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INTERPOLATING SEQUENCES OF COMPLEX HYPERPLANES IN THE UNIT BALL OF C^n

by
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This paper gives a sufficient condition for the existence of a solution to the following problem :

Given a sequence of complex hyperplanes, $\{L_j\}_{j\in \mathbf{Z}_+}$, all intersecting $\mathbf{B}^n$ (the unit ball of $\mathbf{C}^n$), and given a sequence of holomorphic functions $\{f_j\}_{j\in \mathbf{Z}_+} \subseteq H^\infty(\mathbf{B}^{n-1})$ is there a function $f \in H^\infty(\mathbf{B}^n)$ such that $f|_{L_j} \equiv f_j \circ \phi_j^{-1}, j \in \mathbf{Z}_+$, where ϕ_j is a complex-linear map from $\mathbf{B}^{n-1}$ onto $L_j \cap \mathbf{B}^n$? If there is such an f, we shall say that $\{L_j\}_{j\in \mathbf{Z}_+}$ is *interpolating.*

Notations. – If $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$, $w = (w_1, \ldots, w_n) \in \mathbf{C}^n$, then $z \cdot \overline{w} = \sum_{j=1}^{n} z_j \overline{w}_j$ and $|z| = (z \cdot \overline{z})^{1/2}$ (modulus of z),

$$z^* = \frac{z}{|z|} \in \partial \mathbf{B}^n = \{z : |z| = 1\}.$$

For all $j \in \mathbf{Z}_+, a_j =$ point of smallest modulus in L_j (a_j is the center of the ball $L_j \cap \mathbf{B}^n$). Equivalently,

$$L_j = \{z \in \mathbf{C}^n : (z - a_j) \cdot \overline{a}_j = 0\} \quad (a_j \neq 0).$$

For all $j \in \mathbf{Z}_+$,

$$U_j = \left\{ z \in \mathbf{B}^n : \left| \frac{\overline{a}_j \cdot (a_j - z)}{|a_j|(1 - z \cdot \overline{a}_j)} \right| < \delta_0 \right\}.$$

Key-words: Interpolating sequences – Bounded holomorphic functions – Carleson measures – Extension of functions.

THEOREM 1. — *Given a sequence* $\{L_j\}$ *as above, it is interpolating if the following sufficient conditions are met:*

$$\text{(B)} \sum_{j \in \mathbf{Z}_+} \frac{(1 - |a_j|^2)(1 - |a_k|^2)}{|1 - a_j \cdot \bar{a}_k|^2} \leqslant M < \infty$$

and

$$\text{(U)} \quad \textit{for all } j, k \in \mathbf{Z}_+, j \neq k, \textit{ then } U_j \cap U_k = \emptyset .$$

Remarks. — 1) By applying an element of the unitary group, we can send any a_j to a point of the form $(a, 0)$, $a \in \mathbf{B}^1$. Then

$$U_j = \left\{ (z_1, z_2) : \left| \frac{z_1 - a}{1 - z_1 \bar{a}} \right| < \delta_0 \right\} .$$

Since the definition of U_j is rotation-invariant, we see that for all j, U_j is a tube surrounding the hyperplane L_j, of radius commensurate to $1 - |a_j|$.

In particular, for $\epsilon > 0$ small enough, U_j contains any set of the form $\{z \in \mathbf{B}^n : \exists\, w \in L_j : d_H(z, w) < \epsilon\}$, where

$$d_H(z, w) = \left(1 - \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - z \cdot \bar{w}|^2}\right)^{1/2}$$

is the "hyperbolic" distance, invariant under automorphism of $\mathbf{B}^n$. The regions U_j are not automorphism-invariant, but condition (U) implies in particular that the lines are separated in the metric d_H, so that if $j \neq k$, we can find $f \in H^\infty(\mathbf{B}^n)$ such that $f|_{L_j} \equiv 1$ and $f|_{L_k} \equiv 0$ (explicit computation omitted).

2) Trivially, if $\{L_j\}_{j \in \mathbf{Z}_+}$ is interpolating, then the sequence $\{a_j\}_{j \in \mathbf{Z}_+}$ associated to it is.

In [3], Berndtsson gives a sufficient condition for a sequence $\{a_j\}_{j \in \mathbf{Z}_+}$ to be interpolating:

$$\prod_{j : j \neq k} |\phi_{a_j}(a_k)| \geqslant \epsilon > 0 ,$$

where $\phi_a(z)$ is the automorphism of $\mathbf{B}^n$ defined in ([7], 2.2.1, p. 25):

$$\phi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - z \cdot \bar{a}} ,$$

$P_a(z) = (z \cdot \bar{a}/|a|^2)\, a$ is the projection of z onto the complex line through a and 0, $Q_a(z) = z - P_a(z)$ is the projection of z onto the complex hyperplane through 0 orthogonal to a, and $s_a = (1 - |a|^2)^{1/2}$.

$|\phi_{a_j}(a_k)|^2 = d_H(a_j, a_k)^2$, so that the convergence of the above product is equivalent to (B) together with the requirement that the points a_j are *separated*, i.e. $d_H(a_j, a_k) \geqslant \delta > 0$ for $j \neq k$. (U) implies, of course, that a_j are separated. We are now ready for the following

DEFINITION. – *Given a function* $f_k : L_k \longrightarrow \mathbf{C}$, *define an extension* $\widetilde{f_k} : \mathbf{B}^n \longrightarrow \mathbf{C}$ *by*

$$\widetilde{f_k} = f_k \circ \phi_{a_k} \circ Q_{a_k} \circ \phi_{a_k}.$$

This definition makes sense, since

$$\begin{aligned}\phi_{a_k}(L_k) = \phi_{a_k}^{-1}(L_k) &= \{z : \phi_{a_k}(z) \cdot \bar{a}_k = |a_k|^2\} \\ &= \left\{ z : 1 - \frac{1 - |a_k|^2}{1 - z \cdot \bar{a}_k} = |a_k|^2 \right\} \\ &= \{z : z \cdot \bar{a}_k = 0\} = \text{Range}\,(Q_{a_k}),\end{aligned}$$

and consequently $\phi_{a_k}(R(Q_{a_k})) = L_k$, so $\widetilde{f_k}$ is indeed defined on $\mathbf{B}^n$. Furthermore,

$$\begin{aligned}\widetilde{f_k}\,|_{L_k} &= f_k \circ \phi_{a_k} \circ Q_{a_k}|_{R(Q_{a_k})} \circ \phi_{a_k}|_{L_k} \\ &= f_k \circ \phi_{a_k} \circ \phi_{a_k}|_{L_k}, \text{ since } Q \text{ is a projection,} \\ &= f_k, \text{ since } \phi = \phi^{-1}.\end{aligned}$$

In other words, $\widetilde{f_k} \circ \phi_{a_k} = (f_k \circ \phi_{a_k}) \circ Q_{a_k}$, i.e. first we pull back the situation to the case where f_k is defined on a complex hyperplane through 0, and extend it trivially to be independent of the last coordinate.

Clearly, $\|\widetilde{f_k}\|_{H^\infty(\mathbf{B}^n)} = \|f_k\|_{H^\infty(L_k)}$; ($f_k$ is what was denoted in the introduction $f_k \circ \phi_k^{-1}$).

3) Suppose that for all $j \in \mathbf{Z}_+$, $a_j = (\alpha_j, 0)$, $\alpha_j \in \mathbf{B}^1$. Then all the L_j are parallel, $L_j = \{z_1 = \alpha_j\}$, and $\{L_j\}$ is an interpolating sequence if and only if $\{\alpha_j\}_{j \in \mathbf{Z}_+}$ is an interpolating sequence in $\mathbf{B}^1$.

Conditions (U) reduces to

$$\left|\frac{\alpha_j - \alpha_k}{1 - \alpha_j \bar{\alpha}_k}\right| \leqslant c < 1 \quad \text{for} \quad j \neq k,$$

and condition (B) reduces to:

$$\sum_{j:j\neq k} \frac{(1 - |\alpha_j|^2)(1 - |\alpha_k|^2)}{|1 - \alpha_j \bar{\alpha}_k|^2} \leqslant c.$$

In the case $n = 1$, it is well known (see Carleson [4] or Garnett [5]) that if the points are separated (i.e. (U)), then (B) $\Leftrightarrow$ $\{\alpha_j\}$ is interpolating, so from that point of view the result is sharp.

4) Of course the points a_j cannot cluster at any interior point of $\mathbf{B}^n$. We will, without loss of generality, remove a finite number of hyperplanes from our sequence and henceforth assume $|a_j| \geqslant 1/2$, $j \in \mathbf{Z}_+$, for technical reasons.

The main step in the proof of the theorem is the following:

PROPOSITION 1. – *Under the assumptions* (U) *and* (B), *there exist two positive constants* C_1 *and* C_2, *and analytic functions* $\{F_k\}_{k\in\mathbf{Z}_+}$ *such that*

(i) $\forall\, z \in \mathbf{B}, \sum_k |F_k(z)| \leqslant c_1$

(ii) $\forall\, k \in \mathbf{Z}, \left|F_k|_{L_k}\right| \geqslant c_2$

(iii) $\forall\, j \neq k, \left|F_k|_{L_j}\right| \leqslant \frac{c_2}{2}$

(*the* F_k *are "pseudo P. Beurling functions"*).

Proof of the Theorem (assuming Proposition 1). – We will show that one can construct from the F_k true P. Beurling functions, i.e. $E_k(z)$ verifying:

(i) $\forall\, z \in B, \sum_k |E_k(z)| \leqslant c < \infty$

(ii)′ $E_k|_{L_k} \equiv 1$

(iii)′ $E_k|_{L_j} \equiv 0,\ j \neq k.$

Then our interpolating function will be $f = \sum_k \widetilde{f}_k(z)\, E_k(z)$. $f\,|_{L_k} = \widetilde{f}_k\,|_{L_k} = f_k$, and $\|f\|_\infty \leqslant c(\sup_k \|\widetilde{f}_k\|_\infty) = c \sup_k \|f_k\|_\infty < \infty$.

To construct the E_k:

First let $G_k = \dfrac{F_k}{(F_k\,|_{L_k})^{\sim}}$ where $\sim$ is the extension discussed above.

Then $\sum_k |G_k(z)| \leqslant c_1/c_2$, $G_k\,|_{L_k} \equiv 1$, $\left|G_k\,|_{L_j}\right| \leqslant \frac{1}{2}$, $j \neq k$.

Let $H_k = G_k \prod_{j:j\neq k} (1 - G_j)$.

Since every factor is bounded below by $1/2$,

$$\Big|\prod_{j:j\neq k} (1 - G_j)\Big| \geqslant e^{-2c_1/c_2} \quad \text{on } L_k \quad \text{and} \quad \left|H_k\,|_{L_k}\right| \geqslant e^{-2c_1/c_2},$$

while $H_k\,|_{L_j} \equiv 0$, $j \neq k$.

$$\forall\, z \in B,\ \sum_k |H_k(z)| \leqslant e^{c_1/c_2} \sum_k |G_k(z)| \leqslant \frac{c_1}{c_2}\, e^{c_1/c_2}.$$

Finally, let $E_k = H_k/(H_k\,|_{L_k})^{\sim}$;

$E_k\,|_{L_j} \equiv 0$, $j \neq k$, $E_k\,|_{L_k} \equiv 1$, and $\sum_k |E_k(z)| \leqslant \dfrac{c_1}{c_2}\, e^{3c_1/c_2}$, q.e.d.

Proof of Proposition 1. – Let

$$F_k(z) = (1 - |a_k|^2/1 - z\cdot \bar{a}_k)^p\, W(a_k, z) \prod_{\substack{j:j\neq k\\ |1-a_k\cdot \bar{a}_j| \leqslant C_0(1-|a_k|^2)}} \phi_{a_j}(z)\cdot \bar{a}_j$$

where $p \geqslant 4$ and $C_0 = C_0(\delta_0) > 1$ will be specified, and following [3],

$$W(a_k, z) = \exp - \sum_j \left[\left(\frac{1 + z\cdot \bar{a}_j}{1 - z\cdot \bar{a}_j} - \frac{1 + a_k\cdot \bar{a}_j}{1 - a_k\cdot \bar{a}_j}\right) \frac{(1 - |a_j|^2)(1 - |a_k|^2)}{1 - |a_j\cdot \bar{a}_k|^2}\right].$$

Convergence of the infinite product will be proved below. Note that $|\phi_{a_j}(z) \cdot \bar{a}_j| \leqslant |\phi_{a_j}(z)|\,|a_j| \leqslant 1$, so

$$|F_k(z)| \leqslant 2^{p-4}(1 - |a_k|^2/|1 - z \cdot \bar{a}_k|)^4\, |W(a_k, z)|.$$

The main step in the proof of [3] is that

$$\forall z \in B, \sum_k (1 - |a_k|^2/|1 - z \cdot \bar{a}_k|)^4\, |W(a_k, z)| \leqslant M_1$$

so $\sum_k |F_k(z)| \leqslant 2^{p-4} M_1 = c_1$, which proves (i).

Proof of (iii). – Case 1 : j is such that

$$|1 - a_j \cdot \bar{a}_k| \leqslant C_0(1 - |a_k|^2).$$

Then $\phi_{a_j}(z) \cdot \bar{a}_j = (a_j - z) \cdot \bar{a}_j/1 - z \cdot \bar{a}_j = 0$ for $z \in L_j$ is a factor in the infinite product, so $|F_k(z)| = 0 \leqslant c_2/2$.

Case 2 : j is such that $|1 - a_j \cdot \bar{a}_k| \geqslant C_0(1 - |a_k|^2)$.

LEMMA 1. – *If* $\{L_k\}_{k \in \mathbf{Z}_+}$ *satisfy* (U), *and* $z \in L_j$, $j \neq k$, *then* $C_3|1 - z \cdot \bar{a}_k| \geqslant |1 - a_j \cdot \bar{a}_k|$, *where* C_3 *is a constant depending only on* δ_0.

Thus for all $z \in L_j$,

$$\frac{1 - |a_k|^2}{|1 - z \cdot \bar{a}_k|} \leqslant \frac{C_3(1 - |a_k|^2)}{|1 - a_j \cdot \bar{a}_k|} \leqslant \frac{C_3}{C_0} = \frac{1}{2}$$

if we pick $C_0 = 2C_3$.

So for $z \in L_j$, $|F_k(z)| \leqslant (1/2)^p\, |W(a_k, z)|$. But

$$|W(a_k, z)| = \left(\exp - \sum_j \frac{1 - |z \cdot \bar{a}_j|^2}{|1 - z \cdot \bar{a}_j|^2} \frac{(1 - |a_j|^2)(1 - |a_k|^2)}{|1 - a_j \cdot \bar{a}_k|^2}\right)$$

$$\times \left(\exp \sum_j \frac{(1 - |a_j|^2)(1 - |a_k|^2)}{|1 - a_j \cdot \bar{a}_k|^2}\right) \leqslant e^{M} \text{ (see [3])}.$$

So it will be enough to take

$$p \geqslant \log_2\left(\frac{2e^{M}}{C_2}\right) \quad \text{to get (iii).}$$

Proof of (ii). – First note that

$$F_k\,|_{L_k} \equiv W(a_k, z) \prod_{\substack{j:j\neq k \\ |1-a_j\cdot \bar{a}_k| \leq C_0(1-|a_k|^2)}} \phi_{a_j}(z)\cdot \bar{a}_j$$

$z \in L_k \subset U_k$, hence $z \notin U_j$, so

$$|\phi_{a_j}(z)\cdot \bar{a}_j| = \left|\frac{(a_j - z)\cdot \bar{a}_j}{1 - z\cdot \bar{a}_j}\right| \geq \delta_0 |a_j| \geq \frac{\delta_0}{2};$$

each term in the infinite product is bounded below, so we only have to consider

$$\sum_{\substack{j:|1-a_j\cdot a_k| \leq C_0(1-|a_k|^2) \\ j\neq k}} |1 - \phi_{a_j}(z)\cdot \bar{a}_j| = \sum_{\substack{j:|1-a_j\cdot \bar{a}_k| \leq C_0(1-|a_k|^2) \\ j\neq k}} \frac{1-|a_j|^2}{|1 - z\cdot \bar{a}_j|}.$$

By Lemma 1, exchanging k and j,

$$C_3 |1 - z\cdot \bar{a}_j| \geq |1 - a_k\cdot \bar{a}_j|.$$

Thus our sum is

$$\leq C_3 \sum_{j:|1-a_j\cdot \bar{a}_k| \leq C_0(1-|a_k|^2)} \frac{1-|a_j|^2}{|1 - a_k\cdot \bar{a}_j|}$$

$$\leq C_3 \sum_j \frac{C_0(1-|a_k|^2)}{|1 - a_j\cdot \bar{a}_k|}\,\frac{(1-|a_j|^2)}{|1 - a_k\cdot \bar{a}_j|}$$

$$\leq C_3\, C_0\, M,$$

so the infinite product in F_k converges and is bounded below by $e^{-(2/\delta_0)C_0C_3M}$.

On the other hand,

$$|W(a_k, z)| \geq \exp - \sum_j \frac{1-|z\cdot \bar{a}_j|^2}{|1 - z\cdot \bar{a}_j|^2}\,\frac{(1-|a_j|^2)(1-|a_k|^2)}{1 - |a_j\cdot \bar{a}_k|^2}$$

$$\geq \exp - \sum_j \frac{1-|z\cdot \bar{a}_j|^2}{|1 - z\cdot \bar{a}_j|}\,\frac{C_3}{|1 - a_k\cdot \bar{a}_j|}\,\frac{(1-|a_j|^2)(1-|a_k|^2)}{1 - |a_j\cdot \bar{a}_k|^2}$$

by lemma 1.

LEMMA 2. – *Given any two points* $a_j, a_k \in \mathbf{B}^n, z \in L_k$, *then*

$$\frac{1-|z\cdot \bar{a}_j|^2}{|1-z\cdot \bar{a}_j|} \leqslant 18\,\frac{1-|a_k\cdot \bar{a}_j|^2}{|1-a_k\cdot \bar{a}_j|}.$$

Thus

$$|W(a_k, z)| \geqslant \exp - \sum_j 18\,C_3\,\frac{(1-|a_j|^2)(1-|a_k|^2)}{|1-a_j\cdot \bar{a}_k|^2} = e^{-18 C_3 M}.$$

So we may take $c_2 = e^{-2M(c_0/\delta_0 + 9)C_3}$, which concludes the proof of (ii).

Proof of the Lemmas

Proof of Lemma 1. – Choose coordinates so that $a_j = (a, 0)$. Let $a_k = (b_1, b')$, $b' \in \mathbf{C}^{n-1}$. $a_k \notin U_j$ means

$$|b_1 - a| \geqslant \delta_0 |1 - b_1 \bar{a}|,$$

so it will be enough to show

$$C|1 - a\bar{b}_1 - z'\cdot \bar{b}'| \geqslant |b_1 - a|,$$

for $z = (a, z') \in L_j \cap B$, i.e.

$$|z'|^2 \leqslant 1 - |a|^2.$$

$$|1 - a\bar{b}_1 - z'\cdot\bar{b}'| \geqslant |1 - a\bar{b}_1| - \sqrt{1-|a|^2}\,\sqrt{1-|b_1|^2}$$

$$= \frac{|b_1 - a|^2}{|1 - a\bar{b}_1| + \sqrt{1-|a|^2}\,\sqrt{1-|b_1|^2}}.$$

However,

$$1 - |a|^2 \leqslant 2(1-|a|) \leqslant 2|1 - b_1\bar{a}| \leqslant \frac{2}{\delta_0}|b_1 - a|$$

and

$$1 - |b_1|^2 \leqslant 2(1 - |b_1|) \leqslant 2(1 - |a| + |b_1 - a|)$$

$$\leqslant 2\left(1 + \frac{1}{\delta_0}\right)|b_1 - a|.$$

So the last expression is

$$\geqslant \frac{|b_1 - a|^2}{\left(\dfrac{1}{\delta_0} + \sqrt{\dfrac{2}{\delta_0}\cdot 2\left(1+\dfrac{1}{\delta_0}\right)}\right)|b_1 - a|}$$

and $C_3 = (\delta_0^2/(1 + 2\sqrt{1+\delta_0}))^{-1}$ will do.

Proof of Lemma 2. – Note first that

$$\frac{1 - |z \cdot \overline{a}_j|^2}{|1 - z \cdot \overline{a}_j|} \leqslant (1 + |z \cdot \overline{a}_j|) \frac{1 - |z \cdot \overline{a}_j|}{|1 - z \cdot \overline{a}_j|} \leqslant 2 .$$

So that if $1 - |a_k \cdot \overline{a}_j|^2 / |1 - a_k \cdot \overline{a}_j| \geqslant 1/9$, we have

$$\frac{1 - |z \cdot \overline{a}_j|^2}{|1 - z \cdot \overline{a}_j|} \leqslant 2(9) \frac{1 - |a_k \cdot \overline{a}_j|^2}{|1 - a_k \cdot \overline{a}_j|}, \text{ q.e.d.}$$

If on the contrary

$$(1 - |a_k \cdot \overline{a}_j|^2) \leqslant \frac{1}{9} |1 - a_k \cdot \overline{a}_j| ,$$

then

$$(1 - |a_k|^2) \leqslant \frac{1}{9} |1 - a_k \cdot \overline{a}_j| .$$

So

$$|1 - z \cdot \overline{a}_j|^{1/2} \geqslant |1 - a_k \cdot \overline{a}_j|^{1/2} - |1 - z \cdot \overline{a}_k|^{1/2}$$
$$= |1 - a_k \cdot \overline{a}_j|^{1/2} - (1 - |a_k|^2)^{1/2} \geqslant \left(1 - \frac{1}{3}\right) |1 - a_k \cdot \overline{a}_j|^{1/2} ;$$

and ([3], lemma 5)

$$1 - |z \cdot \overline{a}_j|^2 \leqslant 2(1 - |z \cdot \overline{a}_j|) \leqslant 4(1 - |z \cdot \overline{a}_k| + 1 - |a_k \cdot \overline{a}_j|)$$
$$\leqslant 4(1 - |a_k|^2 + 1 - |a_k \cdot \overline{a}_j|^2) .$$

Hence

$$\frac{1 - |z \cdot \overline{a}_j|^2}{|1 - z \cdot a_j|} \leqslant \frac{4(1 - |a_k|^2 + 1 - |a_k \cdot \overline{a}_j|^2)}{\left(\frac{2}{3}\right)^2 |1 - a_k \cdot \overline{a}_j|}$$
$$\leqslant \frac{(9)(4)(2)(1 - |a_k \cdot \overline{a}_j|^2)}{4 |1 - a_k \cdot \overline{a}_j|}, \text{ q.e.d.}$$

More Remarks. – 5) The interpolation problem is invariant under automorphisms of the ball. Condition (U) is not. An optimal (but not very practical) statement of the theorem would be: if there exists

$\psi \in \mathrm{Aut(B)}$ such that $\{\psi(\mathrm{L}_j)\}_{j\in \mathbf{B}_+}$ satisfies (B) and (U), then $\{\mathrm{L}_j\}_{j\in \mathbf{Z}_+}$ is an interpolating sequence.

It is natural to ask whether the theorem can be proved if one substitutes for (U) the weaker, invariant requirement that the hyperplanes L_j be separated in the metric d_{H}. Unfortunately, it seems to require some new idea, since U_j is precisely the region where $|\phi_{a_j}(z) \cdot \overline{a}_j|$ is small.

6) Amar [1] has put to use (essentially) the same infinite product $\mathrm{P}(z) = \prod_{j\in \mathbf{Z}_+} \phi_{a_j}(z) \cdot \overline{a}_j$ to prove similar results; specifically, if $f_j \in \mathrm{H}^\infty$, $f \in \mathrm{BMOA}$ is obtained, and if f_j verify:

$$(\mathrm{H}^p) \quad \sum_{j\in \mathbf{Z}_+} (1-|a_j|^2) \int_{\mathrm{L}_j} |f_j|^p \, d\lambda_{2n-2} < \infty$$

where $p \geqslant 1$, and $d\lambda_{2n-2}$ is $2n-2$-dimensional Lebesgue measure on L_j, then $f \in \mathrm{H}^p(\mathbf{B}^n)$ is obtained.

This is done by solving a certain $\overline{\partial}$ problem, namely, if g is a C^∞ solution to the interpolation problem, let $f = g + u\mathrm{P}$ with $\overline{\partial} u = -(1/\mathrm{P})\, \overline{\partial} g$. One then needs:

$$(\mathrm{US})\ \exists\ \delta_0, \delta_1 > 0 \text{ such that } \forall\, z \in \mathrm{U}_k(\delta_0), \prod_{j:j\neq k} |\phi_{a_j}(z) \cdot \overline{a}_j| \geqslant \delta_1 .$$

Clearly, (US) $\Longrightarrow$ (B), and by Remark 5, (US) $\Longrightarrow$ (U) (cf. [1], lemma 2.1). Applying (US) to $z = a_k$, one see that it implies in fact

$$(\mathrm{P}) \quad \forall\, k \in \mathbf{Z}_+, \sum_{j:j\neq k} \frac{(1-|a_k \cdot \overline{a}_j^*|^2)(1-|a_j|^2)}{|1-a_k \cdot \overline{a}_j|^2} \leqslant c.$$

With the help of lemmas 1 and 2, one can show that (U) and (P) $\Leftrightarrow$ (US).

Under those assumptions, one can use Berndtsson's L^∞ solution to the $\overline{\partial}$ equation [2] to obtain an interpolating $f \in \mathrm{H}^\infty$, but one has to require a further condition involving "C1 measures" (see [2]), which is also more restrictive than (B), and not equivalent to (P). It gives rise to unwieldy computation, even for $n = 2$.

But we are now in a position to strengthen Amar's results; Theorem 1 implies that under (US), bounded data can be interpolated by a bounded function, and we have :

THEOREM 2. – *If* $\{L_k\}_{k\in \mathbf{Z}_+}$ *verifies* (U) *and* (B), *and* $\{f_k\}$ *verifies* (H^p), *then there exists* $f \in H^p(B)$ *such that*

$$f|_{L_k} = f_k\,,\ \forall\, k \in \mathbf{Z}_+ \quad (1 \leqslant p < \infty).$$

Note that, since $\sum_k (1-|a_k|^2)\int_{L_k} \cdot\, d\lambda_{2n-2}$ is a Carleson measure in $\mathbf{B}^n$, condition (H^p) must be verified if there is an interpolating function f.

Theorem 2 is a consequence of:

LEMMA 4. – *If there are P. Beurling functions for a sequence of hyperplanes* $\{L_k\}$, *then it is* H^p *-interpolating.*

This implies in particular that any H^∞ -interpolating sequence will be H^p-interpolating, since one can show it will necessarily have P. Beurling functions (follow Varopoulos' proof [9] or [5], p. 298).

Proof of lemma 4. – Let $f(z) = \sum_{k\in\mathbf{Z}_+} \hat{f}_k(z)\, E_k(z)$, where E_k are the P. Beurling functions and $\hat{f}_k|_{L_k} = f_k$.

Let $S = \partial B^n$, $d\sigma = 2n-1$-dimensionnal Lebesgue measure on S

$$\int_S |f|^p d\sigma = \int_S \Big|\sum_k \hat{f}_k E_k\Big|^p d\sigma$$

$$\leqslant \int_S \Big(\sum_k |\hat{f}_k|^p\Big)\Big(\sum_k |E_k|^q\Big)^{p/q} d\sigma$$

$$\leqslant c \sum_k \int_S |\hat{f}_k|^p\, d\sigma\,, \ \text{(where } 1/p + 1/q = 1).$$

It is enough to show that, for an appropriate choice of $\hat{f}_k$, the last series is convergent (which will retroactively prove that the integrals we wrote down were making sense).

Let $\hat{f}_k(z) = (1-|a_k|^2/1 - z\cdot\bar{a}_k)^{2n}\tilde{f}_k(z)\,;\hat{f}_k|_{L_k} = \tilde{f}_k|_{L_k}$, but $\hat{f}_k$ drops off more rapidly away from L_k.

$$\int_S |\hat{f}_k(z)|^p\, d\sigma(z) = \int_S \left(\frac{1-|a_k|^2}{|1-z\cdot\bar{a}_k|}\right)^{2pn} |f_k|^p \circ \phi\circ Q \circ \phi(z)\, d\sigma(z)$$

where $\phi = \phi_{a_k}$, $Q = Q_{a_k}$. Since $\phi(S) = S$, we make the change of variable $w = \phi(z)$, to get

$$\int_S |\hat{f_k}|^p \, d\sigma = \int_S |1 - w \cdot \bar{a}_k|^{2pn} |f_k|^p \circ \phi \circ Q(w) \, J_\phi(w) \, d\sigma(w)$$

where $J_\phi(w)$ is the real Jacobian of $\phi|_s$ at w.

The Jacobian matrix of ϕ as a map from $\mathbf{B}^n$ to $\mathbf{B}^n$ can be computed with no difficulty (e.g. in the case $a_k = (0, a)$) and the real Jacobian of ϕ as a map from $\mathbf{B}^n$ to $\mathbf{B}^n$ is

$$(1 - |a_k|^2)^{n+1} / |1 - w \cdot \bar{a}_k|^{2(n+1)}.$$

$$|J_\phi(w)| = \left(\frac{\partial |\phi(w)|}{\partial |w|}\right)^{-1} \frac{(1 - |a_k|^2)^{n+1}}{|1 - w \cdot \bar{a}_k|^{2(n+1)}}$$

$$= \left(\frac{1 - |a_k|^2}{|1 - w \cdot \bar{a}_k|^2}\right)^{-1} \frac{(1 - |a_k|^2)^{n+1}}{|1 - w \cdot \bar{a}_k|^{2(n+1)}}$$

$$= \frac{(1 - |a_k|^2)^n}{|1 - w \cdot \bar{a}_k|^{2n}}.$$

So

$$\begin{aligned} \int_S |\hat{f_k}|^p \, d\sigma &= (1 - |a_k|^2)^n \int_S |1 - w \cdot \bar{a}_k|^{2n(p-1)} |f_k|^p \circ \phi \circ Q(w) \, d\sigma(w) \\ &\leqslant 2^{2n(p-1)} (1 - |a_k|^2)^n \int_S |f_k|^p \circ \phi \circ Q(w) \, d\sigma(w) \\ &= 2^{2n(p-1)} (1 - |a_k|^2)^n \int_{R(Q)} |f_k|^p \circ \phi(w') \, d\lambda_{2(n-1)}(w'), \end{aligned}$$

where $d\lambda_{2(n-1)}$ is $2n - 2$-dimensional Lebesgue measure on $R(Q)$, because $|f_k|^p \circ \phi \circ Q$ is a function depending on $n - 1$ variables only. Notice that

$$\phi_{a_k} : R(Q_{a_k}) \cong \mathbf{B}^{n-1}(0, 1) \longrightarrow L_k \cong \mathbf{B}^{n-1}(0, (1 - |a_k|^2)^{1/2})$$

is given by $\phi_{a_k}(z) = a_k - s_{a_k} z \, (z \cdot \bar{a}_k = 0\,!)$ so that ϕ simply induces a dilation with ratio $(1 - |a_k|^2)^{1/2}$ and

$$\int_{R(Q_k)} |f_k|^p \circ \phi(w') \, d\lambda_{2(n-1)}(w')$$

$$= (1 - |a_k|^2)^{-(n-1)} \int_{L_k} |f_k|^p (w'') \, d\lambda_{2(n-1)}(w''),$$

hence $\int_s |\hat{f_k}|^p \, d\sigma \leqslant C(n, p)(1 - |a_k|^2) \int_{L_k} |f_k|^p \, d\lambda_{2(n-1)}$, which by (H^p) is a term in a convergent series, q.e.d.

7) In the other direction (finding *necessary* conditions), the "trivial" result cannot be improved.

Namely, if $\{L_j\}$ is an interpolating sequences of hyperplanes, then $\{a_j\}$ is an interpolating sequence of points, so they must satisfy Varopoulos's necessary condition (cf. [10]) :

$$\text{(V)} \quad \sum_{j \in \mathbf{Z}_+} \left(\frac{(1-|a_j|^2)(1-|a_k|^2)}{|1-a_j \cdot \bar{a}_k|^2} \right)^n \leqslant \mathrm{C}$$

where C is a constant (independent of k).

On the other hand, using the fact that $\bigcup_{j \in \mathbf{Z}_+} L_j$ must be a zero-set for an H^∞ function, and Skoda's Blaschke condition for the Nevanlinna class [8] (which cannot be quantitatively improved for H^∞, cf. Hakim & Sibony [6], or again [3]), we find:

$$\text{(S)} \quad \sum_{j \in \mathbf{Z}_+} (1-|a_j|^2)^n \leqslant \mathrm{C}.$$

(S) is a consequence of (V) (which is the invariant version of (S)). No stronger condition of the same type can be substituted for (S) without some geometrical requirement (e.g. all L_j are parallel !), as shown by :

PROPOSITION 2. — *For all* $n \geqslant 1$, *for all* $\epsilon > 0$, *there is an interpolating sequence of* C*-hyperplanes,* $\{L_j\}_{j \in \mathbf{Z}_+}$ in $\mathbf{B}^n$ *such that*

$$(6) \quad \sum_{j \in \mathbf{Z}_+} (1-|a_j|)^{n-\epsilon} = +\infty .$$

Proof. — We shall use as "centers" of the hyperplanes L_j the points a_j given by Berndtsson ([3], Theorem 4) which satisfy (6) (refer to [3] for the precise details of the construction).

Berndtsson shows that there are "pseudo P. Beurling functions", $F_j \in H^\infty(\mathbf{B}^n)$ satisfying (i) and:

(ii)″ $F_j(a_j) = 1$

(iii)″ $|F_j(a_k)| \leqslant 1/2$, $j \neq k$.

Since in fact

$$F_j(z) = \left(\frac{1-|a_j|^2}{1-z \cdot \bar{a}_j} \right)^{n+1}$$

we have (ii) since $F_j|_{L_j} \equiv 1$, LEMMA 5. – *With Berndtsson's choice of* a_j, *we also have:*

$$\text{(iii)}\quad |F_j(z)| \leqslant \frac{1}{2},\ z \in L_k\,,\ j \neq k\,.$$

Proposition 2 then follows in the same way as Theorem 1 (with $c_2 = 1$).

Proof of Lemma 5. – Recall that $1 - R_m \ll r_m$ are two sequences of positive numbers, and that Berndtsson's sequence is indexed a_j^m, $m \in \mathbf{Z}_+$, $1 \leqslant j \leqslant C_m$.

$$|1 - a_j^m \cdot \bar{a}_k^m| \geqslant 100(1 - R_m)\,,\ j \neq k\,,$$

and

$$|1 - a_j^m \cdot \bar{a}_k^n| \geqslant 50\max(r_m\,, r_n)\,,\ m \neq n.$$

If $z \in L_{a_k^m}$,

$$1 - z \cdot \bar{a}_k^m = 1 - |a_k^m|^2 = 1 - R_m^2\,.$$

For $j \neq k$,

$$2(|1 - z \cdot \bar{a}_j^m| + |1 - z \cdot \bar{a}_k^m|) \geqslant |1 - a_j^m \cdot \bar{a}_k^m|$$

so

$$|1 - z \cdot \bar{a}_j^m| \geqslant \frac{1}{2}(100)(1 - R_m) - (1 - R_m^2) \geqslant 20(1 - R_m^2)\,,$$

so that

$$|F_{a_j^m}(z)| \leqslant \frac{1}{20^{n+1}} \leqslant \frac{1}{2}.$$

For $F_{a_k^n}, n \neq m$, things are even easier:

$$\begin{aligned} |1 - z \cdot \bar{a}_k^n| &\geqslant \frac{1}{2}|1 - a_j^m \cdot \bar{a}_k^n| - (1 - R_m^2) \\ &\geqslant \frac{50}{2}\max(r_n\,, r_m) - (1 - R_m^2) \\ &\geqslant 10(1 - R_m^2)\,, \text{ q.e.d.} \end{aligned}$$

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