## HIROHIKO SHIMA Vanishing theorems for compact hessian manifolds

Annales de l'institut Fourier, tome 36, nº 3 (1986), p. 183-205 <http://www.numdam.org/item?id=AIF\_1986\_\_36\_3\_183\_0>

© Annales de l'institut Fourier, 1986, tous droits réservés.

L'accès aux archives de la revue « Annales de l'institut Fourier » (http://annalif.ujf-grenoble.fr/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

# $\mathcal{N}$ umdam

Article numérisé dans le cadre du programme Numérisation de documents anciens mathématiques http://www.numdam.org/

### VANISHING THEOREMS FOR COMPACT HESSIAN MANIFOLDS

#### by Hirohiko SHIMA

Let M be a flat affine manifold with a locally flat affine connection D. Among the Riemannian metrics on M there is an important class of Riemannian metrics which are compatible with the flat affine structure on M. A Riemannian metric g on M is said to be *Hessian* if g has an expression  $g = D^2 u$  where u is a local C<sup> $\infty$ </sup>-function. A flat affine manifold provided with a Hessian metric is called a *Hessian manifold*. A certain geometry of Hessian manifolds has been studied in Shima [10]-[14]. See also Cheng and Yau [2] and Yagi [15].

Hessian manifolds have in a certain sense some analogy with Kählerian manifolds. In this paper, being motivated by the theory of cohomology for Kählerian manifolds we study cohomology groups for Hessian manifolds.

Let F be a locally constant vector bundle over M. We denote by  $\Omega^{p,q}(F)$  the space of all sections of  $\begin{pmatrix} p \\ \wedge T^* \end{pmatrix} \otimes \begin{pmatrix} q \\ \wedge T^* \end{pmatrix} \otimes F$ , where T\* is the cotangent bundle over M. Since the vector bundle  $\begin{pmatrix} q \\ \wedge T^* \end{pmatrix} \otimes F$  is locally constant, we can naturally define a complex

 $\dots \xrightarrow{\delta} \Omega^{p-1,q}(\mathbf{F}) \xrightarrow{\delta} \Omega^{p,q}(\mathbf{F}) \xrightarrow{\delta} \Omega^{p+1,q}(\mathbf{F}) \xrightarrow{\delta} \dots$ 

We denote by  $H^{p,q}(F)$  the *p*-th cohomology group of the complex. Then we have the following duality theorem analogous to that of Serre [9].

THEOREM. – Let M be a compact oriented flat affine manifold of dimension n. Then we have

Key-words: Hessian manifolds - Cohomology - Vanishing theorems.

$$\mathrm{H}^{p,q}(\mathrm{F}) \stackrel{\sim}{=} \mathrm{H}^{n-p,n-q}((\mathrm{K} \otimes \mathrm{F})^*),$$

where K is the canonical line bundle over M and  $(K \otimes F)^*$  is the dual bundle of  $K \otimes F$ .

Let F be a locally constant line bundle over M. Choose an open covering  $\{U_{\lambda}\}$  of M such that the local triviality holds on each  $U_{\lambda}$ . Denote by  $\{f_{\lambda\mu}\}$  the constant transition functions with respect to  $\{U_{\lambda}\}$ . A fiber metric  $a = \{a_{\lambda}\}$  on F is a collection of positive C<sup>°</sup>-functions  $a_{\lambda}$  on  $U_{\lambda}$  such that

$$a_{\mu}=f_{\lambda\mu}^{2}a_{\lambda}.$$

Using this we can define a globally defined closed 1-form A and a symmetric bilinear form B by

$$A = -D \log a_{\lambda} ,$$
  
$$B = -D^2 \log a_{\lambda} ,$$

and we call them the first Koszul form and the second Koszul form of F with respect of the fiber metric  $a = \{a_{\lambda}\}$  respectively.

A locally constant line bundle F is said to be *positive* (resp. *negative*) if the second Koszul form is positive (resp. negative) definite with respect to a certain fiber metric. It should be remarked that if a compact connected flat affine manifold M admits a locally constant positive (resp. negative) line bundle, then by a theorem of Koszul [6] M is a hyperbolic affine manifold, that is, the universal covering of M is an open convex cone not containing any full straight line.

Kodaira-Nakano's vanishing theorem for compact Kählerian manifolds plays an essential role in the theory of compact Kählerian manifolds. In this paper we prove the following vanishing theorem for a compact Hessian manifold analogous to that of Kodaira-Nakano.

THEOREM. – Let M be a compact connected oriented Hessian manifold. Denote by K the canonical line bundle over M. Let F be a locally constant line bundle over M.

(i) If 2F + K is positive, then

$$H^{p, q}(F) = 0$$
 for  $p + q > n$ .

(ii) If 2F + K is negative, then

 $H^{p, q}(F) = 0$  for p + q < n.

As to vanishing theorem for compact hyperbolic affine manifolds we should mention the following theorem due to Koszul [7].

THEOREM. – Let M be a compact oriented hyperbolic affine manifold. Then we have

$$H^{p, q}(1) = 0$$
 for  $p, q > 0$ ,

where 1 is the trivial line bundle over M.

In § 1 and § 2 a Riemannian metric g is not assumed to be Hessian. We define in § 1 fundamental operators e(g), i(g),  $\Pi$ , \*,  $\partial$ ,  $\delta$ and  $\Box$ . In § 2 we define the Laplacian  $\Box_a$  on  $\Omega^{p,q}(F)$ , and prove the duality theorem  $H^{p,q}(F) \cong H^{n-p,n-q}((K \otimes F)^*)$  and the cohomology isomorphisms  $\mathcal{H}^{p,q}(F) \cong H^{p,q}(F) \cong H^p(P^q(F))$ . In § 3 we give the local expressions for geometric concepts on Hessian manifolds. In § 4 and § 5 the formulae of Weitzenböck type for  $\Box$  and  $\Box_a$  are obtained. In § 6 we prove a vanishing theorem analogous to that of Kodaira-Nakano. In § 7 we mention a vanishing theorem of Koszul type.

The author would like to thank Professor J.L. Koszul for his kind suggestions.

#### 1. The Laplacian $\Box$ on $\Omega^{p, q}$ .

Let M be a flat affine manifold with a locally flat affine connection D. Then there exist local coordinate systems  $\{x^1, \ldots, x^n\}$  such that  $Ddx^i = 0$ , which will be called *affine local* coordinate systems. Throughout this paper the local expressions for geometric concepts on M will be given in terms of affine local coordinate system. From now on we assume further that M is compact, connected and oriented.

Choose an arbitrary Riemannian metric g on M. Let  $\Omega^{p,q}$ be the space of all sections of  $(\Lambda T^*) \otimes (\Lambda T^*)$ . We denote the local expression of  $\phi \in \Omega^{p, q}$  by

$$\phi = \frac{1}{p \mid q \mid} \Sigma \phi_{i_1 \dots i_p \, \overline{j_1} \dots \overline{j_q}} (dx^{i_1} \wedge \dots \wedge dx^{i_p}) \otimes (dx^{\overline{j_1}} \wedge \dots \wedge dx^{\overline{j_q}}).$$

For simplicity let us fix some notation. We denote as follows :

$$I_{p} = (i_{1}, \dots, i_{p}), \quad i_{1} < i_{2} < \dots < i_{p}, \quad i \le i_{\sigma} \le n,$$
  
$$I_{n-p} = (i_{p+1}, \dots, i_{n}), \quad i_{p+1} < \dots < i_{n}, \quad 1 \le i_{\tau} \le n,$$

and  $(i_1, \ldots, i_p, i_{p+1}, \ldots, i_n)$  is a permutation of  $(1, \ldots, n)$ . Then with this notation we write

$$\phi = \sum_{\mathbf{I}_p, \, \overline{\mathbf{J}_q}} \phi_{\mathbf{I}_p \, \overline{\mathbf{J}_q}} \, dx^{\mathbf{I}_p} \otimes dx^{\overline{\mathbf{J}_q}} \,,$$

where  $dx^{I_p} = dx^{i_1} \wedge \ldots \wedge dx^{i_p}$ .

For  $\phi, \psi \in \Omega^{p, q}$  we set  $h(\phi, \psi) = \frac{1}{p! q!} \phi_{i_1 \dots i_p \overline{j_1} \dots \overline{j_q}} \psi^{i_1 \dots i_p \overline{j_1} \dots \overline{j_q}} (*) \qquad (1.1)$   $= \phi_{I_p \overline{J_q}} \psi^{I_p \overline{J_q}} . (**)$ 

DEFINITION 1.1. – The inner product of  $\phi, \psi \in \Omega^{p, q}$  is

$$(\phi,\psi)=\int_{\mathbf{M}} h(\phi,\psi) v,$$

where v is the volume element determined by g.

DEFINITION 1.2. - We define \*-operation

$$*: \Omega^{p, q} \longrightarrow \Omega^{n-p, n-q}$$

by  $(*\phi)_{I_{n-p}}\overline{J}_{n-q} = (-1)^{pq} \operatorname{sgn}(I_p I_{n-p}) \operatorname{sgn}(\overline{J}_q \overline{J}_{n-q}) G \phi^{I_p \overline{J}_q}$ , where  $\operatorname{sgn}(I_p I_{n-p})$  is the signature of the permutation  $(I_p I_{n-p})$  of  $(1, \ldots, n)$  and  $G = \operatorname{det}(g_{ij})$ .

(\*\*) 
$$\phi_{\mathbf{I}_p \overline{\mathbf{J}}_q} \psi^{\mathbf{I}_p \overline{\mathbf{J}}_q}$$
 means  $\sum_{\mathbf{I}_p, \mathbf{J}_q} \phi_{\mathbf{I}_p \overline{\mathbf{J}}_q} \psi^{\mathbf{I}_p \overline{\mathbf{J}}_q}$ .

186

<sup>(\*)</sup> Throughout this paper we use Einstein's convention on indices.

DEFINITION 1.3. - Let  $\phi = \Sigma \phi_{\mathbf{l}_p \, \overline{\mathbf{j}}_q} \, dx^{\mathbf{l}_p} \otimes dx^{\overline{\mathbf{l}}_q}$  and  $\psi = \Sigma \psi_{\mathbf{K}_r \, \overline{\mathbf{L}}_g} \, dx^{\mathbf{K}_r} \otimes dx^{\overline{\mathbf{L}}_g}.$ 

We set 
$$\phi \wedge \psi = \Sigma \phi_{\mathbf{I}_p \, \overline{\mathbf{J}}_q} \psi_{\mathbf{K}_r \, \overline{\mathbf{L}}_s} (dx^{\mathbf{I}_p} \wedge dx^{\mathbf{K}_r}) \otimes (dx^{\overline{\mathbf{J}}_q} \wedge dx^{\overline{\mathbf{L}}_s}).$$

A straightforward calculation shows

**PROPOSITION** 1.1. - Let  $\phi, \psi \in \Omega^{p, q}$ . Then

(i) 
$$**\phi = (-1)^{n+p+q} \phi$$
,

(ii)  $\phi \wedge * \psi = (-1)^{pq} h(\phi, \psi) v \otimes v$ .

DEFINITION 1.4. – Considering the Riemannian metric g as an element in  $\Omega^{1,1}$  we define

$$e(g): \Omega^{p, q} \longrightarrow \Omega^{p+1, q+1},$$
  
$$i(g): \Omega^{p, q} \longrightarrow \Omega^{p-1, q-1},$$

by  $e(g) \phi = g \wedge \phi$  for  $\phi \in \Omega^{p,q}$  and  $i(g) = (-1)^{n+p+q+1} * e(g) *$ .

Then i(g) is the adjoint operator of e(g) with respect to the inner product given in Definition 1.1:

 $(i(g) \phi, \psi) = (\phi, e(g) \psi)$  for  $\phi \in \Omega^{p, q}, \psi \in \Omega^{p-1, q-1}$ .

DEFINITION 1.5. - We set

$$\Pi = \sum_{p,q} (n-p-q) \pi_{p,q},$$

where  $\pi_{p,q}$  is the projection from  $\sum_{r,s} \Omega^{r,s}$  onto  $\Omega^{p,q}$ .

**PROPOSITION** 1.2. – We have

$$[\Pi, e(g)] = -2e(g), \quad [\Pi, i(g)] = 2i(g), \quad [i(g), e(g)] = \Pi.$$

The proof is carried out by a direct calculation and so it is omitted.

**DEFINITION 1.6.** – Define

$$\partial:\Omega^{p,q}\longrightarrow\Omega^{p+1,q}$$

by  $\partial = \sum_{k} (e(dx^{k}) \otimes id) D_{k}$ , where  $e(dx^{k})$  is a linear map from  $\bigwedge^{p} T^{*}$  to  $\bigwedge^{p+1} T^{*}$  given by  $e(dx^{k}) \omega = dx^{k} \wedge \omega$ , id is the identity map on  $\bigwedge^{p} T^{*}$  and  $D_{k}$  is the covariant derivation with respect to  $\partial/\partial x^{k}$  for the locally flat affine connection D.

Then we have

$$\partial \partial = 0. \tag{1.2}$$

DEFINITION 1.7. – Define

 $\delta: \Omega^{p,q} \longrightarrow \Omega^{p-1,q}$   $by \quad \delta = (-1)^{n+1} \sqrt{G} * \partial \left(\frac{1}{\sqrt{G}}*\right).$ 

**PROPOSITION** 1.3.  $-\delta$  is the adjoint operator of  $\partial$  with respect to the inner product given in Definition 1.1;

 $(\partial \phi, \psi) = (\phi, \delta \psi)$  for  $\phi \in \Omega^{p, q}$ ,  $\psi \in \Omega^{p+1, q}$ .

In Proposition 2.1 we prove the above fact in more general situation and so we omit the proof.

DEFINITION 1.8. - We define

 $\Box \colon \Omega^{p, q} \longrightarrow \Omega^{p, q}$ 

by  $\Box = \partial \delta + \delta \partial$ , and call it the Laplacian.  $\phi \in \Omega^{p,q}$  is said to be  $\Box$ -harmonic if  $\Box \phi = 0$ .

### 2. The Laplacian $\Box_{q}$ on $\Omega^{p,q}(\mathbf{F})$ .

Let F be a locally constant vector bundle over M. Choose an open covering  $\{U_{\lambda}\}$  of M such that the local triviality holds

188

on each  $U_{\lambda}$ . Let  $\{\xi_{\lambda}^1, \ldots, \xi_{\lambda}^m\}$  be fiber coordinate systems such that the transition functions  $\{f_{\lambda\mu}\}$  defined by

$$\xi^i_{\lambda} = \sum_j f_{\lambda \mu \ j} \ \xi^j_{\mu}$$

are constants. A fiber metric  $a = \{a_{\lambda}\}$  on F is a collection of  $m \times m$  positive definite symmetric matrices  $a = (a_{\lambda ij})$  such that each  $a_{\lambda ij}$  is a C<sup>°</sup>-function on U<sub> $\lambda$ </sub> and

$$a_{\lambda} = {}^{r} f_{\mu\lambda} a_{\mu} f_{\mu\lambda}$$

holds.

Let  $\Omega^{p, q}(F)$  denote the space of all sections of  $(\bigwedge^{p} T^{*}) \otimes (\bigwedge^{q} T^{*}) \otimes F$ .

Using fiber coordinate systems  $\{\xi_{\lambda}^{i}\}$  we express an element  $\phi \in \Omega^{p, q}(F)$  as  $\phi = \{\phi_{\lambda}^{i}\}$ .

DEFINITION 2.1. - Define

$$\partial: \Omega^{p, q}(F) \longrightarrow \Omega^{p+1, q}(F)$$

by  $\partial \{\phi^i\} = \{\partial \phi^i\} . (*)$ 

We have then

$$\partial \partial = 0. \tag{2.1}$$

DEFINITION 2.2. – The inner product of  $\phi$ ,  $\psi \in \Omega^{p, q}(F)$  is

$$(\phi, \psi) = \int_{\mathbf{M}}^{*} \Sigma a_{ij} h(\phi^{i}, \psi^{j}) v.$$

**DEFINITION 2.3** – Define

$$\delta_{q}: \Omega^{p, q}(\mathbf{F}) \longrightarrow \Omega^{p-1, q}(\mathbf{F})$$

by  $\delta_a \{\phi^i\} = \left\{ (-1)^{n+1} \sum_{j,k} \sqrt{G} a^{ij} * \partial \left( \frac{a_{jk}}{\sqrt{G}} * \phi^k \right) \right\}$ , where  $a^{ij}$  is the (i,j)-component of  $(a_{ij})^{-1}$ .

<sup>(\*)</sup> For brevity the subscripts  $\lambda, \mu, \ldots$  are droped where no confusion will arise.

**PROPOSITION** 2.1.  $-\delta_a$  is the adjoint operator of  $\partial$  with respect to the inner product given in Definition 2.2;

 $(\partial \phi, \psi) = (\phi, \delta_a \psi) \quad for \quad \phi \in \Omega^{p-1, q}(\mathbf{F}), \ \psi \in \Omega^{p, q}(\mathbf{F}).$ 

*Proof.* – Since  $\sum_{i,j} a_{ij} \phi^i \wedge * \psi^j$  is globally defined on M, there exists (n-1)-form  $\omega$  on M such that  $\omega \otimes v = \sum a_{ij} \phi^i \wedge * \psi^j$ . Then

$$\partial(\omega \otimes v) = (\alpha \wedge \omega + d\omega) \otimes v,$$

where  $\alpha = d \log \sqrt{G}$ , and

$$\begin{split} \partial \left( \sum a_{ij} \phi^i \wedge * \psi^j \right) \\ &= (-1)^{pq} \sum a_{ij} h(\partial \phi^i, \psi^j) v \otimes v + (-1)^{n-q} \sum \phi^i \wedge * * \partial (a_{ij} * \psi^j). \end{split}$$

Since

$$\delta_a \psi^i = -(-1)^{n+1} * (\alpha \wedge * \psi^i) + (-1)^{n+1} \Sigma a^{ij} * \partial(a_{jk} * \psi^k),$$

we have

$$\begin{aligned} (\alpha \wedge \omega + d\omega) \otimes v \\ &= (-1)^{pq} \sum a_{ij} h(\partial \phi^i, \psi^j) v \otimes v + (-1)^{n-q} \sum a_{ij} \phi^i \wedge ** (\alpha \wedge *\psi^j) \\ &+ (-1)^{q+1} \sum a_{ij} \phi^i \wedge * \delta_a \psi^j \\ &= (-1)^{pq} \sum a_{ij} h(\partial \phi^i, \psi^j) v \otimes v + (\alpha \wedge \omega) \otimes v \\ &+ (-1)^{pq-1} \sum a_{ij} h(\phi^i, \delta_a \psi^j) v \otimes v , \end{aligned}$$

and so

$$d\omega = (-1)^{pq} \left( \sum a_{ij} h(\partial \phi^i, \psi^j) - \sum a_{ij} h(\phi^i, \delta_a \psi^j) \right) v.$$

Therefore

$$0 = \int_{\mathbf{M}}^{\cdot} d\omega = (-1)^{pq} \left( (\partial \phi, \psi) - (\phi, \delta_a \psi) \right).$$
Q.E.D.

DEFINITION 2.4. - We define

$$\Box_{a}: \Omega^{p, q}(\mathbf{F}) \longrightarrow \Omega^{p, q}(\mathbf{F})$$

by  $\Box_a = \partial \delta_a + \delta_a \partial$ , and call it the Laplacian.  $\phi \in \Omega^{p,q}(F)$  is said to be  $\Box_a$ -harmonic if  $\Box_a \phi = 0$ .

DEFINITION 2.5. - We set

$$\mathcal{H}^{p,q}(\mathbf{F}) = \{ \phi \in \Omega^{p,q}(\mathbf{F}) \mid \Box_{p} \phi = 0 \}.$$

THEOREM 2.2. - We have the following duality:

$$\mathcal{H}^{p,q}(\mathbf{F}) \cong \mathcal{H}^{n-p,n-q}((\mathbf{K} \otimes \mathbf{F})^*),$$

where K is the canonical line bundle over M and  $(K \otimes F)^*$  is the dual bundle of  $K \otimes F$ .

*Proof.* - For  $\psi = \{\psi^j\} \in \Omega^{p, q}(F)$  we set

$$\psi_i^* = \sum_j \frac{a_{ij}}{\sqrt{G}} * \psi^j. \tag{2.2}$$

Then we have  $\psi^* = \{\psi_i^*\} \in \Omega^{n-p, n-q} ((K \otimes F)^*)$ . It follows from Proposition 1.1 (i)

$$\psi^{j} = (-1)^{n+p+q} \sum_{i} \sqrt{G} \, a^{ji} * \psi^{*}_{i} \,. \tag{2.3}$$

Thus the map  $\psi \longrightarrow \psi^*$  is a linear isomorphism from  $\Omega^{p, q}(F)$  onto  $\Omega^{n-p, n-q}((K \otimes F)^*)$ .

Let  $\phi \in \Omega^{p, q}(F)$  and  $\psi^* \in \Omega^{n-p, n-q}((K \otimes F)^*)$ . Then  $\sum_i \sqrt{G} \phi^i \wedge \psi_i^*$  is globally defined on M. Hence there exists a  $C^{\infty}$ -function  $k(\phi, \psi^*)$  on M such that

$$\sum_{i} \sqrt{\mathbf{G}} \phi^{i} \Lambda \psi_{i}^{*} = k(\phi, \psi^{*}) v \otimes v.$$

We set

$$\langle \phi, \psi^* \rangle = (-1)^{pq} \int_{\mathbf{M}} k(\phi, \psi^*) v.$$

Since

$$k(\phi,\psi^*)\,v\,\otimes\,v\,=\,\sum_{i,j}\,a_{ij}\,\phi^i\,\Lambda\,*\,\psi^j=(-\,1)^{pq}\,\sum_{i,j}a_{ij}\,h(\phi^i,\psi^j)\,v\,\otimes\,v\,,$$

we have

$$\langle \phi, \psi^* \rangle = (\phi, \psi) \quad \text{for} \quad \phi, \psi \in \Omega^{p, q}(\mathbf{F}).$$

Define the inner product of  $\psi^*$ ,  $\phi^* \in \Omega^{n-p, n-q} ((K \otimes F)^*)$  by

$$(\psi^*,\phi^*) = \int_{\mathsf{M}} \Sigma \operatorname{G} a^{ij} h(\psi_i^*,\phi_j^*) v.$$

Since

$$\begin{split} \sum_{i,j} \mathcal{G} \, a^{ij} h(\psi_i^*, \phi_j^*) \, v \otimes v &= \sum_{i,j} a_{ij} h(* \, \psi^i, * \, \phi^j) \, v \otimes v \\ &= (-1)^{pq} \sum_{i,j} a_{ij} \phi^j \wedge * \psi^i = \sum_{i,j} a_{ij} h(\phi^j, \psi^i) \, v \otimes v \,, \end{split}$$

we obtain

$$(\psi^*, \phi^*) = (\phi, \psi) \text{ for } \phi, \psi \in \Omega^{p, q}(\mathbf{F}).$$

Let  $\phi \in \Omega^{p-1, q}(F)$  and  $\psi^* \in \Omega^{n-p, n-q}((K \otimes F)^*)$ . Then  $\sum_i \sqrt{G} \phi^i \wedge \psi_i^*$  is globally defined on M and hence there exists (n-1)-form  $\omega$  on M such that

$$\sum_i \sqrt{\mathbf{G}} \phi^i \wedge \psi_i^* = \omega \otimes v.$$

Since

$$\partial \left(\sum_{i} \sqrt{G} \phi^{i} \wedge \psi_{i}^{*}\right)$$

$$= \sum_{i} \left\{ \alpha \wedge \sqrt{G} \phi^{i} \wedge \psi_{i}^{*} + \sqrt{G} \partial \phi^{i} \wedge \psi_{i}^{*} + (-1)^{p-1} \sqrt{G} \phi^{i} \wedge \partial \phi_{i}^{*} \right\}$$

$$= (\alpha \wedge \omega) \otimes v + \sum_{i} \left\{ k(\partial \phi^{i}, \psi_{i}^{*}) + (-1)^{p-1} k(\phi^{i}, \partial \psi_{i}^{*}) \right\} v \otimes v,$$

and

$$\partial(\omega \otimes v) = (\alpha \wedge \omega + d\omega) \otimes v,$$

we obtain

$$d\omega = \sum_{i} \{k(\partial \phi^{i}, \psi_{i}^{*}) + (-1)^{p-1} k(\phi^{i}, \partial \psi_{i}^{*})\} v.$$

Therefore

$$0 = \int_{M} d\omega$$
  
=  $(-1)^{pq} \langle \partial \phi, \psi^* \rangle + (-1)^{p-1+(p-1)q} \langle \phi, \partial \psi^* \rangle.$ 

This implies

$$\langle \partial \phi, \psi^* \rangle = (-1)^{p+q} \langle \phi, \partial \psi^* \rangle.$$

Using these facts we obtain

$$\begin{split} (\phi^*, \partial\psi^*) &= \langle \phi, \partial\psi^* \rangle = (-1)^{p+q} \langle \partial\phi, \psi^* \rangle = (-1)^{p+q} (\partial\phi, \psi) \\ &= (-1)^{p+q} (\phi, \delta_a \psi) = (-1)^{p+q} (\phi^*, (\delta_a \psi)^*), \end{split}$$

hence

$$\partial \psi^* = (-1)^{p+q} (\delta_a \psi)^* \quad \text{for} \quad \psi \in \Omega^{p,q}(\mathbf{F}).$$
 (2.4)

By the same way we have

$$\begin{aligned} (\psi^*, \delta_a \phi^*) &= (\partial \psi^*, \phi^*) = \langle \phi, \partial \psi^* \rangle = (-1)^{p+q} \langle \partial \phi, \psi^* \rangle \\ &= (-1)^{p+q} (\partial \phi, \psi) = (-1)^{p+q} ((\partial \phi)^*, \psi^*), \end{aligned}$$

hence

$$\delta_a \phi^* = (-1)^{p+q} (\partial \phi)^*.$$

Thus

$$\delta_a \psi^* = (-1)^{p+q+1} (\partial \psi)^* \quad \text{for} \quad \psi \in \Omega^{p,q}(\mathbf{F}).$$
 (2.5)

(2.4) and (2.5) imply that  $\psi^*$  is harmonic if and only if  $\psi$  is harmonic.

DEFINITION 2.6. - We set  $H^{p, q}(F) = \{ \phi \in \Omega^{p, q}(F) \mid \partial \phi = 0 \} / \{ \partial \psi \mid \psi \in \Omega^{p-1, q}(F) \}.$ 

A q-form  $\omega$  on M is said to be *D*-parallel if  $D\omega = 0$ . Let us denote by  $P^{q}(F)$  the sheaf over M of germs of F-valued D-parallel q-forms.

DEFINITION 2.7. – We denote by  $H^{p}(P^{q}(F))$  the p-th cohomology group of M with coefficients on  $P^{q}(F)$ .

THEOREM 2.3. – We have the following isomorphisms:

$$\mathcal{H}^{p,q}(\mathbf{F}) \cong \mathbf{H}^{p,q}(\mathbf{F}) \cong \mathbf{H}^{p}(\mathbf{P}^{q}(\mathbf{F})).$$

H. SHIMA

Proof. – By the theory of harmonic integral we have

$$\mathcal{\mathcal{H}}^{p,\,q}(\mathbf{F})\cong \mathbf{H}^{p,\,q}(\mathbf{F})\,.$$

Let  $A^{p,q}(F)$  denote the sheaf over M of germs of sections of  $(\stackrel{p}{\wedge}T^*) \otimes (\stackrel{q}{\wedge}T^*) \otimes F$ . Then

 $0 \longrightarrow \mathbf{P}^{q}(\mathbf{F}) \longrightarrow \mathbf{A}^{0,q}(\mathbf{F}) \xrightarrow{\partial} \mathbf{A}^{1,q}(\mathbf{F}) \xrightarrow{\partial} \mathbf{A}^{2,q}(\mathbf{F}) \xrightarrow{\partial} \dots$ is a fine resolution of  $\mathbf{P}^{q}(\mathbf{F})$ . Thus we have  $\mathbf{H}^{p,q}(\mathbf{F}) \cong \mathbf{H}^{p}(\mathbf{P}^{q}(\mathbf{F}))$ .

Q.E.D.

#### 3. Hessian metrics on affine local coordinate systems.

Let M be a Hessian manifold with a locally flat affine connection D and a Hessian metric g. We denote by  $\nabla$  the Riemannian connection for g. In this section we shall express various geometric concepts on the Hessian manifold M in terms of affine local coordinate systems. Let us denote by  $D_k$  and  $\nabla_k$  the covariant derivations with respect to  $\partial/\partial x^k$  for D and  $\nabla$ respectively. Since the Christoffel symbol  $\Gamma^i_{ijk}$  for g is the difference between the components of affine connections  $\nabla$  and D, we may consider that  $\Gamma^i_{ik}$  is a tensor field. We have then

$$\Gamma_{jk}^{i} = \frac{1}{2} g^{is} D_{k} g_{gj}, \qquad (3.1)$$
$$D_{k} g_{ij} = 2\Gamma_{ijk}, \qquad D_{k} g^{ij} = -2\Gamma_{k}^{ij},$$
$$\Gamma_{ijk} = \Gamma_{jik} = \Gamma_{ikj}.$$

DEFINITION 3.1. – We define a 1-form  $\alpha$  and a symmetric bilinear form  $\beta$  by

$$\alpha = D \log \sqrt{G},$$
  
$$\beta = D^2 \log \sqrt{G},$$

where  $G = det(g_{ij})$ , and call them the first Koszul form and the second Koszul form of M respectively.

Then we have

$$\alpha_{i} = \Gamma^{r}_{ir}, \qquad (3.2)$$
  
$$\beta_{ij} = D_{j}\Gamma^{r}_{ir}.$$

DEFINITION 3.2. – Let  $\gamma_k$  be the derivation of the algebra of tensor fields defined by

$$\gamma_k = \nabla_k - \mathbf{D}_k.$$

Let  $T_q^p$  be the space of tensor fields of type (p,q) defined on M.

DEFINITION 3.3. – We define certain covariant derivations  $\nabla'_k$ ,  $\overline{\nabla}'_k$  on  $T^p_q \otimes T^r_s$  by

$$\nabla'_{\mathbf{k}} = (2\gamma_{\mathbf{k}}) \otimes \mathrm{id} + \mathrm{D}_{\mathbf{k}},$$

$$\overline{\nabla}'_{\overline{k}} = \mathrm{id} \otimes (2\gamma_{\overline{k}}) + \mathrm{D}_{\overline{k}},$$

where id are the identity transformations.

Notice that

$$\nabla_{k} = \frac{1}{2} (\nabla'_{k} + \overline{\nabla}'_{\overline{k}}), \text{ where } k = \overline{k}.$$

LEMMA 3.1. – For the Hessian metric g we have

$$\nabla'_{k} g_{i\bar{i}} = 0, \qquad \overline{\nabla}'_{\bar{k}} g_{i\bar{i}} = 0,$$
$$\nabla'_{k} g^{i\bar{i}} = 0, \qquad \overline{\nabla}'_{\bar{k}} g^{i\bar{i}} = 0.$$

*Proof.* - By (3.1) we obtain

$$\nabla'_{k} g_{i\bar{j}} = \mathcal{D}_{k} g_{i\bar{j}} - 2\Gamma^{m}_{ki} g_{m\bar{j}} = 2\Gamma_{i\bar{j}k} - 2\Gamma_{\bar{j}ki} = 0.$$

Similarly we can prove the other equalities.

Q.E.D.

DEFINITION 3.4. – Considering  $\gamma_i$  as tensor fields of type (1.1) we define tensor fields  $\gamma$  and S by

$$\gamma = \sum_{i} \gamma_{i} \otimes dx^{i},$$
$$S = D\gamma.$$

The component of S is given by

$$\mathbf{S}^{i}_{jkl} = \mathbf{D}_{k} \, \Gamma^{i}_{jl} \, .$$

LEMMA 3.2.  $-S_{ijkl} = S_{kjl} = S_{klij} = S_{ilkj}$ . Proof. - Let  $g_{ij} = D_i D_j u$ . By (3.1) we have  $S_{ijkl} = g_{ip} D_k \Gamma_{jl}^p = g_{ip} D_k (g^{pq} \Gamma_{qjl}) = g_{ip} (D_k g^{pq}) \Gamma_{qjl} + g_{ip} g^{pq} D_k \Gamma_{qjl}$   $= -2\Gamma_i^q \Gamma_k \Gamma_{qjl} + D_k \Gamma_{ijl} = -2g^{qr} \Gamma_{irk} \Gamma_{qjl} + D_k \Gamma_{ijl}$  $= \frac{1}{2} D_i D_j D_k D_l u - \frac{1}{2} g^{qr} (D_r D_l D_k u) (D_q D_j D_l u)$ .

This proves the Lemma.

Q.E.D.

LEMMA 3.3. 
$$-\beta_{ij} = S_{rij}^r = S_{ij}^r$$
.  
*Proof.*  $-\beta_{ij} = D_j \alpha_i = D_i \alpha_j = D_i \Gamma_{rj}^r = S_{rij}^r$ . By Lemma 3.2 we have  $S_{rij}^r = g^{rp} S_{prij} = g^{rp} S_{ijpr} = S_{ij}^r$ .  
Q.E.D.

#### 4. The local expression for $\Box$ .

From now on we always assume that M is a compact connected oriented Hessian manifold.

PROPOSITION 4.1. - Let 
$$\phi \in \Omega^{p,q}$$
. Then we have  
 $(\partial \phi)_{i_1 \dots i_{p+1}} \overline{J}_q = \sum_{\sigma} (-1)^{\sigma-1} \nabla'_{i_{\sigma}} \phi_{i_1 \dots \hat{i}_{\sigma} \dots i_{p+1}} \overline{J}_q$ ,  
where  $\hat{i}_{\sigma}$  means "omit  $i_{\sigma}$ ".

8 0

Proof. - By Definition 1.6 we have

$$(\partial \phi)_{\mathbf{I}_{p+1}\overline{\mathbf{J}}_q} = \sum_{\sigma=1}^{p+1} (-1)^{\sigma-1} \mathcal{D}_{i_{\sigma}} \phi_{i_1...\hat{i}_{\sigma}...i_{p+1}\overline{\mathbf{J}}_q}.$$
 (4.1)

Using this and (3.1) we obtain the proposition.

Q.E.D.

196

**PROPOSITION** 4.2. - Let  $\phi \in \Omega^{p, q}$ . Then we have

$$(\delta\phi)_{\mathbf{I}_{p-1}\overline{\mathbf{J}}_{q}} = -g^{s\overline{r}} \,\overline{\nabla}'_{\overline{r}} \phi_{s\overline{\mathbf{J}}_{p-1}\overline{\mathbf{J}}_{q}} + \alpha^{s} \phi_{s\overline{\mathbf{I}}_{p-1}\overline{\mathbf{J}}_{q}}.$$

**Proof.** - Let  $\psi \in \Omega^{p-1,q}$ . By (4.1) and Green's theorem we have

$$(\phi, \partial \psi) = -\int_{\mathbf{M}} D_r(\phi^{r\mathbf{I}_{p-1}\bar{\mathbf{J}}_q}\sqrt{\mathbf{G}}) \frac{1}{\sqrt{\mathbf{G}}} \psi_{\mathbf{I}_{p-1}\bar{\mathbf{J}}_q} v.$$

Thus we obtain

$$(\delta\phi)^{\mathbf{I}_{p-1}\overline{\mathbf{J}}_q} = -\mathbf{D}_r \ \phi^{r\mathbf{I}_{p-1}\overline{\mathbf{J}}_q} - \alpha_r \ \phi^{r\mathbf{I}_{p-1}\overline{\mathbf{J}}_q}$$
$$= -\nabla_r \ \phi^{r\mathbf{I}_{p-1}\overline{\mathbf{J}}_q} + \alpha_r \ \phi^{r\mathbf{I}_{p-1}\overline{\mathbf{J}}_q}.$$

This completes the proof.

Q.E.D.

THEOREM 4.1. - Let 
$$\phi \in \Omega^{p, q}$$
. Then we have  
 $(\Box \phi)_{\mathbf{I}_p \, \overline{\mathbf{J}}_q} = -g^{s\overline{r}} \, \overline{\nabla}'_{\overline{r}} \, \nabla'_{s} \, \phi_{\mathbf{I}_p \, \overline{\mathbf{J}}_q} + \alpha^{s} \, \nabla'_{s} \, \phi_{\mathbf{I}_p \, \overline{\mathbf{J}}_q} - \sum_{\sigma} \beta^{s}_{i_{\sigma}} \phi_{i_{1} \dots (s)_{\sigma} \dots i_{p} \, \overline{\mathbf{J}}_q} + 2 \sum_{\sigma, \tau} S^{\overline{i}s}_{i_{\sigma} \, \overline{j}_{\tau}} \, \phi_{i_{1} \dots (s)_{\sigma} \dots i_{p} \, \overline{j}_{1} \dots (\overline{t})_{\tau} \dots \overline{j}_q},$ 

where  $(s)_{\sigma}$  means "substitute s for  $\sigma$ -th place".

*Proof.* – Using Proposition 4.1, Proposition 4.2 and  $\nabla'_i \alpha^j = \beta^i_i$  we obtain

$$(\partial \delta \phi)_{\mathbf{I}_{p} \overline{\mathbf{J}_{q}}} = -g^{s\overline{r}} \sum_{\sigma} \nabla'_{i_{\sigma}} \overline{\nabla}'_{\overline{r}} \phi_{i_{1} \dots (s)_{\sigma} \dots i_{p} \overline{\mathbf{J}}_{q}} + \sum_{\sigma} \beta^{s}_{i_{\sigma}} \phi_{i_{1} \dots (s)_{\sigma} \dots i_{p} \overline{\mathbf{J}}_{q}} + \sum_{\sigma} \alpha^{s} \nabla'_{i_{\sigma}} \phi_{i_{1} \dots (s)_{\sigma} \dots i_{p} \overline{\mathbf{J}}_{q}},$$

 $(\delta \partial \phi)_{\mathbf{I}_{p}\overline{\mathbf{J}}_{q}} = -g^{s\overline{r}} \overline{\nabla}'_{\overline{r}} (\nabla'_{s} \phi_{\mathbf{I}_{p}\overline{\mathbf{J}}_{q}} - \sum_{\sigma} \nabla'_{i_{\sigma}} \phi_{i_{1} \dots (s)_{\sigma} \dots i_{p}\overline{\mathbf{J}}_{q}})$ 

$$+ \alpha^{s} (\nabla'_{s} \phi_{\mathbf{I}_{p} \overline{\mathbf{J}}_{q}} - \sum_{\sigma} \nabla'_{i_{\sigma}} \phi_{i_{1} \cdots (s)_{\sigma} \cdots i_{p} \overline{\mathbf{J}}_{q}}) ,$$

and so

$$(\Box \phi)_{\mathbf{I}_{p} \overline{\mathbf{J}}_{q}} = -g^{s\overline{r}} \overline{\nabla}'_{\overline{r}} \nabla'_{s} \phi_{\mathbf{I}_{p} \overline{\mathbf{J}}_{q}} + \alpha^{s} \nabla'_{s} \phi_{\mathbf{I}_{p} \overline{\mathbf{J}}_{q}} -g^{s\overline{r}} \sum_{\sigma} [\nabla'_{i\sigma}, \overline{\nabla}'_{\overline{r}}] \phi_{i_{1} \dots (s)_{\sigma} \dots i_{p} \overline{\mathbf{J}}_{q}} + \sum_{\sigma} \beta^{s}_{i_{\sigma}} \phi_{i_{1} \dots (s)_{\sigma} \dots i_{p} \overline{\mathbf{J}}_{q}}.$$

Let us calculate the third term on the right-hand of the above formula. Since  $[\nabla'_i, \overline{\nabla}'_i]$  is a derivation of the algebra of tensor fields which maps every function to 0 and since

$$\begin{split} [\nabla_i', \overline{\nabla}_j'] \, \xi_k &= 2 \mathbf{S}_{i\bar{j}k}^p \, \xi_p \, , \\ [\nabla_i', \overline{\nabla}_j'] \, \xi_{\bar{k}} &= -2 \mathbf{S}_{\bar{j}\bar{k}\bar{k}}^p \, \xi_{\bar{p}} \, , \end{split}$$

we have

$$\begin{bmatrix} \nabla'_{i_{\sigma}}, \overline{\nabla'_{r}} \end{bmatrix} \phi_{i_{1}...(s)_{\sigma}...i_{p}} \overline{\mathbf{J}}_{q} = \sum_{\tau} 2\mathbf{S}^{m}_{i_{\sigma}\overline{r}i_{\tau}} \phi_{i_{1}...(s)_{\sigma}...(m)_{\tau}...i_{p}} \overline{\mathbf{J}}_{q} + 2\mathbf{S}^{m}_{i_{\sigma}\overline{r}s} \phi_{i_{1}...(m)_{\sigma}...i_{p}} \overline{\mathbf{J}}_{q} - \sum_{\tau} 2\mathbf{S}^{\overline{m}}_{\overline{r}i_{\sigma}\overline{j}_{\tau}} \phi_{i_{1}...(s)_{\sigma}...i_{p}} \overline{j}_{1}...(\overline{m})_{\tau}...\overline{j}_{q} \cdot \mathbf{I}_{q}$$

Thus, by Lemma 3.2 and 3.3 we obtain

$$g^{s\bar{r}} \sum_{\sigma} \left[ \nabla'_{i_{\sigma}}, \overline{\nabla}'_{\bar{r}} \right] \phi_{i_{1}...(s)_{\sigma}...i_{p}\overline{J}_{q}} = 2 \sum_{\sigma} \beta^{m}_{i_{\sigma}} \phi_{i_{1}...(m)_{\sigma}...i_{p}\overline{J}_{q}} - 2 \sum_{\sigma,\tau} S^{\overline{m}s}_{i_{\sigma}\overline{J}_{\tau}} \phi_{i_{1}...(s)_{\sigma}...i_{p}\overline{J}_{1}...(\overline{m})_{\tau}...\overline{J}_{q}}$$

This completes the proof.

Q.E.D.

Example. - For the Hessian metric g we have

$$(\Box g)_{i\bar{j}} = -\beta_{i\bar{j}}.$$

198

Thus the Hessian metric g is  $\Box$ -harmonic if and only if the second Koszul form  $\beta = 0$ . Therefore, by [12] the following conditions are equivalent:

(i) g is  $\Box$ -harmonic.

(ii) The first Koszul form  $\alpha = 0$ .

(iii) The second Koszul form  $\beta = 0$ .

(iv) g is locally flat.

#### 5. The local expression for $\Box_a$ .

Let F be a locally constant line bundle over a compact connected oriented Hessian manifold M, and let a be a fiber metric on F.

**PROPOSITION** 5.1. - We have

$$\delta_a = \delta + i(\mathbf{A}),$$

where  $A = -D \log a$  and  $(i(A) \phi)_{I_{p-1}\overline{J}_q} = A^r \phi_{rI_{p-1}\overline{J}_q}$  for  $\phi \in \Omega^{p,q}(F)$ .

Proof. - By Definition 1.2, 1.7 and 2.3 we have

$$\delta_a = (-1)^{n+1} \frac{\sqrt{G}}{a} * \partial \left(\frac{a}{\sqrt{G}}*\right)$$
$$= (-1)^n * e(A) * (-1)^{n+1} \sqrt{G} * \partial \left(\frac{1}{\sqrt{G}}*\right)$$
$$= i(A) + \delta,$$

where

for

$$(e(\mathbf{A}) \phi)_{i_1 \dots i_{p+1} \overline{J}_q} = \sum_{\sigma} (-1)^{\sigma-1} \mathbf{A}_{i_{\sigma}} \phi_{i_1 \dots \hat{i}_{\sigma} \dots i_{p+1} \overline{J}_q}$$
$$\phi \in \Omega^{p, q} (\mathbf{F}).$$

Q.E.D.

DEFINITION 5.1. – For  $\phi \in \Omega^{p, q}(F)$  we set

$$\overline{\nabla}_{\overline{r}}^{\prime(a)} \phi = \frac{1}{a} \overline{\nabla}_{\overline{r}}^{\prime}(a\phi).$$

THEOREM 5.1. - Let  $\phi \in \Omega^{p, q}$  (F). Then we have  $(\Box_a \phi)_{\mathbf{1}_p \overline{\mathbf{j}}_q} = -g^{s\overline{r}} \overline{\nabla}_{\overline{r}}^{r(a)} \nabla'_s \phi_{\mathbf{1}_p \overline{\mathbf{j}}_q} + \alpha^s \nabla'_s \phi_{\mathbf{1}_p \overline{\mathbf{j}}_q} + \sum_{\sigma} (-\beta^s_{i_{\sigma}} + \mathbf{B}^s_{i_{\sigma}} \phi_{i_1}...(s)_{\sigma}...i_p \overline{\mathbf{j}}_q + 2\sum_{\sigma, \tau} \mathbf{S}^{\overline{rs}}_{i_{\sigma} \overline{l}_{\tau}} \phi_{i_1}...(s)_{\sigma}...i_p \overline{l}_1...(\overline{r})_{\tau}...\overline{l}_q.$ 

**Proof.** - By Proposition 5.1 we have

$$\Box_{a} = \Box + i(\mathbf{A}) \,\partial + \partial i(\mathbf{A}).$$

A straightforward calculation shows

$$(i(\mathbf{A}) \ \partial \phi)_{\mathbf{I}_{p} \overline{\mathbf{J}}_{q}} + (\partial \ i(\mathbf{A}) \ \phi)_{\mathbf{I}_{p} \overline{\mathbf{J}}_{q}}$$
$$= g^{s\overline{r}} \ \mathbf{A}_{\overline{r}} \ \nabla_{s}' \ \phi_{\mathbf{I}_{p} \overline{\mathbf{J}}_{q}} + \sum_{\sigma=1}^{p} \mathbf{B}_{s_{\sigma}}' \phi_{i_{1} \dots (r)_{\sigma} \dots i_{p} \overline{\mathbf{J}}_{q}}.$$

Thus our assertion follows from the above facts and Theorem 4.1.

Q.E.D.

#### 6. A vanishing theorem of Kodaira-Nakano type.

Let  $\theta$  be a symmetric covariant tensor field of degree 2. Considering  $\theta$  as an element in  $\Omega^{1,1}$  we define

$$e(\theta): \Omega^{p, q} \longrightarrow \Omega^{p+1, q+1},$$
$$i(\theta): \Omega^{p, q} \longrightarrow \Omega^{p-1, q-1},$$

by  $e(\theta) \phi = \theta \wedge \phi$  for  $\phi \in \Omega^{p, q}$  and  $i(\theta) = (-1)^{n+p+q+1} * e(\theta) * .$ 

Then  $i(\theta)$  is the adjoint operator of  $e(\theta)$  with respect to the inner product in Definition 1.1 and 2.2.

In this section we always assume that F is a locally constant line bundle over M.

**PROPOSITION 6.1.** – We have

(i) 
$$[\Box_n, e(g)] = e(B + \beta),$$

(ii)  $[\Box_a, i(g)] = -i(\mathbf{B} + \beta).$ 

The proof follows from a straightforward calculation and so it is omitted.

**PROPOSITION** 6.2. – Suppose  $\Box_{\alpha}\phi = 0$ . Then we have

(i)  $(e(B + \beta) i(g) \phi, \phi) \le 0$ .

- (ii)  $(i(g) e (\mathbf{B} + \beta) \phi, \phi) \ge 0$ .
- (iii)  $([i(g), e(B + \beta)] \phi, \phi) \ge 0$ .

*Proof.* – By Proposition 6.1 (i) we have  $\Box_a e(g) \phi = e(B + \beta) \phi$ . Thus we have

$$0 \leq (\Box_a e(g) \phi, e(g) \phi) = (e(\mathbf{B} + \beta) \phi, e(g) \phi) = (i(g) e(\mathbf{B} + \beta) \phi, \phi),$$

which implies (ii). By the same way, since  $\Box_a i(g) \phi = -i(B + \beta) \phi$ we obtain

$$0 \leq (\Box_a i(g) \phi, i(g) \phi) = (-i(B + \beta) \phi, i(g) \phi)$$

 $= (\phi, -e(\mathbf{B} + \beta) i(g) \phi),$ 

which shows (i). (iii) follows from (i) and (ii).

Q.E.D.

. 8

THEOREM 6.1. – Let M be a compact connected oriented Hessian manifold. Denote by K the canonical line bundle over M. Let F be a locally constant line bundle over M.

(i) If 2F + K is positive, then

$$H^{p, q}(F) = 0$$
 for  $p + q > n$ .

(ii) If 2F + K is negative, then

$$\mathrm{H}^{p, q}(\mathrm{F}) = 0 \quad for \quad p + q < n.$$

**Proof.** – Suppose 2F + K is negative. Then  $B + \beta$  is negative definite. Therefore  $g' = -(B + \beta)$  gives a Hessian metric on M. If we denote by  $\beta'$  the Koszul form on M with respect to g', then there exists a positive  $C^{\infty}$ -function f on M such that

$$\beta' \doteq \beta + D^2 \log f.$$

If B is a Koszul form of F with respect to a fiber metric  $a = \{a_{\lambda}\}$ , then the Koszul form B' of F with respect to the fiber metric  $a' = \{fa_{\lambda}\}$  satisfies

$$\mathbf{B}' + \boldsymbol{\beta}' = \mathbf{B} + \boldsymbol{\beta} = -\mathbf{g}'.$$

Therefore if we use  $-(B + \beta)$  as a Hessian metric, the formula in Proposition 6.2 (iii) is reduced to

$$([i(g), -e(g)]\phi, \phi) \ge 0$$
 for  $\phi \in \mathcal{H}^{p, q}(F)$ .

Thus by Proposition 1.2 we have

$$(n-p-q) (\phi, \phi) \leq 0$$
 for  $\phi \in \mathcal{H}^{p, q}$  (F).

Therefore, if n - p - q > 0 then  $\phi = 0$ . Hence (ii) is proved. (i) follows from (ii) and Theorem 2.2

Q.E.D.

#### 7. A vanishing theorem of Koszul type.

In this section we mention a vanishing theorem of Koszul type. Let M be a compact oriented hyperbolic affine manifold. Then there exists a canonical Hessian metric g and a unique Killing vector field H on M such that

$$D_{\mathbf{x}} \mathbf{H} = \mathbf{X},\tag{7.1}$$

for all vector field X on M[7]. The following theorem is essentially due to Koszul.

THEOREM 7.1. – Let F be a locally constant vector bundle over a compact hyperbolic affine manifold. If there exist a fiber metric  $a = \{(a_{ij})\}$  and a constant  $c \neq -2q$  such that

$$Ha_{ii} = ca_{ii}$$

then we have

$$H^{p,q}(F) = 0, \quad for \quad p > 0 \quad and \quad q \ge 0.$$

The proof of this theorem is nearly the same as Koszul [7], and so we omit the proof.

COROLLARY 7.1. – Let M be a compact oriented hyperbolic affine manifold. Then we have

$$H^{p, q}(1) = 0, \text{ for } p, q > 0,$$

where 1 is the trivial vector bundle over M.

The tensor bundle  $\overset{r}{\otimes} T \overset{s}{\otimes} T^*$  satisfies the condition of Theorem 7.1 if  $q - r + s \neq 0$ .

We give another example of locally constant vector bundle over  $\mathbf{M}$  which satisfies the conditions of Theorem 7.1. Let  $\Omega$  be an open convex cone in  $\mathbf{R}^n$  with vertex 0 not containing any full straight line. Suppose that a discrete subgroup  $\Gamma$  of  $GL(n, \mathbf{R})$  acts properly discontinuously and freely on  $\Omega$  such that  $\mathbf{M} = \Gamma \setminus \Omega$  is compact. Assume further that there exist a linear mapping from  $\Omega$  to the space of all  $m \times m$  positive definite real symmetric matrices and a homomorphism from  $\Gamma$  to  $GL(m, \mathbf{R})$ , which are denoted by the same letter  $\rho$ , such that

$$\rho(\gamma x) = \rho(\gamma) \rho(x) {}^t \rho(\gamma) \quad \text{for} \quad \gamma \in \Gamma, \ x \in \Omega.$$

We denote by  $F_{\rho}$  the vector bundle over M associated with the universal covering  $\Omega \longrightarrow M$  and  $\rho$ . Let U be an evenly covered open set in M. Choosing a section  $\sigma$  on U we set

$$a = (\rho \circ \sigma)^{-1}.$$

Then a is a fiber metric on  $F_{o}$  and we have

$$Ha^{\dagger} = -a$$

Therefore

COROLLARY 7.2. – We have

$$\mathrm{H}^{p, q}(\mathrm{F}_{\rho}) = 0 \quad for \quad p > 0 \quad and \quad q \ge 0.$$

#### H. SHIMA

#### BIBLIOGRAPHY

- [1] Y. AKIZUKI and S. NAKANO, Note on Kodaira-Spencer's proof of Lefschetz theorems, *Proc. Japan Acad.*, 30 (1954), 266-272.
- [2] S.Y. CHENG and S.T. YAU, The real Monge-Ampère equation and affine flat structures, *Proceedings of the 1980 Beijing* symposium of differential geometry and differential equations, Science Press, Beijing, China, 1982, Gordon and Breach, Science Publishers, Inc., New York, 339-370.
- [3] K. KODAIRA, On cohomology groups of compact analytic varieties with coefficients in some analytic faisceaux, Proc. Nat. Acad. Sci., U.S.A, 39 (1953), 865-868.
- [4] K. KODAIRA, On a differential-geometric method in the theory of analytic stacks, Proc. Nat. Acad. Sci., U.S.A., 39 (1953), 1268-1273.
- [5] J.L. KOSZUL, Domaines bornés homogènes et orbites de groupes de transformations affines, Bull. Soc. Math. France, 89 (1961), 515-533.
- [6] J.L. KOSZUL, Variétés localement plates et convexité, Osaka J. Math., 2 (1965), 285-290.
- [7] J.L. KOSZUL, Déformations de connexions localement plates, Ann. Inst. Fourier, Grenoble, 18-1 (1968), 103-114.
- [8] J. MORROW and K. KODAIRA, *Complex manifolds*, Holt, Rinehart and Winston, Inc., 1971.
- [9] J.P. SERRE, Une théorème de dualité, Comm. Math. Helv., 29 (1955), 9-26.
- [10] H. SHIMA, On certain locally flat homogeneous manifolds of solvable Lie groups, Osaka J. Math., 13 (1976), 213-229.
- [11] H. SHIMA, Symmetric spaces with invariant locally Hessian structures, J. Math. Soc. Japan, 29 (1977), 581-589.
- [12] H. SHIMA, Compact locally Hessian manifolds, Osaka J. Math., 15 (1978), 509-513.

- [13] H. SHIMA, Homogeneous Hessian manifolds, Ann. Inst. Fourier, Grenoble, 30-3 (1980), 91-128.
- [14] H. SHIMA, Hessian manifolds and convexity, in Manifolds, and Lie groups, Papers in honor of Y. Matsushima, *Progress in Mathematics*, vol. 14, Birkhäuser, Boston, Basel, Stuttgart, 1981, 385-392.
- [15] K. YAGI, On Hessian structures on an affine manifold, in Manifolds and Lie groups. Papers in honor of Y. Matsushima, *Progress in Mathematics*, vol. 14, Birkhäuser, Boston, Basel, Stuttgart, 1981, 449-459.

Manuscrit reçu le 17 juillet 1985 révisé le 14 avril 1986.

Hirohiko SHIMA, Department of Mathematics Yamaguchi University Yamaguchi 753 (Japan).