EDWARD BIERSTONE
P. D. MILMAN

Relations among analytic functions. I

Annales de l'institut Fourier, tome 37, nº 1 (1987), p. 187-239
<http://www.numdam.org/item?id=AIF_1987__37_1_187_0>

Ann. Inst. Fourier, Grenoble
37, 1 (1987), 187-239

RELATIONS AMONG ANALYTIC FUNCTIONS I

by

E. BIERSTONE (1) and P.D. MILMAN (2)

Contents

---

(1) Research partially supported by NSERC operating grant A 9070.

(2) Supported by NSERC University Research Fellowship and operating grant U 0076.

(*) Will be published in the second issue of volume 37 (1987).

Introduction.

Let $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Let X and Y denote analytic spaces over $\mathbf{K}$, and let $\phi : X \longrightarrow Y$ be a morphism. Let $\phi^* : \mathcal{O}_Y \longrightarrow \mathcal{O}_X$ denote the associated homomorphism of the structure sheaves. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are coherent $\mathcal{O}_X$- and $\mathcal{O}_Y$-modules, respectively, and that $\Psi : \mathcal{G} \longrightarrow \mathcal{F}$ is a module homomorphism over the ring homomorphism ϕ^*.

Let $a \in X$. Then ϕ^* determines a homomorphism of local rings $\phi_a^* : \mathcal{O}_{Y,\phi(a)} \longrightarrow \mathcal{O}_{X,a}$ and Ψ determines a module homomorphism $\Psi_a : \mathcal{G}_{\phi(a)} \longrightarrow \mathcal{F}_a$ over ϕ_a^*. We write $\mathcal{O}_a = \mathcal{O}_{X,a}$, etc., when there is no possibility of confusion. Let $\hat{\phi}_a^* : \hat{\mathcal{O}}_{\phi(a)} \longrightarrow \hat{\mathcal{O}}_a$ and $\hat{\Psi}_a : \hat{\mathcal{G}}_{\phi(a)} \longrightarrow \hat{\mathcal{F}}_a$ denote the induced homomorphisms of the completions.

Let $s \in \mathbf{N}$. Let X_ϕ^s denote the s-fold fiber product

$$X_\phi^s = \{a = (a^1, \ldots, a^s) \in X^s : \phi(a^1) = \ldots = \phi(a^s)\},$$

and let $\boldsymbol{\varphi} : X_\phi^s \longrightarrow Y$ denote the induced morphism.

We study the variation with respect to $\boldsymbol{a} = (a^1, \ldots, a^s) \in X_\phi^s$ of the *module of formal relations*

$$\mathcal{R}_{\boldsymbol{a}} = \bigcap_{i=1}^{s} \operatorname{Ker} \hat{\Psi}_{a^i},$$

and of associated invariants such as the *Hilbert-Samuel function* $H_{\boldsymbol{a}}$ of $\hat{\mathcal{G}}_{\boldsymbol{\varphi}(\boldsymbol{a})}/\mathcal{R}_{\boldsymbol{a}}$:

$$H_{\boldsymbol{a}}(k) = \dim_{\mathbf{K}} \frac{\hat{\mathcal{G}}_{\boldsymbol{\varphi}(\boldsymbol{a})}}{\mathcal{R}_{\boldsymbol{a}} + \mathfrak{m}_{\boldsymbol{\varphi}(\boldsymbol{a})}^{k+1} \cdot \hat{\mathcal{G}}_{\boldsymbol{\varphi}(\boldsymbol{a})}}$$

(where $\mathfrak{m}_{\boldsymbol{\varphi}(\boldsymbol{a})}$ denotes the maximal ideal of $\mathcal{O}_{\boldsymbol{\varphi}(\boldsymbol{a})}$).

We show (cf. Theorem C) that the Hilbert-Samuel function $H_{\boldsymbol{a}}$ is upper semicontinuous in the (analytic) Zariski topology of X_ϕ^s in each of the following cases:

(a) In the algebraic category. (Here we can use the (algebraic) Zariski topology.)

Key-words: Morphism of analytic spaces – Module of formal relations – Hilbert-Samuel function – Diagram of initial exponents – Zariski semicontinuity – Division and composition of $\mathcal{C}^\infty$ functions.

(b) If X is smooth, $\Psi = \phi^* : \mathcal{O}_Y \longrightarrow \mathcal{O}_X$, and ϕ is *regular* in the sense of Gabrielov [14]; i.e., the Krull dimension of $\mathcal{O}_{\phi(a)}/\mathrm{Ker}\ \phi_a^*$ is locally constant on X.

(c) If X is Cohen-Macauley and ϕ is *locally finite*; i.e., for all $a \in X$, $\mathcal{O}_a$ is a finite $\mathcal{O}_{\phi(a)}$-module via the homomorphism ϕ_a^*.

(d) In the coherent case: $X = Y$, $\phi =$ identity.

Semicontinuity in the coherent case (d) is, of course, known classically. The image of a morphism $\phi : X \longrightarrow Y$ is not, in general coherent. Nor, in general, is the Hilbert-Samuel function of $\hat{\mathcal{G}}_{\phi(a)} / \mathcal{R}_{*\phi(a)}$, where $\mathcal{R}_{*b} = \bigcap_{a \in \phi^{-1}(b)} \mathrm{Ker}\ \hat{\Psi}_a$, semicontinuous as a function of $a \in X$. Nevertheless, we conjecture that the Hilbert-Samuel function H_a is always Zariski semicontinuous as a function of $a \in X_\phi^s$.

We prove that, in general, Zariski semicontinuity of H_a is equivalent to two other important conditions on the variation of the module of formal relations $\mathcal{R}_a$ (Theorem A):

(1) A uniform version of a lemma of Chevalley [10]. In the coherent case, this is equivalent to a uniform version of the Artin-Rees theorem.

(2) Zariski semicontinuity of a diagram of "initial exponents" associated to $\mathcal{R}_a$, introduced by Hironaka (cf. [8], [15], [23]). This diagram gives a combinatorial picture of the module $\mathcal{R}_a$, in the spirit of the classical Newton diagram of a formal power series.

The diagram of initial exponents and the condition (2) depend on a local embedding of Y in affine space $\mathbf{K}^n$ and on a presentation $\mathcal{O}_Y^q \longrightarrow \mathcal{G} \longrightarrow 0$ of $\mathcal{G}$. We can assume that $\mathcal{R}_a$ is a submodule of $\mathbf{K}[[y]]^q$, where $\mathbf{K}[[y]]$ denotes the ring of formal power series in $y = (y_1, \ldots, y_n)$. Using the condition (2), we prove that X_ϕ^s admits an analytic stratification with the property that, along each stratum, $\mathcal{R}_a$ is generated by finitely many q-tuples of formal power series in y whose coefficients are functions analytic on the stratum and meromorphic through its frontier (cf. Theorem B). We conclude that H_a is Zariski semicontinuous on X_ϕ^s, for a given positive integer s, if and only if it is Zariski semicontinuous in the case $s = 1$.

Our results on the variation of $\mathcal{R}_a$ have important applications to the solution of equations involving differentiable functions. Suppose that X and Y are smooth real analytic spaces. Then ϕ induces a homomorphism $\phi^* : \mathscr{C}^\infty(\mathrm{Y}) \longrightarrow \mathscr{C}^\infty(\mathrm{X})$ between the rings of infinitely differentiable functions. Let A and B denote $p \times q$ and $p \times r$ matrices, respectively, whose entries are analytic functions on X. Let $\Phi : \mathscr{C}^\infty(\mathrm{Y})^q \longrightarrow \mathscr{C}^\infty(\mathrm{X})^p$ denote the module homomorphism over ϕ^* defined by $\Phi(g)(x) = \mathrm{A}(x) \cdot g(\phi(x))$, where $g = (g_1, \ldots, g_q) \in \mathscr{C}^\infty(\mathrm{Y})^q$, and let

$$\mathrm{B} \cdot : \mathscr{C}^\infty(\mathrm{X})^r \longrightarrow \mathscr{C}^\infty(\mathrm{X})^p$$

denote the $\mathscr{C}^\infty(\mathrm{X})$-homomorphism induced by multiplication by the matrix B.

There is also an induced homomorphism $\Phi : \mathcal{O}_\mathrm{Y}^q \longrightarrow \mathcal{O}_\mathrm{X}^p$ over $\phi^* : \mathcal{O}_\mathrm{Y} \longrightarrow \mathcal{O}_\mathrm{X}$, and an induced $\mathcal{O}_\mathrm{X}$-homomorphism

$$\mathrm{B} : \mathcal{O}_\mathrm{X}^r \longrightarrow \mathcal{O}_\mathrm{X}^p .$$

Let $\Psi : \mathcal{O}_\mathrm{Y}^q \longrightarrow \mathrm{Coker\,B}$ denote the homomorphism over ϕ^* induced by Φ. (Locally, any ϕ^*-homomorphism from $\mathcal{O}_\mathrm{Y}^q$ to a coherent $\mathcal{O}_\mathrm{X}$-module has this form.)

For every $a \in \mathrm{X}$, there is a Taylor series homomorphism $f \longmapsto \hat{f}_a$ from $\mathscr{C}^\infty(\mathrm{X})^p$ onto $\hat{\mathcal{O}}_a^p$. Let

$$(\Phi\, \mathscr{C}^\infty(\mathrm{Y})^q + \mathrm{B} \cdot \mathscr{C}^\infty(\mathrm{X})^r)\hat{}$$

denote the elements of $\mathscr{C}^\infty(\mathrm{X})^p$ which formally belong to $\Phi\, \mathscr{C}^\infty(\mathrm{Y})^q + \mathrm{B} \cdot \mathscr{C}^\infty(\mathrm{X})^r$; i.e., $\{f \in \mathscr{C}^\infty(\mathrm{X})^p :$ for all $b \in \phi(\mathrm{X})$, there exists $\mathrm{G}_b \in \hat{\mathcal{O}}_b^q$ such that $\hat{f}_a - \hat{\Phi}_a(\mathrm{G}_b) \in \mathrm{Im}\,\hat{\mathrm{B}}_a$, for all $a \in \phi^{-1}(b)\}$. Then $(\Phi\, \mathscr{C}^\infty(\mathrm{Y})^q + \mathrm{B} \cdot \mathscr{C}^\infty(\mathrm{X})^r)\hat{}$ contains the closure of $\Phi\, \mathscr{C}^\infty(\mathrm{Y})^q + \mathrm{B} \cdot \mathscr{C}^\infty(\mathrm{X})^r$ in the $\mathscr{C}^\infty$ topology on $\mathscr{C}^\infty(\mathrm{X})^p$.

Suppose that ϕ is proper. We prove that if the Hilbert-Samuel function H_a of $\hat{\mathcal{O}}_{\phi(a)}^q/\mathrm{Ker}\,\hat{\Psi}_a$ is Zariski semicontinuous on X, then

$$\Phi\, \mathscr{C}^\infty(\mathrm{Y})^q + \mathrm{B} \cdot \mathscr{C}^\infty(\mathrm{X})^r = (\Phi\, \mathscr{C}^\infty(\mathrm{Y})^q + \mathrm{B} \cdot \mathscr{C}^\infty(\mathrm{X})^r)\hat{} \qquad (*)$$

(Theorem D). Because ϕ is proper, then (locally in Y) there is a bound s on the number of distinct submodules $\mathrm{Ker}\,\hat{\Psi}_a$ of $\hat{\mathcal{O}}_b^q$, where $a \in \phi^{-1}(b)$. The conditions (1) and (2) above are applied with this s to prove (*).

It follows from (a) above that (*) holds if $\phi : X \longrightarrow Y$ is a proper morphism of Nash manifolds and A and B are matrices of Nash functions on X (Theorem E). This seems to be the first general result for modules over a ring of composite differentiable functions.

From (a) - (d), we also recover several classical results, including Malgrange's theorem on ideals generated by analytic functions [27, Ch. VI] and his $\mathscr{C}^\infty$ version of the Weierstrass division theorem [27, Ch. V]. The solution [5] of the composition problem of Glaeser [16] follows from (b).

This article was distributed in preprint form in the spring of 1984. Some of our results were announced in [6]. We gratefully acknowledge valuable discussions we have had with Herwig Hauser and Gerald W. Schwarz.

CHAPTER 0

MAIN THEOREMS, PROBLEMS, EXAMPLES

We continue to use the notation of the introduction. Let $\mathbf{N}$ denote the nonnegative integers.

1. Preliminaries on local analytic invariants.

Let $\boldsymbol{a} \in X_\phi^s$, $\boldsymbol{a} = (a^1, \ldots, a^s)$, and let $G \in \mathscr{G}_{\varphi(\boldsymbol{a})}$. The following lemma of Chevalley [10, § II, Lemma 7](cf. Lemma 8.2.2) estimates the order of vanishing of G in terms of the order of vanishing of $\hat{\Psi}_{a^i}(G)$, $i = 1, \ldots, s$.

LEMMA 1.1. – *Let* $\boldsymbol{a} \in X_\phi^s$, $\boldsymbol{a} = (a^1, \ldots, a^s)$. *For each* $k \in \mathbf{N}$, *there exists* $\ell \in \mathbf{N}$ *such that if* $G \in \hat{\mathscr{G}}_{\varphi(\boldsymbol{a})}$ *and* $\hat{\Psi}_{a^i}(G) \in \mathfrak{m}_{a^i}^{\ell+1} \cdot \hat{\mathscr{F}}_{a^i}$, $i = 1, \ldots, s$, *then* $G \in \mathscr{R}_{\boldsymbol{a}} + \mathfrak{m}_{\varphi(\boldsymbol{a})}^{k+1} \cdot \hat{\mathscr{G}}_{\varphi(\boldsymbol{a})}$.

DEFINITION 1.2. – *Let* $a \in X^s_\phi$. *For each* $k \in \mathbf{N}$, *let* $\ell(k, a)$ *denote the smallest* $\ell \in \mathbf{N}$ *satisfying the conclusion of Lemma 1.1.*

Let $\mathbf{N}^{\mathbf{N}}$ denote the set of functions from $\mathbf{N}$ to itself. $\mathbf{N}^{\mathbf{N}}$ is partially ordered as follows: Let H, $H' \in \mathbf{N}^{\mathbf{N}}$. Then $H < H'$ if $H(k) \leqslant H'(k)$ for all k, and $H(k) < H'(k)$ for some k.

As in the introduction, $H_a \in \mathbf{N}^{\mathbf{N}}$ denotes the Hilbert-Samuel function of $\hat{\mathcal{G}}_{\varphi(a)}/\hat{\mathcal{R}}_a$.

1.3. Locally, we can assume that Y is a closed analytic subspace of an open subspace V of $\mathbf{K}^n$, and that $\mathcal{G}$ is a quotient of $\mathcal{O}^q_V$ restricted to Y. Let $\phi' : X \longrightarrow V$ denote the composition of ϕ with the inclusion $Y \hookrightarrow V$, and let $\Psi' : \mathcal{O}^q_V \longrightarrow \mathcal{F}$ be the module homomorphism over ϕ' induced by Ψ. Clearly, if $a \in X^s_\phi$, then the Hilbert-Samuel functions H_a as well as the Chevalley estimates $\ell(k, a)$ associated to Ψ and to Ψ' coincide. In order to study the local variation of these invariants, we can, therefore, assume that Y is an open subspace of $\mathbf{K}^n$ and that $\mathcal{G}$ is a free $\mathcal{O}_Y$-module.

1.4. *Hironaka's diagram of initial exponents.*

The notation of this subsection will be used throughout the article. Let $\mathbf{K}[[y]] = \mathbf{K}[[y_1, \ldots, y_n]]$ denote the ring of formal power series in n variables. Let R be a submodule of $\mathbf{K}[[y]]^q$. Following Hironaka [8], [15], [23], we associate to R a subset $\mathfrak{N}(R)$ of $\mathbf{N}^n \times \{1, \ldots, q\}$.

If $\beta = (\beta_1, \ldots, \beta_n) \in \mathbf{N}^n$, put $|\beta| = \beta_1 + \ldots + \beta_n$. We order the $(n+2)$-tuples $(\beta_1, \ldots, \beta_n, j, |\beta|)$, where

$$(\beta, j) \in \mathbf{N}^n \times \{1, \ldots, q\},$$

lexicographically from the right. This induces a total ordering of $\mathbf{N}^n \times \{1, \ldots, q\}$.

Let $G \in \mathbf{K}[[y]]^q$, $G = (G_1, \ldots, G_q)$. Write $G_j = \sum_{\beta \in \mathbf{N}^n} g_{\beta,j} y^\beta$, $j = 1, \ldots, q$, where $g_{\beta,j} \in \mathbf{K}$ and y^β denotes $y_1^{\beta_1} \ldots y_n^{\beta_n}$. We also let $y^{\beta,j}$ denote the q-tuple $(0, \ldots, y^\beta, \ldots, 0)$ with y^β in the j'th place, so that $G = \sum_{\beta,j} g_{\beta,j} y^{\beta,j}$. Let

$$\operatorname{supp} G = \{(\beta, j) \in \mathbf{N}^n \times \{1, \dots, q\} : g_{\beta, j} \neq 0\}$$

and let $\nu(G)$ denote the smallest element of $\operatorname{supp} G$. Let in g denote $g_{\nu(G)} y^{\nu(G)}$.

We define the diagram of initial exponents $\mathfrak{N}(R)$ as $\{\nu(G) : G \in R\}$. Clearly, $\mathfrak{N}(R) + \mathbf{N}^n = \mathfrak{N}(R)$, where addition is defined by

$$(\beta, j) + \gamma = (\beta + \gamma, j), (\beta, j) \in \mathbf{N}^n \times \{1, \dots, q\}, \gamma \in \mathbf{N}^n.$$

Put $\mathscr{D}(n, q) = \{\mathfrak{N} \subset \mathbf{N}^n \times \{1, \dots, q\} : \mathfrak{N} + \mathbf{N}^n = \mathfrak{N}\}$. Let $\mathfrak{N} \in \mathscr{D}(n, q)$. Then there is a smallest finite subset $\mathfrak{V}$ of $\mathfrak{N}$ such that $\mathfrak{N} = \mathfrak{V} + \mathbf{N}^n$. We call $\mathfrak{V}$ the *vertices* of $\mathfrak{N}$.

The set $\mathscr{D}(n, q)$ is totally ordered as follows: Let $\mathfrak{N}^1, \mathfrak{N}^2 \in \mathscr{D}(n, q)$. For each $i = 1, 2$, let (β_k^i, j_k^i), $k = 1, \dots, t_i$, denote the vertices of $\mathfrak{N}^i$ indexed in ascending order. After perhaps interchanging $\mathfrak{N}^1$ and $\mathfrak{N}^2$, there exists $t \in \mathbf{N}$ such that $(\beta_k^1, j_k^1) = (\beta_k^2, j_k^2)$, $k = 1, \dots, t$, and either (1) $t_1 = t = t_2$, (2) $t_1 > t = t_2$, or (3) $t_1, t_2 > t$ and $(\beta_{t+1}^1, j_{t+1}^1) < (\beta_{t+1}^2, j_{t+1}^2)$. In case (1), $\mathfrak{N}^1 = \mathfrak{N}^2$. In case (2) or (3), we say that $\mathfrak{N}^1 < \mathfrak{N}^2$.

Clearly, if $\mathfrak{N}^1 \supset \mathfrak{N}^2$, then $\mathfrak{N}^1 < \mathfrak{N}^2$.

Assume that Y is an open subspace of $\mathbf{K}^n$ and that $\mathscr{G} = \mathscr{O}_Y^q$. If $b \in Y$, then $\hat{\mathscr{O}}_b$ identifies with the ring of formal power series $\mathbf{K}[[y]]$ in the affine coordinates $y = (y_1, \dots, y_n)$ of $\mathbf{K}^n$. Let $a \in X_\phi^s$. We put

$$\mathfrak{N}_a = \mathfrak{N}(\mathscr{R}_a).$$

2. Semicontinuity: Theorems A, B, C.

Let Σ be a partially ordered set and let Z be an analytic space.

DEFINITION 2.1. – *A function $\mathscr{H} : Z \longrightarrow \Sigma$ is (analytic) Zariski (upper-) semicontinuous if, for every $a \in Z$ and every irreducible germ of an analytic subset V of Z at a, there exists a germ of a proper analytic subset W of V such that*

(1) $\mathscr{H}(x) = \mathscr{H}(y)$ if $x, y \in V - W$;

(2) $\mathscr{H}(x) < \mathscr{H}(y)$ if $x \in V - W, y \in W$.

(We use the same notation for a germ at a point and a representative of the germ in a suitable neighborhood.)

Remark 2.2. – If $\mathscr{H} : Z \longrightarrow \Sigma$ is Zariski semicontinuous, then, for all $\sigma \in \Sigma$, $\{x \in Z : \mathscr{H}(x) \geqslant \sigma\}$ is a closed analytic subset of Z. The converse is true provided that Σ is totally ordered.

By Zariski semicontinuity of the Hilbert-Samuel function H_a (respectively, of $H_a(k)$ for fixed k, or of the diagram of initial exponents $\mathfrak{N}_a$), we understand Zariski semicontinuity of the corresponding function on X_ϕ^s with values in $\mathbf{N}^{\mathbf{N}}$ (respectively, in $\mathbf{N}$, or in $\mathscr{D}(n, q)$).

We conjecture that H_a is always Zariski semicontinuous on X_ϕ^s.

Remark 2.3. – For each $b \in Y$, let $\mathscr{R}_{*b} = \bigcap_{a \in \phi^{-1}(b)} \operatorname{Ker} \hat{\Psi}_a$.
Suppose that $\mathbf{K} = \mathbf{C}$ and ϕ is proper. Then the direct image $\phi_* \mathscr{F}$ is a coherent sheaf of $\mathscr{O}_Y$-modules [17], and Ψ induces an $\mathscr{O}_Y$-homomorphism $\Psi_* : \mathscr{G} \longrightarrow \phi_* \mathscr{F}$. If $b \in Y$, then $\operatorname{Ker} \Psi_{*b} = \bigcap_{a \in \phi^{-1}(b)} \operatorname{Ker} \Psi_a$. It follows from a theorem of Siu [35, Thm. 2] that $\mathscr{R}_{*b} = (\operatorname{Ker} \Psi_{*b})^{\wedge}$. Hence the Hilbert-Samuel function of $\hat{\mathscr{G}}_b / \mathscr{R}_{*b}$ is Zariski semicontinuous on Y.

On the other hand, if $\mathbf{K} = \mathbf{R}$ and ϕ is finite, the Hilbert-Samuel function of $\hat{\mathscr{G}}_{\phi(a)} / \mathscr{R}_{*\phi(a)}$ need not be Zariski semicontinuous even as a function of $a \in X$. For example, take $X = \mathbf{R}^2$, $Y = \mathbf{R}^3$ and define ϕ by $\phi(x_1, x_2) = (x_1, x_2(x_2^3 + x_1 x_2), x_2^3 + x_1 x_2)$. Let $\Psi = \phi^* : \mathscr{O}_Y \longrightarrow \mathscr{O}_X$. Then the Hilbert-Samuel function of $\hat{\mathscr{G}}_{\phi(a)} / \mathscr{R}_{*\phi(a)}$ is constant on the half-lines $\{x_2 = 0, x_1 > 0\}$ and $\{x_2 = 0, x_1 < 0\}$ but has different values on the two half-lines.

If ϕ is not proper, the Hilbert-Samuel function of $\hat{\mathscr{G}}_{\phi(a)} / \mathscr{R}_{*\phi(a)}$ need not even be topologically semicontinuous. For example, with $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, define $\phi : \mathbf{K} - \{0\} \longrightarrow \mathbf{K}^2$ by

$$\phi(t) = (\cos t, \sin t + \sin(1/t)) \text{ and take } \Psi = \phi^*.$$

THEOREM A. – *Let* X *and* Y *denote analytic spaces over* $\mathbf{K}$, and let $\phi : X \longrightarrow Y$ *be a morphism. Suppose that* $\mathcal{F}$ *and* $\mathcal{G}$ *are coherent* $\mathcal{O}_X$- *and* $\mathcal{O}_Y$*-modules, respectively, and that* $\Psi : \mathcal{G} \longrightarrow \mathcal{F}$ *is a homomorphism over the ring homomorphism* $\phi^* : \mathcal{O}_Y \longrightarrow \mathcal{O}_X$. *Let* $s \in \mathbf{N}$. *Then the following conditions are equivalent:*

(1) *The Hilbert-Samuel function* H_a *is Zariski semicontinuous on* X^s_ϕ.

(2) $H_a(k)$ *is Zariski semicontinuous on* X^s_ϕ, *for each fixed* $k \in \mathbf{N}$.

(3) Uniform Chevalley estimate. *Let* K *be a compact subset of* X^s_ϕ. *Then, for every* $k \in \mathbf{N}$, *there exists* $\ell = \ell(k, K) \in \mathbf{N}$ *such that* $\ell(k, a) \leqslant \ell$ *for all* $a \in K$.

Assume, moreover, that Y *is an open subspace of* $\mathbf{K}^n$ *and that* $\mathcal{G} = \mathcal{O}^q_Y$. *Then each of the conditions above is equivalent to:*

(4) *The diagram of initial exponents* $\mathfrak{N}_a$ *is Zariski semicontinuous on* X^s_ϕ.

2.4. *Special generators.*

Although the modules $\mathcal{R}_a$ are not, in general, the completions of stalks of a coherent sheaf, we can deduce from the semicontinuity of $\mathfrak{N}_a$ a precise description of the variation of $\mathcal{R}_a$ with respect to a, which replaces Oka's theorem in the coherent case.

Let R be a submodule of $\mathbf{K}[[y]]^q$, where $y = (y_1, \ldots, y_n)$, as in 1.4. Let (β_i, j_i), $i = 1, \ldots, t$, denote the vertices of $\mathfrak{N}(R)$. By the formal division algorithm of Grauert [18] and Hironaka [1] (cf. Theorem 6.2), for each $i = 1, \ldots, t$, there is a uniquely determined element $G^i \in R$ such that $\operatorname{in} G^i = y^{\beta_i, j_i}$ and $\operatorname{supp} G^i \cap \mathfrak{N}(R) = \{(\beta_i, j_i)\}$. Then $G^1, \ldots, G^t$ generate R (cf. Corollary 6.8). Following Hironaka, we call this canonical choice of generators the *standard basis* of R.

THEOREM B. – *Let* $X, Y, \phi, \mathcal{F}, \mathcal{G}$ *and* Ψ *be as in Theorem A. Assume that* Y *is an open subspace of* $\mathbf{K}^n$, $\mathcal{G} = \mathcal{O}^q_Y$, *and* $\mathfrak{N}_a = \mathfrak{N}(\mathcal{R}_a)$ *is Zariski semicontinuous on* X^s_ϕ. *Let* $a_0 \in X^s_\phi$. *Then there is a neighborhood* U *of* a_0 *in* X^s_ϕ *and a filtration of* U *by closed analytic subsets,*

$$U = X_0 \supset X_1 \supset \ldots \supset X_{\mu+1} = \phi,$$

such that, for each $\lambda = 0, \ldots, \mu$:

(1) $\mathfrak{N}_a$ *is constant on* $X_\lambda - X_{\lambda+1}$.

(2) *Let* G_a^i, $i = 1, \ldots, t$, *denote the standard basis of* $\mathscr{R}_a \subset \mathbf{K}[[y]]^q$, *where* $a \in X_\lambda - X_{\lambda+1}$. *Write*

$$G_a^i(y) = \Sigma\, g_{\beta,j}^i(a)\, y^{\beta,j}.$$

Then each $g_{\beta,j}^i$ *is a meromorphic function on* X_λ *with poles in* $X_{\lambda+1}$.

Remark 2.5. – Although the coefficients of the elements of the standard bases are meromorphic on each X_λ in Theorem B, the elements of the standard bases themselves need not be meromorphic, even in the coherent case. For example, let $\mathscr{I} \subset \mathcal{O}_{\mathbf{K}^3}$ be the sheaf of principal ideals generated by $x_1^2 - x_2 x_3^2$. Then, for each $a = (a_1, a_2, a_3) \in \mathbf{K}^3$ such that $a_1^2 - a_2 a_3^2 = 0$ and $a_1 \neq 0$, $\mathfrak{N}_a = (1, 0, 0) + \mathbf{N}^3$. The standard basis of $\hat{\mathscr{I}}_a$ has one element $G_a(x) = a_1 + x_1 - (a_2 + x_2)^{1/2}(a_3 + x_3)$, where the square root is determined by $a_1 = a_2^{1/2} a_3$. Of course, $G_a(x)$ can be rewritten

$$G_a(x) = x_1 + a_1\left[1 - \left[1 + \frac{x_2}{a_2}\right]^{1/2}\left[1 + \frac{x_3}{a_3}\right]\right],$$

so that the coefficients of its power series expansion are rational functions of a.

Remark 2.6. – Using Theorem B, we prove that the diagram of initial exponents $\mathfrak{N}_a$ is Zariski semicontinuous on X_ϕ^s, for a given integer s, if and only if it is Zariski semi-continuous in the case $s = 1$ (Proposition 9.6). Nevertheless, in applications of our theorems on the variation of $\mathscr{R}_a$ (e.g., in Theorem D below), the choice of a suitable s plays a critical part.

2.7. Invariants of Gabrielov [14].

Let $\phi : X \longrightarrow Y$ be a morphism of analytic spaces. Assume that X is smooth. Let $a \in X$. Let $r_1(a)$ denote the generic rank of ϕ near a. Put

$$r_2(a) = \dim \frac{\hat{\mathcal{O}}_{\phi(a)}}{\operatorname{Ker} \hat{\phi}_a^*}, \quad r_3(a) = \dim \frac{\mathcal{O}_{\phi(a)}}{\operatorname{Ker} \phi_a^*}$$

(where dim denotes the Krull dimension). Then $r_1(a) \leqslant r_2(a) \leqslant r_3(a)$. The formal rank $r_2(a)$ is the degree of the Hilbert-Samuel polynomial of $\hat{\mathcal{O}}_{\phi(a)}/\mathrm{Ker}\,\hat{\phi}_a^*$; in particular, if the Hilbert-Samuel function of $\hat{\mathcal{O}}_{\phi(a)}/\mathrm{Ker}\,\hat{\phi}_a^*$ is Zariski semi-continuous on X, then so is $r_2(a)$.

We say that ϕ is *regular at* a if $r_1(a) = r_3(a)$. Clearly, regularity at a is an open condition. We say that ϕ is *regular* if it is regular at each $a \in \mathrm{X}$. For example, if ϕ is algebraic, then it is regular.

Example 2.8 (Grauert-Remmert [19, II.5.2], Gabrielov [13]). – Let X be the open unit disk in $\mathbf{K}^2$. Define $\phi : \mathrm{X} \longrightarrow \mathbf{K}^3$ by $\phi(x_1, x_2) = (x_1, x_1 x_2, x_1 x_2 e^{x_2})$ (Osgood). Let $a \in \mathrm{X} \cap \{x_1 = 0\}$. Then $r_1(a) = 2$, but $r_2(a) = r_3(a) = 3$. Put

$$f(x_1, x_2) = \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} \frac{j!}{(j+k)!} x_1^j x_2^{j+k+1},$$

$$\mathrm{G}(y_1, y_2, y_3) = \sum_{j=1}^{\infty} j! \left[y_1^{j-1} y_3 - \sum_{i=1}^{j} \frac{1}{(i-1)!} y_1^{j-i} y_2^i \right].$$

Then f converges in X, G is divergent, $f = \hat{\phi}_0^*(\mathrm{G})$ but $f \notin \phi_0^* \mathcal{O}_{\mathbf{K}^3,0}$. Define $\psi : \mathrm{X} \longrightarrow \mathbf{K}^4$ by $y_i = \phi_i(x)$, $i = 1, 2, 3$, where $\phi = (\phi_1, \phi_2, \phi_3)$, and $y_4 = f(x)$. Then $y_4 - \mathrm{G}(y_1, y_2, y_3)$ generates $\mathrm{Ker}\,\hat{\psi}_a^*$, but $\mathrm{Ker}\,\psi_a^* = 0$. In particular, for ψ we get $r_1(a) = 2$, $r_2(a) = 3$ and $r_3(a) = 4$.

Remark 2.9. – Let $\phi : \mathrm{X} \longrightarrow \mathrm{Y}$ be as in 2.7. Gabrielov [14] proves that if $r_1(a) = r_2(a)$, then

(i) $r_2(a) = r_3(a)$; i.e., $\mathrm{Ker}\,\hat{\phi}_a^*$ is generated by $\mathrm{Ker}\,\phi_a^*$;

(ii) $\mathcal{O}_a \cap \hat{\phi}_a^*(\hat{\mathcal{O}}_{\phi(a)}) = \phi_a^*(\mathcal{O}_{\phi(a)})$.

In fact, by [4] and [30], (ii) is equivalent to the regularity of ϕ at a. (These results are not used in our theorems ; but see Remarks 2.11 and 4.3 below.)

THEOREM C. – *Let* $\mathrm{X}, \mathrm{Y}, \phi, \mathcal{F}, \mathcal{G}$ *and* Ψ *be as in Theorem A. Let* $s \in \mathbf{N}$. *Then the conditions of Theorem A are satisfied in each of the following cases:*

(1) Algebraic case. $\phi : X \longrightarrow Y$ *is a morphism of algebraic spaces (cf.* [3], [24]), $\mathcal{F}$ *and* $\mathcal{G}$ *are coherent modules over the structure sheaves* $\mathcal{O}_X$ *and* $\mathcal{O}_Y$ *of* X *and* Y, *respectively, and* $\Psi : \mathcal{G} \longrightarrow \mathcal{F}$ *is a homomorphism over the ring homomorphism* $\phi^* : \mathcal{O}_Y \longrightarrow \mathcal{O}_X$.

(2) Regular case. X *is smooth and* ϕ *is regular,* $\Psi = \phi^* : \mathcal{O}_Y \longrightarrow \mathcal{O}_X$.

(3) Finite case. X *is Cohen-Macauley and* ϕ *is locally finite.*

(4) Coherent case. $X = Y$, $\phi =$ *identity.*

Remark 2.10. – In the coherent case C(4), the uniform Chevalley estimate A(3) is equivalent to a uniform version of the Artin-Rees theorem: Let X be an analytic space and let $\mathcal{H} \subset \mathcal{F}$ be coherent sheaves of $\mathcal{O}_X$-modules. Let K be a compact subset of X. Then there exists $\lambda = \lambda(K) \in \mathbf{N}$ such that, for all $k \in \mathbf{N}$ and all $a \in K$, $\mathcal{H}_a \cap \mathfrak{m}_a^{k+\lambda} \cdot \mathcal{F}_a \subset \mathfrak{m}_a^k \cdot \mathcal{H}_a$; cf. [39, Thme. 3.8]. In fact, there exists $\lambda = \lambda(K)$ such that, for all $k \in \mathbf{N}$ and all $a \in K$, $\mathcal{H}_a \cap \mathfrak{m}_a^{k+\lambda} \cdot \mathcal{F}_a = \mathfrak{m}_a^k \cdot (\mathcal{H}_a \cap \mathfrak{m}_a^{\lambda} \cdot \mathcal{F}_a)$ (see Remark 7.6).

Remarks 2.11. – (1) In each case of Theorem C, the analogues of Gabrielov's results (2.9) hold: Let $\boldsymbol{a} \in X_\phi^s$, $\boldsymbol{a} = (a^1, \dots, a^s)$. Let $\boldsymbol{\Psi}_{\boldsymbol{a}} : \mathcal{G}_{\varphi(\boldsymbol{a})} \longrightarrow \bigoplus_{i=1}^{s} \mathcal{F}_{a^i}$ denote the composition of $\bigoplus_{i=1}^{s} \Psi_{a^i}$ with the diagonal injection $\mathcal{G}_{\varphi(\boldsymbol{a})} \longrightarrow \bigoplus_{i=1}^{s} \mathcal{G}_{\varphi(\boldsymbol{a})}$, and let $\hat{\boldsymbol{\Psi}}_{\boldsymbol{a}}$ denote the induced homomorphism of the completions. Then:

(i) Ker $\hat{\boldsymbol{\Psi}}_{\boldsymbol{a}}$ is generated by Ker $\boldsymbol{\Psi}_{\boldsymbol{a}}$.

(ii) $\hat{\boldsymbol{\Psi}}_{\boldsymbol{a}}(\hat{\mathcal{G}}_{\varphi(\boldsymbol{a})}) \cap \bigoplus_{i=1}^{s} \mathcal{F}_{a^i} = \boldsymbol{\Psi}_{\boldsymbol{a}}(\mathcal{G}_{\varphi(\boldsymbol{a})})$.

In the coherent case, these follow from Krull's theorem. For the algebraic and finite cases, see Lemma 12.13 and Corollary 12.17, and Remark 14.12, respectively.

(2) Suppose that (ii) above holds at each $\boldsymbol{a} \in X_\phi^s$. Then the conclusion of Theorem B can be strengthened as follows: Let $\boldsymbol{a}_0 \in X_\lambda - X_{\lambda+1}$ and let $g^i = G_{\boldsymbol{a}_0}^i$, $i = 1, \dots, t$. Then, for each

i, g^i is convergent and $\hat{g}^i_{\varphi(a)} = \mathrm{G}^i_a$ for all a in a suitable neighborhood of a_0 in $\mathrm{X}_\lambda - \mathrm{X}_{\lambda+1}$ (cf. Theorem 9.1).

2.12. *Local formulation of Theorems A - C.*

The assertions of Theorems A - C are local in X^s_ϕ. Replacing our analytic spaces and coherent sheaves by appropriate local models, we can assume that the module of formal relations $\mathscr{R}_a$ is given as follows:

Let M and N denote analytic manifolds over $\mathbf{K}$, and let $\phi : \mathrm{M} \longrightarrow \mathrm{N}$ be an analytic mapping. Let A (respectively, B) be a $p \times q$ (respectively, $p \times r$) matrix whose entries are analytic functions on M. If $a \in \mathrm{M}$, let A_a (respectively, B_a) denote the matrix of elements of $\mathscr{O}_a$ induced by A (respectively, B). If $\mathrm{G} = (\mathrm{G}_1, \ldots, \mathrm{G}_q) \in \hat{\mathscr{O}}^q_{\phi(a)}$, we write $\mathrm{G} \circ \hat{\phi}_a$ for $(\hat{\phi}^*_a(\mathrm{G}_1), \ldots, \hat{\phi}^*_a(\mathrm{G}_q))$. If $a \in \mathrm{M}^s_\phi$, $a = (a^1, \ldots, a^s)$, put $\mathscr{R}_a = \bigcap_{i=1}^{s} \mathscr{R}_{a^i}$, where $\mathscr{R}_{a^i} = \{\mathrm{G} \in \hat{\mathscr{O}}^q_{\varphi(a)} : \mathrm{A}_{a^i} \cdot (\mathrm{G} \circ \hat{\phi}_{a^i}) + \mathrm{B}_{a^i} \cdot \mathrm{H}^i = 0, \text{ for some } \mathrm{H}^i \in \hat{\mathscr{O}}^r_{a^i}\}$.

Theorems A - C will be reformulated and proved in this local context. Our problems, from this point of view, concern the solution of a system of equations of the form

$$f(x) = \mathrm{A}(x) \cdot g(\phi(x)) + \mathrm{B}(x) \cdot h(x),$$

where $f = (f_1, \ldots, f_p)$ is given and $g = (g_1, \ldots, g_q)$ and $h = (h_1, \ldots, h_r)$ are the unknown functions.

3. Geometry of subanalytic sets.

Our conjecture that the Hilbert-Samuel function H_a is always semicontinuous has the following consequence: Let N be a real analytic manifold (i.e., a smooth real analytic space) and let E be a closed subanalytic subset of N. Then the points of E near which E is not semianalytic form a closed subanalytic subset C of E.

To prove this, we consider the class of *Nash subanalytic* subsets of N; i.e., the images of proper real analytic mappings $\phi : M \longrightarrow N$ such that M is smooth and ϕ is regular [5], [7]. Every closed semianalytic set is Nash. The non-semianalytic (respectively, non-Nash) points of E, i.e., the points which do not admit neighborhoods in which E is semianalytic (respectively, Nash) form a closed subset of E.

Let $\phi : M \longrightarrow N$ be a proper real analytic mapping, where M is smooth. If $r_2(a) = \dim \hat{\mathcal{O}}_{\phi(a)}/\mathrm{Ker}\,\hat{\phi}_a^*$ is Zariski semi-continuous on M, then $\Sigma = \{a \in M : r_2(a) > r_1(a)\}$ is analytic. By 2.9(*i*), $\Sigma = \{a \in M : \phi$ is not regular at $a\}$. If $r_1(a)$ is constant on M, then $\phi(\Sigma)$ is the non-Nash points of $\phi(M)$.

If E has pure dimension k, then there exists ϕ as above such that $\phi(M) = E$ and $r_1(a) = k$ for all $a \in M$ [21](3); therefore, the subset of non-Nash points of E is subanalytic. The same conclusion follows in general(4).

Put $E_0 = E$. Inductively, let $E_{\lambda+1}$ denote the complement in E_λ of the smooth points of E_λ of the highest dimension. Then each E_λ is subanalytic [36]. Let $b \in E$. We claim that E is semianalytic near b if and only if each E_λ is Nash near b. Indeed, if E is semianalytic near b, then so is each E_λ [26]. Suppose each E_λ is Nash near b. In a suitable neighborhood of b, there are closed analytic sets Z_λ such that $E_\lambda \subset Z_\lambda$ and $\dim E_\lambda = \dim Z_\lambda$. Let Sing Z_λ denote the singular points of Z_λ. Then $D_\lambda = E_\lambda - (Z_{\lambda+1} \cup \mathrm{Sing}\, Z_\lambda)$ is open and closed in $Z_\lambda - (Z_{\lambda+1} \cup \mathrm{Sing}\, Z_\lambda)$. Thus D_λ is semi-analytic [26]. If $E_{\lambda+1}$ is semianalytic at b, then $E_\lambda = \overline{D}_\lambda \cup E_{\lambda+1}$ is too. By induction, $E = E_0$ is semianalytic near b.

For each λ, let C_λ denote the non-Nash point of E_λ; C_λ is closed and subanalytic. The subset of non-semianalytic points of E is $C = \cup C_\lambda$.

4. Differentiable functions: Theorems D, E.

Suppose that X and Y are smooth real analytic spaces and that $\phi : X \longrightarrow Y$ is a morphism. Let A and B be $p \times q$ and $p \times r$

(3) For a simple proof, see E. Bierstone and P.D. Milman, Semianalytic and subanalytic sets (to appear).

(4) This has been proved by W. Pawlucki, Points de Nash des ensembles sous-analytiques, *Mem. Amer. Math. Soc.* (to appear).

matrices, respectively, whose entries are real analytic functions on X. Let $\Phi : \mathscr{C}^\infty(Y)^q \longrightarrow \mathscr{C}^\infty(X)^p$ denote the homomorphism of modules over $\phi^* : \mathscr{C}^\infty(Y) \longrightarrow \mathscr{C}^\infty(X)$ defined by

$$\Phi(g)(x) = A(x) \cdot g(\phi(x)),$$

where $g \in \mathscr{C}^\infty(Y)^q$, and let $B \cdot : \mathscr{C}^\infty(X)^r \longrightarrow \mathscr{C}^\infty(X)^p$ denote the $\mathscr{C}^\infty(X)$-homomorphism induced by multiplication by the matrix B.

There is also an induced homomorphism $\Phi : \mathcal{O}_Y^q \longrightarrow \mathcal{O}_X^p$ over $\phi^* : \mathcal{O}_Y \longrightarrow \mathcal{O}_X$, and an induced $\mathcal{O}_X$-homomorphism $B : \mathcal{O}_X^r \longrightarrow \mathcal{O}_X^p$. Let $\mathscr{F} = \text{Coker } B$, $\mathscr{G} = \mathcal{O}_Y^q$, and let $\Psi : \mathscr{G} \longrightarrow \mathscr{F}$ denote the ϕ^*-homomorphism induced by Φ. Let $(\Phi\, \mathscr{C}^\infty(Y)^q + B \cdot \mathscr{C}^\infty(X)^r)^\wedge$ be as in the introduction.

If Z is a closed subset of Y, let $\mathscr{I}(Y\,;Z)$ denote the ideal in $\mathscr{C}^\infty(Y)$ of functions which vanish on Z together with their partial derivatives of all orders.

THEOREM D. – *Let* X, Y, ϕ, A, B, Φ *and* Ψ *be as above. Suppose that* ϕ *is proper. Let* s *be a positive integer, and assume any of the equivalent conditions of Theorem A on* Ψ. *Then, if* $f \in (\Phi\mathscr{C}^\infty(Y)^q + B \cdot \mathscr{C}^\infty(X)^r)^\wedge$, *there exists* $g \in \mathscr{C}^\infty(Y)^q$ *and* $h \in \mathscr{C}^\infty(X)^r$ *such that*

$$f(x) = A(x) \cdot g(\phi(x)) + B(x) \cdot h(x), \qquad (4.1)$$

for all $x \in X$. *If, moreover,* Z *is a closed subanalytic subset of* Y *and* $\hat{f}_a \in \operatorname{Im} \hat{B}_a$ *for all* $a \in \phi^{-1}(Z)$, *then (4.1) is satisfied with* $g \in \mathscr{I}(Y\,;Z)^q$.

Remark 4.2. – According to Remark 2.6, any positive integer s, e.g., $s = 1$, serves in the hypotheses of Theorem D. Nevertheless, our proof depends on a suitable choice of s: Let $\mathscr{B}$ denote the image of $B : \mathcal{O}_X^r \longrightarrow \mathcal{O}_X^p$. Then $\mathscr{B}$ is a coherent $\mathcal{O}_X$-module. By Theorem C(4), each point of X admits a coordinate neighborhood U and a filtration of U by closed analytic subsets,

$$U = X_0 \supset X_1 \supset \ldots \supset X_{t+1} = \phi,$$

such that the diagram of initial exponents $\mathfrak{N}(\mathscr{B}_a)$ is constant on $X_\lambda - X_{\lambda+1}$, $\lambda = 0, \ldots, t$. For U small enough, there is a bound on the number of connected components E of each

$$(X_\lambda - X_{\lambda+1}) \cap \phi^{-1}(b),\ b \in Y$$

(cf. Corollary 11.6). Moreover, the module of formal relations $\operatorname{Ker}\hat{\Psi}_a$ is constant on each such E (Proposition 11.1). (For example, if $B = 0$, then $\operatorname{Ker}\hat{\Psi}_a$ is constant on each connected component of a fiber $\phi^{-1}(b)$.)

It follows that, if ϕ is proper, then (locally in Y) there is a bound s on the number of distinct submodules $\operatorname{Ker}\hat{\Psi}_a$ of $\hat{\mathcal{O}}_b^q$, where $a \in \phi^{-1}(b)$. It is with this s that we prove Theorem D.

With s as above, $(\Phi\,\mathscr{C}^\infty(Y)^q + B \cdot \mathscr{C}^\infty(X)^r)^\wedge = \{f \in \mathscr{C}^\infty(X)^p :$ for all $\boldsymbol{a} \in X_\phi^{s+1}$, $\boldsymbol{a} = (a^1, \ldots, a^{s+1})$, there exists $G_{\boldsymbol{a}} \in \hat{\mathcal{O}}^q_{\phi(a^1)}$ such that $\hat{f}_{a^i} - \hat{\phi}_{a^i}(G_{\boldsymbol{a}}) \in \operatorname{Im}\hat{B}_{a^i}$, $i = 1, \ldots, s+1\}$.

As a consequence of Theorems D and C(1), we obtain:

THEOREM E. – *Suppose that* M *and* N *are Nash manifolds and that* $\phi : M \longrightarrow N$ *is a Nash mapping. Let* A *and* B *be* $p \times q$ *and* $p \times r$ *matrices of Nash functions on* M, *respectively. Define* $\Phi : \mathscr{C}^\infty(N)^q \longrightarrow \mathscr{C}^\infty(M)^p$ *by* $\Phi(g) = A \cdot (g \circ \phi)$, *where* $g \in \mathscr{C}^\infty(N)^q$, *and* $B\,\cdot : \mathscr{C}^\infty(M)^r \longrightarrow \mathscr{C}^\infty(M)^p$ *by multiplication by* B. *If* ϕ *is proper, then*

$$\Phi\mathscr{C}^\infty(N)^q + B \cdot \mathscr{C}^\infty(M)^r = (\Phi\mathscr{C}^\infty(N)^q + B \cdot \mathscr{C}^\infty(M)^r)^\wedge .$$

From Theorems D and C(2), we get the composition theorem of [5], which generalizes earlier results of Glaeser [16], Schwarz [34] and Tougeron [38]. From C(4) and (3), we recover, respectively, Malgrange's theorem on ideals generated by analytic functions [27, Ch. VI] and a result of Merrien [28].

Remarks 4.3. – We use the notation of the beginning of this section. Let $\boldsymbol{a} \in X_\phi^s$, $\boldsymbol{a} = (a^1, \ldots, a^s)$.

(1) There is an analogue of Theorem D for germs of $\mathscr{C}^\infty$ functions at $\{a^1, \ldots, a^s\}$, without the hypothesis that ϕ is proper. The conclusion is a $\mathscr{C}^\infty$ version of 2.11 (1) (ii).

(2) Let $\mathscr{R}_{\boldsymbol{a}}^\infty$ denote the submodule of $\operatorname{Ker}\hat{\boldsymbol{\Psi}}_{\boldsymbol{a}}$ generated by $\mathscr{C}^\infty$ relations at $\boldsymbol{a}$; i.e., q-tuples $g = (g_1, \ldots, g_q)$ of germs of $\mathscr{C}^\infty$ functions at $\boldsymbol{\varphi}(\boldsymbol{a}) \in Y$ such that $A \cdot (g \circ \phi)$ vanishes modulo $\operatorname{Im} B\,\cdot$ as a germ at $\{a^1, \ldots, a^s\}$. Then

$$\operatorname{Ker}\boldsymbol{\Psi}_{\boldsymbol{a}} \subset \mathscr{R}_{\boldsymbol{a}}^\infty \subset \operatorname{Ker}\hat{\boldsymbol{\Psi}}_{\boldsymbol{a}} . \tag{4.4}$$

The conclusion of Theorem D implies: Every formal relation (i.e., every element of $\operatorname{Ker} \hat{\Psi}_a$) is the formal Taylor expansion at $\varphi(a)$ of some $\mathscr{C}^\infty$ relation. In each case of Theorem C, this is also a consequence of 2.11 (1) (i) and (4.4).

Example 4.5. – The Malgrange preparation theorem. Let $P(t, \lambda)$ denote the polynomial $t^d + \sum_{j=1}^{d} \lambda_j t^{d-j}$ of degree d in t, with generic coefficients $\lambda = (\lambda_1, \ldots, \lambda_d)$. Then every $\mathscr{C}^\infty$ function $f(x, t) = f(x_1, \ldots, x_n, t)$ can be written

$$f(x, t) = P(t, \lambda) \cdot Q(x, t, \lambda) + \sum_{j=1}^{d} R_j(x, \lambda) t^{d-j},$$

where $Q, R_1, \ldots, R_d$ are $\mathscr{C}^\infty$ [27, Ch. V]. This follows from Theorem E, where $M = N = \mathbf{R}^{n+d}$, $\phi : M \longrightarrow N$ is the mapping

$$\phi(x, t, \lambda_1, \ldots, \lambda_{d-1}) = \left(x, \lambda_1, \ldots, \lambda_{d-1}, -t^d - \sum_{j=1}^{d-1} \lambda_j t^{d-j}\right),$$

$B = 0$ and $A(x, t, \lambda_1, \ldots, \lambda_{d-1})$ is the $1 \times d$ matrix $(1 \ t \ldots t^{d-1})$. Indeed, by the formal Weierstrass division theorem and an easy interpolation argument, $\mathscr{C}^\infty(M) = (\Phi \mathscr{C}^\infty(N)^d)^\wedge$. Therefore, given $f(x, t)$ $\mathscr{C}^\infty$, there exist $\mathscr{C}^\infty$ functions $R_j(x, \lambda)$, $j = 1, \ldots, d$, such that $f = \sum_{j=1}^{d} t^{d-j} \cdot (R_j \circ \phi)$. Hence $f(x, t) - \sum_{j=1}^{d} t^{d-j} R_j(x, \lambda)$ is divisible by $t^d + \sum_{j=1}^{d-1} \lambda_j t^{d-j} + \lambda_d = P(t, \lambda)$.

In this example, Zariski semicontinuity of the diagram of initial exponents $\mathfrak{N}_a$ is not difficult to show directly: We can use $s = d$. Then M_ϕ^d can be identified with

$$\{\boldsymbol{x} = (x, t_1, \ldots, t_d, \lambda_1, \ldots, \lambda_{d-1}) \in \mathbf{R}^{n+2d-1} :$$

$$t_k^d + \sum_{j=1}^{d-1} \lambda_j t_k^{d-j} = t_\ell^d + \sum_{j=1}^{d-1} \lambda_j t_\ell^{d-j} \text{ for each } k, \ell\}.$$

If $x = (x, t_1, \ldots, t_d, \lambda_1, \ldots, \lambda_{d-1}) \in M^d_\phi$, put

$$c(x) = -t_1^d - \sum_{j=1}^{d-1} \lambda_j t_1^{d-j}$$

and let $m(x)$ denote the sum of the multiplicities of the distinct roots t_i of the polynomial $p(z, x) = z^d + \sum_{j=1}^{d-1} \lambda_j z^{d-j} + c(x)$; i.e., $m(x)$ is the degree of the greatest common divisor in $\mathbf{R}[z]$ of the polynomials $p(z, x)$ and $q(z, x) = \prod_{i=1}^{d} (z - t_i)^d$. It follows from the Euclidean division algorithm that, for all $m \in \mathbf{N}$,

$$\{x \in M^d_\phi : m(x) \geqslant m\}$$

is a closed algebraic subset of M^d_ϕ. On the other hand, if $x \in M^d_\phi$, then the vertices of $\mathfrak{N}_x \subset \mathbf{N}^{n+d} \times \{1, \ldots, d\}$ are precisely

$$(\beta_i, j_i) = (0, d - i + 1), \quad i = 1, \ldots, d - m(x)$$

(cf. Example 8.5.1).

Remark 4.6. – The conclusion of Theorem D implies that $\Phi\, \mathscr{C}^\infty(Y)^q + B \cdot \mathscr{C}^\infty(X)^r$ is a closed subspace of $\mathscr{C}^\infty(X)^p$, with the $\mathscr{C}^\infty$ topology. Using Theorem B and techniques of [7], we can prove, moreover, that, under the hypotheses of Theorem D, the canonical surjection

$$\mathscr{C}^\infty(Y)^q \oplus \mathscr{C}^\infty(X)^r \longrightarrow \Phi\, \mathscr{C}^\infty(Y)^q + B \cdot \mathscr{C}^\infty(X)^r$$

admits a continuous linear splitting. Details of this result will appear elsewhere ([5]).

5. Organization of the paper.

Throughout this article, we exploit the elementary but powerful formal division algorithm of Grauert [18] and Hironaka [1], [8], which is recalled in § 6.

(5) E. Bierstone and P.D. Milman, Local analytic invariants and splitting theorems in differential analysis, *Israel J. Math.* (to appear).

Section 7 contains a variant of the classical coherent case C(4). Let X be a closed analytic set and let $\mathcal{M}(X, Z)$ denote the ring of meromorphic functions on X with poles in a proper closed analytic subset Z (i.e., analytic functions on $X - Z$ whose germs at each $a \in Z$ are induced by complex meromorphic functions (cf. [32, Ch. IV, § 5]) on a local complexification $X_a^{\mathbf{C}}$ of X whose poles lie in $Z_a^{\mathbf{C}}$). For modules generated by power series with coefficients in $\mathcal{M}(X, Z)$, we give elementary proofs of the conditions analogous to A(1)-(4). Power series with meromorphic coefficients arise, for example, in Theorem B. The results of § 7 are needed in §§ 9, 10, 13 and 14. The techniques illustrate the utility of the diagram of initial exponents: consequences of Lemma 7.2 include Zariski semicontinuity of the Hilbert-Samuel functions associated to a coherent sheaf or to the fibers of an analytic morphism (Lejeune-Teissier [25, Ch. I, Thme. 8.2.9]) as well as the generic flatness theorem of Hironaka [22, Rmk. 2.6].

The constructions (and the notation) of § 8 are central to the article. Theorems A and B are proved here and in the following section.

Chapters II and III are independent. Our results on $\mathscr{C}^\infty$ functions are placed in Chapter II; Theorem D is proved in § 11. The proof uses Malgrange's theorem on ideals generated by analytic functions. However, we give an elementary proof of the latter in § 10, as an immediate application of § 7 and the formal division algorithm. Consequently, the only results used to prove Theorem D, apart from the techniques in analytic geometry developed here, are Whitney's extension theorem [27, 1.4.1], Łojasiewicz's inequality [27, IV.4.1] and an estimate of Glaeser [16, §§ 4,5], [37, pp. 180-181].

The combinatorics of the diagram of initial exponents bears on questions of convergence or differentiability in the following way (cf. Corollary 7.7 and § 10): In the notation of 1.4, let $G \in \hat{\mathcal{O}}^q_{\varphi(a)} = \mathbf{K}[[y]]^q$, $G = (G_1, \ldots, G_q)$. By the formal division algorithm, G has a unique representative modulo $\mathscr{R}_a$ such that $\operatorname{supp} G \cap \mathfrak{N}_a = \emptyset$. If $\beta = (\beta_1, \ldots, \beta_n) \in \mathbf{N}^n$, let D^β denote formal differentiation of order β. If $\operatorname{supp} G \cap \mathfrak{N}_a = \emptyset$, then $D^\beta G_j = 0$ for all $(\beta, j) \in \mathfrak{N}_a$; in particular, $\operatorname{supp} D^\beta G \cap \mathfrak{N}_a = \emptyset$ for all $\beta \in \mathbf{N}^n$.

Chapter III includes cases (1) - (3) of Theorem C. Our proof, in each case, is presented for arbitrary $s \in \mathbf{N}$. Although $s = 1$ suffices, by Remark 2.6, the general setting involves no extra cost and provides the most direct route to the corresponding result on differentiable functions.

The algebraic hypothesis in Theorem C(1) is essential only to the following point in our proof: With reference to 2.11 (1) (i) above, we show that any $G \in \mathscr{R}_a = \operatorname{Ker} \hat{\Psi}_a$ can be approximated to any order by algebraic relations. In the local representation of 2.12, this amounts to considering a system of equations of the form

$$A(x) \cdot g(y) + B(x) \cdot h(x, y) = \sum_{i=1}^{n} q_i(x, y)(y_i - \phi_i(x)),$$

where A, B and ϕ are algebraic, and finding an algebraic approximation $g(y)$, $h(x, y)$, $q_i(x, y)$ to a given formal solution. Since the equations are linear in h and the q_i, this special case of "Artin approximation with respect to nested subrings" follows from Artin's theorem [2]; cf. Theorem 12.6. (A general version has recently been proved by Popescu [33]). Example 2.8 shows that the corresponding assertion in the analytic category is false (cf. Remark 12.8).

CHAPTER I

VARIATION OF FORMAL RELATIONS

6. Preliminaries: the formal division algorithm.

Let $\mathbf{K}$ be a field and let $\mathbf{K}[[y]]$ denote the ring of formal power series in $y = (y_1, \ldots, y_n)$ with coefficients in $\mathbf{K}$. We use the notation of 1.4.

Let $g^1, \ldots, g^t \in \mathbf{K}[[y]]^p$ and let $(\beta_i, j_i) = \nu(g^i)$, $i = 1, \ldots, t$. We associate to $g^1, \ldots, g^t$ the following decomposition of $\mathbf{N}^n \times \{1, \ldots, p\}$: Set $\Delta_0 = \emptyset$ and define

$$\Delta_i = ((\beta_i, j_i) + \mathbf{N}^n) - \bigcup_{0 \leq k < i} \Delta_k, \quad i = 1, \ldots, t.$$

Put $\Delta = \mathbf{N}^n \times \{1, \ldots, p\} - \bigcup_{1 \leq i \leq t} \Delta_i$.

Clearly, for every $f \in \mathbf{K}[[y]]^p$, there exist unique

$$q_i^0 \in \mathbf{K}[[y]],\ i = 1, \ldots, t, \text{ and } r^0 \in \mathbf{K}[[y]]^p$$

such that

$$\begin{aligned} (\beta_i, j_i) + \operatorname{supp} q_i^0 &\subset \Delta_i, \quad i = 1, \ldots, t, \\ \operatorname{supp} r^0 &\subset \Delta, \end{aligned} \tag{6.1}$$

and
$$f = \sum_{i=1}^{t} q_i^0\, y^{\beta_i, j_i} + r^0.$$

THEOREM 6.2 (Grauert [18, § 2], Hironaka [1, Ch. 1, § 1], [8]). – *Let* $g^1, \ldots, g^t \in \mathbf{K}[[y]]^p$ *and let* $(\beta_i, j_i) = \nu(g^i)$, $i = 1, \ldots, t$. *Then, for every* $f \in \mathbf{K}[[y]]^p$, *there exist unique* $q_i \in \mathbf{K}[[y]]$, $i = 1, \ldots, t$, *and* $r \in \mathbf{K}[[y]]^p$ *such that*

$$\begin{aligned} (\beta_i, j_i) + \operatorname{supp} q_i &\subset \Delta_i,\ i = 1, \ldots, t, \\ \operatorname{supp} r &\subset \Delta, \end{aligned} \tag{6.3}$$

and
$$f = \sum_{i=1}^{t} q_i g^i + r.$$

Proof. – *Uniqueness.* By (6.3), $\nu(q_i g^i) \subset \Delta_i$, $i = 1, \ldots, t$, and $\nu(r) \subset \Delta$. Thus $\operatorname{in} f$ is one of the $\operatorname{in}(q_i g^i) = \operatorname{in} q_i \cdot \operatorname{in} g^i$ or $\operatorname{in} r$, since their exponents lie in disjoint regions of $\mathbf{N}^n \times \{1, \ldots, p\}$. Therefore, if the $\operatorname{in} q_i$ and $\operatorname{in} r$ do not all vanish, neither does f.

Algorithm. Write $g^i = \sum_{\beta, j} g^i_{\beta, j}\, y^{\beta, j}$, $i = 1, \ldots, t$. Suppose $f \in \mathbf{K}[[y]]^p$. Let q_i^0, r^0 be as in (6.1). Put

$$q_i(f) = (g^i_{\beta_i, j_i})^{-1} \cdot q_i^0 \in \mathbf{K}[[y]], \quad i = 1, \ldots, t,$$

and $r(f) = r^0 \in \mathbf{K}[[y]]^p$. Let

$$\mathrm{E}(f) = f - \sum_{i=1}^{t} q_i(f)\, g^i - r(f)$$

$$= \sum_{i=1}^{t} q_i(f) \cdot (g^i_{\beta_i, j_i}\, y^{\beta_i, j_i} - g^i).$$

For each $i = 1, \ldots, t$,

$\nu(q_i(f) \cdot (g^i_{\beta_i, j_i}\, y^{\beta_i, j_i} - g^i))$

$$= \nu(g^i_{\beta_i, j_i}\, y^{\beta_i, j_i} - g^i) + \nu(q_i(f)) > (\beta_i, j_i) + \nu(q_i(f)) \geqslant \nu(f).$$

Therefore, $\nu(\mathrm{E}(f)) > \nu(f)$. Define

$$q_i = \sum_{j=0}^{\infty} q_i(\mathrm{E}^j(f)) \quad \text{and} \quad r = \sum_{j=0}^{\infty} r(\mathrm{E}^j(f)), \tag{6.4}$$

where $\mathrm{E}^0(f) = f$ and $\mathrm{E}^j(f) = \mathrm{E}(\mathrm{E}^{j-1}(f))$, $j \geqslant 1$. It follows that these series converge in the Krull topology and that $\nu(f - \Sigma\, q_i g^i - r) > \nu(\mathrm{E}^j(f))$ for every $j \in \mathbf{N}$; thus $f = \Sigma\, q_i g^i + r$.

□

Remark 6.5 – Let A be an integral domain. Suppose that $\mathbf{K}$ is the field of fractions of A. Let $\mathrm{A}[[y]]$ denote the subring of $\mathbf{K}[[y]]$ of formal power series with coefficients in A. Suppose that $g^1, \ldots, g^t \in \mathrm{A}[[y]]^p$. Let S denote the multiplicative subset of A generated by the $g^i_{\beta_i, j_i}$, and let $\mathrm{S}^{-1}\mathrm{A}$ denote the corresponding localization of A; i.e., the subring of $\mathbf{K}$ comprising quotients with denominators in S. Then $\mathrm{S}^{-1}\mathrm{A}[[y]] \subset \mathbf{K}[[y]]$. By (6.4), if $f \in \mathrm{A}[[y]]^p$, then $q_i \in \mathrm{S}^{-1}\mathrm{A}[[y]]$, $i = 1, \ldots, t$, and $r \in \mathrm{S}^{-1}\mathrm{A}[[y]]^p$.

In fact, if A is any ring and each $g^i_{\beta_i, j_i} = 1$, the formal division algorithm applies to give quotients and remainder with coefficients in A.

Remark 6.6. – From the proof of Theorem 6.2, $\nu(r) \geqslant \nu(f)$ and $\nu(g^i) + \nu(q_i) \geqslant \nu(f)$, $i = 1, \ldots, t$. Let $\hat{\mathfrak{m}}$ denote the maximal ideal of $\mathbf{K}[[y]]$. It follows that, for every $k \in \mathbf{N}$, if $f \in \hat{\mathfrak{m}}^k \cdot \mathbf{K}[[y]]^p$, then $r \in \hat{\mathfrak{m}}^k \cdot \mathbf{K}[[y]]^p$ and each $q_i \in \hat{\mathfrak{m}}^{k - |\beta_i|}$ (where $\hat{\mathfrak{m}}^\ell = \mathbf{K}[[y]]$ if $\ell \leqslant 0$).

Remark 6.7. – Assume that $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Suppose that f and $g^1, \ldots, g^t$ converge. Then the q_i and r all converge [1, Ch. 1, § 1], [8], [15, Ch. 1, § 3].

COROLLARY 6.8. – *Let* R *be a submodule of* $\mathbf{K}[[y]]^p$. *Let* $\mathfrak{N} = \mathfrak{N}(\mathrm{R})$ *be the diagram of initial exponents of* R, *and let* (β_i, j_i), $i = 1, \ldots, t$, *denote the vertices of* $\mathfrak{N}$ *(without repetitions). Choose* $g^i \in \mathrm{R}$ *such that* $\nu(g^i) = (\beta_i, j_i)$, $i = 1, \ldots, t$. *Then:*

(1) $\mathfrak{N} = \bigcup_{i=1}^{t} \Delta_i$, *and* $g^1, \ldots, g^t$ *generate* R.

(2) *There is a unique set of generators* $f^1, \ldots, f^t$ *of* R *such that, for each* i, in $f^i = y^{\beta_i, j_i}$ *and* $\operatorname{supp}(f^i - y^{\beta_i, j_i}) \cap \mathfrak{N} = \emptyset$. *If* $\mathbf{K} = \mathbf{R}$ *or* $\mathbf{C}$, *and* R *is generated by convergent elements, then each* f^i *converges.*

We call $f^1, \ldots, f^t$ in (2) the *standard basis* of R.

Proof of Corollary 6.8. – (1) is clear from Theorem 6.2. For (2), divide y^{β_i, j_i} by $g^1, \ldots, g^t$: By Theorem 6.2, there exist unique $q_k^i \in \mathbf{K}[[y]]$, $k = 1, \ldots, t$, and $r^i \in \mathbf{K}[[y]]^p$ such that

$$y^{\beta_i, j_i} = \sum_{k=1}^{t} q_k^i g^k + r^i \quad \text{and} \quad (\beta_k, j_k) + \operatorname{supp} q_k^i \subset \Delta_k, \ \operatorname{supp} r^i \subset \dot{\Delta}.$$

Then $f^i = y^{\beta_i, j_i} - r^i$ satisfies the required conditions. The second assertion of (2) follows from Remark 6.7. □

COROLLARY 6.9. – *Let* R *be a submodule of* $\mathbf{K}[[y]]^p$. *Put* $\mathrm{E} = \mathbf{K}[[y]]^p/\mathrm{R}$. *Let* $\mathrm{H_E}$ *denote the Hilbert-Samuel function of* E; *i.e.,* $\mathrm{H_E}(k) = \dim_{\mathbf{K}} \mathrm{E}/\hat{\mathfrak{m}}^{k+1} \cdot \mathrm{E}$, $k \in \mathbf{N}$. *Then* $\mathrm{H_E}(k)$ *is the number of elements* $(\beta, j) \in \mathbf{N}^n \times \{1, \ldots, p\}$ *such that* $(\beta, j) \notin \mathfrak{N}(\mathrm{R})$ *and* $|\beta| \leqslant k$.

Proof. – By Remark 6.6. □

Remark 6.10. – (1) Let R and E be as in Corollary 6.9. Let $\mathfrak{V}$ denote the vertices of $\mathfrak{N}(\mathrm{R})$. It follows from Corollary 6.9 that $\mathrm{H_E}(k)$ coincides with a polynomial in k, for

$$k \geqslant \sum_{i=1}^{n} \max \{\beta_i : (\beta, j) \in \mathfrak{V}, \ \beta = (\beta_1, \ldots, \beta_n)\}$$

(the "Hilbert-Samuel polynomial").

(2) In particular, let $f(y) \in \mathbf{K}[[y]]$ and let $E = \mathbf{K}[[y]]/(f)$, where (f) is the principal ideal generated by $f(y)$. Let $\mu = |\nu(f)|$. Then $H_E(k) = \binom{k+n}{n}$ if $0 \leqslant k < \mu$, and

$$H_E(k) = \binom{k+n}{n} - \binom{k+n-\mu}{n} \text{ if } k \geqslant \mu .$$

Thus, $H_E(k)$ coincides with a polynomial of degree $n-1$ in k, for $k \geqslant \mu$, and the coefficient of k^{n-1} in this polynomial is $\mu/(n-1)!$.

7. Power series with meromorphic coefficients.

Let $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Let $\mathbf{K}[[y]]$ denote the ring of formal power series in $y = (y_1, \ldots, y_n)$, and let $\hat{\mathfrak{m}}$ denote the maximal ideal of $\mathbf{K}[[y]]$.

Let U be an open subset of $\mathbf{K}^m$ and let X be a closed analytic subset of U. Let Z be a proper closed analytic subset of X.

Let $\mathcal{M}(X;Z)$ denote the ring of meromorphic functions on X with poles in Z (cf. § 5). It is easy to see that $h \in \mathcal{M}(X;Z)$ if and only if $h \in \mathcal{O}(X - Z)$ and, locally in X, there exist finitely many pairs of analytic functions f_i, g_i such that $\cap g_i^{-1}(0) \subset Z$ and $h = f_i/g_i$ outside $Z \cup g_i^{-1}(0)$. If $Z = \phi$, then $\mathcal{M}(X;Z)$ is the ring $\mathcal{O}(X)$ of analytic functions on X. Let $a \in X - Z$. There is an evaluation mapping $h \longmapsto h(a)$ of $\mathcal{M}(X;Z)$ onto $\mathbf{K}$. If $g = \sum_{\beta,j} g_{\beta,j}\, y^{\beta,j} \in \mathcal{M}(X;Z)[[y]]^p$, we write $g(x;y) = \Sigma\, g_{\beta,j}(x)\, y^{\beta,j}$, and $g(a,y) = \Sigma\, g_{\beta,j}(a)\, y^{\beta,j}$ when the coefficients are evaluated at $x = a$. Let $\nu(g) \in \mathbf{N}^n \times \{1, \ldots, p\}$ denote the smallest (β,j) such that $g_{\beta,j} \in \mathcal{M}(X;Z)$ is non-zero. Put $\mathrm{in}\, g = g_{\beta,j}(x)\, y^{\beta,j}$, where $(\beta,j) = \nu(g)$.

Let $f^1, \ldots, f^q \in \mathcal{M}(X;Z)[[y]]^p$. Let $\mathcal{B}$ denote the submodule of $\mathcal{M}(X;Z)[[y]]^p$ generated over $\mathcal{M}(X;Z)[[y]]$ by $f^1, \ldots, f^q$. Put $\mathfrak{N} = \{\nu(g) : g \in \mathcal{B}\}$. Clearly, $\mathfrak{N} + \mathbf{N}^n = \mathfrak{N}$. If $a \in X - Z$, let $\mathcal{A}_a$ denote the submodule of $\mathbf{K}[[y]]^p$ generated by the $f^k(a;y)$; put $\mathfrak{N}_a = \mathfrak{N}(\mathcal{A}_a)$ and let H_a be the Hilbert-Samuel function of $\mathbf{K}[[y]]^p/\mathcal{A}_a$.

LEMMA 7.1. – *For all* $a \in X - Z$, $\mathfrak{N} \leqslant \mathfrak{N}_a$.

LEMMA 7.2. – *Assume that* $0 \in X$, *that the germ of* X *at* 0 *is irreducible, and that every connected component of the smooth points of* X *is adherent to* 0. *Then there is a proper analytic subset* Y *of* X *containing* Z *such that:*

(1) $\mathfrak{N}_a = \mathfrak{N}$ *for all* $a \in X - Y$. *In fact, for every vertex* (β, j) *of* $\mathfrak{N}$, *there exists* $g \in \mathcal{B}$ *such that* $\nu(g) = (\beta, j) = \nu(g(a;\cdot))$ *for all* $a \in X - Y$.

(2) H_a *is constant on* $X - Y$.

(3) *There exists* $\lambda \in \mathbf{N}$ *such that*

$$\mathcal{A}_a \cap \hat{\mathfrak{m}}^{\ell+\lambda} \cdot \mathbf{K}[[y]]^p = \hat{\mathfrak{m}}^{\ell} \cdot (\mathcal{A}_a \cap \hat{\mathfrak{m}}^{\lambda} \cdot \mathbf{K}[[y]]^p)$$

for all $\ell \in \mathbf{N}$ *and* $a \in X - Y$.

Proof of Lemma 7.1. – Let $a \in X - Z$. Let (α_i, j_i), $i = 1, \ldots, s$ (respectively, (β_i, k_i), $i = 1, \ldots, t$) denote the vertices of $\mathfrak{N}_a$ (respectively, $\mathfrak{N}$) indexed in ascending order.

Consider $h \in \mathcal{A}_a$ such that $\nu(h) = (\alpha_1, j_1)$. Say

$$h(y) = \sum_{\ell=1}^{q} c_\ell(y) f^\ell(a;y), \quad c_\ell(y) \in \mathbf{K}[[y]].$$

Define $g \in \mathcal{B}$ by $g(x,y) = \Sigma\, c_\ell(y) f^\ell(x,y)$. Then $\nu(g) \leqslant (\alpha_1, j_1)$ since, in any case, the coefficient of y^{α_1, j_1} is nonzero. Thus $(\beta_1, k_1) \leqslant \nu(g) \leqslant (\alpha_1, j_1)$. If $(\beta_1, k_1) = (\alpha_1, j_1)$, then $\nu(g) = (\alpha_1, j_1) = \nu(g(a;\cdot))$.

Now suppose that, for each $i = 1, \ldots, r$, we have:

(i) $(\beta_i, k_i) = (\alpha_i, j_i)$, and (ii) there exists $g^i(x;y) \in \mathcal{B}$ such that $\nu(g^i) = (\alpha_i, j_i) = \nu(g^i(a;\cdot))$. If $s = r$, we are done. Otherwise, consider $h(y) = \Sigma\, c_\ell(y) f^\ell(a;y) \in \mathcal{A}_a$ such that $\nu(h) = (\alpha_{r+1}, j_{r+1})$;

say in $h = y^{\alpha_{r+1}, j_{r+1}}$. Put $g(x;y) = \Sigma\, c_\ell(y) f^\ell(x;y) \in \mathscr{B}$. Then $\nu(g) \leqslant (\alpha_{r+1}, j_{r+1})$. If $\nu(g) = (\alpha_{r+1}, j_{r+1})$, then $(\beta_{r+1}, j_{r+1}) \leqslant (\alpha_{r+1}, j_{r+1})$. If $\nu(g) < (\alpha_{r+1}, j_{r+1})$, then either: (i) $\nu(g) \notin \bigcup_{i=1}^{r} (\alpha_i, j_i) + \mathbf{N}^n$ and $(\beta_{r+1}, j_{r+1}) < (\alpha_{r+1}, j_{r+1})$, or (ii) $\nu(g) \in \bigcup_{i=1}^{r} (\alpha_i, j_i) + \mathbf{N}^n$.

In the latter case, $\nu(g) = (\alpha_i + \gamma, j_i)$ for some $j = 1, \ldots, r$ and some $\gamma \in \mathbf{N}^n$. Then in $g = g_{\alpha_i+\gamma, j_i}(x) \cdot y^{\alpha_i+\gamma, j_i}$, where $g_{\alpha_i+\gamma, j_i}(a) = 0$ since $g(a;y) = h(y)$ and in $h = y^{\alpha_{r+1}, j_{r+1}}$. On the other hand, in $g^i = g^i_{\alpha_j, j_i}(x) \cdot y^{\alpha_j, j_i}$, where $g^i_{\alpha_j, j_i}(a) \neq 0$ since $\nu(g^i(a;\cdot)) = (\alpha_i, j_i)$. Let

$$g'(x;y) = g^i_{\alpha_i, j_i}(x) \cdot g(x;y) - g_{\alpha_i+\gamma, j_i}(x) \cdot y^\gamma \cdot g^i(x;y).$$

Then $g'(a;y) = g^i_{\alpha_i, j_i}(a) \cdot g(a;y)$, so that $\nu(g'(a;\cdot)) = (\alpha_{r+1}, j_{r+1})$ and $\nu(g) < \nu(g') \leqslant (\alpha_{r+1}, j_{r+1})$. The result follows by induction.

$\square$

Proof of Lemma 7.2. – Let (β_i, k_i), $i = 1, \ldots, t$, denote the vertices of $\mathfrak{N}$. For each i, choose $g^i \in \mathscr{B}$ such that $\nu(g^i) = (\beta_i, k_i)$. Put

$$\mathrm{Y} = \mathrm{Z} \cup \bigcup_{i=1}^{t} \{x \in \mathrm{X} : g^i_{\beta_i, k_i}(x) = 0\}.$$

Let $a \in \mathrm{X} - \mathrm{Y}$. Then $g'(a;y) \in \mathscr{A}_a$ and $\nu(g^i(a;\cdot)) = (\beta_i, k_i)$. Thus $\mathfrak{N} \subset \mathfrak{N}_a$. By Lemma 7.1, $\mathfrak{N}_a = \mathfrak{N}$. (2) follows, by Corollary 6.9.

Let $a \in \mathrm{X} - \mathrm{Y}$. Let $h \in \mathscr{A}_a$. By (1) and Remark 6.6,

$$h(y) = \sum_{i=1}^{t} c_i(y) g^i(a;y),$$

where, if $h \in \hat{\mathfrak{m}}^{\ell} \cdot \mathbf{K}[[y]]^p$, then each $c_i \in \hat{\mathfrak{m}}^{\ell - |\beta_i|}$. Put $\lambda = \max_i |\beta_i|$. Then each $c_i \in \hat{\mathfrak{m}}^{\ell-\lambda} \cdot \hat{\mathfrak{m}}^{\lambda - |\beta_i|}$. Thus

$$\mathscr{A}_a \cap \hat{\mathfrak{m}}^{\ell} \cdot \mathbf{K}[[y]]^p \subset \hat{\mathfrak{m}}^{\ell-\lambda} \cdot (\mathscr{A}_a \cap \hat{\mathfrak{m}}^{\lambda} \cdot \mathbf{K}[[y]]^p).$$

The opposite inclusion is trivial.

□

Let U be an open subset of $\mathbf{K}^n$ and let $\mathcal{O} = \mathcal{O}_{\mathrm{U}}$. Let $a \in \mathrm{U}$. We identify $\mathcal{O}_a$ (respectively, $\hat{\mathcal{O}}_a$) with the ring of convergent power series $\mathbf{K}\{y\}$ (respectively, the ring of formal power series $\mathbf{K}[[y]]$), where $y = (y_1, \ldots, y_n)$.

Remark 7.3. – Let $\mathscr{A} \subset \mathcal{O}^p$ be a coherent sheaf of $\mathcal{O}$-modules. Suppose there are $f^1, \ldots, f^q \in \mathcal{O}(\mathrm{U})^p$ which generate $\mathscr{A}_a$ for all $a \in \mathrm{U}$. For each $j = 1, \ldots, q$,

$$f^j(x+y) = \sum_{\beta} \mathrm{D}^{\beta} f^j(x) \cdot y^{\beta}/\beta!,$$

where $\mathrm{D}^{\beta} = \partial y^{|\beta|}/\partial y_1^{\beta_1} \ldots \partial y_n^{\beta_n}$ and $\beta! = \beta_1! \ldots \beta_n!$. Thus the f^j induce elements of $\mathcal{O}(\mathrm{U})[[y]]^p$ which, when evaluated at $a \in \mathrm{U}$, generate $\mathscr{A}_a$. In this case, Lemma 7.2 (1) holds with each $g(a; y) \in \mathbf{K}\{y\}$. Let $\hat{\mathscr{A}}_a$ denote the completion of $\mathscr{A}_a$; $\hat{\mathscr{A}}_a = \hat{\mathcal{O}}_a \cdot \mathscr{A}_a$.

THEOREM 7.4. – *Let* U *be an open subset of* $\mathbf{K}^n$, *and let* $\mathcal{O} = \mathcal{O}_{\mathrm{U}}$. *Let* $\mathscr{A} \subset \mathcal{O}^p$ *be a coherent sheaf of* $\mathcal{O}$*-modules. Then:*

(1) *The diagram of initial exponents* $\mathfrak{N}_a = \mathfrak{N}(\hat{\mathscr{A}}_a)$ *is Zariski semicontinuous on* U.

(2) *The Hilbert-Samuel function* H_a *of* $\mathcal{O}_a^p/\mathscr{A}_a$ *is Zariski semicontinuous on* U.

(3) Uniform Artin-Rees theorem. *For every compact subset* K *of* U, *there exists* $\lambda \in \mathbf{N}$ *such that*

$$\mathscr{A}_a \cap \mathfrak{m}_a^{\ell+\lambda} \cdot \mathcal{O}_a^p = \mathfrak{m}_a^{\ell} \cdot (\mathscr{A}_a \cap \mathfrak{m}_a^{\lambda} \cdot \mathcal{O}_a^p)$$

for all $\ell \in \mathbf{N}$ *and all* $a \in \mathrm{K}$.

Remark 7.5. – The Hilbert-Samuel function of $\hat{\mathcal{O}}_a^p/\hat{\mathscr{A}}_a$ equals H_a, by Krull's therorem. Likewise, the formal and convergent versions of the Artin-Rees theorem are equivalent. The uniform Artin-Rees theorem, as stated in Remark 2.10, clearly follows from 7.4 (3).

Remark 7.6. – Let A and B be submodules of $\mathbf{K}[[y]]^p$ and $\mathbf{K}[[y]]^q$, respectively. Put $F = \mathbf{K}[[y]]^p/A$ and $G = \mathbf{K}[[y]]^q/B$. Let $\Phi : G \longrightarrow F$ be a $\mathbf{K}[[y]]$-homomorphism, and let $H = \operatorname{Im}\Phi$, $R = \operatorname{Ker}\Phi$. Let $\ell, \lambda \in \mathbf{N}$. It is easy to see that the following conditions are equivalent:

(1) (Artin-Rees estimate) $H \cap \hat{\mathfrak{m}}^{\ell+\lambda} \cdot F \subset \hat{\mathfrak{m}}^{\ell} \cdot H$.

(2) (Chevalley estimate) $\Phi^{-1}(\hat{\mathfrak{m}}^{\ell+\lambda} \cdot F) \subset R + \hat{\mathfrak{m}}^{\ell} \cdot G$.

Proof of Theorem 7.4. – We can assume there are $f^1, \ldots, f^q \in \mathcal{O}(U)^p$ which generate $\mathcal{A}_a$ for all $a \in U$. Let X be a closed analytic subset of U. Assume that $0 \in X$, that the germ of X at 0 is irreducible, and that every connected component of the smooth points of X is adherent to 0. We apply Lemmas 7.1 and 7.2 with $Z = \phi$, and $\mathcal{B} \subset \mathcal{O}(X)[[y]]^p$ generated by the elements induced by the f^i, as in Remark 7.3.

Let $a \in Y$, where Y is given by Lemma 7.2. Let $\mathcal{B}_a$ denote the submodule of $\mathcal{O}_{X,a}[[y]]^p$ induced by the f^j. Then $\mathfrak{N} = \mathfrak{N}(\mathcal{B}_a)$, by 7.2 (1), and $\mathfrak{N}(\mathcal{B}_a) \leqslant \mathfrak{N}_a$, by Lemma 7.1. The assertion (1) follows.

Since $\mathcal{A}$ is coherent, then $H_a(k)$ is topologically upper-semicontinuous, for each fixed k (cf. [37, II.5.3]). Then (2) follows from (1) and Corollary 6.9. Finally, (3) follows from 7.2 (3).

□

COROLLARY 7.7. – *Let* U *and* $\mathcal{A}$ *be as in Theorem 7.4. Let* $a_0 \in U$. *Then there is a neighborhood* V *of* a_0 *and a filtration of* V *by closed analytic subsets,* $V = X_0 \supset X_1 \supset \ldots \supset X_{\ell+1} = \phi$, *such that, for each* $k = 0, \ldots, \ell$:

(1) $\mathfrak{N}_a = \mathfrak{N}(\hat{\mathcal{A}}_a)$ *is constant on* $X_k - X_{k+1}$.

(2) *Let* g_a^i, $i = 1, \ldots, t$, *denote the standard basis of* $\hat{\mathcal{A}}_a \subset \mathbf{K}[[y]]^p$, *where* $a \in X_k - X_{k+1}$. *Then each* g_a^i *converges. Write* $g_a^i(y) = \sum_{\beta, j} g_{\beta,j}^i(a)\, y^{\beta,j}$. *Then each* $g_{\beta,j}^i$ *is a meromorphic function on* X_k *with poles in* X_{k+1}.

(3) *There exist (p-tuples of) analytic functions* g^i *defined in a neighborhood of* $X_k - X_{k+1}$, *whose power series expansions at each* $a \in X_k - X_{k+1}$ *are the* g^i_a.

Proof. – (1) and (2) follow from 7.4 (1), Remark 6.5 and Corollary 6.8. For b in a sufficiently small neighborhood of a in $X_k - X_{k+1}$, $g^i_a(b - a + y) \in \mathscr{A}_b$. If in $g^i_a = y^{\beta_i, j_i}$, then $\operatorname{supp}(g^i_a(b - a + y) - y^{\beta_i, j_i}) \cap \mathfrak{N}_b = \phi$, since $\mathfrak{N}_a = \mathfrak{N}_b$ and $\operatorname{supp}(g^i_a(y) - y^{\beta_i, j_i}) \cap \mathfrak{N}_a = \phi$. Hence in $g^i_a(b - a + y) = y^{\beta_i, j_i}$ and, by the uniqueness of formal division, $g^i_a(b - a + y) = g^i_a(y)$. (3) follows. $\sqcup$

Let U be an open subset of $\mathbf{K}^m$, and let $Z \subset X$ denote closed analytic subsets of U. We conclude this section with some remarks on relations among elements of $\mathscr{M}(X ; Z)[[y]]^p$, where $y = (y_1, \ldots, y_n)$.

Let $f^1, \ldots, f^q \in \mathscr{M}(X ; Z)[[y]]^p$. For each $a \in X - Z$, let $\mathscr{R}_a \subset \mathbf{K}[[y]]^q$ denote the module of relations among the $f^j(a ; y)$; i.e., $\mathscr{R}_a = \{g(y) = (g_1(y), \ldots, g_q(y)) \in \mathbf{K}[[y]]^q$ such that $\sum_j g_j(y) f^j(a ; y) = 0\}$.

PROPOSITION 7.8. – *Let* $a_0 \in X$. *Then there is a neighborhood* V *of* a_0 *in* U, *and a filtration of* $X \cap V$ *by closed analytic subsets,*

$$X \cap V = X_0 \supset X_1 \supset \ldots \supset X_{t+1} = Z \cap V,$$

satisfying the following property: For each $k = 0, \ldots, t$, *there are finitely many elements* $g^j_k \in \mathscr{M}(X_k ; X_{k+1})[[y]]^q$ *such that the* $g^j_k(a ; y)$ *generate* $\mathscr{R}_a$, *for all* $a \in X_k - X_{k+1}$.

Proposition 7.8 can be proved by a straightforward modification of the standard proof of Oka's theorem [32, Ch. IV, § 2]. It can also be proved using the techniques of § 8 below; see Proposition 9.4.

Proposition 7.8 has the usual functorial consequences. We will need the following :

COROLLARY 7.9. – *Let* $f^1, \dots, f^q, g^1, \dots, g^r \in \mathcal{M}(X;Z)[[y]]^p$. *For each* $a \in X - Z$, *Let* $\mathcal{R}_a$ *(respectively,* $\mathcal{S}_a$*) denote the submodule of* $\mathbf{K}[[y]]^p$ *generated by the* $f^i(a,y)$ *(respectively, by the* $g^j(a;y)$*). Let* $a_0 \in X$. *Then there is a neighborhood* V *of* a_0 *in* U, *and a filtration of* $X \cap V$ *by closed analytic subsets,* $X \cap V = X_0 \supset X_1 \supset \dots \supset X_{t+1} = Z \cap V$, *satisfying the following property: For each* $k = 0, \dots, t$, *there are finitely many elements* $h_k^\ell \in \mathcal{M}(X_k;X_{k+1})[[y]]^p$ *such that the* $h_k^\ell(a;y)$ *generate* $\mathcal{R}_a \cap \mathcal{S}_a$ *for all* $a \in X_k - X_{k+1}$.

8. Local invariants of an analytic morphism.

The notation of this section will be used in the remainder of the article.

8.1. *A lemma in linear algebra.*

Let R denote a commutative ring with identity. Consider a diagram of R-modules and homomorphisms:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E' & \longrightarrow & E & \xrightarrow{\pi} & E'' & \longrightarrow & 0 \\ & & {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle A} & & \downarrow & & \\ 0 & \longrightarrow & F' & \longrightarrow & F & \longrightarrow & F'' & \longrightarrow & 0, \end{array}$$

where the sequences are exact and the squares are commutative. We regard E′ and F′ as submodules of E and F, respectively. Let $\rho \in \mathbf{N}$. We define

$$\mathrm{Ad}^\rho D \in \mathrm{Hom}_R(F, \mathrm{Hom}_R(\Lambda^\rho E', \Lambda^{\rho+1} F))$$

by the formula

$$\mathrm{Ad}^\rho D(\omega)(\eta_1 \wedge \dots \wedge \eta_\rho) = \omega \wedge D\eta_1 \wedge \dots \wedge D\eta_\rho,$$

where $\omega \in F$ and $\eta_i \in E'$, $i = 1, \dots, \rho$. ($\mathrm{Ad}^0 D$ means the identity mapping of F.) Let $\mathrm{ad}^\rho D \in \mathrm{Hom}_R(F', \mathrm{Hom}_R(\Lambda^\rho E', \Lambda^{\rho+1} F'))$

denote the homomorphism obtained by restricting $\mathrm{Ad}^{\rho}\,\mathrm{D}$ to F' ($\mathrm{ad}^{\rho}\,\mathrm{D}$ depends only on D).

If $\rho > \operatorname{rank} \mathrm{D}$, then $\mathrm{Ad}^{\rho}\,\mathrm{D} = 0$. If $\rho = \operatorname{rank} \mathrm{D}$, then $\mathrm{Ad}^{\rho}\,\mathrm{D} \circ \mathrm{A}$ vanishes on E', hence factors through E''; say $\mathrm{Ad}^{\rho}\,\mathrm{D} \circ \mathrm{A} = \mathrm{C} \circ \pi$. (The rank of D means the least σ such that $\Lambda^{\tau}\,\mathrm{D} = 0$ for all $\tau > \sigma$).

Suppose that $\mathrm{B} : \mathrm{G} \longrightarrow \mathrm{F}$ is a homomorphism of R-modules. Let $\rho = \operatorname{rank} \mathrm{D}$ and let $\mathrm{S} = \mathrm{Ad}^{\rho}\,\mathrm{D} \circ \mathrm{B}$. Let $\sigma \in \mathbf{N}$ and let $\mathrm{T} = \mathrm{ad}^{\sigma}\,\mathrm{S} \circ \mathrm{C}$, where $\mathrm{Ad}^{\rho}\,\mathrm{D} \circ \mathrm{A} = \mathrm{C} \circ \pi$. (If $\sigma = 0$, then $\mathrm{T} = \mathrm{C}$.) If $\sigma = \operatorname{rank} \mathrm{S}$ and $\mathrm{A}\,\xi \in \operatorname{Im} \mathrm{B}$, where $\xi \in \mathrm{E}$, then $\mathrm{Ad}^{\rho}\,\mathrm{D} \circ \mathrm{A}\,\xi \in \operatorname{Im} \mathrm{S}$, so that $\mathrm{T} \circ \pi\xi = 0$. Thus $\operatorname{Ker} \mathrm{T} \supset \pi\,\{\xi \in \mathrm{E} : \mathrm{A}\,\xi \in \operatorname{Im} \mathrm{B}\}$.

LEMMA 8.1.1. – *Let the notation be as above. Suppose that* R *is a field* $\mathbf{K}$. *If* $\rho = \operatorname{rank} \mathrm{D}$, *then:*

(1) $\operatorname{Im} \mathrm{D} = \operatorname{Ker} \mathrm{Ad}^{\rho}\,\mathrm{D} = \operatorname{Ker} \mathrm{ad}^{\rho}\,\mathrm{D}$.

(2) $\operatorname{Ker} \mathrm{C} = \pi(\operatorname{Ker} \mathrm{A})$.

If, moreover, $\sigma = \operatorname{rank} \mathrm{S}$, *where* $\mathrm{S} = \mathrm{Ad}^{\rho}\,\mathrm{D} \circ \mathrm{B}$, *then:*

(3) $\operatorname{Ker} \mathrm{T} = \pi\,\{\xi \in \mathrm{E} : \mathrm{A}\xi \in \operatorname{Im} \mathrm{B}\}$.

Proof. – Since $\operatorname{rank} \mathrm{D} = \dim_{\mathbf{K}} \operatorname{Im} \mathrm{D}$, it follows that if $\rho = \operatorname{rank} \mathrm{D}$ and $\omega \in \operatorname{Ker} \mathrm{Ad}^{\rho}\,\mathrm{D}$, then $\omega \in \operatorname{Im} \mathrm{D}$; i.e., (1) holds. (2) is (3) with $\mathrm{B} = 0$. To prove (3): Let $\sigma = \operatorname{rank} \mathrm{S}$ and let $\xi \in \mathrm{E}$. Suppose $\xi'' = \pi\xi \in \operatorname{Ker} \mathrm{T}$. Then $\mathrm{C}\,\xi'' \in \operatorname{Ker} \mathrm{ad}^{\sigma}\,\mathrm{S} = \operatorname{Im} \mathrm{S}$; i.e., there exists $\eta \in \mathrm{G}$ such that $\mathrm{Ad}^{\rho}\,\mathrm{D} \circ \mathrm{A}\,\xi = \mathrm{C}\xi'' = \mathrm{S}\eta = \mathrm{Ad}^{\rho}\,\mathrm{D} \circ \mathrm{B}\eta$. Therefore $\mathrm{A}\xi - \mathrm{B}\eta \in \operatorname{Ker} \mathrm{Ad}^{\rho}\,\mathrm{D} = \operatorname{Im} \mathrm{D}$, so that $\mathrm{A}\xi - \mathrm{B}\eta = \mathrm{D}\zeta$, where $\zeta \in \mathrm{E}'$. Hence $\mathrm{A}(\xi - \zeta) = \mathrm{B}\eta$ and $\xi'' = \pi\xi = \pi(\xi - \zeta)$.

□

8.2. Let $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Let M and N denote analytic manifolds over $\mathbf{K}$, and let $\phi : \mathrm{M} \longrightarrow \mathrm{N}$ be an analytic mapping. Let A and B be $p \times q$ and $p \times r$ matrices, respectively, whose entries are analytic functions on M.

Let $a \in \mathrm{M}$. We write $\mathcal{O}_a = \mathcal{O}_{\mathrm{M},a}$, etc., when there is no possibility of confusion. Then ϕ determines homomorphisms of local rings $\phi_a^* : \mathcal{O}_{\phi(a)} \longrightarrow \mathcal{O}_a$ and $\hat{\phi}_a^* : \hat{\mathcal{O}}_{\phi(a)} \longrightarrow \hat{\mathcal{O}}_a$, and A (respectively, B) induces module homomorphisms $\mathrm{A}_a : \mathcal{O}_a^q \longrightarrow \mathcal{O}_a^p$

and $\hat{A}_a : \hat{\mathcal{O}}^q_a \longrightarrow \hat{\mathcal{O}}^p_a$ (respectively, $B_a : \mathcal{O}^r_a \longrightarrow \mathcal{O}^p_a$ and $\hat{B}_a : \hat{\mathcal{O}}^r_a \longrightarrow \hat{\mathcal{O}}^p_a$). Let $\Phi_a : \mathcal{O}^q_{\phi(a)} \longrightarrow \mathcal{O}^p_a$ denote the module homomorphism over ϕ^*_a defined by $\Phi_a(g) = A_a \cdot \phi^*_a(g)$, where $g = (g_1, \ldots, g_q) \in \mathcal{O}^q_{\phi(a)}$ and $\phi^*_a(g) = (\phi^*_a(g_1), \ldots, \phi^*_a(g_q))$. Let $\hat{\Phi}_a : \hat{\mathcal{O}}^q_{\phi(a)} \longrightarrow \hat{\mathcal{O}}^p_a$ be the analogous module homomorphism over $\hat{\phi}^*_a$.

Let $s \in \mathbf{N}$. Let M^s_ϕ denote the fiber product

$$M^s_\phi = \{\boldsymbol{a} = (a^1, \ldots, a^s) \in M^s : \phi(a^1) = \ldots = \phi(a^s)\},$$

and let $\boldsymbol{\varphi} : M^s_\phi \longrightarrow N$ be the induced morphism.

Let $\boldsymbol{a} = (a^1, \ldots, a^s) \in M^s_\phi$. Let $\boldsymbol{\Phi}_{\boldsymbol{a}} : \mathcal{O}^q_{\boldsymbol{\varphi}(\boldsymbol{a})} \longrightarrow \bigoplus_{i=1}^{s} \mathcal{O}^p_{a^i}$ denote the composition of $\bigoplus_{i=1}^{s} \Phi_{a^i}$ with the diagonal injection $\mathcal{O}^q_{\boldsymbol{\varphi}(\boldsymbol{a})} \longrightarrow \bigoplus_{i=1}^{s} \mathcal{O}^q_{\boldsymbol{\varphi}(\boldsymbol{a})}$, and let

$$\mathbf{B}_{\boldsymbol{a}} = \bigoplus_{i=1}^{s} B_{a^i} : \bigoplus_{i=1}^{s} \mathcal{O}^r_{a^i} \longrightarrow \bigoplus_{i=1}^{s} \mathcal{O}^p_{a^i}.$$

Likewise, $\hat{\boldsymbol{\Phi}}_{\boldsymbol{a}} : \hat{\mathcal{O}}^q_{\boldsymbol{\varphi}(\boldsymbol{a})} \longrightarrow \bigoplus_{i=1}^{s} \hat{\mathcal{O}}^p_{a^i}$ and

$$\hat{\mathbf{B}}_{\boldsymbol{a}} : \bigoplus_{i=1}^{s} \hat{\mathcal{O}}^r_{a^i} \longrightarrow \bigoplus_{i=1}^{s} \hat{\mathcal{O}}^p_{a^i}.$$

For each $k \in \mathbf{N}$, $\hat{\boldsymbol{\Phi}}_{\boldsymbol{a}}$ and $\hat{\mathbf{B}}_{\boldsymbol{a}}$ induce linear mappings

$$A_k(\boldsymbol{a}) : \frac{\hat{\mathcal{O}}^q_{\boldsymbol{\varphi}(\boldsymbol{a})}}{\mathfrak{m}^{k+1}_{\boldsymbol{\varphi}(\boldsymbol{a})} \cdot \hat{\mathcal{O}}^q_{\boldsymbol{\varphi}(\boldsymbol{a})}} \longrightarrow \bigoplus_{i=1}^{s} \frac{\hat{\mathcal{O}}^p_{a^i}}{\mathfrak{m}^{k+1}_{a^i} \cdot \hat{\mathcal{O}}^p_{a^i}}$$

$$B_k(\boldsymbol{a}) : \bigoplus_{i=1}^{s} \frac{\hat{\mathcal{O}}^r_{a^i}}{\mathfrak{m}^{k+1}_{a^i} \cdot \hat{\mathcal{O}}^r_{a^i}} \longrightarrow \bigoplus_{i=1}^{s} \frac{\hat{\mathcal{O}}^p_{a^i}}{\mathfrak{m}^{k+1}_{a^i} \cdot \hat{\mathcal{O}}^p_{a^i}},$$

respectively. If $\ell \geqslant k$, we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \dfrac{\mathfrak{m}^{k+1}_{\varphi(a)} \cdot \hat{\mathcal{O}}^q_{\varphi(a)}}{\mathfrak{m}^{\ell+1}_{\varphi(a)} \cdot \hat{\mathcal{O}}^q_{\varphi(a)}} \arrow[r] \arrow[d, "D_{\ell,k}(a)"] & \dfrac{\hat{\mathfrak{m}}^q_{\varphi(a)}}{\mathfrak{m}^{\ell+1}_{\varphi(a)} \cdot \hat{\mathcal{O}}^q_{\varphi(a)}} \arrow[r, "\Pi_{\ell,k}(\varphi(a))"] \arrow[d, "A_\ell(a)"] & \dfrac{\hat{\mathcal{O}}^q_{\varphi(a)}}{\mathfrak{m}^{k+1}_{\varphi(a)} \cdot \hat{\mathcal{O}}^q_{\varphi(a)}} \arrow[r] \arrow[d, "A_k(a)"] & 0 \\
0 \arrow[r] & \displaystyle\bigoplus_{i=1}^{s} \dfrac{\mathfrak{m}^{k+1}_{a^i} \cdot \hat{\mathcal{O}}^p_{a^i}}{\mathfrak{m}^{\ell+1}_{a^i} \cdot \hat{\mathcal{O}}^p_{a^i}} \arrow[r] & \displaystyle\bigoplus_{i+1}^{s} \dfrac{\hat{\mathcal{O}}^p_{a^i}}{\mathfrak{m}^{\ell+1}_{a^i} \cdot \hat{\mathcal{O}}^p_{a^i}} \arrow[r, "\Pi^p_{\ell,k}(a)"] & \displaystyle\bigoplus_{i=1}^{s} \dfrac{\hat{\mathcal{O}}^p_{a^i}}{\mathfrak{m}^{k+1}_{a^i} \cdot \hat{\mathcal{O}}^p_{a^i}} \arrow[r] & 0,
\end{tikzcd} \tag{8.2.1}$$

where the rows are exact and $\Pi^p_{\ell,k}(a)$, $\Pi_{\ell,k}(\varphi(a))$ denote the canonical projections. Likewise, $\Pi^p_{\ell,k}(a) \circ B_\ell(a) = B_k(a) \circ \Pi^r_{\ell,k}(a)$.

Put $\mathscr{R}_a = \{G \in \hat{\mathcal{O}}^q_{\varphi(a)} : \hat{\Phi}_a(G) \in \operatorname{Im} B_a\}$. Write

$$J_{\varphi(a)}(k) = \frac{\hat{\mathcal{O}}^q_{\varphi(a)}}{\mathfrak{m}^{k+1}_{\varphi(a)} \cdot \hat{\mathcal{O}}^q_{\varphi(a)}}, \quad \mathscr{R}_a(k) = \frac{\mathscr{R}_a + \mathfrak{m}^{k+1}_{\varphi(a)} \cdot \hat{\mathcal{O}}^q_{\varphi(a)}}{\mathfrak{m}^{k+1}_{\varphi(a)} \cdot \hat{\mathcal{O}}^q_{\varphi(a)}}.$$

Let $E_k(a) = \{\xi \in J_{\varphi(a)}(k) : A_k(a)\xi \in \operatorname{Im} B_k(a)\}$. If $\ell \geqslant k$, let $E_{\ell,k}(a) = \Pi_{\ell,k}(\varphi(a))(E_\ell(a))$.

The following is a reformulation of Lemma 1.1 (Chevalley estimate):

LEMMA 8.2.2. – *Let* $a \in M^s_\varphi$, $a = (a^1, \ldots, a^s)$. *For each* $k \in \mathbf{N}$, *there exists* $\ell \in \mathbf{N}$ *such that if* $G \in \mathcal{O}^q_{\varphi(a)}$ *and*

$$\hat{\Phi}_a(G) \in \operatorname{Im} B_a + \bigoplus_{i=1}^{s} \mathfrak{m}^{\ell+1}_{a^i} \cdot \hat{\mathcal{O}}^p_{a^i},$$

then $G \in \mathscr{R}_a + \mathfrak{m}^{k+1}_{\varphi(a)} \cdot \hat{\mathcal{O}}^q_{\varphi(a)}$.

Proof. – Let $k \in \mathbf{N}$. If $\ell_2 \geqslant \ell_1 \geqslant k$, then

$$\mathscr{R}_a(k) \subset E_{\ell_2,k}(a) \subset E_{\ell_1,k}(a)$$

and the projection $\bigcap_{\ell \geqslant \ell_2} \mathrm{E}_{\ell,\ell_2}(a) \longrightarrow \bigcap_{\ell \geqslant \ell_1} \mathrm{E}_{\ell,\ell_1}(a)$ is onto. It follows that $\mathscr{R}_a(k) = \bigcap_{\ell \geqslant k} \mathrm{E}_{\ell,k}(a)$. Since $\dim_{\mathbf{K}} \mathrm{J}_{\varphi(a)}(k) < \infty$, there exists $\ell \in \mathbf{N}$ such that $\mathscr{R}_a(k) = \mathrm{E}_{\ell,k}(a)$.

□

DEFINITION 8.2.3. – *Let* $a \in \mathrm{M}^s_\phi$. *For each* $k \in \mathbf{N}$, *let* $\ell(k,a)$ *denote the smallest* $\ell \in \mathbf{N}$ *satisfying the conclusion of Lemma 8.2.2.*

If $a \in \mathrm{M}^s_\phi$, let H_a denote the Hilbert-Samuel function of $\hat{\mathcal{O}}^q_{\varphi(a)}/\mathscr{R}_a$; thus $\mathrm{H}_a(k) = \dim_{\mathbf{K}} \mathrm{J}_{\varphi(a)}(k)/\mathscr{R}_a(k)$. If $k \leqslant \ell$, define $d_{\ell,k}(a)$ by

$$d_{\ell,k}(a) = \dim_{\mathbf{K}} \frac{\mathrm{J}_{\varphi(a)}(k)}{\mathrm{E}_{\ell,k}(a)} .$$

Remark 8.2.4. – $d_{\ell,k}(a) \leqslant \mathrm{H}_a(k)$ and $\mathscr{R}_a(k) \subset \mathrm{E}_{\ell,k}(a)$, with equality in each case if and only if $\ell \geqslant \ell(k,a)$.

Our main theorem A can be reformulated as follows:

THEOREM 8.2.5. – *Let* $s \in \mathbf{N}$. *Then the following conditions are equivalent:*

(1) Uniform Chevalley estimate. *Let* K *be a compact subset of* M^s_ϕ. *Then, for every* $k \in \mathbf{N}$, *there exists* $\ell = \ell(k,\mathrm{K}) \in \mathbf{N}$ *such that* $\ell(k,a) \leqslant \ell$ *for all* $a \in \mathrm{K}$.

(2) $\mathrm{H}_a(k)$ *is Zariski semicontinuous on* M^s_ϕ, *for each fixed* $k \in \mathbf{N}$.

(3) *The Hilbert-Samuel function* H_a *is Zariski semicontinuous on* M^s_ϕ.

Assume, moreover, that N *is an open submanifold of* $\mathbf{K}^n$. *Then each of the above conditions is equivalent to:*

(4) *The diagram of initial exponents* $\mathfrak{N}_a = \mathfrak{N}(\mathscr{R}_a)$ *is Zariski semicontinuous on* M^s_ϕ.

Suppose that X is (a representative in a small neighborhood of) an irreducible germ of a closed analytic subset of M^s_ϕ, at some point. Our proof of Theorem 8.2.5 will be based on a construction which associates to X and $k \in \mathbf{N}$, a linear transformation $\mathrm{T}^{\mathrm{X}}_k(a)$

defined on $J_{\varphi(a)}(k)$, depending analytically on a and satisfying the following condition: there are countably many proper analytic subsets of X such that, for each a in their complement D_k, $\mathscr{R}_a(k) = \operatorname{Ker} T_k^X(a)$.

Let $a_0 \in M_\phi^s$, $a_0 = (a_0^1, \ldots, a_0^s)$. Let X denote a germ at a_0 of a closed analytic subset of M_ϕ^s.

Let $U = \prod_{i=1}^{s} U^i$ be a product coordinate neighborhood of a_0 in M^s, and let V be a coordinate neighborhood of $\phi(a_0)$ in N, such that $\phi(U^i) \subset V$, $i = 1, \ldots, s$. Shrinking U if necessary, we can assume that X is a closed analytic subset of U such that each connected component of its smooth points is adherent to a_0.

We use α (respectively, β) to denote multiindices in $\mathbf{N}^m$ (respectively, $\mathbf{N}^n$). If $g = (g_1, \ldots, g_q) \in \mathcal{O}(V)^q$, write $(D^\beta g)_{|\beta| \leq \ell}$ for the vector whose components are the derivatives

$$D^\beta g_j = \partial^{|\beta|} g_j / \partial y_1^{\beta_1} \ldots \partial y_n^{\beta_n} \in \mathcal{O}(V)$$

(with respect to the coordinates of V), $|\beta| \leq \ell$, $j = 1, \ldots, q$, ordered by $(\beta, j, |\beta|)$ lexicographically from the right.

For each $i = 1, \ldots, s$, define $\Phi^i : \mathcal{O}(V)^q \longrightarrow \mathcal{O}(U^i)^p$ by $\Phi^i(g) = A \cdot (g \circ \phi)$, where $g \in \mathcal{O}(V)^q$, and

$$B^i : \mathcal{O}(U^i)^r \longrightarrow \mathcal{O}(U^i)^p$$

by $B^i(h) = B \cdot h$, where $h \in \mathcal{O}(U^i)^r$. Let $\ell \in \mathbf{N}$. By the chain rule, there is a commutative diagram

$$\begin{array}{ccc} \mathcal{O}(V)^q & \longrightarrow & \bigoplus_{|\beta| \leq \ell} \mathcal{O}(U^i)^q \\ \Big\downarrow \Phi^i & & \Big\downarrow A_\ell^i \\ \mathcal{O}(U^i)^p & \longrightarrow & \bigoplus_{|\alpha| \leq \ell} \mathcal{O}(U^i)^p , \end{array}$$

where the upper horizontal arrow is $g \longmapsto (D^\beta g \circ (\phi \mid U^i))_{|\beta| \leq \ell}$, the lower is $f \mapsto (D^\alpha f)_{|\alpha| \leq \ell}$, and A_ℓ^i is a matrix with entries in (U^i). Likewise, there is a commutative diagram

$$
\begin{tikzcd}
\mathcal{O}(\mathrm{U}^i)^r \arrow[r] \arrow[d, "\mathrm{B}^i"] & \bigoplus_{|\alpha|\leq \ell} \mathcal{O}(\mathrm{U}^i)^r \arrow[d, "\mathrm{B}^i_\ell"] \\
\mathcal{O}(\mathrm{U}^i)^p \arrow[r] & \bigoplus_{|\alpha|\leq \ell} \mathcal{O}(\mathrm{U}^i)^p ,
\end{tikzcd}
$$

where the upper horizontal arrow is $h \longmapsto (\mathrm{D}^\alpha h)_{|\alpha|\leq \ell}$, etc.

For each $i = 1, \ldots, s$, the composition $\mathrm{X} \hookrightarrow \mathrm{U} \longrightarrow \mathrm{U}^i$ of inclusion and projection induces a mapping $\mathcal{O}(\mathrm{U}^i) \longrightarrow \mathcal{O}(\mathrm{X})$, where $\mathcal{O}(\mathrm{X})$ denotes the ring of analytic functions on X. Let $a \in \mathrm{X}$. We write $\mathrm{A}^i_{\ell,a}$ (respectively, $\mathrm{B}^i_{\ell,a}$) for the matrix of elements of $\mathcal{O}_{\mathrm{X},a}$ induced by A^i_ℓ (respectively, B^i_ℓ), and for the induced $\mathcal{O}_{\mathrm{X},a}$-homomorphism $\bigoplus_{|\beta|\leq \ell} \mathcal{O}^q_{\mathrm{X},a} \longrightarrow \bigoplus_{|\alpha|\leq \ell} \mathcal{O}^p_{\mathrm{X},a}$ (respectively, $\bigoplus_{|\alpha|\leq \ell} \mathcal{O}^r_{\mathrm{X},a} \longrightarrow \bigoplus_{|\alpha|\leq \ell} \mathcal{O}^p_{\mathrm{X},a}$). We get commutative diagrams

$$
\begin{tikzcd}
\mathcal{O}^q_{\varphi(a)} \arrow[r] \arrow[d, "\Phi_a"] & \bigoplus_{|\beta|\leq \ell} \mathcal{O}^q_{\mathrm{X},a} \arrow[d, "\mathrm{A}_{\ell,a}"] \\
\bigoplus_{i=1}^{s} \mathcal{O}^p_{a^i} \arrow[r] & \bigoplus_{i=1}^{s} \bigoplus_{|\alpha|\leq \ell} \mathcal{O}^p_{\mathrm{X},a}
\end{tikzcd}
\qquad
\begin{tikzcd}
\bigoplus_{i=1}^{s} \mathcal{O}^r_{a^i} \arrow[r] \arrow[d, "\mathrm{B}_a"] & \bigoplus_{i=1}^{s} \bigoplus_{|\alpha|\leq \ell} \mathcal{O}^r_{\mathrm{X},a} \arrow[d, "\mathrm{B}_{\ell,a}"] \\
\bigoplus_{i=1}^{s} \mathcal{O}^p_{a^i} \arrow[r] & \bigoplus_{i=1}^{s} \bigoplus_{|\alpha|\leq \ell} \mathcal{O}^p_{\mathrm{X},a}
\end{tikzcd}
\tag{8.2.6}
$$

where $\mathrm{A}_{\ell,a}$ (respectively, $\mathrm{B}_{\ell,a}$) is the matrix with vertical blocks $\mathrm{A}^i_{\ell,a}$ (respectively, diagonal blocks $\mathrm{B}^i_{\ell,a}$), $i = 1, \ldots, s$. If $\ell \geq k$, there is a commutative diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & \bigoplus_{k<|\beta|\leq \ell} \mathcal{O}^q_{X,a} & \longrightarrow & \bigoplus_{|\beta|\leq \ell} \mathcal{O}^q_{X,a} & \xrightarrow{\Pi_{\ell,k,a}} & \bigoplus_{|\beta|\leq k} \mathcal{O}^q_{X,a} & \longrightarrow 0 \\
& \Big\downarrow D_{\ell,k,a} & & \Big\downarrow A_{\ell,a} & & \Big\downarrow A_{k,a} & \\
0 \longrightarrow & \bigoplus_{i=1}^{s} \bigoplus_{k<|\alpha|\leq \ell} \mathcal{O}^p_{X,a} & \longrightarrow & \bigoplus_{i=1}^{s} \bigoplus_{|\alpha|\leq \ell} \mathcal{O}^p_{X,a} & \longrightarrow & \bigoplus_{i=1}^{s} \bigoplus_{|\alpha|\leq k} \mathcal{O}^p_{X,a} & \longrightarrow 0
\end{array}$$

(8.2.7)

Let $\hat{A}_{\ell,a}$, $\hat{B}_{\ell,a}$, $\hat{D}_{\ell,k,a}$, $\hat{\Pi}_{\ell,k,a}$ denote the morphisms of the completions induced by $A_{\ell,a}$, etc.

Let $\mathfrak{m}_{X,a}$ denote the maximal ideal of $\mathcal{O}_{X,a}$. Let

$$\mathcal{O}_{X,a} \longrightarrow \mathcal{O}_{X,a}/\mathfrak{m}_{X,a} = \mathbf{K}$$

be the canonical projection ("evaluation at a"). The horizontal arrows in (8.2.6) induce identifications

$$\frac{\mathcal{O}^q_{\varphi(a)}}{\mathfrak{m}^{\ell+1}_{\varphi(a)} \cdot \mathcal{O}^q_{\varphi(a)}} \xrightarrow{\sim} \bigoplus_{|\beta|\leq \ell} \left[\frac{\mathcal{O}_{X,a}}{\mathfrak{m}_{X,a}}\right]^q ,$$

$$\bigoplus_{i=1}^{s} \frac{\mathcal{O}^t_{a^i}}{\mathfrak{m}^{\ell+1}_{a^i} \cdot \mathcal{O}^t_{a^i}} \xrightarrow{\sim} \bigoplus_{i=1}^{s} \bigoplus_{|\alpha|\leq \ell} \left[\frac{\mathcal{O}_{X,a}}{\mathfrak{m}_{X,a}}\right]^t ,$$

where $t = r, p$. Using these identifications, evaluation at a transforms the diagram (8.2.7) (or the analogous diagram of completions) into the diagram (8.2.1).

8.3. *The Hilbert-Samuel function and the Chevalley estimate.* Let $\ell, k \in \mathbf{N}$, $\ell \geq k$. Let $a \in M^s_\phi$. We apply the formalism of 8.1 to the diagram (8.2.1) and the linear mapping $B_\ell(a)$. Put

$$\rho_{\ell,k}(a) = \operatorname{rank} D_{\ell,k}(a).$$

Then $\mathrm{Ad}^{\rho_{\ell,k}(a)} D_{\ell,k}(a) \circ A_\ell(a)$ factors as

$$\mathrm{Ad}^{\rho_{\ell,k}(a)} D_{\ell,k}(a) \circ A_\ell(a) = C_{\ell,k}(a) \circ \Pi_{\ell,k}(\varphi(a)).$$

Put $\sigma_{\ell,k}(a) = \operatorname{rank} S_{\ell,k}(a)$, where

$$S_{\ell,k}(a) = \mathrm{Ad}^{\rho_{\ell,k}(a)} D_{\ell,k}(a) \circ B_\ell(a),$$

and put $T_{\ell,k}(a) = \mathrm{ad}^{\sigma_{\ell,k}(a)} S_{\ell,k}(a) \circ C_{\ell,k}(a)$.

Remark 8.3.1. – By Lemma 8.1.1, $E_{\ell,k}(a) = \operatorname{Ker} T_{\ell,k}(a)$ and $d_{\ell,k}(a) = \operatorname{rank} T_{\ell,k}(a)$.

Let $a_0 \in M^s_\phi$ and let X, etc., be as in 8.2 above. Let

$$\rho_{\ell,k}(X) = \max_{a \in X} \rho_{\ell,k}(a).$$

If $a \in X$, then $\mathrm{Ad}^{\rho_{\ell,k}(X)} D_{\ell,k}(a) \circ A_\ell(a)$ factors as

$$\mathrm{Ad}^{\rho_{\ell,k}(X)} D_{\ell,k}(a) \circ A_\ell(a) = C^X_{\ell,k}(a) \circ \Pi_{\ell,k}(\varphi(a)).$$

Put $\sigma^X_{\ell,k}(a) = \operatorname{rank} S^X_{\ell,k}(a)$, where

$$S^X_{\ell,k}(a) = \mathrm{Ad}^{\rho_{\ell,k}(X)} D_{\ell,k}(a) \circ B_\ell(a),$$

and let $\sigma_{\ell,k}(X) = \max_{a \in X} \sigma^X_{\ell,k}(a)$. Put

$$T^X_{\ell,k}(a) = \mathrm{ad}^{\sigma_{\ell,k}(X)} S^X_{\ell,k}(a) \circ C^X_{\ell,k}(a).$$

Let

$$d^X_{\ell,k}(a) = \operatorname{rank} T^X_{\ell,k}(a) \quad \text{and} \quad d_{\ell,k}(X) = \max_{a \in X} d^X_{\ell,k}(a).$$

Let

$$\begin{aligned}
Y_{\ell,k} &= \{a \in X : \rho_{\ell,k}(a) < \rho_{\ell,k}(X)\},\\
Z_{\ell,k} &= \{a \in X : \sigma^X_{\ell,k}(a) < \sigma_{\ell,k}(X)\},\\
X_{\ell,k} &= Y_{\ell,k} \cup Z_{\ell,k} \cup \{a \in X : d^X_{\ell,k}(a) < d_{\ell,k}(X)\}.
\end{aligned}$$

Then $Y_{\ell,k}$, $Z_{\ell,k}$ and $X_{\ell,k}$ are closed analytic subsets of X. If X is irreducible, then $X_{\ell,k}$ is a proper analytic subset of X and, for all $a \in X - X_{\ell,k}$, $\sigma_{\ell,k}(a) = \sigma^X_{\ell,k}(a) = \sigma_{\ell,k}(X)$ and

$$d_{\ell,k}(a) = d^X_{\ell,k}(a) = d_{\ell,k}(X).$$

Let $k \in \mathbf{N}$. Put

$$D_k = X - \bigcup_{\ell > k} X_{\ell,k}. \tag{8.3.2}$$

If $\mathbf{K} = \mathbf{C}$ and X is irreducible, then D_k is dense in X.

LEMMA 8.3.3. – *For all* $a, b \in D_k$, $H_a(k) = H_b(k)$ *and* $\ell(k, a) = \ell(k, b)$.

Proof. – Let $a \in D_k$. Then $d_{\ell,k}(a) = d_{\ell,k}(X)$, $\ell > k$. If $\ell \geqslant \ell(k, a)$, then, by Remark 8.2.4, $H_a(k) = d_{\ell,k}(X)$. Let $a, b \in D_k$. Choosing $\ell \geqslant \ell(k, a)$, $\ell \geqslant \ell(k, b)$, we get $H_a(k) = H_b(k)$. The second assertion follows from Remark 8.2.4.

□

DEFINITION 8.3.4. – *We write* $H_X(k) = H_a(k)$ *and* $\ell(k, X) = \ell(k, a)$, *for any* $a \in D_k$. ($H_X(k)$ *is the "generic Hilbert-Samuel functions").*

Remark 8.3.5. – (1) If $\ell \geqslant \ell(k, X)$, then $H_X(k) = d_{\ell,k}(X)$.

(2) If $a \in X$ and $\ell > k$, then, by Remarks 8.2.4. and 8.3.1, $\mathscr{R}_a(k) \subset E_{\ell,k}(a) = \operatorname{Ker} T_{\ell,k}(a) \subset \operatorname{Ker} T^X_{\ell,k}(a)$. Suppose $a \in D_k$. Then $T_{\ell,k}(a) = T^X_{\ell,k}(a)$. If, moreover, $\ell \geqslant \ell(k, X)$, then

$$\mathscr{R}_a(k) = E_{\ell,k}(a) = \operatorname{Ker} T^X_{\ell,k}(a).$$

PROPOSITION 8.3.6. – *Let* $a_0 \in M^s_\phi$, *and let* X, *etc., be as above. Let* $k \in \mathbf{N}$. *Then:*

(1) *If* $\ell \geqslant \ell(k, a)$ *for all* $a \in X$, *then* $H_a(k)$ *is constant on* $X - X_{\ell,k}$.

(2) *Suppose* X *is irreducible. Let* $\ell = \ell(k, X)$. *Let* Y *be a proper analytic subset of* X. *If* $H_a(k)$ *is constant on* $X - Y$, *then* $\ell \geqslant \ell(k, a)$ *for all* $a \in X - (X_{\ell,k} \cup Y)$.

Proof. – (1) If $\ell \geqslant \ell(k, a)$ for all $a \in X$, then, by Remark 8.2.4, $H_a(k) = d_{\ell,k}(a) = d_{\ell,k}(X)$ for all $a \in X - X_{\ell,k}$.

(2) By Remark 8.3.5, $H_a(k) = d_{\ell,k}(X)$ for all $a \in D_k$. Since $H_a(k)$ is constant on $X - Y$ and $d_{\ell,k}(a) = d_{\ell,k}(X)$ on $X - X_{\ell,k}$, then $H_a(k) = d_{\ell,k}(a)$ on $X - (X_{\ell,k} \cup Y)$. By Remark 8.2.4, $\ell \geqslant \ell(k, a)$ for all $a \in X - (X_{\ell,k} \cup Y)$.

□

Let $\ell \geqslant k$. Let $a \in X$. Clearly, rank $D_{\ell,k,a} \leqslant \rho_{\ell,k}(X)$, with equality if D_k is adherent to a. Hence $\operatorname{Ad}^{\rho_{\ell,k}(X)} D_{\ell,k,a} \circ A_{\ell,a}$

factors as $C_{\ell,k,a} \circ \Pi_{\ell,k,a}$ (cf. (8.2.7)). Put

$$T_{\ell,k,a} = \mathrm{ad}^{\sigma_{\ell,k}(X)} S_{\ell,k,a} \circ C_{\ell,k,a},$$

where $S_{\ell,k,a} = \mathrm{Ad}^{\rho_{\ell,k}(X)} D_{\ell,k,a} \circ B_{\ell,a}$. Let $\hat{C}_{\ell,k,a}$ and $\hat{T}_{\ell,k,a}$ denote the analogous homomorphisms of the completions. If X is irreducible, then $\sigma_{\ell,k}(X) = \operatorname{rank} S_{\ell,k,a}$. Evaluation at a transforms $C_{\ell,k,a}$ and $\hat{C}_{\ell,k,a}$ (respectively, $T_{\ell,k,a}$ and $\hat{T}_{\ell,k,a}$) into $C^X_{\ell,k}(a)$ (respectively, $T^X_{\ell,k}(a)$).

PROPOSITION 8.3.7. – *Suppose that* X *is irreducible. Let* $k \in \mathbf{N}$. *Then, for all* $a \in X$, $H_a(k) \geq H_X(k)$.

Remark 8.3.8. – Suppose that, for all $a \in M^s_\phi$,

$$\operatorname{Ker} \hat{\Phi}_a = \hat{\mathcal{O}}_{\phi(a)} \cdot \operatorname{Ker} \Phi_a$$

(cf. Remarks 2.11). It is easy to see, then, that $H_a(k)$ is topologically semicontinuous for each fixed k (cf. [37, II.5.3]). In this case, Proposition 8.3.7 is immediate.

Proof of Proposition 8.3.7. – We can assume that $\mathbf{K} = \mathbf{C}$. Let $\ell = \ell(k, X)$. Let $a \in X$ and let $\xi_1, \ldots, \xi_t$ denote a basis of $\mathcal{R}_a(k)$. For each $i = 1, \ldots, t$, choose a representative G_i of ξ_i in $\mathcal{R}_a$, and let $H_i = (H_i^\beta)_{|\beta| \leq \ell}$ denote the image of G_i in $\bigoplus_{|\beta| \leq \ell} \hat{\mathcal{O}}^q_{X,a}$ (i.e., the image by the upper horizontal arrow in the left-hand diagram of (8.2.6), for the completions). Then

$$\hat{A}_{\ell,a}(H_i) \in \operatorname{Im} \hat{B}_{\ell,a},$$

so that $\hat{\Pi}_{\ell,k,a}(H_i) \in \operatorname{Ker} \hat{T}_{\ell,k,a}$, $i = 1, \ldots, t$.

By Krull's theorem, there exist convergent generators $\eta_1, \ldots, \eta_u \in \operatorname{Ker} T_{\ell,k,a}$ of $\operatorname{Ker} \hat{T}_{\ell,k,a}$. Since the $\xi_i = \hat{\Pi}_{\ell,k,a}(H_i)(a)$ are linearly independent, then the $\eta_j(a) \in \operatorname{Ker} T^X_{\ell,k}(a)$ span a linear subspace of dimension $\geq t$. If $x \in X$ is close enough to a, then the η_i can be evaluated at x, and the $\eta_j(x) \in \operatorname{Ker} T^X_{\ell,k}(x)$ span a linear subspace of dimension $\geq t$. Taking $x \in D_k$, we have $H_X(k) = H_x(k) \leq H_a(k)$, by Remark 8.3.5.

□

8.4. *The Hilbert-Samuel function and the diagram of initial exponents.* Assume that $N = V$ is an open submanifold *of* $\mathbf{K}^n$. If $a \in M_\phi^s$, let $\mathfrak{N}_a \subset \mathbf{N}^n \times \{1, \dots, q\}$ denote the diagram of initial exponents $\mathfrak{N}_a = \mathfrak{N}(\mathscr{R}_a)$. We continue to use the notation of 8.2 and 8.3. Assume that X is irreducible. We introduce a "generic diagram of initial exponents" $\mathfrak{N}_X$:

DEFINITION 8.4.1. – *Let* $k(X)$ *denote the smallest* $k \in \mathbf{N}$ *such that* $H_X(\ell)$ *coincides with a polynomial in* ℓ, *if* $\ell \geqslant k$.

Suppose $k \geqslant k(X)$. Put $\ell = \ell(k, X)$. Let $a \in X$. Let $\xi \in \bigoplus_{|\beta| \leqslant k} \mathscr{O}^q_{X,a}$; say $\xi = (\xi_\beta)_{|\beta| \leqslant k}$, where each

$$\xi_\beta = (\xi_{\beta,1}, \dots, \xi_{\beta,q}) \in \mathscr{O}^q_{X,a}.$$

Let $\nu(\xi)$ denote the smallest $(\beta, j) \in \mathbf{N}^n \times \{1, \dots, q\}$ such that $\xi_{\beta,j} \neq 0$. If $\gamma \in \mathbf{N}^n$, define $S^\gamma \xi \in \bigoplus_{|\beta| \leqslant k} \mathscr{O}^q_{X,a}$ by

$$(S^\gamma \xi)_\beta = \begin{cases} \dfrac{\beta!}{(\beta-\gamma)!}\, \xi^{\beta-\gamma}, & \text{if } \gamma \leqslant \beta \\ 0, & \text{otherwise.} \end{cases}$$

LEMMA 8.4.2. – *Let* $a \in X$ *such that* D_k *is adherent to* a (i.e., any $a \in X$ if $\mathbf{K} = \mathbf{C}$). *Let* $\xi \in \operatorname{Ker} T_{\ell,k,a}$ *and* $\gamma \in \mathbf{N}^n$. *Then* $S^\gamma \xi \in \operatorname{Ker} T_{\ell,k,a}$.

Proof. – Consider the evaluation $(S^\gamma \xi)(x)$ of $S^\gamma \xi$ at $x \in X$ a. If $\operatorname{Ker} T^X_{\ell,k}(x) = \mathscr{R}_x(k)$, then $(S^\gamma \xi)(x) \in \operatorname{Ker} T^X_{\ell,k}(x)$: dentifies with

$$\sum_{|\beta| \leqslant k} \xi_\beta(x)\, y^\beta/\beta! \in \mathscr{R}_x + \mathfrak{m}^{k+1}_{\varphi(x)} \cdot \hat{\mathscr{O}}^q_{\varphi(x)} \subset \mathbf{K}[[y]]^q,$$

and $(S^\gamma \xi)(x)$ identifies with

$$y^\gamma \cdot \Big(\sum_{|\beta| \leqslant k} \xi_\beta(x)\, y^\beta/\beta! \Big) \in \mathscr{R}_x + \mathfrak{m}^{k+1}_{\varphi(x)} \cdot \hat{\mathscr{O}}^q_{\varphi(a)}.$$

In particular $T^X_{\ell,k}(x) \cdot (S^\gamma \xi)(x) = 0$ for all $x \in D_k$ near a. Thus $T_{\ell,k,a}(S^\gamma \xi) = 0$.

□

DEFINITION AND REMARK 8.4.3. – *Set*

$$\mathfrak{N}_X = \{\nu(\xi) : \xi \in \operatorname{Ker} T_{\ell,k,a_0}\} + \mathbf{N}^n .$$

According to Lemma 8.4.2,

$$\mathfrak{N}_X \cap \{(\beta, j) : |\beta| \leqslant k\} = \{\nu(\xi) : \xi \in \operatorname{Ker} T_{\ell,k,a_0}\} .$$

The definition of $\mathfrak{N}_X$ *is independent of* $k \geqslant k(X)$.

Remark 8.4.4. – Suppose that $k \geqslant k(X)$. Put $\ell = \ell(k, X)$. It is easy to see that there is a proper closed analytic subset Z' of X such that, for all $a \in X - Z'$,

$$\operatorname{Ker} T^X_{\ell,k}(a) = \{\xi(a) : \xi \in \operatorname{Ker} T_{\ell,k,a_0}\} .$$

Let (β_i, k_i), $i = 1, \ldots, t_X$, denote the vertices of $\mathfrak{N}_X$. For each i, let $\xi^i \in \operatorname{Ker} T_{\ell,k,a_0}$ such that $\nu(\xi^i) = (\beta_i, k_i)$;

$\xi^i = (\xi^i_{\beta,j})_{|\beta| \leqslant k,\, 1 \leqslant j \leqslant q}$. Put

$$Z = Z' \cup \bigcup_{i=1}^{t_X} \{x \in X : \xi^i_{\beta_i, k_i}(x) = 0\} .$$

Since X is irreducible, Z is a proper closed analytic subset of X. From Remark 8.3.5, we obtain :

LEMMA 8.4.5. – $\mathfrak{N}_a = \mathfrak{N}_X$ *for all* $a \in D_k \cap (X - Z)$.

PROPOSITION 8.4.6. – (1) *For all* $a \in X$, $\mathfrak{N}_X \leqslant \mathfrak{N}_a$.

(2) *Let* Z *be as in Remark 8.4.4. If* $a \in X - Z$, *then* $\mathfrak{N}_a \subset \mathfrak{N}_X$.

Proof. – We can assume that $\mathbf{K} = \mathbf{C}$. Let $a \in X$. Let (α_i, j_i), $i = 1, \ldots, t_a$ (respectively, (β_i, k_i), $i = 1, \ldots, t_X$) denote the vertices of $\mathfrak{N}_a$ (respectively, $\mathfrak{N}_X$), indexed in ascending order. Let $k \in \mathbf{N}$ such that $k \geqslant k(X)$ and $H_a(\ell)$ coincides with a polynomial for $\ell \geqslant k$. Let $\ell = \ell(k, X)$.

The arguments will be similar to those for Lemma 7.1. To prove (1), first consider $G \in \mathcal{R}_a$ such that $\nu(G) = (\alpha_1, j_1)$. Then G induces an element $\zeta \in \operatorname{Ker} \hat{T}_{\ell,k,a}$ such that $\zeta(a) \in \operatorname{Ker} T^X_{\ell,k}(a)$ is the projection of G to $\mathcal{R}_a(k) \subset \operatorname{Ker} T^X_{\ell,k}(a)$. Since

$$\operatorname{Ker} \hat{T}_{\ell,k,a} = \hat{\mathcal{O}}_{X,a} \cdot \operatorname{Ker} T_{\ell,k,a} ,$$

there exists $\eta \in \operatorname{Ker} T_{\ell, k, a}$ such that $\eta(a) = \zeta(a)$. Let Z be as in Remark 8.4.4. Evaluate η at $x \in D_k \cap (X - Z)$ near a: $\eta(x) \in \operatorname{Ker} T^X_{\ell, k}(x)$ and $\nu(\eta(x)) \leqslant (\alpha_1, j_1)$. ($\nu(\eta(x))$ makes sense when $\eta(x) \in J_{\varphi(x)}(k)$ is not zero). By Lemma 8.4.5, $(\beta_1, k_1) \leqslant \nu(\eta(x)) \leqslant (\alpha_1, j_1)$. If $(\beta_1, k_1) = (\alpha_1, j_1)$, then $\nu(\eta(x)) = (\alpha_1, j_1)$ for all $x \in X$ near a.

Now suppose that, for each $i = 1, \ldots, t$, we have: (i) $(\beta_i, k_i) = (\alpha_i, j_i)$; (ii) there exists $\eta^i \in \operatorname{Ker} T_{\ell, k, a}$ such that $\nu(\eta^i(x)) = (\alpha_i, j_i)$ for $x \in X$ near a. If $t_a = t$, we are done. Otherwise, consider $G \in \mathscr{R}_a$ such that $\nu(G) = (\alpha_{t+1}, j_{t+1})$. As above, there exists $\eta \in \operatorname{Ker} T_{\ell, k, a}$ such that $\eta(a) \in \mathscr{R}_a(k)$ is the image of G. Evaluate η at $x \in D_k \cap (X - Z)$ near a: $\eta(x) \in \operatorname{Ker} T^X_{\ell,k}(x)$ and $\nu(\eta(x)) \leqslant (\alpha_{t+1}, j_{t+1})$. If $\nu(\eta(x)) = (\alpha_{t+1}, j_{t+1})$, then $(\beta_{t+1}, k_{t+1}) \leqslant (\alpha_{t+1}, j_{t+1})$. If $\nu(\eta(x)) < (\alpha_{t+1}, j_{t+1})$, then either: (i) $\nu(\eta(x)) \notin \bigcup_{i=1}^{t} (\alpha_i, j_i) + \mathbf{N}^n$ and $(\beta_{t+1}, k_{t+1}) < (\alpha_{t+1}, j_{t+1})$, or (ii) $\nu(\eta(x)) \in \bigcup_{i=1}^{t} (\alpha_i, j_i) + \mathbf{N}^n$.

In the latter case, $\nu(\eta(x)) = (\alpha_i + \gamma, j_i)$, for some $i = 1, \ldots, t$ and some $\gamma \in \mathbf{N}^n$; thus $\eta_{\alpha_i+\gamma, j_i}(a) = 0$ and $\eta_{\alpha_i+\gamma, j_i}(x) \neq 0$. On the other hand, $\eta^i_{\alpha_i, j_i}(a) \neq 0$. Let

$$\eta' = \eta^i_{\alpha_i, j_i} \cdot \eta - \frac{\alpha_i!}{(\alpha_i + \gamma)!} \eta_{\alpha_i+\gamma, j_i} \cdot S^\gamma \eta^i .$$

Then $\eta' \in \operatorname{Ker} T_{\ell,k,a}$, $\eta'(a) = \eta^i_{\alpha_i, j_i}(a) \cdot \eta(a)$, and

$$\nu(\eta(x)) < \nu(\eta'(x)) \leqslant (\alpha_{t+1}, j_{t+1}).$$

(1) follows by induction.

To prove (2), consider ξ^i and Z from Remark 8.4.4. Let $a \in X - Z$. Consider $G \in \mathscr{R}_a$ such that $\nu(G) = (\alpha_t, j_t)$, where $1 \leqslant t \leqslant t_a$. As before, there exists $\eta \in \operatorname{Ker} T_{\ell, k, a}$ such that $\eta(a) \in \mathscr{R}_a(k)$ is the image of G. Evaluate η at $x \in D_k \cap (X - Z)$ near a: $\eta(x) \in \operatorname{Ker} T^X_{\ell,k}(x)$ and $\nu(\eta(x)) \leqslant (\alpha_t, j_t)$. If $\nu(\eta(x)) = (\alpha_t, j_t)$, then $(\alpha_t, j_t) \in \mathfrak{N}_X$, by Lemma 8.4.5. Otherwise,

$$\nu(\eta(x)) = (\beta_i + \gamma, k_i) < (\alpha_t, j_t),$$

for some $i = 1, \ldots, t_X$ and some $\gamma \in \mathbf{N}^n$. Thus $\eta_{\beta_i+\gamma, k_i}(a) = 0$

and $\eta_{\beta_i+\gamma,k_i}(x) \neq 0$. On the other hand, $\xi^i_{\beta_i,k_i}(a) \neq 0$, since $a \in X - Z$. Let

$$\eta' = \xi^i_{\beta_i,k_i} \cdot \eta - \frac{\beta_i\,!}{(\beta_i + \gamma)\,!}\,\eta_{\beta_i+\gamma,k_i} \cdot S^\gamma\, \xi^i_{\ell}.$$

Then $\eta'(a) = \xi^i_{\beta_i,k_i}(a) \cdot \eta(a)$ and $\nu(\eta(x)) < \nu(\eta'(x)) \leqslant (\alpha_t, j_t)$. (2) follows by induction.

□

PROPOSITION 8.4.7. – *Suppose that* $k \geqslant k(X)$. *Let* Y *be a closed analytic subset of* X. *Suppose that* $H_a(k) = H_X(k)$ *for all* $a \in X - Y$. *Let* $\mathfrak{L} = \mathfrak{L}(k, X)$ *and let* Z *be as in Remark 8.4.4. Then* $\mathfrak{N}_a = \mathfrak{N}_X$ *for all* $a \in X - (Y \cup Z)$.

Proof. – Let $a \in X - (Y \cup Z)$. Then $\mathfrak{N}_a \subset \mathfrak{N}_X$, by Proposition 8.4.6 (2). Since $H_a(k) = H_X(k)$, $\mathfrak{N}_a = \mathfrak{N}_X$.

□

8.5. *Proof of Theorem 8.2.5.* – (1) and (2) are equivalent, by Propositions 8.3.6 and 8.3.7. (3) trivially implies (2). (2) implies (4) by Propositions 8.4.6 (1) and 8.4.7. (4) implies (3) by Corollary 6.9 and Proposition 8.3.7.

□

Example 8.5.1. – We show that condition (4) of Theorem 8.2.5 is satisfied in Example 4.5. The notation is from 4.5. Let $a \in M^d_\Phi$. We will prove that the vertices of $\mathfrak{N}_a \subset \mathbf{N}^{n+d} \times \{1, \ldots, d\}$ are $(\beta_i, j_i) = (0, d - i + 1), i = 1, \ldots, d - m$, where $m = m(a)$: Write $a = (x^0, t^0_1, \ldots, t^0_d, \lambda^0_1, \ldots, \lambda^0_{d-1})$ and $\varphi(a) = (x^0, \lambda^0)$, where $\lambda^0 = (\lambda^0_1, \ldots, \lambda^0_d)$. It is enough to show :

(1) If $(R_d, \ldots, R_{d-m+1}, 0, \ldots, 0) \in \mathcal{R}_a$, then each $R_j = 0$.

(2) There exist analytic functions $S_j(\lambda)$, $j = 1, \ldots, m$, defined near λ^0, such that, for each $i = 1, \ldots, d - m$,

$$(0, \ldots, 0, S_m, \ldots, S_1, 1, 0, \ldots, 0) \in \mathcal{R}_a$$

(where 1 is in the i'th place from the right).

We use the following observation [29]: Let

$$g(z,\mu) = z^p + \sum_{j=1}^{p} \mu_j z^{p-j} \quad \text{and} \quad h(z,\nu) = z^q + \sum_{j=1}^{q} \nu_j z^{q-j},$$

where $\mu = (\mu_1, \ldots, \mu_p), \nu = (\nu_1, \ldots, \nu_q)$. Write

$$g(z,\mu) \cdot h(z,\nu) = z^{p+q} + \sum_{j=1}^{p+q} \lambda_j(\mu,\nu) z^{p+q-j}.$$

Then the Jacobian determinant $\det(\partial\lambda/\partial(\mu,\nu))$ is the resultant of g and h.

Let $\{t_{i_\alpha}^0\}$ denote the distinct t_i^0, and let m_α denote the multiplicity of the root $t_{i_\alpha}^0$ of $p(z,\boldsymbol{a})$. Then $\Sigma\, m_\alpha = m$ and $p(z,\boldsymbol{a}) = g(z) \cdot \prod_\alpha (z - t_{i_\alpha}^0)^{m_\alpha}$, where $g(t_i^0) \neq 0, i = 1, \ldots, d$. By the preceding observation and the implicit function theorem,

$$z^d + \sum_{j=1}^{d} \lambda_j z^{d-j} = g(z,\lambda) \cdot \prod_\alpha q_\alpha(z,\lambda),$$

where

$$q_\alpha(z,\lambda) = z^{m_\alpha} + \sum_{j=1}^{m_\alpha} a_{\alpha j}(\lambda) z^{m_\alpha - j},$$

$$g(z,\lambda) = z^{d-m} + \sum_{j=1}^{d-m} b_j(\lambda) z^{d-m-j},$$

and

(i) each $a_{\alpha j}(\lambda)$ and $b_j(\lambda)$ is analytic near λ^0;

(ii) $g(z) = g(z,\lambda^0)$;

(iii) each $(z - t_{i_\alpha}^0)^{m_\alpha} = q_\alpha(z,\lambda^0)$.

Define $S_j(\lambda), j = 1, \ldots, m,$ by

$$z^m + \sum_{j=1}^{m} S_j(\lambda) z^{m-j} = \prod_\alpha q_\alpha(z,\lambda).$$

Then the $S_j(\lambda)$ satisfy (2).

To prove (1), consider analytic functions $R_j(x,\lambda)$ near (x^0,λ^0), $j = d-m+1, \ldots, d$, such that

$$(R_d, \ldots, R_{d-m+1}, 0, \ldots, 0) \in \mathcal{R}_a .$$

Let $G(x,t,\lambda) = \sum_{j=d-m+1}^{d} t^{d-j} R_j(x,\lambda)$. Then for each α, G is divisible by $t^d + \sum_{j=1}^{d} \lambda_j t^{d-j}$, and thus by $q_\alpha(t,\lambda)$, in the ring of formal power series centered at $(x^0, t^0_{i_\alpha}, \lambda^0)$. By (iii) and uniqueness in the formal Weierstrass division theorem, the quotient of G by $q_\alpha(t,\lambda)$ belongs to $R[t]$, where R denotes the ring of formal power series centered at (x^0,λ^0). Therefore, G is divisible by $\prod_\alpha q_\alpha(t,\lambda)$ in $R[t]$. Since degree $G < m =$ degree $\prod_\alpha q_\alpha(t,\lambda)$, then $G = 0$; i.e., each $R_j = 0$, as required.

9. Special generators.

Let M, N, ϕ, A and B be as in 8.2. Let $s \in \mathbf{N}$. Let $a_0 = (a_0^1, \ldots, a_0^s) \in M_\phi^s$ and let $b_0 = \varphi(a_0)$. Let X denote an irreducible germ at a_0 of a closed analytic subset of M_ϕ^s. Let $U = \prod_{j=1}^{s} U^j$ be a product coordinate neighborhood of a_0 in M^s, and let V be a coordinate neighborhood of b_0 in N, such that $\phi(U^j) \subset V$, $j = 1, \ldots, s$. Shrinking U if necessary, we can assume that X is a closed analytic subset of U such that each connected component of its smooth points is adherent to a_0.

We continue to use the notation of 8.2-8.4. Assume that V is an open subset of $\mathbf{K}^n$. Let $\mathfrak{N}_X$ denote the generic diagram of initial exponents (cf. 8.4.3), and let (β_i, k_i), $i = 1, \ldots, t$, denote the vertices of $\mathfrak{N}_X$. Our main theorem B and Remark 2.11 (2) are consequences of the following:

THEOREM 9.1. – *Assume there is a proper analytic subset* Y *of* X *such that* $\mathfrak{N}_a = \mathfrak{N}_X$ *for all* $a \in X - Y$. *For each* $a \in X - Y$, *let* $G_a^i(y) = y^{\beta_i, k_i} - r_a^i(y)$, $i = 1, \ldots, t$, *denote the standard* *basis of* $\mathcal{R}_a$, *where, for each* i, *in* $\mathrm{G}_a^i = y^{\beta_i, k_i}$ *and*

$$\operatorname{supp} r_a^i \cap \mathfrak{N}_a = \emptyset$$

(cf. Corollary 6.8). Write $r_a^i(y) = \sum_{\beta, j} r_{\beta,j}^i(a)\, y^{\beta,j}$. *Then:*

(1) *For each* $i = 1, \ldots, t$ *and* $(\beta, j) \in \mathbf{N}^n \times \{1, \ldots, q\}$, $r_{\beta,j}^i \in \mathcal{M}(\mathrm{X}\,;\mathrm{Y})$.

(2) *Suppose that* $\mathrm{Y} = \emptyset$ *and that* $\mathcal{R}_{a_0}$ *is generated by* $\mathcal{R}_{a_0} \cap \mathcal{O}_{b_0}^q$. *Then, for each* $i = 1, \ldots, t$, $g^i = \mathrm{G}_{a_0}^i \in \mathcal{O}_{b_0}^q$ *and* $g_{\varphi(a)}^i = \mathrm{G}_a^i$ *for* $a \in \mathrm{X}$ *sufficiently close to* a_0. *In particular,* $\mathcal{R}_a$ *is generated by* $\mathcal{R}_a \cap \mathcal{O}_{\varphi(a)}^q$ *in some neighborhood of* a_0 *in* X.

Proof. – We can assume that $\mathbf{K} = \mathbf{C}$. For each $i = 1, \ldots, t$, put $f^i(x) = \mathrm{A}(x) \cdot \phi(x)^{\beta_i, k_i}$ (in local coordinates, where $\phi(x)^{\beta_i, k_i}$ denotes the composition of y^{β_i, k_i} with $y = \phi(x)$). Let $a \in \mathrm{X} - \mathrm{Y}$. Put

$$\mathrm{H}_a^i(y) = (\varphi(a) + y)^{\beta_i, k_i} - y^{\beta_i, k_i} + r_a^i(y).$$

Then, for each $i = 1, \ldots, t$,

$$\operatorname{supp} \mathrm{H}_a^i(y) \cap \mathfrak{N}_{\mathrm{X}} = \emptyset \tag{9.2}$$

and $\hat{f}_{aj}^i(x) - \hat{\mathrm{A}}_{aj}(x) \cdot (\mathrm{H}_a^i \circ \hat{\phi}_{aj})(x) \in \operatorname{Im} \hat{\mathrm{B}}_{aj}$, $j = 1, \ldots, s$; i.e., $(\hat{f}_{aj}^i)_{1 \le j \le s} - \hat{\Phi}_a(\mathrm{H}_a^i) \in \operatorname{Im} \hat{\mathrm{B}}_a$.

For each $\ell \in \mathbf{N}$, let ${}^{\ell}\mathrm{F}_a^i$ (respectively, ${}^{\ell}\mathrm{H}_a^i$) denote the image of $(\hat{f}_{aj}^i)_{1 \le j \le s}$ (respectively, of H_a^i) by the lower (respectively, upper) horizontal arrow in the completion of the left-hand diagram (8.2.6); thus,

$${}^{\ell}\mathrm{F}_a^i - \hat{\mathrm{A}}_{\ell,a} \cdot {}^{\ell}\mathrm{H}_a^i \in \operatorname{Im} \hat{\mathrm{B}}_{\ell,a}. \tag{9.3}$$

Recall that ${}^{\ell}\mathrm{H}_a^i$ is the element of $\bigoplus_{|\beta| \le \ell} \hat{\mathcal{O}}_{\mathrm{X},a}^q$ induced by $(\mathrm{D}^{\beta}\mathrm{H}_a^i \circ \hat{\varphi}_a)_{|\beta| \le \ell}$. Write ${}^{\ell}\mathrm{H}_a^i = (\mathrm{H}_{\beta,a}^i)_{|\beta| \le \ell} = (\mathrm{H}_{\beta,j,a}^i)_{|\beta| \le \ell, 1 \le j \le q}$, where each $\mathrm{H}_{\beta,j,a}^i \in \hat{\mathcal{O}}_{\mathrm{X},a}$ and $\mathrm{H}_{\beta,a}^i = (\mathrm{H}_{\beta,j,a}^i)_{1 \le j \le q}$. It follows from (9.2) that $\mathrm{H}_{\beta,j,a}^i = 0$ for all $(\beta, j) \in \mathfrak{N}_{\mathrm{X}}$.

Let $k \in \mathbf{N}$ and let $\ell = \ell(k, \mathrm{X})$. From (9.3), it follows that

$$\mathrm{ad}^{\sigma_{\ell,k}(\mathrm{X})}\, \hat{\mathrm{S}}_{\ell,k,a} \circ \mathrm{Ad}^{\rho_{\ell,k}(\mathrm{X})}\, \hat{\mathrm{D}}_{\ell,k,a} \cdot {}^{\ell}\mathrm{F}^i_a = \hat{\mathrm{T}}_{\ell,k,a} \cdot {}^{k}\mathrm{H}^i_a ,$$

where $\hat{\mathrm{S}}_{\ell,k,a} = \mathrm{Ad}^{\rho_{\ell,k}(\mathrm{X})}\, \hat{\mathrm{D}}_{\ell,k,a} \circ \hat{\mathrm{B}}_{\ell,a}$.

Let $e(k)$ denote the number of exponents $(\beta, j) \in \mathbf{N}^n \times \{1, \ldots, q\}$ such that $(\beta, j) \notin \mathfrak{N}_{\mathrm{X}}$ and $|\beta| \leqslant k$. Suppose that $a \in \mathrm{X} - (\mathrm{X}_{\ell,k} \cup \mathrm{Y})$. By the formal division theorem 6.2 and Corollary 6.9, Remarks 8.2.4 and 8.3.1, and Proposition 8.3.6 (2), rank $\mathrm{T}^{\mathrm{X}}_{\ell,k}(a) = e(k)$; moreover, if $\mathrm{W}_a(k)$ denotes the subspace $\{\mathrm{H} = (\mathrm{H}_{\beta,j})_{|\beta| \leqslant k,\, 1 \leqslant j \leqslant q} \in \bigoplus_{|\beta| \leqslant k} (\hat{\mathcal{O}}_{\mathrm{X},a} / \mathfrak{m}_{\mathrm{X},a} \cdot \hat{\mathcal{O}}_{\mathrm{X},a})^q :$ $\mathrm{H}_{\beta,j} = 0$ if $(\beta, j) \in \mathfrak{N}_{\mathrm{X}}\}$, then rank $\mathrm{T}^{\mathrm{X}}_{\ell,k}(a) \,|\, \mathrm{W}_a(k) = e(k)$. Then, by Cramer's rule, for all $\beta \in \mathbf{N}^n$, $|\beta| \leqslant k$, and all $j = 1, \ldots, q$, we obtain $\xi^i_{\beta,j}, \eta^i_{\beta,j} \in \mathcal{O}(\mathrm{U})$ such that $\eta^i_{\beta,j}(a) \neq 0$ if
This gives (1). $a \in \mathrm{X} - (\mathrm{X}_{\ell,k} \cup \mathrm{Y})$, and $\mathrm{H}^i_{\beta,j,a} = \hat{\xi}^i_{\beta,j,a} / \hat{\eta}_{\beta,j,a}$.

Now suppose $\mathrm{Y} = \emptyset$ and $\mathscr{R}_{a_0}$ is generated by $\mathscr{R}_{a_0} \cap \mathcal{O}^q_{b_0}$. Then $g^i = \mathrm{G}^i_{a_0} \in \mathcal{O}^q_{b_0}$, $i = 1, \ldots, t$, by Corollary 6.8 (2). For a sufficiently close to a_0 in X, $\hat{g}^i_{\varphi(a)}(y) = g^i(\varphi(a) - b_0 + y) \in \mathscr{R}_a$. But $\operatorname{supp}(\hat{g}^i_{\varphi(a)} - y^{\beta_i, k_i}) \cap \mathfrak{N}_{\mathrm{X}} = \emptyset$. By uniqueness of the standard basis, $\mathrm{G}^i_a(y) = \hat{g}^i_{\varphi(a)}(y)$. We have proved (2). □

PROPOSITION 9.4. – *Let* M *be an analytic manifold over* $\mathbf{K}$, *and let* $\mathrm{Z} \subset \mathrm{X}$ *denote closed analytic subsets of* M. *Let* $f^1, \ldots, f^q \in \mathscr{M}(\mathrm{X}\,;\mathrm{Z})\,[[y]]^p$, *where* $y = (y_1, \ldots, y_n)$. *For each* $a \in \mathrm{X} - \mathrm{Z}$, *let* $\mathscr{R}_a \subset \mathbf{K}\,[[y]]^q$ *denote the module of relations among the* $f^j(a\,;y)$. *Let* $a_0 \in \mathrm{X}$. *Then there is a neighborhood* U *of* a_0 *in* M, *and a filtration of* $\mathrm{X} \cap \mathrm{U}$ *by closed analytic subsets,*

$$\mathrm{X} \cap \mathrm{U} = \mathrm{X}_0 \supset \mathrm{X}_1 \supset \ldots \supset \mathrm{X}_{r+1} = \mathrm{Z} \cap \mathrm{U},$$

such that, for each $k = 0, \ldots, r$:

(1) $\mathfrak{N}(\mathscr{R}_a)$ *is constant on* $\mathrm{X}_k - \mathrm{X}_{k+1}$.

(2) *Let* $\mathrm{G}^i_a(y) = y^{\beta_i, k_i} - r^i_a(y)$, $i = 1, \ldots, t$, *denote the standard basis of* $\mathscr{R}_a$ *(as in 9.1), where* $a \in \mathrm{X}_k - \mathrm{X}_{k+1}$. *Write* $r^i_a(y) = \sum_{\beta,j} r^i_{\beta,j}(a)\, y^{\beta,j}$. *Then each* $r^i_{\beta,j} \in \mathscr{M}(\mathrm{X}_k\,;\mathrm{X}_{k+1})$.

Proof. – If $a \in X - Z$, let $\Phi_a : \mathbf{K}[[y]]^q \longrightarrow \mathbf{K}[[y]]^p$ denote the homomorphism given by the $p \times q$ matrix whose columns are $f^1(a;y), \ldots, f^q(a;y)$, and let $\mathscr{A}_a \subset \mathbf{K}[[y]]^p$ denote the image of Φ_a. By Lemma 7.2 (3), there exist a neighborhood U of a_0 and $\lambda \in \mathbf{N}$ such that $\mathscr{A}_a \cap \mathfrak{m}^{\ell+\lambda} \cdot \mathbf{K}[[y]]^p \subset \mathfrak{m}^\ell \cdot \mathscr{A}_a$, for all $\ell \in \mathbf{N}$ and $a \in (X - Z) \cap U$ (where $\mathfrak{m}$ denotes the maximal ideal of $\mathbf{K}[[y]]$). Then, by Remark 7.6,

$$\Phi_a^{-1}(\mathfrak{m}^{\ell+\lambda} \cdot \mathbf{K}[[y]]^p) \subset \mathscr{R}_a + \mathfrak{m}^\ell \cdot \mathbf{K}[[y]]^q ,$$

for all $\ell \in \mathbf{N}$ and $a \in (X - Z) \cap U$. With this uniform Chevalley estimate, the arguments used to prove Theorems 8.2.5 and 9.1 can be repeated to obtain the result.

□

We conclude this section with two "glueing" results, which provide alternative reductions of the problem of verifying the conditions of Theorem 8.2.5. The first reduces the problem to the case that the source M is connected, and the second to the case $s = 1$. We use the notation of 8.2.

LEMMA 9.5. – *Suppose that* M *is a disjoint union*

$$M = M_1 \cup M_2 .$$

Let $\phi_i = \phi \mid M_i$, $i = 1, 2$. *Let* $s \in \mathbf{N}$. *Suppose that, for each* $t \in \mathbf{N}, t \leqslant s$, *one of the conditions of Theorem 8.2.5 is satisfied when* ϕ *is replaced by* ϕ_1 *or* ϕ_2 *and* s *is replaced by* t. *Then the conditions of Theorem 8.2.5 are satisfied.*

Proof. – Let $t \in \mathbf{N}$, $t \leqslant s$. Suppose that I is an ordered subset of $\{1, \ldots, s\}$ containing t elements (perhaps $I = \emptyset$). Put

$${}^I M_\phi^s = \{a = (a^1, \ldots, a^s) \in M_\phi^s : a^i \in M_1 \text{ if and only if } i \in I\} .$$

Then ${}^I M_\phi^s$ identifies with the fiber product

$$M_{1,\phi_1}^t \times_N M_{2,\phi_2}^{s-t} = \{(a_1, a_2) \in M_{1,\phi_1}^t \times M_{2,\phi_2}^{s-t} : \varphi_1(a_1) = \varphi_2(a_2)\} .$$

Clearly, M_ϕ^s is the disjoint union $\cup\, {}^I M_\phi^s$ over all ordered subsets I of $\{1, \ldots, s\}$.

Suppose $I = (1, \ldots, t)$ ($I = \emptyset$ if $t = 0$). Then it is enough to show that 8.2.5 (4) holds in ${}^I M_\phi^s = M_{1,\phi_1}^t \times_N M_{2,\phi_2}^{s-t}$. Let $a \in {}^I M_\phi^s$; say $a = (a^1, \ldots, a^s)$, where $a_1 = (a^1, \ldots, a^t) \in M_{1,\phi_1}^t$

and $a_2 = (a^{t+1}, \ldots, a^s) \in M^{s-t}_{2,\phi_2}$. Let $s_1 = t$, $s_2 = s - t$. Then $\mathscr{R}_a = \mathscr{R}_{a_1} \cap \mathscr{R}_{a_2}$, where for each $i = 1, 2$, $\mathscr{R}_{a_i}$ is the module of formal relations at $a_i \in M^{s_i}_{i,\phi_i}$ associated to the mapping $\phi_i : M_i \longrightarrow N$ and the matrices of analytic functions A and B restricted to M_i. Therefore, the result follows from Theorem 9.1, Corollary 7.9, Lemma 7.2 (1) and Proposition 8.4.6 (1) (cf. the proof of 9.6 below).

□

PROPOSITION 9.6. – *Assume that* N *is an open subset of* $\mathbf{K}^n$. *Then the diagram of initial exponents* $\mathfrak{N}_a = \mathfrak{N}(\mathscr{R}_a)$ *is Zariski semicontinuous on* M^s_ϕ, *for a given positive integer* s, *if and only if it is Zariski semicontinuous in the case* $s = 1$.

Proof. – Let s be a positive integer. Since $M = M^1_\phi$ is embedded in M^s_ϕ by the diagonal mapping, Zariski semicontinuity of $\mathfrak{N}_a$ on M^s_ϕ implies semicontinuity in the case $s = 1$. On the other hand, suppose that $\mathfrak{N}(\mathscr{R}_a)$ is Zariski semicontinuous on M, where $\mathscr{R}_a = \{G \in \hat{\mathscr{O}}^q_{\phi(a)} : \Phi_a(G) \in \operatorname{Im} \hat{B}_a\}$, $a \in M$. Let $a_0 \in M^s_\phi$, and let $a_0 \in X \subset U$ as at the beginning of this section. By Proposition 8.4.6 (1), it suffices to find a proper analytic subset Y of X such that $\mathfrak{N}_a$ is constant on $X - Y$. Let $\mu^i : M^s_\phi \longrightarrow M$ denote the projection onto the i'th coordinate; i.e., $\mu^i(a) = a^i$, where $a = (a^1, \ldots, a^s) \in M^s_\phi$, $i = 1, \ldots, s$. Then $\mathscr{R}_a = \bigcap_{i=1}^{s} \mathscr{R}_{a^i}$. By the hypothesis and Theorem 9.1, there is a proper closed analytic subset Z of X such that, for each $i = 1, \ldots, s$, $\mathfrak{N}(\ _{\mu^i(a)})$ is constant on $X - Z$, and the coefficients of the standard $\mathscr{R}$asis of $\mathscr{R}_{\mu^i(a)}$, as functions of $a \in X - Z$, belong to $\mathscr{M}(X; Z)$. The result follows from Corollary 7.9 and Lemma 7.2.

□

Remark 9.7. – It follows from Theorem 8.2.5 that the same assertion is true with M a (possibly singular) analytic space. The proofs of Lemma 9.5 and Proposition 9.6 apply also to the case that M is singular.

BIBLIOGRAPHY

[1] J.M. AROCA, H. HIRONAKA and J.L. VICENTE, The theory of the maximal contact, *Mem. Mat. Inst. Jorge Juan*, No. 29, Consejo Superior de Investigaciones Científicas, Madrid, 1975.

[2] M. ARTIN, Algebraic approximation of structures over complete local rings, *Inst. Hautes Etudes Sci. Publ. Math.*, 36 (1969), 23-58

[3] M. ARTIN, Algebraic spaces, *Yale Math. Monographs*, No. 3, Yale University Press, New Haven, 1971.

[4] J. BECKER and W.R. ZAME, Applications of functional analysis to the solution of power series equations, *Math. Ann.*, 243 (1979), 37-54.

[5] E. BIERSTONE and P.D. MILMAN, Composite differentiable functions, *Ann. of Math.*, 116 (1982), 541-558.

[6] E. BIERSTONE and P.D. MILMAN, The Newton diagram of an analytic morphism, and applications to differentiable functions, *Bull. Amer. Math. Soc. (N.S.)*, 9 (1983), 315-318.

[7] E. BIERSTONE and G.W. SCHWARZ, Continuous linear division and extension of $\mathscr{C}^\infty$ functions, *Duke Math. J.*, 50 (1983), 233-271.

[8] J. BRIANCON, Weierstrass préparé à la Hironaka, *Astérisque*, 7, 8 (1973), 67-73.

[9] D.A. BUCHSBAUM and D. EISENBUD, Some structure theorems for finite free resolutions, *Adv. in Math.*, 12 (1974), 84-139.

[10] C. CHEVALLEY, On the theory of local rings, *Ann. of Math.*, 44 (1943), 690-708.

[11] Z. DENKOWSKA, S. ŁOJASIEWICZ and J. STASICA, Sur le nombre des composantes connexes de la section d'un sous-analytique, *Bull. Acad. Polon. Sci. Sér. Sci. Math.*, 30 (1982), 333-335.

[12] A.M. GABRIELOV, Projections of semi-analytic sets, *Functional Anal. Appl.*, 2 (1968), 282-291 = *Funkcional. Anal. i Priložen.*, 2, No. 4 (1968), 18-30.

[13] A.M. GABRIELOV, Formal relations between analytic functions, *Functional Anal. Appl.*, 5 (1971), 318-319 = *Funkcional. Anal. i Priložen.*, 5, No. 4 (1971), 64-65.

[14] A.M. GABRIELOV, Formal relations between analytic functions, *Math. USSR Izvestija*, 7 (1973), 1056-1088 = *Izv. Akad. Nauk SSSR Ser. Mat.*, 37 (1973), 1056-1090.

[15] A. GALLIGO, Théorème de division et stabilité en géométrie analytique locale, *Ann. Inst. Fourier, Grenoble*, 29-2 (1979), 107-184.

[16] G. GLAESER, Fonctions composées différentiables, *Ann. of Math.*, 77 (1963), 193-209.

[17] H. GRAUERT, Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen, *Inst. Hautes Etudes Sci. Publ. Math.*, 5 (1960).

[18] H. GRAUERT, Über die Deformation isolierter Singularitäten analytischer Mengen, *Invent. Math.*, 15 (1972), 171-198.

[19] H. GRAUERT and R. REMMERT, *Analytische Stellenalgebren*, Springer, Berlin-Heidelberg-New York, 1971.

[20] R.M. HARDT, Stratification of real analytic mappings and images, *Invent. Math.*, 28 (1975), 193-208.

[21] H. HIRONAKA, Subanalytic sets, *Number Theory, Algebraic Geometry and Commutative Algebra*, pp. 453-493, Kinokuniya, Tokyo, 1973.

[22] H. HIRONAKA, Introduction to the theory of infinitely near singular points, *Mem. Mat. Inst. Jorge Juan*, No. 28, Consejo Superior de Investigaciones Científicas, Madrid, 1974.

[23] H. HIRONAKA, Stratification and flatness, *Real and Complex Singularities, Oslo 1976*, Proc. Nineth Nordic Summer School/NAVF Sympos. Math., pp. 199-265, Sijthoff and Noordhoff, Alphen aan den Rijn, 1977.

[24] D. KNUTSON, Algebraic spaces, *Lecture Notes in Math.*, No. 203, Springer, Berlin-Heidelberg-New York, 1971.

[25] M. LEJEUNE and B. TEISSIER, Contribution à l'étude des singularités du point de vue du polygone de Newton, Thèse, Université Paris VII, 1973.

[26] S. ŁOJASIEWICZ, Ensembles semi-analytiques, Inst. Hautes Etudes Sci., Bures-sur-Yvette, 1964.

[27] B. MALGRANGE, *Ideals of Differentiable Functions*, Oxford University Press, Bombay, 1966.

[28] J. MERRIEN, Applications des faisceaux analytiques semi-cohérents aux fonctions différentiables, *Ann. Inst. Fourier, Grenoble*, 31-1 (1981), 63-82.

[29] P.D. MILMAN, The Malgrange-Mather division theorem, *Topology*, 16 (1977), 395-401.

[30] P.D. MILMAN, Analytic and polynomial homomorphisms of analytic rings, *Math. Ann.*, 232 (1978), 247-253.

[31] D. MUMFORD, *Algebraic Geometry I. Complex Projective Varieties*, Springer, Berlin-Heidelberg-New York, 1976.

[32] R. NARASIMHAN, Introduction to the theory of analytic spaces, *Lecture Notes in Math.*, No. 25, Springer, Berlin-Heidelberg-New York, 1966.

[33] D. POPESCU, General Néron desingularization and approximation. I, II (preprints, National Institute for Scientific and Technical Creation, Bucharest, 1983).

[34] G.W. SCHWARZ, Smooth functions invariant under the action of a compact Lie group, *Topology*, 14 (1975), 63-68.

[35] Y.-T. SIU, $\mathcal{O}^N$-Approximable and holomorphic functions on complex spaces, *Duke Math. J.*, 36 (1969), 451-454.

[36] M. TAMM, Subanalytic sets in the calculus of variations, *Acta Math.*, 146 (1981), 167-199.

[37] J.Cl. TOUGERON, *Idéaux de Fonctions Différentiables*, Springer, Berlin-Heidelberg-New York, 1972.

[38] J.Cl. TOUGERON, Fonctions composées différentiables : cas algébrique, *Ann. Inst. Fourier, Grenoble*, 30-4 (1980), 51-74.

[39] J.Cl. TOUGERON, Existence de bornes uniformes pour certaines familles d'idéaux de l'anneau des séries formelles $k[[x]]$. Applications (to appear).

[40] O. ZARISKI and P. SAMUEL, *Commutative Algebra*, Vol. II, Springer, New York-Heidelberg-Berlin, 1975.

Manuscrit reçu le 16 décembre 1985.

E. BIERSTONE & P.D. MILMAN,
University of Toronto
Department of Mathematics
Toronto, Ontario, Canada M5S 1A1.