ANNALES DE L'INSTITUT FOURIER

JEAN-LIN JOURNÉ

Two problems of Calderón-Zygmund theory on product-spaces

Annales de l'institut Fourier, tome 38, nº 1 (1988), p. 111-132

<http://www.numdam.org/item?id=AIF_1988__38_1_111_0>

Ann. Inst. Fourier, Grenoble
38, 1 (1988), 111-132

TWO PROBLEMS OF CALDERON-ZYGMUND THEORY ON PRODUCT-SPACES

par Jean-Lin JOURNÉ*

Robert Fefferman introduced in [1] the notion of a rectangle atom on $\mathbb{R}^n \times \mathbb{R}^m$ and proved the following theorem.

THEOREM A. – *Let T be a bounded linear operator on $L^2(\mathbb{R}^n \times \mathbb{R}^m)$. Suppose that for any $H^p(\mathbb{R}^n \times \mathbb{R}^m)(0 < p \leq 1)$ rectangle atom a supported on the rectangle R we have*

$$\int_{(\gamma R)^c} |Ta|^p dx_1 dx_2 \leq c\gamma^{-\delta}$$

for some fixed $\delta > 0$ and all $\gamma \geq 2$. Then T is a bounded operator from $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ to $L^p(\mathbb{R}^n \times \mathbb{R}^m)$.

The definitions and tools involved in this theorem and its proof have been generalized to product spaces with an arbitrary number of factors [2], [3], but the question of whether Theorem A extends for three or more factors or not, raised implicitely in [4] and explicitely in [2] was open. Our purpose is to show that in the case $p = 1$, and of the space $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$, Theorem A does not extend without any further assumptions on the nature of T. If, however, one supposes that T is

---

* Supported by N.S.F.

Key-words : Calderón-Zygmund operators – Product spaces – Bounded mean oscillation.

a convolution operator and if $\delta > (1/8)$, then Theorem A extends. As will be apparent from the proof 1/8 is probably not sharp and it is reasonable to conjecture that $\delta > 0$ should suffice.

The second question which we shall answer has been raised by Raphy Coifman and concerns the L^2-boundedness of the operator c_a defined for $a \in L^\infty_{\mathbb{C}}(\mathbb{R}^2)$ and $||a||_\infty < 1$, by the kernel

$$c_a(x,y) = \frac{1}{(x_1 - y_1)(x_2 - y_2) + \int_{x_1}^{y_1}\int_{x_2}^{y_2} a(u_1,u_2)du_1du_2}.$$

The case $||a||_\infty < \varepsilon$ was handled in [3] and was a consequence of the estimate

$$||L_{k,a}||_{2,2} \leq c^k ||a||_\infty^k$$

where $L_{k,a}$ is the operator defined by the kernel

$$\frac{1}{(x_1 - u_1)(x_2 - y_2)}\left[\frac{\int_{x_1}^{y_1}\int_{x_2}^{y_2} a(u_1,u_2)du_1du_2}{(x_1 - y_1)(x_2 - y_2)}\right]^k.$$

Here we improve this estimate and obtain $||L_{k,a}||_{2,2} \leq c_\delta(1+k)^{2+\delta}$ for all $\delta > 0$, which yields the general case $||a||_\infty < 1$.

In Section 1 we recall some facts about bounded mean oscillation over rectangles, and state Theorem A, restricted to $p = 1$, in this dual setting. In Section 2 we present the counterexample to the extension of Theorem A for $\mathbf{R}\times\mathbf{R}\times\mathbf{R}$ and $p = 1$. In Section 3 we show how the positive result for convolution operators can be reduced to a problem on finite families of convolution operators, which is handled in Section 4. In Section 5 we treat the operators $L_{k,a}$. This section essentially combines ideas already contained elsewhere, and for this reason, is rather sketchy.

I wish to thank Robert Fefferman and Jill Pipher whose papers brought the first question treated here to my attention, Elias Stein for encouraging me to clarify it and Michael Christ for discouraging me from clarifying it further.

1. Bounded mean oscillation over rectangles.

The space $B(\mathbb{R} \times \mathbb{R})$, introduced in [5], is the dual space of the atomic H^1-space, constructed from rectangle atoms. In other words, let $b \in L^2_{\text{loc}}(\mathbb{R}^2)$. For every rectangle $R = I \times J$ in $\mathbb{R}^2$ let $\text{Osc}_R b = \inf_{b_1,b_2} \left(\frac{1}{|R|} \int_R |b(x_1,x_2) - b_1(x_1) - b_2(x_2)|^2 dx_1 dx_2 \right)^{1/2}$, where the inf is taken over all b_1, b_2 respectively in $L^2(I)$ and $L^2(J)$. Then $b \in B$ if and only if

$$\text{1.1} \qquad \sup_R \text{Osc}_R b < +\infty.$$

The left hand side of 1.1 is denoted $||b||_B$.

An equivalent definition can be given in terms of Carleson measures over rectangles. Let ψ be a real even $C_0^\infty(\mathbb{R})$ function such that $\int \psi \, dx = 0$. For $t > 0$ and $i \in \{1,2\}$, let Q_{t_i} be the convolution operator on $\mathbb{R}^2$ of symbol $\hat{\psi}(t\xi_i)$. We normalize ψ so that $\int_0^{+\infty} Q_{t_i}^2 \frac{dt_i}{t_i} = I$. For each rectangle R, the set $S(R)$ denotes the subset of $\mathbb{R}^2_+ \times \mathbb{R}^2_+$ of those $(x_1, t_1, x_2, t_2) = (x, t)$ such that $]x_1 - t_1, x_1 + t_1[\times]x_2 - t_2, x_2 + t_2[\subset R$.

LEMMA 1. – *A function $b \in L^2_{loc}(\mathbb{R}^2)$ is in B if and only if for some constant c_b*

$$\text{1.2} \qquad \int_{s(R)} |Q_{t_1} Q_{b_2} b|^2 dx_1 \frac{dt_1}{t_1} dx_2 \frac{dt_2}{t_2} \leq c_b |R|.$$

Moreover if c_b is optimal, $c_b \approx ||b||_B^2$.

Notice that 1.1 $\Longrightarrow$ 1.2 is clear since ψ has compact support. We shall prove the converse in the non-product setting but the proof we give extends easily.

It is enough to show that if $b \in B(\mathbb{R}) = \text{BMO}(\mathbb{R})$, and for all interval $I \subseteq \mathbb{R}$

$$\text{1.3} \qquad \int_{(x,t),]x-t,x+t[\subseteq I} |Q_t b|^2 dx \frac{dt}{t} \leq |I|$$

then $||b||_B = ||b||_{\mathrm{BMO}} \leq c$ where c depends only on ψ. For all $t > 0$, let P_t be the operator $I - \int_0^t Q_s^2 \frac{ds}{s}$. Then for all $t > 0$, $P_t b$ is C^∞ and $||(P_t b)'||_\infty \leq c||b||_B t^{-1}$. It follows that if I is an interval of center x_o and $t_I = K|I|$ for K fixed large enough, $\left(\frac{1}{|I|}\int_I |P_{t_I}b(x) - P_{t_I}b(x_0)|^2 dx\right)^{1/2} \leq \frac{1}{2}||b||_B$. Therefore

$$\left(\frac{1}{|I|}\int_I |b(x) - P_{t_I}b(x_0)|^2 dx\right)^{1/2}$$
$$\leq \left(\frac{1}{|I|}\int_I |b(x) - P_{t_I}b(x)|^2 dx\right)^{1/2} + \frac{1}{2}||b||_B.$$

By taking the sup over I, we see that $||b||_B \leq 2 \sup_I \left(\frac{1}{|I|}\right.$ $\left.\int_I |b(x) - P_{t_I}b(x)|^2 dx\right)^{1/2}$. To estimate the right hand side we let g be in $L^2(I)$, with $||g||_2 = 1$ and dominate $< g, b - P_{t_I}b >$ by

$$\int_{s \leq t_I} |Q_s g(x)|\, |Q_s b(x)| \frac{ds}{s} dx. \tag{1.5}$$

The conditions $Q_s g \neq 0$, $s \leq t_I$, and $g \in L^2(I)$, imply $x\, K'I$ for some K' fixed. Using Cauchy-Schwarz, 1.3 and $||g||_2 = 1$, we can dominate 1.5 by an absolute constant, which proves the lemma.

In the following lemma the notations and definitions are those of [3].

LEMMA 2. – *Let T be a translation invariant δ –* CZO *on* $\mathbf{R}\times\mathbf{R}$. *Then T is bounded on B.*

This lemma is an easy consequence of lemma 1. For simplicity we shall consider the non-product situation, but give a proof which extends trivially.

Let T be a translation invariant δ – CZO on $\mathbf{R}$. The kernel $(Q_t\, TQ_{t'})(x-y)$ of T is easily seen to satisfy

1.6 $$|(Q_tTQ_{t'})(x-y)| \leq c\, w_{\delta',t\vee t'}(x-y)\left(\frac{t\wedge t'}{t\vee t'}\right)^{\delta'}$$

for some $\delta' < \delta$, where $w_{\delta',t}(z) = \dfrac{t^{\delta'}}{t^{1+\delta'}+|z|^{1+\delta'}}$. For $(x,t) \in \mathbf{R}^2_+$ and $b \in B$ $|Q_tTb(x)| = \left|\int_{\mathbf{R}^2_+} (Q_tTQ_{t'})\ (x,y)\ (Q_{t'}b)\ (y)\ dy\ \dfrac{dt'}{t'}\right|$. By Cauchy-Schwarz and because of 1.6 this is less than

$$c\left[\int_{\mathbf{R}^2_+} w_{\delta',t\vee t'}(x-y)\ \left(\frac{t\wedge t'}{t\vee t'}\right)^{\delta'}\ |Q_{t'}b(y)|^2 dy \frac{dt'}{t'}\right]^{1/2}.$$

It follows that if $|Q_{t'}b(y)|^2\ dy' \frac{dt'}{t'}$ is a Carleson measure, then $|Q_tTb(x)|^2 dx\ \frac{dt}{t}$ is a Carleson measure. The same proof using Carleson measures over rectangles yields the result in the product case. Lemma 2 is proved, by Lemma 1.

We conclude this section by stating Theorem A, restricted to $p=1$, in dual form [4].

THEOREM A$'$. – *Let T be a linear operator bounded on $L^2(\mathbf{R}\times\mathbf{R})$. Suppose that for any rectangle R, and any L^∞-function a supported out of γR,*

1.7 $$\mathrm{Osc}_R\ Ta \leq c\gamma^{-\delta}$$

for some $\delta > 0$ and all $\gamma > 2$. Then T maps L^∞ to BMO$(\mathbf{R}\times\mathbf{R})$.

2. A Counterexample.

Any counterexample in this kind of question has to be related to the counterexample of Carleson [6] showing that rectangular Carleson measures are not, on the bidisc, a good substitute for classical Carleson-measures. As shown by R. Fefferman in [5], this counterexample implies that there can be no a priori estimate $||b||_{\mathrm{BMO}} \le c||b||_B$ on the bidisc. We shall denote for each $k \ge 0$ by b_k a function on $\mathbb{R}\times\mathbb{R}$ such that $||b_k||_{\mathrm{BMO}} = 1$ and $||b||_B \le 2^{-k}$. Using b_k we form a δ – CZO on T_k on $\mathbb{R}\times\mathbb{R}$ by letting $T_k f = \int\int Q_{t_1}Q_{t_2}\{(Q_{t_1}Q_{t_2}b_k)\,(P_{t_1}P_{t_2}f)\}\,\frac{dt_1}{t_1}\,\frac{dt_2}{t_2}$ for $f \in C_0^\infty(\mathbb{R})\otimes C_0^\infty(\mathbb{R})$. The fact that T_k is a $\delta-$CZO is clear, and proved in [3]. Let also S_k be defined on $\mathbb{R}$ as the operator of convolution by $\frac{1}{x}\{-\phi(x)+\phi(2^{-k}x)\}$, where ϕ is a $C_0^\infty(\mathbb{R})$ function equal to 1 near 0, followed by the multiplication by e^{ix}. Finally let $U_k = T_k \otimes S_k$.

We claim that U_k satisfies uniformly the assumptions of Theorem A', adapted to the case of three factors. Clearly $||U_k||_{2,2} \le c$. To check 1.7, only the oscillation of b_k over rectangles is used, introducting a gain of 2^{-k}. On the other hand the kernel $s_k(x,y)$ of S_k satisfies $|\nabla_x\, s_k(x,y)| \le \frac{c}{|x-y|}$, but since on the support of $s_k(.,.)$, $|x-y| \le c2^k$, one also has $|\nabla_x\, s_k(x,y)| \le \frac{c2^k}{|x-y|^2}$, and $|\nabla_y\, s_k(x,y)| \le \frac{c}{|x-y|^2}$. Therefore, writing $U_k = (2^kT_k) \otimes (2^{-k}S_k)$, we see that U_k satisfies 1.7 as any tensor product of a $\delta-$CZO on $\mathbb{R}\times\mathbb{R}$ with a $1-$ CZO on $\mathbb{R}$. The contradiction comes from the fact that the functions $u_k = U_k(1\otimes \operatorname{sgn} x) = b_k \otimes S_k \operatorname{sgn} x$, are not a bounded sequence in $\mathrm{BMO}(\mathbb{R}\times\mathbb{R}\times\mathbb{R})$. Indeed $||u_k||_{\mathrm{BMO}} \approx ||S_k \operatorname{sgn}(x)||_{\mathrm{BMO}} \approx k$.

3. Extension of Theorem A in the convolution case.

We wish to prove the following.

THEOREM 1. – *Let T be a bounded convolution operator on $L^2(\mathbb{R}^3)$. Suppose that for any rectangle R, and any L^∞-function supported out of γR,*

3.1 $$\mathrm{Osc}_R\, Ta \le c\,\gamma^{-\delta}$$

for some $\delta > 1/8$ and all $\gamma > 2$. Then T maps $L^\infty(\mathbb{R}^3)$ to BMO $(\mathbb{R}\times\mathbb{R}\times\mathbb{R})$.

In [3] the L^∞ – BMO boundedness of operators bounded on L^2, whose kernels satisfy some assumptions of Calderón- Zygmund type, is proved using the characterization of BMO in terms of Carleson measures [7] and some geometrical lemmas. Assumption 3.1 can be thought of as a very weak Calderón-Zygmund type of assumption and is reminiscent of the weak vector-valued standard estimates satisfied by the kernel of the square function operator of J.-L. Rubio de Francia [8]. It turns out that the technique used in [3] to extend Rubio de Francia's theorem in several parameters also applies here. Indeed this technique can be summarized in the following lemma.

LEMMA 3. – *Let T be a bounded operator on $L^2(\mathbb{R}^n)$. Let $i \in \{1,2,\ldots,n\}$ and let $(\ell_1,\ldots,\ell_i) \in \mathbb{Z}^i$. Let $(x_1,\ldots,x_i) \in \mathbb{R}^i$. Let $b \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ be supported in $\{(z_1,\ldots,z_i) \in \mathbb{R}^n, 2^{\ell_k} \leq |z_k - x_k| \leq 2^{\ell_k+1}$ for $1 \leq k \leq i\}$, such that for all $(z_1,\ldots,z_i)$,*

$$3.2 \qquad \int_{z_{i+1},\ldots,z_n} |b(z_1,\ldots,z_n)|^2 dz_{i+1},\ldots,dz_n \leq 1.$$

Suppose that for $(t_1,\ldots,t_i)$ such that $t_k \leq 2^{\ell_k-1}$ for $1 \leq k \leq i$,
3.3

$$\int_{x_{i+1},\ldots,x_n} |Q_{t_1}\ldots Q_{t_i} Tb(x_1,\ldots,x_n)|^2 dx_{i+1},\ldots,dx_n \leq c \prod_{1\leq k\leq i} \left(\frac{t_k}{2^{\ell_k}}\right)^{\varepsilon}$$

for some $\varepsilon > 0$ and that all of this remains true if the set $\{1,\ldots,i\}$ is replaced by any other non-empty subset of $\{1,\ldots,n\}$.

Then T maps $L^\infty(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}\times\ldots\times\mathbb{R})$.

Of course when $i = n$, then 3.2 and 3.3 simply mean that when

$$||b||_\infty \leq 1, |Q_{t_i},\ldots,Q_{t_n} T\, b(x_1,\ldots,x_n)| \leq c \prod_{1\leq i\leq n} \left(\frac{t_k}{2^{\ell_k}}\right)^{\varepsilon},$$

which follows trivially from 3.1.

Now we are going to see how to reduce the proof of Theorem 1 to a problem on finite families of convolution operators. In this reduction we shall suppose that $n = 2$ and $i = 1$. We want to prove that if T satisfies the assumptions of Theorem 1, then it satisfies the assumptions of Lemma 3.

We shall assume that the function ψ defining Q_{t_i}, $i \in \{1,2\}$, is of the form $\widetilde{\psi} * \widetilde{\psi}$, where $\widetilde{\psi}$ is real even C^∞, supported in $\left[-\frac{1}{2}, \frac{1}{2}\right]$ and with mean-value 0. Then if $|x_1 - z_1| \geq 2t_1$ we can define an operator $(Q_{t_1}T)_{x_1-z_1}$ acting on functions of the second variable by letting, for

$$f, g \in C_0^\infty(\mathbf{R}) < g, (Q_{t_1}T)_{x_1-z_1} f >=< \widetilde{\psi}_{t_1}^{x_1} \otimes g, T\widetilde{\psi}_{t_1}^{z_1} \otimes f >$$

where $\widetilde{\psi}_{t_1}^{x_1}(u)$ is defined as $\frac{1}{t_1}\,\widetilde{\psi}\left(\frac{u-x_1}{t_1}\right)$ and similarly for $\widetilde{\psi}_{t_1}^{z_1}(u)$.

Let $(x_1, t_1) \in \mathbf{R}_+^2$; $\ell_1 \in \mathbf{Z}$ such that $2^{\ell_1} \geq 2t_1$ and $b \in L^2_{\text{loc}}(\mathbf{R}^2)$ such that $\operatorname{supp} b \subseteq \{(z_1, z_2) \in \mathbf{R}^2,\ 2^{\ell_1} \leq |x_1 - z_1| \leq 2^{\ell_1+1}\}$ and for all $z_1 \in \mathbf{R}$, $\int |b(z_1, z_2)|^2\, dz_2 \leq 1$. Then $Q_{t_1}T\, b(x_1, \cdot) = \int (Q_{t_1}T)_{x_1-z_1}$ $b(z_1, \cdot)\, dz_1$. In order to prove that $||Q_{t_1}T\ b(x_1,\cdot)||_2 \leq c\left(\frac{t_1}{2^{\ell_1}}\right)^\varepsilon$ for some $\varepsilon > 0$, it suffices to prove that for all finite sequence $(z_{1,k})_{1\leq k\leq N}$ such that $|z_{1,k} - z_{1,k+1}| = 2t_1$ and $2^{\ell_1} \leq |x_1 - z_{1,k}| \leq 2^{\ell_1+1}$ for all $1 \leq k \leq N$,

$$\left\|\sum_{k=1}^{N} (Q_{t_1}T)_{x_1-z_{1,k}}\ b(z_{1,k}, \cdot)\right\|_2 \leq \frac{c}{t_1}\left(\frac{t_1}{2^{\ell_1}}\right)^\varepsilon. \tag{3.4}$$

On the other hand we are going to see that if $||b||_\infty \leq 1$, 3.1 implies

$$\left\|\sum_{k=1}^{N} (Q_{t_1}T)_{x_1-z_{1,k}}\ b(z_{1,k}, \cdot)\right\|_{B(\mathbf{R})} \leq \frac{c}{t_1}\left(\frac{t_1}{2^{\ell_1}}\right)^\delta. \tag{3.5}$$

Indeed, using the factorization of Q_{t_1} as $\widetilde{Q}_{t_1}^2$, we can rewrite the sum

$$\sum_{k=1}^{N} (Q_{t_1}T)_{x_1-z_{1,k}}\ b(z_{1,k}, \cdot)$$

as

$$\int (\widetilde{Q}_{t_1}T)_{x_1-y_1}\left[\sum_k \widetilde{\psi}_{t_1}(y_1 - z_{1,k})b(z_{1,k}, \cdot)\right] dy_1.$$

As a function of $(y_1, \cdot)$, $\sum_k \widetilde{\psi}_{t_1}(y_1 - z_{1,k})b(z_{1,k}, \cdot)$ is bounded of norm $\frac{c}{t_1}$ and supported in a strip $\{(y_1, \cdot) \in \mathbf{R}^2;\ |x_1 - y_1| \approx 2^{\ell_1}\}$. It follows that 3.1 implies 3.5.

Observe that now we just have to show that 3.5 for $\delta > 1/8$ implies 3.4 for some $\varepsilon > 0$. Since $N \leq \frac{2^{\ell_1}}{t_1}$ it will be a consequence of the following.

PROPOSITION 1. – *Let N be an integer and $(T_j)_{1\leq j\leq N}$ be a family of convolution operators on $L^2(\mathbf{R})$. Suppose that if $(b_j)_{1\leq j\leq N}$ is a sequence of bounded functions*

3.6 $$||\sum T_j b_j||_B \leq \sup_j ||b_j||_\infty.$$

Then if $(f_j)_{1\leq j\leq N}$ is a sequence of L^2-functions

3.7 $$||\sum T_j f_j||_2 \leq C N^{\frac{1}{8}+\eta} \sup_j ||f_j||_2$$

for all $\eta > 0$.

Of course to prove Theorem 1 we need an analogue of Proposition 1 in a higher number of parameters. The extension of the proof of Proposition 1 which we shall give in the next section relies only on Lemma 2, and on the characterization of BMO in terms of L^∞ and partial Hilbert transforms [7]. Therefore we shall leave it to the reader. We shall however use in our proof the symbols $||\cdot||_B$ and $||\cdot||_{\mathrm{BMO}}$, even though the norms they denote coincide in the one-parameter case, in order to indicate which one should be used in several parameters, at which place.

4. Proof of Proposition 1.

LEMMA 4. – *Let $(\xi_m)_{1\leq m\leq M}$ be a finite collection of distinct real numbers and let $(c_m)_{1\leq m\leq M}$ be a finite collection of complex numbers. The function $b = \sum c_m e^{i<\xi_m,\cdot>}$ is in B with a norm at* *least*

$$\left(\sum_{m=1}^{M} |c_m|^2\right)^{1/2}.$$

To see this, it suffices to test the oscillation of b on intervals whose length tends to ∞. We omit the details.

From this lemma it follows that, under the assumption 3.6 applied when the b_j's are characters,

4.1 $$\sum_{i\leq j\leq N} ||T_j||_{2,2}^2 \leq 1.$$

Let us prove that 4.1 allows us to make the assumption that $||T_j||_{2,2} \leq \frac{1}{\sqrt{N}}$ for all j [1,N], without loss of generality. Indeed let $c(N)$ be the best constant for which 3.7 holds with $c(N)$ instead of $cN^{\frac{1}{8}+\eta}$, and similarly for $c'(N)$ but assuming $||T_j||_{2,2} \leq \frac{1}{\sqrt{N}}$ for $j \in [1, N]$. Let $\alpha > 0$. Let $(T_j)_{1\leq j\leq N}$ be a collection of operators satisfying to 3.6. By 4.1, the number of j's for which $||T_j||_{2,2} \geq N^{\alpha-\frac{1}{2}}$ is less than $[N^{1-2\alpha}]$, where [ ] denotes the integer part. It follows, by considering the set of j's for which $||T_j||_{2,2} \leq N^{\alpha-\frac{1}{2}}$, and those for which $||T_j||_{2,2} \geq N^{\alpha-\frac{1}{2}}$, that

4.2 $$c(N) \leq c'(N)N^{\alpha} + c([N^{1-2\alpha}]).$$

Hence if $c'(N)$ grows at most like $N^{\frac{1}{8}+\eta}$ for all $\eta > 0$, so does $c(N)$.

We now suppose that for all $1 \leq j \leq N$, $||T_j||_{2,2} \leq \frac{1}{\sqrt{N}}$. Since the Hilbert transform is bounded on B and any BMO function b can be written as $a_1 + Ha_2$, where a_1 and a_2 are in L^∞ and satisfy $||a_1||_\infty + ||a_2||_\infty \leq c||b||_{\mathrm{BMO}}$, we see that 3.6 implies

4.3 $$||\sum T_j b_j||_B \leq c \sup_j ||b_j||_{\mathrm{BMO}}.$$

We can also assume that the symbol of each T_j vanishes on $\cup_{k\in\mathbb{Z}} [2^k, \frac{5}{4}, 2^k] \cup \mathbb{R}_-$. We let Δ_k be the multiplier of symbol $X_{\left[\frac{5}{4}2^k, 2^{k+1}\right]}$ and let $T_{j,k}$ be $T_j\Delta_k$.

LEMMA 5. – *For all* $k \in \mathbb{Z}$, $\sum \|T_{j,k}\|_{2,2} \le c$.

In order to prove this lemma it suffices to show that if $(S_j)_{1\le j\le N}$ is a family of convolution operators whose symbols σ_j are supported in $\{1 \le \xi \le 1+\beta\}$ for some $\beta > 0$ and if for all sequences $(\xi_j)_{1\le j\le N}$ of real numbers and all sequence $(c_j)_{1\le j\le N}$ of complex numbers

4.4
$$\Big\|\sum_j S_j c_j e^{i<\xi_j,\cdot>}\Big\|_B \le \sup_j |c_j|$$

then

4.5
$$\sum_j \|S_j\|_{2,2} = \sum_j \|\sigma_j\|_\infty \le c.$$

To prove 4.5 it suffices to show that for any sequence $(\xi_j)_{1\le j\le N}$,

4.6
$$\sum_j |\sigma_j(\xi_j)| < c.$$

We may assume that the ξ_j's take their values in $[1, 1+\beta]$ and that the $\sigma_j(\xi_j)$'s are valued in [0,1]. Then we just have to show that

4.7
$$\int_0^1 \Big|\sum_j \sigma_j(\xi_j)e^{i<\xi_j,\cdot>} - m_{[0,1]}\Big(\sum_j \sigma_j(\xi_j)e^{i<\xi_j,\cdot>}\Big)\Big|^2 dx \ge c\Big(\sum_j \sigma_j(\xi_j)\Big)^2.$$

Equivalently it suffices to prove for all j

4.8
$$\int_0^1 |e^{i<\xi_j,\cdot>} - m_{[0,1]}e^{i<\xi_j,\cdot>}|^2\, dx \ge c$$

and for $j,\ \ell,\ j \ne \ell$,

4.9
$$\mathrm{Re}\int_0^1 (e^{i<\xi_j,\cdot>} - m_{[0,1]}e^{i<\xi_j,\cdot>})\,(e^{i<\xi_k,\cdot>} - m_{[0,1]}e^{i<\xi_k,\cdot>})\, dx \ge c.$$

This is clear if β is small enough, which proves Lemma 5. We may therefore assume that $\sum_j \|T_{j,k}\|_{2,2} \le 1$ for all $k \in \mathbb{Z}$.

The last lemma we shall need is classical.

LEMMA 6. – *Let* $(c_k)_{k\in\mathbb{Z}} \in \ell^2{}_{\mathbb{C}}(\mathbb{Z})$ *and* $(\xi_k)_{k\in\mathbb{Z}}$ *be a sequence of real numbers such that* $\xi_k \in \left[\frac{5}{4}2^k,\ 2^{k+1}\right]$ *for all* $k \in \mathbb{Z}$. *Then* $\sum_k c_k e^{i<\xi_k,\cdot>}$ *is in* BMO *and* $\left\|\sum c_k e^{i<\xi_k,\cdot>}\right\|_{\mathrm{BMO}} \leq c\left(\sum |c_k|^2\right)^{1/2}$.

Let us now indicate the strategy to go from 3.6 to 3.7, assuming that $\|T_j\|_{2,2} \leq \frac{1}{\sqrt{N}}$ and $T_j = T_j \sum_{k\in\mathbb{Z}} \Delta_k$, for all j, and that $\sum_j \|T_{j,k}\|_{2,2} \leq 1$ for all $k \in \mathbb{Z}$.

Suppose there exists for each k a number ξ_k in $\left[\frac{5}{4}2^k,\ 2^{k+1}\right]$, such that for all j,

$$\|T_{j,k}\|_{2,2} \leq c|m_j(\xi_k)|. \tag{4.10}$$

For all $j \in [1,N]$, let $(c_{j,k})_k$ be such that $\sum |c_{j,k}|^2 \leq 1$. Then by Lemmas 4 and 6 and by 4.3 we obtain $\sum_k \left|\sum_j m_j(\xi_k)\ c_{j,k}\right|^2 \leq c$ and even

$$\sum_k \left(\sum_j |m_j(\xi_k)|\ |c_{j,k}|\right)^2 \leq c. \tag{4.11}$$

Now if $(f_j)_{1\leq j\leq N}$ are L^2-functions with norm 1, by 4.10 and 4.11,

$$\begin{aligned}\left\|\sum T_j f_j\right\|_2^2 &= \sum_k \left\|\sum_j T_{j,k}\ \Delta_k f_j\right\|_2^2\\ &\leq \sum_k \left(\sum_j \|T_{j,k}\|_{2,2}\|\Delta_k f_j\|_2\right)^2\\ &\leq c\ \sum_k \left(\sum_j |m_j(\xi_k)|\ \|\Delta_k f_j\|_2\right)^2 \leq c,\end{aligned}$$

since for all j, $\sum \|\Delta_k f_j\|_2^2 \leq 1$.

Unfortunately the existence of these miraculous ξ_k's is not guaranteed in general and matters are slightly more complicated. The point will be to select a small number of $\xi_{k,\ell}$'s for each k, in such a way that $\sup_{j,k}\left(\frac{\|T_{j,k}\|_{2,2}}{\sup_\ell |\, m_j(\xi_{k,\ell})\,|}\right)$ be not too large, and then apply essentially the previous argument. We are now going to describe how to select these $\xi_{k,\ell}$'s.

Let k be fixed p be a large fixed integer, and $\mu = \frac{3}{4p}$. Let r and s be two integers such that $0 \leq s \leq r \leq p-1$. We pick up a ξ, denoted $\xi_{k,1}^{r,s}$ such that

4.12 $$\sum_j |m_j(\xi)|^2 \geq \frac{1}{p^2} n^{-\frac{3}{4}}$$

where the sum runs over those j's such that

4.13 $$N^{-\frac{3}{4}+\mu r} \leq \|T_{j,k}\|_{2,2} \leq N^{-\frac{3}{4}+\mu(r+1)}$$

and

4.14 $$N^{-\frac{3}{4}+\mu s} \leq |m_j(\xi)| \leq N^{-\frac{3}{4}+\mu(s+1)}.$$

We take off all the j's satisfying 4.13 and 4.14 and select another ξ, denoted $\xi_{k,2}^{r,s}$, in the same fashion. When we cannot go on for a fixed r, s, we choose another couple r', s', and obtain a collection of $(\xi_{k,\ell}^{r',s'})_\ell$. Finally when the process stops we can conclude that for all ξ's in $\left[\frac{5}{4}2^k,\ 2^{k+1}\right]$,

4.15 $$\sum_j |m_j(\xi)|^2 \leq N^{-\frac{3}{4}},$$

where the sum is restricted to those j's which have not been taken off during the selection process. We call this set E_k. So we have a decomposition of [1,N] as $E_k \cup \bigcup_{r,s,\ell} E_{k,\ell}^{r,s}$ where $E_{k,\ell}^{r,s}$ is the set of j's which have been taken off after selecting $\xi_{k,\ell}^{r,s}$. Notice that all these sets are pairwise disjoints. We define $T_j^{r,s} = T_j\left(\sum_k \Delta_k\right)$, where the sum is extended to those k's such that j belongs to $\cup_\ell\, E^{r,s}_{k,\ell}$. Notice that for each (r,s), the collection $(T^{r,s}_j)_{1\le j\le N}$ satisfy 4.3 uniformly.

LEMMA 7. – *Let $(f_j)_{1\le j\le N}$ be a sequence of L^2-functions of norm less than 1. Then* $\left\|\sum_j T^{r,s}_j f_j\right\|_2 \le c\, p\, N^{\mu\left(\frac{r}{2}+2\right)}$.

To prove this lemma we first observe that for each ℓ, the set $E^{r,s}_{k,\ell}$ has at least $p^{-2}N^{\frac{3}{4}-2\mu(s+1)}$ elements because of 4.12 and 4.14. On the other hand, since $\sum_j \|T_{j,k}\|_{2,2} \le 1$, the set $U_\ell\, E^{r,s}_{k,\ell}$ has at most $N^{\frac{3}{4}-\mu r}$ elements by 4.13. Hence there is at most $p^2 N^{\mu(2s+2-r)}$ distinct values of $\xi^{r,s}_{k,\ell}$, for r,s and k fixed. For each j we consider a sequence $(c_{j,k})_{k\in\mathbb{Z}}$ in ℓ^2 $(\mathbb{Z})$ such that $\sum_k |c_{j,k}|^2 \le 1$, and $c_{j,k}=0$ if $T^{r,s}_j\,\Delta_k = 0$. Let b_j be the BMO function $\sum_k c_{j,k} e^{i<\xi^{r,s}_{k,\ell},\cdot>}$, where $\xi^{r,s}_{k,\ell}$ is the element of $\{\xi^{r,s}_{k,\ell},\ 1\le \ell\le p^2N^{\mu(2s+2-r)}\}$ for which $j\in E^{r,s}_{k,\ell}$. By Lemmas 4 and 6 and by 4.3 and 4.14

$$\sum_{k,\ell}\Big(\sum_j |c_{j,k}|\Big)^2 \le c\,N^{\frac{3}{2}-2\mu s}, \tag{4.16}$$

where the sum in j runs over $E^{r,s}_{k,\ell}$. If $(f_j)_{1\le j\le N}$ are L^2-functions of norms less than 1, $\left\|\sum_j T^{r,s}_j f_j\right\|_2^2 \le \sum_k\Big(\sum_j \|T^{r,s}_j\|_{2,2}\|\Delta_k f_j\|_2\Big)^2$, which is, by 4.13, dominated by $\sum_k\Big(\sum_j \|\Delta_k f_j\|_2\Big)^2 N^{-\frac{3}{2}+2\mu(r+1)}$, the sum in j running over $\bigcup_\ell E^{r,s}_{k,\ell}$. Hence, for each k,

$$\Big(\sum_{j\in\bigcup_\ell E^{r,s}_{k,\ell}} \|\Delta_k f_j\|_2\Big)^2 \le p^2N^{\mu(2s+2-r)}\sum_\ell\Big(\sum_{j\in E^{r,s}_{k,\ell}}\|\Delta_k f_j\|_2\Big)^2.$$

Summing over k and using 4.16 we obtain $\left\|\sum_j T^{r,s}_j f_j\right\|_2^2$

$\leq c\, p^2 N^{\mu(r+4)}$ and Lemma 7 is proved.

By the assumption $||T_j||_{2,2} \leq \frac{1}{\sqrt{N}}$ for all j, we see that $\mu(r+1) \leq \frac{1}{4}$. We then deduce from Lemma 7 that $\left|\left|\sum_{r,s}\sum_j T_j^{r,s} fj\right|\right|_2 \leq c\ p^3 N^{\frac{3}{2}\mu+\frac{1}{8}} \sup_j ||f_j||_2$. Since μ can be made arbitrarily small, we just have to estimate the remainder term $\left|\left|\sum_j T_j f_j - \sum_{r,s} \left(\sum_j T_j^{r,s} f_j\right)\right|\right|_2$. Using 4.15, Plancherel and Cauchy-Schwarz we easily obtain a domination by $c\, N^{\frac{1}{8}}$. This concludes the proof of Proposition 1.

5. Tensor-products of multilinear singular integral operators.

We wish to prove the following.

THEOREM 2. – *For all $\delta > 0$, there exists $c_\delta > 0$, such that for all $a \in L^\infty_{\mathbb{C}}(\mathbb{R}^2)$ and $k \in \mathbb{N}$*

$$||L_{k,a}||_{2,2} \leq c_\delta(1+k)^{2+\delta}||a||_\infty^k.$$

This theorem will essentially follow from a general result on multilinear singular integral operators.

Let T_1 and T_2 be two bounded operators on $L^2(\mathbb{R})$. Then it is a simple consequence of Fubini's Theorem that $T_1 \otimes T_2$, defined on $L^2(\mathbb{R} \times \mathbb{R})$, extends boundedly to all of $L^2(\mathbb{R} \times \mathbb{R})$. If however one considers two bilinear operators bounded from $L^\infty(\mathbb{R}) \times L^2(\mathbb{R})$ to $L^2(\mathbb{R})$, then their tensor-product, defined on $[L^\infty(\mathbb{R}) \otimes L^\infty(\mathbb{R})] \times [L^2(\mathbb{R}) \otimes L^2(\mathbb{R})]$, is not in general bounded from $L^\infty(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$ [9]. It is a surprising fact that it is bounded when the bilinear operators are of Coifman-Meyer type.

We shall see that this is also true for multilinear singular operators.

As in [10] we shall deal with multilinear singular integral forms instead of multilinear operators. We refer to [10] for the precise definition of a multilinear singular integral form, $\delta - n$SIF, and for a boundedness criterion concerning them. We shall denote by U a $\delta - n$SIF on $\mathbf{R}$ and refer to [10] for the notations $U_i 1$, $i \in [1,n]$, $|U|_\delta$, $|U|_W$, $||U||_{i}, j$ and $|U_{ij}|_\delta$. We recall that U is bounded if for $1 \leq i \leq j \leq n$,

$$5.1 \qquad |U(h_1, h_2, \ldots, h_n)| \leq c_{ij}\Big(\prod_{k \neq i,j} ||h_k||_\infty\Big) ||h_i||_2 \; ||h_j||_2$$

for all h_ℓ, $1 \leq \ell \leq n$, in $C_0^\infty(\mathbf{R})$.

If U and U' are two bounded $\delta - n$SIF's, then their tensor-product $U \otimes U'$ is well defined on $[C_0^\infty(\mathbf{R}) \otimes C_0^\infty(\mathbf{R})]^n$. We then say that U is bounded if for $1 \leq i \leq j \leq n$,

$$5.2 \qquad |U \otimes U'(h_1, h_2, \ldots, h_n)| \leq c_{ij}\Big(\prod_{k \neq i,j} ||h_k||_\infty\Big) ||h_i||_2 \; ||h_j||_2$$

for all h_ℓ, $1 \leq \ell \leq n$, $C_0^\infty(\mathbf{R})$, and we denote by $|||U \otimes U'|||$, $\sup_{i,j} \widetilde{c}_{ij}$, where $\widetilde{c}_{ij}$ is the best constant in 5.2.

THEOREM 3. – *If U and U' are two bounded $\delta - n$SIF's, then $U \otimes U'$ is bounded and*

$$5.3 \qquad \begin{aligned} |||U \otimes U'||| \;\leq\; & c\Big\{\sum_i ||U_i 1||_{\mathrm{BMO}} + n^2(|U|_\delta + |U|_W)\Big\} \\ & \Big\{\Big(\sum_i ||U_i' 1||_{\mathrm{BMO}} + n^2(|U|_\delta + |U|_W\Big)\Big\}. \end{aligned}$$

Notice that the constant c appearing in 5.3 is independent of n.

An application of Theorem 3 in the case where $U = U'$ is the form determined by the $(n-2) - nd$ Calderón-commutator (see [10] section 4) yields $||L_{k,a}||_{2,2} \leq c_\delta(1+n)^{4+\delta}$ for all $n \in \mathbf{N}$ and $\delta > 0$. As in [10] the antisymmetry of the kernel $\dfrac{1}{x-y}$ permits to improve this estimate and to obtain Theorem 2. Since this will be clear from the proof of Theorem 3 which we shall now outline, we omit the details.

The proof of Theorem 3 is along the same lines as the proof of Theorem 2 in [10], which we shall assume familiar to the reader. Let us recall however that the main ingredients of this proof are Carleson measures, quadratic estimates that have been developed in [5], [7], and [11] in the context of product-spaces. Another important element is the equivalence between $H^1 \to L^1$ and $L^\infty \to$ BMO boundedness for a singular integral operator. The proof of this equivalence given in [12, p. 49] relies on the atomic decomposition of H^1 in $H^{1,\infty}$ -atoms. Such a decomposition does not seem to exist on product-spaces where the atoms are only in L^2. However this equivalence is still true on product spaces.

PROPOSITION 2. – *Let T be a δ – SIO on $\mathbb{R} \times \mathbb{R}$. Then T is bounded on L^2 if and only if T maps L^∞ to* BMO.

We refer to [3] for the definition of a δ–SIO on $\mathbb{R} \times \mathbb{R}$. Notice that by applying Proposition 2 simultaneously to T and T^* we see that a δ–SIO maps L^∞ to BMO if and only if it maps H^1 to L^1.

The fact that the L^2-boundedness of T implies its L^∞–BMO boundedness is already known [3]. The converse is then an easy consequence. Suppose that T is a δ–SIO bounded from L^∞ to BMO, and let us also assume that $||T||_{2,2} < +\infty$. Then, by the direct part of Proposition 2 applied to T^* we obtain $||T||_{H^1,L^1} \leq c||T||_{2,2} + c(T)$, where $c(T)$ depends only on the constants for the standard estimates of the kernel of T. By interpolation [13] $||T||_{2,2} \leq c(||T||_{L^\infty,\mathrm{BMO}}||T||_{H^1,L^1})^{1/2}$. It follows that, $||T||_{2,2} \leq c(||T||_{L^\infty,\mathrm{BMO}} + c(T))$, which easily implies Proposition 2.

The connection between δ–SIO's on $\mathbb{R}\times\mathbb{R}$ and tensor products of $\delta - n$SIF's on $\mathbb{R}$ is provided by the following lemma.

LEMMA 8. – *Let U and U' be two bounded $\delta - n$SIF's on $\mathbb{R}$. For all $1 \leq i \leq j \leq n$ and all $h_k \in C_0^\infty(\mathbb{R}^d) \otimes . C_0^\infty(\mathbb{R}^d)$, $k \neq i, j$, the operator $T = (U \otimes U')_{i,j}(h_1, \ldots, h_k, \ldots, h_n)$ defined by $< h_1, Th_j > = (U \otimes U')\ (h_1, \ldots, h_n)$, is a δ–SIO on $\mathbb{R} \times \mathbb{R}$, of norm less than*

$$c[(||U||_{i,j} + |U_{ij}|_\delta)|U'_{ij}|_\delta + (||U'||_{i,j} + |U'_{ij}|_\delta)|U_{ij}|_\delta] \prod_{k \neq i,j} ||h_k||_\infty.$$

The proof is routine and we omit it.

We turn to the proof of Theorem 3. Let ϕ be a non-negative function in $C_0^\infty(\mathbb{R})$ such that $\int \phi\, dx = 1$. For all $t > 0$, we denote by P_t the convolution operator on $\mathbb{R}^2$ of symbol $\sigma(\xi, \xi') = \widehat{\phi}(t\xi)$. Similarly $P'_{t'}$ is the operator of symbol $\widehat{\phi}(t'\xi')$. Finally $Q_t = -t\dfrac{\partial}{\partial t}P_t$ and $Q'_{t'} = -t'\dfrac{\partial}{\partial t'}P'_{t'}$.

As in [10] we choose $h_1, \ldots, h_n$ in $C_0^\infty(\mathbb{R}^d) \otimes C_0^\infty(\mathbb{R}^d)$ and express $U \otimes U'(h_1, \ldots, h_n)$ as the sum of n^2 double integrals of two different types :

$$\text{I} \qquad \iint U \otimes U'(Q_t Q'_{t'}, h_1,\ P_t P'_{t'}, h_2, \ldots, P_t P'_{t'}, h_n)\frac{dt}{t}\,\frac{dt'}{t'}$$

$$\text{II} \quad \iint U \otimes U'(Q_t P'_{t'}, h_1,\ P_t Q'_{t'}, h_2,\ P_t P'_{t'}, h_3, \ldots, P_t P'_{t'}, h_n)\frac{dt}{t}\,\frac{dt'}{t'}.$$

It is clear that the estimates of the n integrals of type I can be reduced to Carleson measure estimates. To see that this is also true for the $n^2 - n$ integrals of type II we need the following.

LEMMA 9. – *Let T be a $\delta-$SIO on $\mathbb{R} \times \mathbb{R}$. If $T1$ and T^*1 vanish (or are in BMO), and the partial adjoints of T are bounded on L^2, then T is a $\delta-$CZO.*

This lemma follows immediately from the T1-Theorem and Theorem 3 of [3].

Let V' be the $\delta - n$SIF obtained from U' by letting $V'(f_1, f_2, \ldots, f_n) = U'\ (f_2, f_1, f_3,\ \ldots, f_n)$. By lemmas 8 and 9 we see that

$$\left|\iint U \otimes V'(Q_t Q'_{t'} h_1,\ P_t P'_{t'} h_2, \ldots)\frac{dt}{t}\,\frac{dt'}{t'}\right|$$
$$\leq c_1 ||h_1||_2 ||h_2||_2 \prod_{k \geq 3} ||h_k||_\infty$$

implies

$$\left|\iint U \otimes U'(Q_t P'_{t'} h_1,\ P_t Q'_{t'} h_2, \ldots)\frac{dt}{t}\,\frac{dt'}{t'}\right|$$

$$\leq c_2||h_1||_2||h_2||_2 \prod_{k\geq 3} ||h_k||_\infty .$$

Since V' has the same properties as U', we are reduced to estimating integrals of type I.

We are going to prove the following estimate :

$$\left| \iint U \otimes U'(Q_t Q'_t h_1, P_t P'_{t'} h_2, \ldots) \frac{dt}{t} \frac{dt'}{t'} \right|$$

5.4
$$\leq c||U_1 1||_{\mathrm{BMO}} + n(|U|_\delta + |U|_W))$$
$$(||U'_1 1||_{\mathrm{BMO}} + n(|U'|_\delta + |U'|_W))\left(\prod_{k\geq 3} ||h_k||_\infty\right)||h_1||_2||h_2||_2 .$$

It is easy to see from the previous remarks that 5.4 implies Theorem 3. In turn 5.4 is itself an immediate consequence, after reduction to a Carleson-measure estimate, of an extension of Theorem 1 of [10] to the setting of product spaces, which we now describe. We first need to recall the notion of an ε-family introduced in [10].

DEFINITION 1. – *A family $\mathcal{S} = (s_t)_{t>0}$ of operators given by kernels satisfying*

5.5
$$|s_t(x,y)| \leq c \, \frac{t^\varepsilon}{t^{1+\varepsilon} + |x-y|^{1+\varepsilon}}$$

5.6
$$|s_t(x,y) - s_t(x,z)| \leq c \, \frac{t^\varepsilon}{t^{1+\varepsilon} + |x-y|^{1+\varepsilon}} \left(\frac{y-z}{t+ \mid x-y \mid}\right)^\varepsilon ,$$

for all x, y, and z such that $|y-z| \leq \frac{1}{2}(t+|x-y|)$, is an ε-family. It is bounded if for all $f \in L^2$,

5.7
$$\left[\int_0^{+\infty} ||s_t f||_2^2 \, \frac{dt}{t}\right]^{1/2} \leq c||f||_2 \, .$$

Following the procedure of [3], to extend this notion to product spaces, we first put a norm on the space of ε -families by letting $||\mathcal{S}||_\varepsilon = ||\mathcal{S}||_2 + |\mathcal{S}|_\varepsilon$, where $||\mathcal{S}||_2$ is the best constant in 5.7 and

$|S|_\varepsilon$ in 5.5 and 5.6. An ε-family on $\mathbb{R} \times \mathbb{R}$ will then be a two-parameter family $(T_{t,t'})_{t,t'>0}$ of operators given by integrable kernels $T_{t,t'}(x,x,y,y')$. For t,x,y fixed, we shall denote by $(T_{t,t'}[x,y,])_{t'>0}$, the one-parameter family of operators acting on the second variable, and of kernels $(T_{t,t'}[x,y])\,(x',y') = T_{t,t'}(x,x',y,y')$, and similarly for $(T_{t,t'}[x',y']')_{t>0}$. Then $(T_{t,t'})_{t,t'>0}$ is an ε-family if

5.8 $$||(T_{t,t'}[x,y]_{t'>0}||_\varepsilon \le c\frac{t^\varepsilon}{|x-y|^{1+\varepsilon}+t^{1+\varepsilon}}$$

5.9 $$||(T_{t,t'}[x,y] - T_{t,t'}[x,z]_{t'>0}||_\varepsilon \le c\left(\frac{|y-z|}{t+|x-y|}\right)^\varepsilon \frac{t^\varepsilon}{t^{1+\varepsilon}+|x-y|^{1+\varepsilon}}$$

when $|y-z| \le \frac{1}{2}(t+|x-y|)$ and similarly for $(T_{t,t'}[x',y'])_{t'>0}$. We denote by $|T_{t,t'}|_\varepsilon$ the best constant in 5.8 and 5.9. The family $(T_{t,t'})_{t,t'>0}$ is bounded if for all f L^2

5.10 $$\left[\iint ||T_{t,t'}f||_2^2 \;\frac{dt}{t}\;\frac{dt'}{t'}\right]^{1/2} \le c||f||_2.$$

We also introduce a "Carleson norm" on functions w from $\mathbb{R}^2_+ \times \mathbb{R}^2_+$ into $\mathbb{C}$, by letting

$$|w|_c = \sup_{\Omega\subseteq\mathbb{R}^2}\left[\frac{1}{|\,\Omega\,|}\right]\iint_{S(\Omega)} |w(x,x',t,t')|^2 dx dx' \;\left[\frac{dt}{t}\;\frac{dt'}{t'}\right]^{1/2},$$

where Ω is an arbitrary bounded open subset of $\mathbb{R}^2$, and $S(\Omega)$ consists of these (x,x',t,t') such that $]x-t,\; x+t[\times]\; x'-t',\; x'+t'[\subseteq \Omega$.

THEOREM 4. – *Let $(T_{t,t'})_{t,t'>0}$ be an ε-family. It is bounded if and only if $|(T_{.,.}1)(.\,,.)|_c < +\infty$. In this case for all $a \in L^\infty(\mathbb{R}^2)$*

5.11 $$|(T_{.,.}a)\,(.\,,.)|_c \le ||a||_\infty\; |(T_{.,.}1)\,(.\,,.)| \;+\; c_\varepsilon||a||_\varepsilon\; |T_{t,t'}|_\varepsilon.$$

By the same argument as for Theorem 1 for [10], we need to consider only the case where $T_{t,t'}1 = 0$ for all $t,t' > 0$. We then decompose $T_{t,t'}$as $X_{t,t'} + Y_{t,t'}$ where $X_{t,t'}f(x,x') =$

$\int\int T_{t,t'}(x,x',y,y')\ (P'_{t'}f)\ (y,x')\ dydy'$. Notice that $(X_{t,t'})_{t,t'>0}$ is itself an ε-family, as well as $(Y_{t,t'})_{t,t'>0}$. Furthermore if f does not depend on the first variable $X_{t,t'}f = 0$, while if it depends only on the first variable $Y_{t,t'}f = 0$. Therefore we are reduced to the case where not only $T_{t,t'}1 = 0$ but also $T_{t,t'}f = 0$ for all functions f of the first variable. To show that $(T_{t,t'})_{t,t'>0}$ is bounded in this case it suffices to show that

$$Z = \iint T^*_{t,t'}\ T_{t,t'}\ \frac{dt}{t}\ \frac{dt'}{t'}$$

is bounded on L^2. But Z is an SIO on $\mathbf{R} \times \mathbf{R}$, to which it is easy to see that the $T1$-Theorem of [3] applies. To deduce 5.11 from the boundedness of $(T_{t,t'})_{t,t'>0}$ one proceeds exactly as in the proof of Theorem 3 on [3]. This proves Theorem 4. Routine arguments, which we shall omit, now yield 5.4 and then Theorem 3.

BIBLIOGRAPHIE

[1] R. FEFFERMAN, Calderón-Zygmund theory for product domains- H^p spaces, P.N.S.A., vol. 83 (1986), 840-843.

[2] J. PIPHER, Journé's covering lemma and its extension to higher dimensions, Duke Journal of Math, 53 n° 3 (1986), 683-690.

[3] J.-L. JOURNE, Calderón-Zygmund operators on product spaces, Mat. Iberoamerica, vol. 1, n° 3 (1985), 55-91.

[4] R. FEFFERMAN, Harmonic analysis on product spaces, Ann. of Math., (2), vol. 126 (1987), 109-130.

[5] R. FEFFERMAN, Functions of bounded mean oscillation on the polydisc, Ann. of Math., (2), 10 (1979), 395-406.

[6] L. CARLESON, A counterexample for measures bounded for H^p for the bi-disc, Mittag Leffer Report n° 7, 1974.

[7] S.-Y. A. CHANG and R. FEFFERMAN, A continuous version of the duality of H^1 and BMO on the bi-disc, Ann. of Math., (2), 112 (1980), 179-201.

[8] J.-L. RUBIO DE FRANCIA, A Littlewood-Paley inequality for arbitrary intervals, Revista Mat. Iberoamericana, vol. 1, n° 2 (1985).

[9] P. KRIKELES, Ph.D. dissertation, Yale University, 1982.

[10] M. CHRIST, and J.-L. JOURNÉ, Polynomial growth estimates for multilinear singular integral operators, Acta Math., 159 (1987), 51-80.

[11] S.-Y. A. CHANG, Carleson measure on the bi-disc, Ann. of Math., (2), 109 (1979), 613-620.

[12] J.-L. JOURNÉ, Calderón-Zygmund operators, pseudo-differential operators and the Cauchy integral of Calderón, L.N. 994, Springer-Verlag.

[13] S.-Y. A. CHANG and R. FEFFERMAN, The Calderón-Zygmund decomposition on product domains, Am. J. of Math., vol. 104, 3, 455-468.

Manuscrit reçu le 7 novembre 1986.

Jean-Lin JOURNÉ,
C.N.R.S.
&
PRINCETON UNIVERSITY
FINE HALL
PRINCETON N.J. 08540 (U.S.A.).