sum_{j=1}^{k} (R(e_i,v_j)\omega)_p(v_1,\dots,v_{j-1},e_i,v_{j+1},\dots,v_k)$$

where $v_j \in T_pM$, $\{e_i\}$ are an orthonormal basis for T_pM, and R is the Riemannian curvature tensor, extended to act on k forms as usual. Then we have the Weitzenböck formula ([8], chapter 4):

$$(2) \qquad \langle\Delta\omega,\omega\rangle = \frac{1}{2}\Delta|\omega|^2 + |\nabla\omega|^2 + \langle\mathcal{R}_k\omega,\omega\rangle .$$

By the well-known Bochner argument, if M is compact, boundariless and orientable, and $\Delta\omega = 0$, while $\mathcal{R}_k$ is positive definite everywhere (actually just positive semidefinite everywhere and positive definite at one point suffices), then (2) implies that ω must be zero. The Hodge-DeRham theorem then implies that $b_k(\mathrm{M};\mathbf{R}) = 0$. Since finite coverings of compact manifolds only decreases $b_k(\mathrm{M};\mathbf{R})$, this result is also true for M nonorientable. Therefore, to prove our result about $b_3(\mathrm{M};\mathbf{R})$, we examine $\mathcal{R}_3$. Throughout the remainder of this paper we will be working at a fixed $p \in \mathrm{M}$. Let $\omega \in \Lambda^k(\mathrm{T}_p\mathrm{M}^*)$ and let $\mathrm{X} \in \Lambda^k(\mathrm{T}_p\mathrm{M})$ be the metric dual to $\omega : \mathrm{X}^b = \omega$. The right hand side of (1) can be written as:

$$(3) \qquad - \sum_{i=1}^{n} \sum_{j=1}^{k} \langle \mathrm{R}(e_i,v_j)(v_1 \wedge \ldots \wedge v_k),\mathrm{X}\rangle ,$$

since the extension of R acts as a derivation. Now examine (3) in the case $k = 3$. A straight forward, tedious argument using the Bianchi identity, shows that (3) equals

$$(4) \qquad \langle \mathcal{R}_3(v_1 \wedge v_2 \wedge v_3),\mathrm{X}\rangle$$

where

$$\mathcal{R}_3 = \widehat{\mathrm{Ric}} - 2\hat{\mathrm{R}},$$

with

$$\widehat{\mathrm{Ric}}\,(a \wedge b \wedge c) = \widehat{\mathrm{Ric}}\,(a) \wedge b \wedge c + a \wedge \mathrm{Ric}\,(b) \wedge c + a \wedge b \wedge \mathrm{Ric}\,(c)$$
$$\hat{\mathrm{R}}(a \wedge b \wedge c) = a \wedge \bar{\mathrm{R}}(b \wedge c) + \bar{\mathrm{R}}(a \wedge b) \wedge c + b \wedge \bar{\mathrm{R}}(c \wedge a)$$

a, b, $c \in \mathrm{T}_p\mathrm{M}$, $\widehat{\mathrm{Ric}}$ = Ricci tensor and $\bar{\mathrm{R}}$ = the curvature operator. In particular, $\mathcal{R}_3$ is symmetric, since both $\widehat{\mathrm{Ric}}$ and $\hat{\mathrm{R}}$ are. We should add that this formula for $\mathcal{R}_3$ is most easy to derive by first considering $\mathrm{X} = \mathrm{X}_1 \wedge \mathrm{X}_2 \wedge \mathrm{X}_3$ and then using linearity.

From the previous discussions we know that, for a compact manifold, if $\mathcal{R}_3$ is positive definite, then (since $p \in \mathrm{M}$ was arbitrary) $b_3(\mathrm{M};\mathbf{R}) = 0$. Let $\mathrm{X} \in \Lambda^3\mathrm{T}_p\mathrm{M}$ be an eigenvector for $\mathcal{R}_3$ with minimum (real) eigenvalue $r : \mathcal{R}_3\mathrm{X} = r\mathrm{X}$. Let M now be six dimensional. To proceed we need to have a nice form for X. This is provided in the following.

THEOREM [4,7]. – *Let* $\mathrm{X} \in \Lambda^3\mathbf{R}^6$. *Then there is an orthonormal basis* $e_1, \ldots, e_6$ *of* $\mathbf{R}^6$ *such that*

$$\mathrm{X} = \lambda_1 e_{123} + \lambda_2 e_{145} + \lambda_3 e_{246} + \lambda_4 e_{356} + \lambda_5 e_{456}$$

where

$$e_{ijk} = e_i \wedge e_j \wedge e_k .$$

This theorem is proved by maximizing the function $\langle X, \frac{a\wedge b\wedge c}{|a\wedge b\wedge c|}\rangle$, and λ_1 is the maximum, attained at e_{123}. This procedure is then iterated. By changing the signs of various e_i, we can assume:

$$(5) \qquad |\lambda_1| \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4 \geqslant 0 \quad \text{and} \quad |\lambda_1| \geqslant \lambda_5 \geqslant 0 .$$

There are examples where λ_1 is positive and examples where λ_1 is negative. In our proof that $\mathscr{R}_3$ is positive definite, we will need various inequalities among the λ_i 's. These inequalities are all derived from:

$$(6) \quad |\lambda_1| \geqslant \langle X, \frac{(a_1e_1+a_6e_6) \wedge (a_2e_2-a_5e_5) \wedge (a_3e_3+a_4e_4)}{\sqrt{(a_1^2+a_6^2)(a_2^2+a_5^2)(a_3^2+a_4^2)}} \rangle$$

which is valid since $|\lambda_1|$ is maximal. Taking $a_i = 1$, $i = 1, \ldots, 6$ in (6), we obtain:

$$(7) \qquad 2\sqrt{2}\,|\lambda_1| \geqslant \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 .$$

If $\lambda_1 > 0$, (7) yields

$$(8) \qquad (2\sqrt{2}-1)\lambda_1 \geqslant \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 .$$

Taking $a_1 = a_2 = a_5 = a_6 = 1$, $a_3 = \lambda_1 + \lambda_4$, $a_4 = \lambda_2 + \lambda_3 + \lambda_5$, (6) yields:

$$(9) \qquad 2|\lambda_1| \geqslant \sqrt{(\lambda_1+\lambda_4)^2 + (\lambda_2+\lambda_3+\lambda_5)^2} .$$

This last together with (5), the following:

$$(10) \quad 2|\lambda_1| \geqslant \lambda_2 + \lambda_3 + \lambda_5 \geqslant \lambda_2 + \lambda_4 + \lambda_5 \geqslant \lambda_3 + \lambda_4 + \lambda_5 .$$

Adding these last three inequalities we obtain:

$$(11) \qquad 3|\lambda_1| \geqslant \lambda_2 + \lambda_3 + \lambda_4 + \frac{3}{2}\lambda_5 .$$

2. Theorem and Conclusion.

In this section we prove our:

THEOREM. — *Let* M *be a compact six manifold with section curvature* K, *satisfying* $1 \geqslant K > \dfrac{4\sqrt{10} - 4}{4\sqrt{10} + 23} \cong .2426$. *Then* $b_3(M;\mathbf{R}) = 0$.

Proof. – From § 1, we need only show that $\mathscr{R}_3$ is positive definite. We assume that $1 \geqslant K \geqslant \delta$, where $1 > \delta$. As in § 1, let $\mathscr{R}_3 X = rX$, where r is the minimum eigenvalue for $\mathscr{R}_3$. We shall show: $r > 0$ if $\delta > \dfrac{4\sqrt{10}-4}{4\sqrt{10}+23}\cdot$ We may assume that X has the form indicated in the theorem of § 1. Now use (4) to examine the components of $\mathscr{R}_3 X$, i.e., write out $\langle \mathscr{R}_3 X, e_{123} \rangle$, $\langle \mathscr{R}_3 X, e_{145} \rangle$, etc. The results are:

$$(12) \quad \lambda_1 r = \lambda_1 K_{123} - 2\lambda_2 R_{4532} - 2\lambda_3 R_{4613} - 2\lambda_4 R_{5621}.$$

(This follows from $\lambda_1 r = \langle rX, e_{123} \rangle = \langle \mathscr{R}_3 X, e_{123} \rangle$.)

$$(13) \quad \lambda_2 r = \lambda_2 K_{145} - 2\lambda_1 R_{4532} - 2\lambda_3 R_{6215} - 2\lambda_4 R_{6341} + \lambda_5 \{R_{2162} + R_{3163} - R_{5615} - R_{4614}\}$$

$$(14) \quad \lambda_3 r = \lambda_3 K_{246} - 2\lambda_1 R_{3164} - 2\lambda_2 R_{5126} - 2\lambda_4 R_{3542} + \lambda_5 \{R_{6526} + R_{4524} - R_{3523} - R_{1521}\}$$

$$(15) \quad \lambda_4 r = \lambda_4 K_{356} - 2\lambda_1 R_{1265} - 2\lambda_2 R_{1436} - 2\lambda_3 R_{2453} + \lambda_5 \{R_{1341} + R_{2342} - R_{5435} - R_{6436}\}$$

$$(16) \quad \lambda_5 r = \lambda_5 K_{123} + \lambda_2 \{R_{2162} + R_{3163} - R_{5165} - R_{4164}\} + \lambda_3 \{R_{6256} + R_{4254} - R_{3253} - R_{1251}\} + \lambda_4 \{R_{1341} + R_{2342} - R_{5345} - R_{6346}\}.$$

Here $R_{ijk\ell} = \langle R(e_i,e_j)e_k,e_\ell \rangle$ and

$$K_{i_1 i_2 i_3} = \sum_{j \neq i_1, i_2, i_3} (K_{ji_2} + K_{ji_2} + K_{ji_3})$$

where $\{e_{i_1}, e_{i_2}, e_{i_3}, e_j\} \subset \{e_1, \ldots, e_6\}$, and K_{ij} is the sectional curvature of the plane $\{e_i, e_j\}$.

The proof of our theorem involves interpolating (12)-(16) along with (5), (8)-(11). We shall be freely using that $|R_{ijk\ell}| \leqslant \frac{2}{3}(1-\delta)$ and $|R_{iij\ell}| \leqslant \frac{1}{2}(1-\delta)$, cf. [5]. We may assume that $\lambda_1 = +1$ or -1.

Case 1. $\lambda_1 = +1$. In this case (12) yields:

$$r \geqslant 9\delta - \frac{4}{3}(1-\delta)(\lambda_2 + \lambda_3 + \lambda_4).$$

From (8), we get $r \geqslant 9\delta - \frac{4}{3}(1-\delta)(2\sqrt{2}-1)$. The right hand side here is positive as soon as $\delta > \dfrac{8\sqrt{2}-4}{8\sqrt{2}+23} \cong .213$, so our theorem is proved in this case.

We may now assume that $\lambda_1 = -1$, so (12) becomes:

$$(17) \qquad r = K_{123} + 2\lambda_2 R_{4532} + 2\lambda_3 R_{4613} + 2\lambda_4 R_{5621}.$$

Case 2. $\lambda_2 + \lambda_3 + \lambda_4 \leqslant 2$. In this case, (17) yields $r \geqslant 9\delta - \frac{8}{3}(1-\delta)$. The right hand side here is positive as soon as $\delta > \frac{8}{35} \cong .229$, so our theorem is proved in this case. Note in particular that if $\lambda_5 \geqslant \lambda_4$, then (10) yields $2 \geqslant \lambda_2 + \lambda_3 + \lambda_4$. Thus we may now assume both that $\lambda_2 + \lambda_3 + \lambda_4 > 2$ and that $\lambda_4 > \lambda_5$. In general, $3 \geqslant \lambda_2 + \lambda_3 + \lambda_4 > 2$, so we need more inequalities to work with. This is achieved by adding (17), to (13) through (15). The result is:

$$\begin{aligned}
&(18) \quad (1+\lambda_2+\lambda_3+\lambda_4)r = K_{123} + \lambda_2 K_{145} + \lambda_3 K_{246} + \lambda_4 K_{356}\\
&a) \; + 2[(1+\lambda_2)R_{4532} - R_{3542}(\lambda_3+\lambda_4)]\\
&b) \; + 2[(1+\lambda_3)R_{4613} - R_{4163}(\lambda_2+\lambda_4)]\\
&c) \; + 2[(1+\lambda_4)R_{5621} - R_{2651}(\lambda_2+\lambda_3)] + \lambda_5[S_1+S_2+S_3]
\end{aligned}$$

where $S_1 = R_{2162} + R_{3163} - R_{5615} - R_{4614}$ and so on. Note that $S_i \geqslant -2(1-\delta)$, $i = 1, 2, 3$. We now estimate from below 18*a*), *b*) and *c*).

$$(19) \qquad 18a) \geqslant \frac{4}{3}(1-\delta)(1+\lambda_2).$$

This is clearly true if $R_{3542} \leqslant 0$. If $R_{3542} > 0$, then since $1 + \lambda_2 \geqslant \lambda_3 + \lambda_4$, $-R_{3542}(\lambda_3+\lambda_4) \geqslant -R_{3542}(1+\lambda_2)$. Using this lower bound and the Bianchi identity we get: $18a) \geqslant 2(1+\lambda_2)R_{4352}$ which again yields (19).

$$(20) \qquad 18b) \geqslant -\frac{4}{3}(1-\delta)(1+\lambda_3).$$

This is proved just like (19), since $1 + \lambda_3 \geqslant \lambda_2 + \lambda_4$.

$$(21) \quad \textit{Case A:} \text{ If } 1+\lambda_4 \geqslant \lambda_2+\lambda_3 \text{ then } 18c) \geqslant -\frac{4}{3}(1-\delta)(1+\lambda_4).$$

$$\textit{Case B:} \text{ If } 1+\lambda_4 \leqslant \lambda_2+\lambda_3 \text{ then } 18c) \geqslant -\frac{4}{3}(1-\delta)(\lambda_2+\lambda_3).$$

Again these are proved as in (19), using the Bianchi identity. Note that either of the two cases could occur. We now use (19)-(21) in (18). Let $\lambda = \lambda_2 + \lambda_3 + \lambda_4$, and, since $K_{ijk} \geqslant 9\delta$, (18) yields

(22) *Case A :* $(1+\lambda_4 \geqslant \lambda_2+\lambda_3)$

$$(1+\lambda)r \geqslant (1+\lambda)9\delta - \frac{4}{3}(1-\delta)(3+\lambda) + \lambda_5(S_1+S_2+S_3)$$

Case B : $(1+\lambda_4 \leqslant \lambda_2+\lambda_3)$

$$(1+\lambda)r \geqslant (1+\lambda)9\delta - \frac{8}{3}(1-\delta)(1+\lambda_2+\lambda_3) + \lambda_5(S_1+S_2+S_3).$$

Case 3. One $S_i \geqslant 0$. In this case $S_1 + S_2 + S_3 \geqslant -4(1-\delta)$. From (11), $\frac{2}{3}(3-\lambda) \geqslant \lambda_5$. Using these facts, (22) Case A yields :

$$(1+\lambda)r \geqslant (1+\lambda)9\delta - \frac{4}{3}(1-\delta)(3+\lambda) - \frac{8}{3}(1-\delta)(3-\lambda), \text{ i.e.}$$

$$(1+\lambda)r \geqslant 21\delta - 12 + \lambda\left(9\delta + \frac{4}{3}(1-\delta)\right). \tag{23}$$

Now (17) yields $r \geqslant 9\delta - \frac{4}{3}(1-\delta)\lambda$. Multiply this last inequality by $(1+\lambda)$ and we obtain

$$(1+\lambda)r \geqslant (1+\lambda)\left(9\delta - \frac{4}{3}(1-\delta)\lambda\right). \tag{24}$$

Now set the right hand side of (24) equal to that of (23) and solve for λ. This yields $\lambda = \sqrt{10} - 1$ $(3 \geqslant \lambda > 2)$. Now at $\lambda = \sqrt{10} - 1$, inequalities (23) and (24) agree. If $\lambda \geqslant \sqrt{10} - 1$ then (23) yields

$$(1+\lambda)r \geqslant 21\delta - 12 + (\sqrt{10}-1)\left(9\delta + \frac{4}{3}(1-\delta)\right),$$

while if $\lambda \leqslant \sqrt{10} - 1$, (24) yields $r \geqslant 9\delta - \frac{4}{3}(1-\delta)(\sqrt{10}-1)$, so the lower bound on r with $\lambda = \sqrt{10} - 1$ is always valid. With this value of λ, the right hand side of (24) is positive as soon as $\delta > \dfrac{4\sqrt{10} - 4}{4\sqrt{10} + 23}$, so our theorem is proved in this case.

In case B, using (10),

$$2 \geqslant \lambda_2 + \lambda_3 + \lambda_5 \Rightarrow 3 \geqslant 1 + \lambda_2 + \lambda_3 + \lambda_5 \Rightarrow -(1+\lambda_2+\lambda_3) \geqslant -(3-\lambda_5),$$

(22) case B yields, since $S_1 + S_2 + S_3 \geqslant -4(1-\delta)$,

$$(25) \qquad (1+\lambda)r \geqslant (1+\lambda)9\delta - 8(1-\delta) - \frac{4}{3}(1-\delta)\lambda_5.$$

Again from (10) and case B, $2 \geqslant \lambda_2 + \lambda_3 + \lambda_5 \geqslant 1 + \lambda_4 + \lambda_5$, so adding, we obtain $4 \geqslant 1 + \lambda + 2\lambda_5 \Rightarrow -\lambda_5 \geqslant -\frac{1}{2}(3-\lambda)$. Using this in (25), we obtain after some simplification,

$$(26) \qquad (1+\lambda)r \geqslant 19\delta - 10 + \lambda\left(9\delta + \frac{2}{3}(1-\delta)\right).$$

Now repeat the argument of case A, setting the right hand side of (26) equal to that of (24), and solving for λ. This yields $\lambda = \frac{\sqrt{129} - 3}{4}$, and the right hand side of (24) is positive as soon as $\delta > \frac{\sqrt{129} - 3}{\sqrt{129} + 24} \cong .236$, so our theorem is proved in this case.

Case 4 (The last case). All $S_i < 0$.

To handle this case, consider the sum:

$$(17) + (13) + (14) + (15) - (16).$$

This is:

$$(27) \quad \begin{aligned}(1+\lambda-\lambda_5)r &= (1-\lambda_5)K_{123} + \lambda_2 K_{145} + \lambda_3 K_{246} + \lambda_4 K_{356} \\ &+ 18a + 18b + 18c + (\lambda_5-\lambda_2)S_1 + (\lambda_5-\lambda_3)S_2 + (\lambda_5-\lambda_4)S_3.\end{aligned}$$

From case 2 and (5), $\lambda_5 - \lambda_i < 0$ for $i = 2, 3, 4$. Thus, if each $S_i < 0$, we can eliminate the last three terms of (27) to obtain

$$(28) \quad \textit{Case A}: \ (1+\lambda-\lambda_5)r \geqslant (1+\lambda-\lambda_5)9\delta - \frac{4}{3}(1-\delta)(3+\lambda)$$

$$= (1-\lambda_5)9\delta - 4(1-\delta) + \lambda\left(9\delta - \frac{4}{3}(1-\delta)\right).$$

$$\textit{Case B}: \ (1+\lambda-\lambda_5)r \geqslant (1+\lambda-\lambda_5)9\delta - \frac{8}{3}(1-\delta)(1+\lambda_2+\lambda_3).$$

Consider case B first. From (10),

$$2 \geqslant \lambda_2 + \lambda_3 + \lambda_5 \Rightarrow 3 - \lambda_5 \geqslant 1 + \lambda_2 + \lambda_3,$$

while from case 2, $1 + \lambda > 3$, so case B yields:

$$(1+\lambda-\lambda_5)r \geqslant (3-\lambda_5)\left(9\delta - \frac{8}{3}(1-\delta)\right). \tag{29}$$

The right hand side of (29) is positive as soon as $\delta > \frac{8}{35}$, so our theorem is proved in this case. To handle case A, assume that $\delta > \frac{4}{31}$, so the coefficient of λ is positive. We also have from (17),

$$(1+\lambda-\lambda_5)r \geqslant (1+\lambda-\lambda_5)\left(9\delta - \frac{4}{3}(1-\delta)\lambda\right). \tag{30}$$

Set the right hand side of (30) equal to that of (28) case A. This yields $\lambda = \frac{\lambda_5 \pm \sqrt{\lambda_5^2 + 12}}{2}$. Now repeat the argument of case 3. For $\lambda = \frac{\lambda_5 - \sqrt{\lambda_5^2 + 12}}{2}$, (30) yields, since $1 \geqslant \lambda_5$, $r \geqslant 9\delta + \frac{2}{3}(1-\delta)(\sqrt{12}-1)$ and this last quantity is positive for $\delta > -.22$, so in this case $r > 0$ for $\delta > \frac{4}{31} \cong .13$ and we're done. If $\lambda = \frac{\lambda_5 + \sqrt{\lambda_5^2 + 12}}{2}$, use the estimate $\frac{2}{3} > \lambda_5$ (which comes from (11) and $\lambda > 2$) in (30) to get $r \geqslant 9\delta - \frac{2}{3}(1-\delta)\left(\frac{2+\sqrt{112}}{3}\right)$. This last is positive for $\delta > \frac{4 + 8\sqrt{7}}{85 + 8\sqrt{7}} \cong .237$, so our theorem is proved in this last case.

Q.E.D.

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Walter Seaman,
University of Iowa
Department of Mathematics
101 Maclean Hall
Iowa City, Iowa 52242 (USA).