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PLURISUBHARMONIC FUNCTIONS WITH LOGARITHMIC SINGULARITIES

by E. BEDFORD and B. A. TAYLOR

1. Introduction.

The set of plurisubharmonic (psh) functions with minimal (i.e. logarithmic) growth is given by

$$\mathscr{L} = \{u \text{ psh on } \mathbb{C}^n : u(z) \leqslant \log^+ |z| + C\}$$

where the constant C depends on u. This class has been studied by several authors, including Leja, Lelong, Sadullaev, Siciak, and Zaharjuta, in connection with problems concerning polynomials in n variables. The functions in $\mathscr{L}$ are bounded by $\log^+ |z|$ at infinity, but they do not necessarily « look like » $\log^+ |z|$ at infinity, and so it was also necessary sometimes to use the more restricted class

$$(1.1) \quad \mathscr{L}_+ = \{u \text{ psh on } \mathbb{C}^n : \log^+ |z| - C \leqslant u(z) \leqslant \log^+ |z| + C\}$$

where again the constant C depends on u.

Of particular interest for the classes $\mathscr{L}$ and $\mathscr{L}_+$ is the *Robin function*, defined by

$$\rho_u(z) = \limsup_{\substack{\lambda \to \infty \\ \lambda \in \mathbb{C}}} (u(\lambda z) - \log^+ |\lambda z|)$$

and its upper semicontinuous (usc) regularization

$$\rho_u^*(z) = \limsup_{\zeta \to z} \rho_u(z).$$

(Neither lim sup can be replaced by lim.) It is clear that ρ_u and ρ_u^* are defined on $\mathbb{P}^{n-1}$ and give a natural generalization of the familiar Robin constant from logarithmic potential theory on $\mathbb{C}^1$. The function $\log|z| + \rho_u^*(z)$ is psh on $\mathbb{C}^n$ and is logarithmically homogeneous on $\mathbb{C}^n$ if it is not identically $-\infty$. For further properties of the Robin function see Levenberg [L].

We may also define

$$P_1(\mathbb{P}^{n-1}) = \{\text{usc functions } h \text{ on } \mathbb{P}^{n-1}: dd^c h \geqslant -\Omega\},$$

where Ω is the Kähler form on $\mathbb{P}^{n-1}$ corresponding to the Fubini-Study metric. A direct calculation shows that $H = \log|z| + h$ is psh if and only if $h \in P_1(\mathbb{P}^{n-1})$. Let us define

$$\begin{aligned}\mathscr{L}_\rho &= \{u \in \mathscr{L}: \rho_u^* \in P_1(\mathbb{P}^{n-1})\}\\&= \{u \in \mathscr{L}: \rho_u^* \not\equiv -\infty\}.\end{aligned}$$

Thus $\mathscr{L}_\rho$ is the subclass of $\mathscr{L}$ for which the Robin function makes sense. A psh function $u \in \mathscr{L}_\rho$ behaves like $\log^+|z|$ at infinity in the following sense:

$$\lim_{r \to \infty} \int_{|\alpha|=1} |u(r\alpha) - \log r| \, d\sigma(\alpha) < +\infty,$$

which is equivalent to

$$\lim_{r \to \infty} \int_{|\alpha|=1} |u(r\alpha) - \log r - \rho_u^*(\alpha)| \, d\sigma(\alpha) = 0.$$

The purpose of the present paper is to show that many of the basic results on psh functions can be carried out within the class $\mathscr{L}_\rho$. Although it seems clear that the basic theory of psh functions cannot be developed using a « potential theory » with respect to some sort of kernel, it seems that the elements of $\mathscr{L}_\rho$ may be able to play the role of logarithmic potentials.

The main tool we develop in this paper is the following integral formula: for $u, v, w \in \mathscr{L}$,

$$(1.2) \qquad \int_{\mathbb{C}^n} (u \, dd^c v - v \, dd^c u) \wedge (dd^c w)^{n-1} = 2\pi \int_{\mathbb{P}^{n-1}} (\rho_u^* - \rho_v^*)(dd^c \rho_w^* + \Omega)^{n-1}$$

(where Ω denotes the Kähler form on $\mathbb{P}^{n-1}$). One step of the derivation of this formula involves an integration by parts that is similar to one of the classical Green formulas. It produces terms of the general form

$$\int_{\mathbb{C}^n} du \wedge d^c u \wedge (dd^c u)^{n-1} = +\infty,$$

so the cancellation of terms in (1.2) is essential. The second step requires a limiting operation analogous to taking a residue along the hyperplane at ∞. It involves finding both the slicing and « residual mass » of a certain current. Each of these operations requires some care in case the functions involved are not continuous. However, the technical effort required to do this is justified since (1.2) is derived for the purpose of making applications to discontinuous functions. For instance, an interesting application, part (i) of the Corollary which follows, is trivial in case the relevant psh function is continuous.

One application we make of this integral formula is to the study of capacities in $\mathbb{C}^n$. For $E \subset \mathbb{C}^n$, the « Green function with logarithmic pole at infinity » is given by the usc regularization L_E^* of

$$L_E(z) = \sup \{v(z) : v \in \mathscr{L}, v \leqslant 0 \text{ on } E\}.$$

Let $\rho_E^*(z) = \rho_{L_E}^*(z)$ be the corresponding Robin function. The associated capacity is

$$C(E) = \exp\left(-\sup_{\mathbb{P}^{n-1}} \rho_E^*\right).$$

Here we establish some convergence properties of ρ_E^* (see Section 6).

THEOREM. — *The correspondence $E \to -\rho_E$ satisfies the following properties of a Choquet capacity :*

   (i) *if $E_1 \subset E_2$, then $-\rho_{E_1} \leqslant -\rho_{E_2}$*

   (ii) *if $K_1 \supset K_2 \supset \cdots$ is a sequence of compact sets, and if $\bigcap_{j=1}^{\infty} K_j = K$, then $\lim_{j \to \infty} \rho_{K_j}(z) = \rho_K(z)$ for almost every $z \in \mathbb{P}^{n-1}$.*

   (iii) *if $E_1 \subset E_2 \subset \cdots$, and if $E = \bigcup_{i=1}^{\infty} E_j$ is bounded then for every $z \in \mathbb{P}^{n-1}$,*

$$\lim_{j \to \infty} \rho_{E_j}^*(z) = \rho_E^*(z).$$

The Robin function ρ_K reflects closely the nature of K. For instance (cf. Theorem 6.9) *if $K_1 \subset K_2$ are polynomially convex compact sets and if $\rho_{K_1} = \rho_{K_2}$ a.e., then $K_1 = K_2 \backslash E$ for some pluripolar set E.*

From Theorem 6.6 we deduce the corresponding properties of the capacity C.

COROLLARY. $-$ C has the properties :

(i) C is right continuous on compact sets.

(ii) C is an outer capacity.

Part (i) of the Corollary was proved earlier by Kolodziej [K 1]. One consequence of it is that the capacity C coincides on all compact sets with a certain set function, τ, defined in terms of Chebyschev constants by Zaharjuta [Z]. Sadullaev [Sa] showed that the capacity C coincides with τ for all regular compact sets K. Since τ is known to converge under decreasing limits, it follows from (i) that C and τ coincide for all compact sets K. Interesting connections between convergence of the Robin functions and properties of the capacity C have been studied by Siciak [Si3].

An underlying question is to give the relation between the convergence of a sequence $u_j \in \mathscr{L}_+$ and the convergence of the corresponding Robin functions. It is an elementary consequence of convexity that : if u_j decreases to u in $\mathscr{L}_+$, then ρ_{u_j} decreases to ρ_u. For a sequence u_j increasing a.e. to u in $\mathscr{L}_+$, we have the following result : ρ_{u_j} increases to ρ_u a.e. if and only if (1.3) holds (Theorem 6.6).

$$(1.3) \quad \lim_{j \to \infty} \int_{\mathbb{C}^n} \log(1+|z|)(dd^c u_j)^n = \int_{\mathbb{C}^n} \log(1+|z|) \ (dd^c u)^n.$$

We also show here that $\mathscr{L}_\rho$ is rich enough to deal with polar sets. It was shown by Josefson [J] that the concepts of « locally » and « globally » (pluri-)polar sets coincide, and Siciak [Si 1] showed that if E is polar, then it is defined by a function of logarithmic growth. Here we obtain (in section 7) a more precise result.

THEOREM. $-$ If $E \subset \mathbb{C}^n$ is polar, then there exists $u \in \mathscr{L}_\rho$ with $\{u = -\infty\} \supset E$.

Similar lines of argument also show that complete polar sets are also complete $\mathscr{L}$ polar (Theorem 7.2). We note that in the theorem it is not possible to take u such that $\rho_u(z) > -\infty$ for all z. For instance, if E contains a non polar portion of a complex line α, then $\rho_u(\alpha) = -\infty$ for the corresponding point $\alpha \in \mathbb{P}^{n-1}$.

The arguments leading to the establishment of the integral formula (1.2) also work in more general situations and can be applied locally to functions like $\log|f|$ for meromorphic f. For M a compact, complex manifold, let D^+, $D^- \subset M$ be disjoint smooth divisors on M and set

$$\mathscr{L}(M,D^+,D^-) = \{u \text{ psh on } (M-D^+\cup D^-):$$
$$u(z) \leqslant -\log \text{dist}\,(z,D^+) + \log \text{dist}\,(z,D^-) + C\}.$$

Since M is compact, $\mathscr{L}(M,D^+,D^-)$ is independent of the choice of smooth distance function. $\mathscr{L}_+(M,D^+,D^-)$ is again defined as in (1.1). If h is a local holomorphic defining function for D^+, and if $u \in \mathscr{L}(M,D^+,D^-)$, then $u + \log|h|$ is locally bounded above and thus has a local continuation $\tilde{u}$ over D^+. Although $\tilde{u}$ depends on the local choice of h, $\tilde{u} - \tilde{v}$ and $dd^c\tilde{u}$ are independent of h. Thus we define $\mathscr{L}_\rho(M,D^+,D^-)$ as the elements u of $\mathscr{L}(M,D^+,D^-)$ such that $\tilde{u}$ is not identically $-\infty$ on any open subset of $D^+ \cup D^-$.

With this, we may extend the integral formula (*) (1.2) to

$$(1.4) \quad \int_{M-(D^+\cup D^-)} (udd^cv - vdd^cu) \wedge (dd^cw)^{n-1}$$
$$= 2\pi \int_{D^+} - \int_{D^-} (\tilde{u}-\tilde{v}) \wedge (dd^c\tilde{w})^{n-1}$$

for $u, v, w \in \mathscr{L}_+(M,D^+,D^-)$.

This formula is applied to the study of the propagation of polar sets. If $E \subset \mathbb{C}^n$ is polar, then we may consider

$$E^* = \{z : \psi(z) = -\infty \text{ for all } \psi \in \mathscr{L}_\rho \text{ with } \psi = -\infty \text{ on } E\}.$$

In general E^* may be dense in $\mathbb{C}^n$ even if E is compact. However we obtain the following result.

THEOREM. — *If D is a smooth algebraic divisor in $\mathbb{P}^n$, and if E is a polar subset of $\mathbb{C}^n - D$, then D is not contained in E^*.*

Our purpose in introducing the general class $\mathscr{L}(M,D^+,D^-)$ is to cover the following sort of situation. We can study psh functions with logarithmic decrease along a smooth complex manifold by considering the blow up along the manifold. Define $\mathscr{L}$ and $\mathscr{L}_+$ in an analogous

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(*) *Note added in proof:* The authors have recently extended (1.4) to the case where $D^+ \cup D^-$ may be singular.

fashion and then we have the Robin function defined along the blow up (which would be essentially the projectivized normal bundle). Two examples of this are : (i) logarithmic growth along a hypersurface (in which case there is nothing to blow up) ; (ii) an isolated singularity, in which case we blow up a point. For the first case, we set $\mathbb{C}^n = \mathbb{P}^n \backslash \mathbb{P}^{n-1}$, where $\mathbb{P}^{n-1}$ is the hyperplane at infinity. Thus, if $M = \mathbb{P}^n$, $D^+ = \mathbb{P}^{n-1}$, $D^- =$ empty set, we have $\mathscr{L} = \mathscr{L}(M, D^+, D^-)$ which is the case we have just discussed.

The other case we wish to cover in this context is the case of functions with logarithmic decrease at isolated singularities. Let $\Omega \subset \mathbb{C}^n$ be a bounded domain containing a point z, and let Ω_z be Ω with the point z blown up. We set $D^+ =$ empty set, $D^- = \mathbb{P}^{n-1}$, the fiber over z, and we work with $\mathscr{L} = \mathscr{L}(\Omega_z, D^+, D^-)$. As before, ρ_u^* is in $P_1(\mathbb{P}^{n-1})$. Much of the previous analysis relating u and ρ_u^* continues to hold in this case. In particular, we have the integral formula (1.2), which holds for all u, v, w in $\mathscr{L}_+(M, D^+, D^-)$ such that $u = v$ in a neighborhood of Ω_z, and the integral is taken over Ω instead of $\mathbb{C}^n$. Similarly, the convergence criterion (1.3) continues to hold for the new class $\mathscr{L}_+(M, D^+, D^-)$ which gives us control on the Robin function.

Using this choice of the class $\mathscr{L}$, we may also study the Green function of a bounded domain with logarithmic singularity at the point z, which has been studied by Lempert [Lp 1, 2], Klimek [Km], and Demailly [D 2]. It is known that if Ω is hyperconvex, then there exists a unique psh function (the psh Green function) u_z on Ω that is continuous up to the boundary, vanishes identically there, and satisfies :

$$(dd^c u_z)^n \equiv 0 \quad \text{on} \quad \Omega_z$$
$$u_z(\zeta) = \log |z - \zeta| + O(1), \qquad z - \zeta \to 0.$$

The last condition is just the condition that u_z belongs to $\mathscr{L}_+$; the corresponding Robin function $\rho_{\Omega, p}$ is defined on $\mathbb{P}^{n-1}$ by

$$\rho_{\Omega, p}(\alpha) = \limsup_{|\zeta| \to 0} u_z(p + \zeta \alpha) - \log |\zeta|$$

where $\alpha \in \mathbb{C}^n$, $|\alpha| = 1$ is identified with a point of $\mathbb{P}^{n-1}$. We conclude in particular that the Robin function of the psh Green function depends continuously on the point z and the domain Ω_z in the sense that if z and Ω are varied so that u_z varies continuously on compact subsets away from z, then the Robin function varies continuously too.

2. Residual mass.

In proving the integral formula (1.2), the important limiting operation involves the behavior of the current $T = (dd^c u)^{n-1}$ near a (pluri-) polar set $\{v = -\infty\}$. We prove the existence of such a limit in this section (Proposition 2.11)

$$\lim_{a \to -\infty} [dd^c \max (v, a)] \wedge T.$$

The restriction of this current to the polar set $\{v = -\infty\}$ is called the residual mass of T. It was already used by Demailly [D 1], who considered the case when v is continuous and the polar set is compact.

The key estimate is provided by the following result. Here Ω is an open set in $\mathbb{C}^n$, $P(\Omega)$ is the space of all psh functions on Ω, and $L_{\text{loc}}^\infty (\Omega)$ is the space of all locally bounded measurable functions on Ω.

THEOREM 2.1. — *Suppose*

$$v \in P(\Omega) \qquad and \qquad u_1, \ldots, u_k \in P(\Omega) \cap L_{\text{loc}}^\infty (\Omega).$$

Then

(i) $v \, dd^c u_1 \wedge \cdots \wedge dd^c u_k$ *is a* (k, k) *current on* Ω *with locally finite mass;*

(ii) $v \, d^c u_1 \wedge dd^c u_2 \cdots \wedge dd^c u_k$ *is a* $2k - 1$ *current on* Ω *with locally finite mass;*

(iii) $dd^c v \wedge dd^c u_1 \wedge \cdots \wedge dd^c u_k = dd^c (v \, dd^c u_1 \wedge \cdots \wedge dd^c u_k)$ *is a closed, positive* $(k+1, k+1)$ *current on* Ω.

Further, if S *denotes any of the currents in* (i), (ii), *or* (iii), *if* K *is compact,* ω *is open, with* $K \subset \omega \subset\subset \Omega$, *then there exists a constant* $C = C(K, \omega)$ *independent of* $v, u_1, \ldots, u_k$ *such that*

$$|S|(K) \leqslant C \|v\|_{L^1(\omega)} \prod_{i=1}^k \|u_i\|_{L^\infty(\omega)}.$$

Remark 2.2. − Note that, in contrast to the case when $v, u_1, \ldots, u_n$ are all uniformly bounded, the currents in Theorem 2.1 need not be continuous under increasing limits. For example, if $n = 1, v(z) = \log|z|$, and $u_\varepsilon(z) = [-\log \varepsilon]^{-1} \log^+ \dfrac{|z|}{\varepsilon}$, then $-1 \leqslant u_\varepsilon \leqslant 0$, $u_\varepsilon \nearrow 0$ as $\varepsilon \to 0$, and

$$-2\pi = \lim_{\varepsilon \to 0} \int v(z)\, dd^c u_\varepsilon \neq 0 = \int v(z)\, dd^c \left(\lim_{\varepsilon \to 0} u_\varepsilon(z) \right)^*.$$

Demailly has shown the currents of Theorem 2.1 are continuous under decreasing limits.

Theorem 2.1 is known. See e.g. Cegrell [Ceg], where an estimate for the mass in (ii) is given. A slightly different estimate is given in [AT]. A careful and complete treatment is given in Demailly [D 1], Chapter 2.

From Theorem 2.1, we have the following corollary.

COROLLARY 2.3. − *Suppose* $u_1, \ldots, u_n$ *are psh functions with* $|u_j| \leqslant 1$ *on a neighborhood* ω *of the closed polydisk,* $|z_j| \leqslant 1$. *If* $\Delta(\varepsilon) = \{|z_1| \leqslant \varepsilon, |z_2| \leqslant 1, \ldots, |z_n| \leqslant 1\}$, *then there is a constant* $C = C(\omega)$ *such that*

$$\int_{\Delta(\varepsilon)} dd^c u_1 \wedge \cdots \wedge dd^c u_n \leqslant C \left(\log \frac{1}{\varepsilon} \right)^{-1}, \qquad 0 < \varepsilon < \cdot 5.$$

If $u_1, \ldots, u_n$ *are fixed, then as* $\varepsilon \to 0$,

$$\int_{\Delta(\varepsilon)} dd^c u_1 \wedge \cdots \wedge dd^c u_n = o \left(\log \frac{1}{\varepsilon} \right)^{-1}.$$

Proof. − Let $T = dd^c u_1 \wedge \cdots \wedge dd^c u_n$. By the Theorem,

$$\int_{\Delta(1)} \log \frac{1}{|z_1|} T < +\infty, \qquad \text{so} \qquad \int_{\Delta(\varepsilon)} \log \frac{1}{|z_1|} \wedge T = o(1),$$

which implies the corollary because $\log \dfrac{1}{|z_1|} \geqslant \operatorname{Log} \dfrac{1}{\varepsilon}$ on $\Delta(\varepsilon)$.

DEFINITION 2.4. − Let $T = dd^c u_1 \wedge \cdots \wedge dd^c u_{n-1}$, where $u_j \in P(\Omega) \cap L_{\text{loc}}^\infty(\Omega)$, $\Omega \subset \mathbb{C}^n$. Let $v \in P(\Omega)$. The *residual mass* of T on $\{v = -\infty\}$ *is the* (n,n) current

$$S(T,v) = S(T) = \chi_{\{v = -\infty\}} dd^c v \wedge T$$

(where χ_E denotes the indicator function of the set E).

Remark 2.5. − It is clear from Theorem 2.1 that

$$S(T,v) = \lim_{a \to -\infty} \chi_{\{v \leqslant a\}} dd^c v \wedge T.$$

LEMMA 2.6. − *If v, T are as in Definition 2.4 and*

$$v_a = \max(v,a)$$

then for all $u \in P(\Omega) \cap L_{loc}^\infty(\Omega)$,

$$\lim_{a \to -\infty} u \, dd^c v_a \wedge T = u \, dd^c v \wedge T.$$

Proof. − We have for any test function ψ on Ω,

$$\int \psi u \, dd^c v_a \wedge T = \int v_a \, dd^c(\psi u \wedge T)$$

$$= \int v_a \, \psi \, dd^c u \wedge T + 2 \int v_a \, d\psi \wedge d^c u \wedge T$$

$$+ \int v_a u \, dd^c \psi \wedge T.$$

The first and third integrals converge to the corresponding integrals with v_a replaced by v, as $a \to -\infty$. The second integral differs from the corresponding integral by

$$\left| \int (v_a - v) \, d\psi \wedge d^c u \wedge T \right|$$
$$\leqslant \left| \int (v_a - v) \, d\psi \wedge d^c \psi \wedge T \right|^{1/2} \left| \int_{\text{Spt}(\psi)} (v_a - v) \, du \wedge d^c u \wedge T \right|^{1/2}$$
$$= I_1(a) I_2(a).$$

Now, $I_1(a) \to 0$ by the dominated convergence theorem since vT has locally finite mass. It is no loss of generality to assume $u \geqslant 0$ on a neighborhood of the support of ψ so that u^2 is psh and $2 \, du \wedge d^c u \leqslant dd^c u^2$. But, $v \, dd^c(u^2) \wedge T$ has locally finite mass so $I_2(a) \to 0$ as $a \to -\infty$. Thus,

$$\lim_{a \to -\infty} \int \psi u \, dd^c v_a \wedge T = \int v \, dd^c(\psi u) \wedge T$$

which is the conclusion of the Lemma.

COROLLARY 2.7. − *On the set* $\{v > a\}$ *we have*

$$dd^c v_a \wedge T = dd^c v \wedge T.$$

Proof. $-$ If $a > b > -\infty$, then it follows from Proposition 4.2 of [BT 2] that the currents $dd^c v_a \wedge T$ and $dd^c v_b \wedge T$ agree on the (fine open) set $\{v > a\}$. Thus,

$$(v-a)^+ \{dd^c v_a \wedge T - dd^c v_b \wedge T\} \equiv 0.$$

By the lemma, this current has limit

$$(v-a)^+ \{dd^c v_a \wedge T - dd^c v \wedge T\}$$

as $b \to -\infty$. *Hence* $dd^c v_a \wedge T = dd^c v \wedge T$ *on the set* $\{v > a\}$.

We can now easily give the limiting properties of the residual mass current.

DEFINITION 2.8. $-$ With v, T as in Definition 2.4,

$$S(T;a,v) = S(T;a) = \chi_{\{v \leqslant a\}} dd^c v_a \wedge T$$

(where $v_a = \max(v,a)$).

PROPOSITION 2.9. $-$ $\lim\limits_{a \to -\infty} S(T;a,v) = S(T;v)$.

Proof. $-$ We have

$$dd^c v \wedge T = \chi_{\{v \leqslant a\}} dd^c v \wedge T + \chi_{\{v > a\}} dd^c v \wedge T$$

and

$$dd^c v_a \wedge T = \chi_{\{v \leqslant a\}} dd^c v_a \wedge T + \chi_{\{v > a\}} dd^c v_a \wedge T.$$

By Proposition 2.7, the last term of each of these equations is equal. From Lemma 2.6 with $u \equiv 1$, the limit of $dd^c v_a \wedge T$ as $a \to -\infty$ is $dd^c v \wedge T$. Thus,

$$S(T,v) = \lim_{a \to -\infty} \chi_{\{v \leqslant a\}} dd^c v \wedge T = \lim_{a \to -\infty} \chi_{\{v \leqslant a\}} dd^c v_a \wedge T$$

as asserted.

PROPOSITION 2.10. $-$ *For any test function* ψ *and* $-\infty < b < a$,

$$\int \psi S(T;a,v) = \int \psi S(;b,v) + \int_{b \leqslant v \leqslant a} (a - v_b) dd^c \psi \wedge T + \int_{b < v \leqslant a} \psi dd^c v \wedge T$$

and

$$\int \psi S(T;a,v) = \int \psi S(T,v) + \int_{v < a} (a - v) dd^c \psi \wedge T \int_{-\infty < v \leqslant a} \psi dd^c v \wedge T.$$

Proof. — From the definition of $S(T; a)$ and Corollary 2.7,

$$(2.1) \qquad dd^c v_a \wedge T = S(T;a) + \chi_{\{v > a\}} dd^c v \wedge T.$$

If we replace a by b in (2.1) and substract the two equations, we obtain

$$dd^c v_a \wedge T - dd^c v_b \wedge T = S(T;a) - S(T;b) - \chi_{\{b < v \leqslant a\}} dd^c v \wedge T.$$

Hence, multiplying by the test form ψ and integrating yields

$$\int (v_a - v_b) \, dd^c \psi \wedge T = \int \psi S(T;a) - \int \psi S(T;b) - \int_{b < v \leqslant a} \psi \, dd^c v \wedge T$$

which is the first equation of the proposition. The second equation follows from Theorem 2.1, the dominated convergence theorem, and Proposition 2.9 by letting $b \rightarrow -\infty$.

We conclude this section by noting that the residual mass of T on $v = -\infty$ depends only on the singularity of v.

PROPOSITION 2.11. — *Let v, T be as in Definition 2.4. Suppose that $u \in P(\Omega)$ and*

$$\liminf_{u(z) \to -\infty} \frac{v(z)}{u(z)} \geqslant C > 0$$

uniformly on compact subsets of Ω. Then

$$CS(T, -\infty, u) \leqslant S(T, -\infty, v).$$

Proof. — The proof is essentially the same as that given by Demailly (see Demailly [D 1], Théorème 4.2, p. 41) so we omit it.

3. Transformation of the singularity at infinity to a local singularity.

We will study the logarithmic singularity of functions in $\mathscr{L}_+$ at ∞ by introducing suitable local coordinates. (This transformation was used by Sadullaev [Sa]; see also Siciak [Si 3].) Consider $\mathbb{C}^n$ imbedded in $\mathbb{P}^n$ in the usual way; i.e. $z = (z_1, \ldots, z_n) \rightarrow [1, z_1, \ldots, z_n]$, where $(z_0, z_1, \ldots, z_n)$ denotes homogeneous coordinates on $\mathbb{P}^n$. Let $\mathscr{U}_i = \{[z_0, \ldots, z_n] \in \mathbb{P}^n : z_i \neq 0\}$, $i = 0, 1, \ldots, n$. On the coordinate

patch $\mathscr{U}_1$, we can introduce local coordinates $\zeta = (s,t) = (s, t_2, \ldots, t_n)$, where in terms of the local coordinates $(z_1, \ldots, z_n) = z$ on $\mathscr{U}_0$,

$$(3.1) \qquad z_1 = 1/s, \qquad z_j = t_j/s, \qquad 2 \leqslant j \leqslant n.$$

Thus, $s = 0$ corresponds to the hyperplane at ∞ intersected with $\mathscr{U}_1$. If $v \in \mathscr{L}_+(\mathbb{C}^n)$, say

$$(3.2) \qquad \frac{1}{2}\log\left(1+|z|^2\right) + \alpha \leqslant v(z) \leqslant \frac{1}{2}\log\left(1+|z|^2\right) + \beta$$

and we define, for $\zeta = (s,t) \in \mathbb{C}^n$,

$$(3.3) \qquad \tilde{v}(\zeta) = \tilde{v}(s,t) = v(1/s, t/s) + \log|s|$$

then direct substitution into (3.2) yields

$$(3.4) \qquad \frac{1}{2}\log\left(1+|\zeta|^2\right) + \alpha \leqslant \tilde{v}(\zeta) \leqslant \frac{1}{2}\log\left(1+|\zeta|^2\right) + \beta$$

for $s \neq 0$. Thus, $\tilde{v}$ is a locally bounded psh function on $\mathbb{C}^n \backslash \{s=0\}$. It therefore can be extended to be psh on all of $\mathbb{C}^n$ by defining

$$\tilde{v}(0,t) = \limsup_{\zeta=(s,t)\to(0,t)} \tilde{v}(s,t).$$

From (3.4), $\tilde{v} \in \mathscr{L}_+$.

Note that $t = (t_2, \ldots, t_n)$ provides local coordinates on the coordinate patch of the hyperplane at ∞ which consists of those lines through the origin in $\mathbb{C}^n$ that can be parametrized by z_1. From the definition of ρ_v, ρ_v^* in Section 1, it follows that

$$(3.5) \qquad \tilde{v}(0,t) = \rho_v^*(t) + \frac{1}{2}\log\left(1+|t|^2\right)$$

where $\Omega = dd^c \frac{1}{2}\log\left(1+|t|^2\right)$ denotes (a multiple of) the Kähler form on $\mathbb{P}^{n-1}$.

Note also that v, $\tilde{v}$ differ by the pluriharmonic function $\log|s|$ on $\mathscr{U}_0 \cap \mathscr{U}_1$. That is, there is a closed, positive $(1,1)$ current on $\mathbb{P}^n$

$$(3.6) \qquad \omega(v) = dd^c v = dd^c \tilde{v}_i,$$

(in this notation, $\tilde{v}_0 = v$, $\tilde{v}_1 = \tilde{v}$). If $v(z) = \frac{1}{2}\log\left(1+|z|^2\right)$, the associated

(1, 1) form is the usual Kähler form (up to a normalizing constant). The functions $\tilde{v}_i \in \mathscr{L}_+$ are unique, up to additive constants.

With the aid of these transformations, we can apply the results of Section 2 to functions in $\mathscr{L}_+$.

PROPOSITION 3.1. — *Let* $v, u_1, \ldots, u_n \in \mathscr{L}_+$ *and suppose* α, β *are constants such that* (3.2) *holds for all of the functions* $v, u_1, \ldots, u_n$. *Then there is a constant* $C > 0$, *depending only on* n, α, β, *such that*

$$\int_{\mathbb{C}^n} |v| \, dd^c u_1 \wedge \cdots \wedge dd^c u_n \leqslant C.$$

Further,

$$\int_{|z| > R} dd^c u_1 \wedge \cdots \wedge dd^c u_n = o(\log R)^{-1}, \quad \text{as} \quad R \to \infty.$$

Proof. — Let $0 \leqslant \chi_i \leqslant 1$, $0 \leqslant i \leqslant n$, be a C^∞ partition of unity subordinate to the cover of $\mathbb{P}^n$, $\mathscr{U}_0, \mathscr{U}_1, \ldots, \mathscr{U}_n$. Then

$$\int_{\mathbb{C}^n} v \, dd^c u_1 \wedge \cdots \wedge dd^c u_n = \sum_{i=0}^{n} \int_{\mathscr{U}_0 \cap \mathbb{P}^n} (\chi_i v) \, dd^c u_1 \wedge \cdots \wedge dd^c u_n.$$

The estimate for the term with $i = 0$ is easy because psh functions are bounded on the support of χ_0 (take $v = $ const. in Theorem 2.1). To get the estimate for the other terms, we take $i = 1$ to simplify the notation. The integral extends only over $\mathscr{U}_0 \cap \mathscr{U}_1$, so from (3.3), it is equal to

$$\int_{\mathscr{U}_0 \cap \mathscr{U}_1} \chi_1(\tilde{v} - \log|s|) \, dd^c \tilde{u}_1 \wedge \cdots \wedge dd^c \tilde{u}_n.$$

But, by Theorem 2.1, $\log|s| \, dd^c \tilde{u}_1 \wedge \cdots \wedge dd^c \tilde{u}_n$ has no mass on $s = 0$. Also, the current $\tilde{v} \, dd^c \tilde{u}_1 \wedge \cdots \wedge dd^c \tilde{u}_n$ puts no mass on the polar set $s = 0$, since $\tilde{v}, \tilde{u}_1, \ldots, \tilde{u}_n$ are all bounded on a neighborhood of the support of χ_1 (see e.g. [BT 1], Theorem 6.9). Thus the integral is equal to

$$\int_{\mathscr{U}_1} \chi_1(\tilde{v} - \log|s|) \, dd^c \tilde{u}_1 \wedge \cdots \wedge dd^c \tilde{u}_n.$$

For this integral, we have the estimates of Theorem 2.1 which shows it is bounded by a constant which depends only on the supremum norms of $\tilde{u}_i, \tilde{v}$ and the L^1-norm of $\log|s|$ on a neighborhood of the support

of χ_1. This proves the first assertion of the Proposition, and the second is a direct consequence of it $\left(\text{take } v = \dfrac{1}{2} \log (1 + |z|^2)\right)$.

Note also that if $\omega(u_i)$ is the positive closed $(1, 1)$ current on $\mathbb{P}^n$ associated to $u_i \in \mathscr{L}_+$, we have shown that the current on $\mathbb{P}^n$,

$$T = v\omega(u_1) \wedge \cdots \wedge \omega(u_n) = v\, dd^c u_1 \wedge \cdots \wedge dd^c u_n$$

has finite mass on $\mathbb{P}^n$, and zero mass on the polar ser in $\mathbb{P}^n$ which is the hyperplane at ∞ (or any other hyperplane, for that matter). Thus

$$\int_{\mathbb{P}^n} T = \int_{\mathbb{C}^n} v\, dd^c u_1 \wedge \cdots \wedge dd^c u_n$$

whenever $v, u_1, \ldots, u_n \in \mathscr{L}_+$.

From Proposition 3.1, it also follows that the integrals mentioned in Section 1 involving the function ρ_u^* can be given in terms of the function $\tilde{u}$.

PROPOSITION 3.2. — *For $u, v, w \in \mathscr{L}_+$, we have*

$$\int_{\mathbb{P}^{n-1}} (\rho_u^* - \rho_v^*)(dd^c \rho_w^* + \Omega)^{n-1} = \int_{\mathbb{C}^{n-1}} (\tilde{u}(0,t) - \tilde{v}(0,t))(dd^c \tilde{w}(0,t))^{n-1}.$$

Proof. — The current $T = (\rho_u^* - \rho_v^*)[dd^c \rho_w^* + \Omega]^{n-1}$ can be written in the local coordinate $t = (t_2, \ldots, t_n)$ given in (3.1) for the coordinate patch of lines in $\mathbb{P}^{n-1}$ which can be parametrized by z_1. From (3.5) we see that

$$T = (\tilde{u}(0,t) - \tilde{v}(0,t))(dd^c \tilde{w}(0,t))^{n-1}.$$

Thus, the current T has no mass on any hyperplane in $\mathbb{P}^{n-1}$, so the integral can be written just over the coordinate patch for t,

$$\int_{\mathbb{P}^{n-1}} T = \int_{\mathbb{C}^{n-1}} (\tilde{u}(0,t) - \tilde{v}(0,t))(dd^c \tilde{w}(0,t))^{n-1}.$$

4. Slicing via residual mass.

By the comparison inequality of Section 2, Proposition 2.11, we see that the residual mass $S(T, -\infty, v)$ must vanish unless the psh function v has the strongest possible singularity. In this section, we identify the

residual mass in the simplest case, $v = \log |z_1|$. For this case, we will show that the residual mass is the slice of the current $T = (dd^c u)^{n-1}$ on the set $z_1 = 0$. Further, this current is the Monge Ampere operator $(dd^c u)^{n-1}$ applied to the locally bounded psh function $(z_2, \ldots, z_n) \to u(0, z_2, \ldots, z_n)$. For our purposes, it is important to know the same result for the currents $w(dd^c u)^{n-1}$, where u, w are locally bounded psh functions. In all these cases, it is easy to see that the slice of the current exists for almost all values of z_1. The point is to know that it exists for all values of z_1.

Recall that the slice of an $(n-1, n-1)$ current T with respect to a hyperplane $z_1 = a$ is the current on $z_1 = a$ given by the formula

$$(4.1) \quad \langle T, z_1, a \rangle (\psi) = \lim_{\varepsilon \to 0} \frac{1}{\pi \varepsilon^2} \int_{|z_1 - a| \leqslant \varepsilon} \psi(z_2, \ldots, z_n) \frac{i}{2} dz_1 \wedge d\bar{z}_1 \wedge T$$

where ψ is a test form on $z_1 = a$. By [F], Section 4.3, p. 435, the slice of a normal (or flat) current exists for almost all $a \in \mathbb{C}$ and is characterized by the formula

$$(4.2) \quad \int \varphi(z_1) \psi(z_2, \ldots, z_n) \wedge T = \int_{a \in \mathbb{C}} \varphi(a) \wedge \{\langle T, z_1, a \rangle (\psi)\}$$

for all test forms $\psi(z_2, \ldots, z_n)$ and $(1, 1)$ test forms $\varphi(z_1)$. From (4.2) and the identity

$$(4.3) \quad \int \varphi(z_1) \psi(z') w(z) (dd^c u)^{n-1}$$
$$= \int_{z_1} \varphi(z_1) \wedge \left\{ \int_{z'} \psi(z') w(z_1, z') (dd^c u(z_1, z'))^{n-1} \right\}$$

where $z' = (z_2, \ldots, z_n)$, it follows that when $T = w(dd^c u)^{n-1}$ with u, w bounded psh functions, then

$$\langle T, z_1, a \rangle = w(a, z')(dd^c u(a, z'))^{n-1}$$

for almost all $a \in \mathbb{C}$. The formula (4.3) is obvious when u, w are smooth and holds by a limiting argument in the general case. Similar remarks apply to the currents $T = u_1 \, dd^c u_2 \wedge \cdots \wedge dd^c u_n$ and $T = w \, du_1 \wedge d^c u_2 \wedge dd^c u_3 \wedge \cdots \wedge dd^c u_n$, where w and the u_i are locally bounded psh functions.

In fact, these slicing formulas hold for all hyperplanes $z_1 = a$, not just almost all hyperplanes.

PROPOSITION 4.1. $-$ *Let u, w be locally bounded* psh *functions on $\Omega \subset \mathbb{C}^n$. If T denotes the current $w(dd^c u)^{n-1}$, then the slice of T with respect to the hyperplane $z_1 = 0$ exists. Further,*

$$\langle T, z_1, 0 \rangle = \frac{1}{2\pi} w(0, z')(dd^c u(0, z'))^{n-1}$$

where $z' = (z_2, \ldots, z_n)$. The analogous formula holds for the currents

$$T = w \, du_1 \wedge d^c u_2 \wedge dd^c u_3 \wedge \cdots \wedge dd^c u_n$$

and

$$T = w \, dd^c u_2 \wedge dd^c u_3 \wedge \cdots \wedge dd^c u_n,$$

where w and the u_i are locally bounded psh *functions.*

Proof. $-$ We will prove the proposition only for the case $T = w(dd^c u)^n$. The other cases may be proved by exactly the same argument. Or, one can use a polarization identity to reduce to the case considered here, along with the identity $dd^c u^2 = 2 \, du \wedge d^c u + 2u \, dd^c u$. For $\varepsilon > 0$, let $v_\varepsilon = \max \{\log |z|, \log \varepsilon\}$. Then we have

$$\int \psi w \, dd^c v_\varepsilon \wedge (dd^c u)^{n-1} = \frac{1}{2\varepsilon^2} \int_{|z_1| \leqslant \varepsilon} (\psi w) \, dd^c |z_1|^2 \wedge (dd^c u)^{n-1}$$

$$+ \frac{1}{2\varepsilon^2} \int_{|z_1| \leqslant \varepsilon} (\varepsilon^2 - |z_1|^2) \, dd^c (\psi w) \wedge (dd^c u)^{n-1}.$$

(The formula is easily verified when w, u are smooth, and the general case is obtained from a limiting argument.) The first term on the right hand side is 2π times the integral on the right hand side of (4.1) when ψ is independent of z_1. The second term on the right hand side tends to zero as $\varepsilon \to 0$, since it involves a bounded function times the currents $dd^c \psi \wedge (dd^c u)^{n-1}$, $d\psi \wedge d^c w \wedge (dd^c u)^{n-1}$, or $\psi \, dd^c w \wedge (dd^c u)^{n-1}$. All of these put zero mass on the polar set $z_1 = 0$ and, hence, have total mass on the « collar », $|z_1| \leqslant \varepsilon$, which goes to zero as $\varepsilon \to 0$. On the left hand side, we have

$$w \, dd^c v_\varepsilon \wedge (dd^c u)^{n-1} = \chi_{\{|z_1| \leqslant \varepsilon\}} w \, dd^c v_\varepsilon \wedge (dd^c u)^{n-1}$$

$$= S(w(dd^c u)^{n-1}, \log \varepsilon, \log |z_1|).$$

By Lemma 2.6 and Corollary 2.7, it follows that the limit of the left hand side exists and equals

$$\chi_{\{z_1 = 0\}} w \, dd^c \log |z_1| \wedge (dd^c u)^{n-1}.$$

Thus, the slice of $w(dd^c u)^{n-1}$ on $z_1 = 0$ exists and is equal to $T_0 = w \, dd^c \log |z_1| \wedge (dd^c u)^{n-1}$. (Note that T_0, a current on $\mathbb{C}^n$, is really supported on $z_1 = 0$ and hence can be identified with a current on $z_1 = 0$.)

We still have to show that

$$(4.5) \quad w \, dd^c \log |z| \wedge (dd^c u)^{n-1} = w(0, z')(dd^c u(0, z'))^{n-1}.$$

When u, w are smooth, this is obvious. The general case then follows by taking smooth u_j, w_j decreasing to u, w. Both sides of (4.5) converge (cf. the convergence theorems of [BT 1] or [D 1], Théorème 2.6), and this completes the proof.

We note that it was actually proved ˙that the slice is the limit of the currents $w(dd^c u)^{n-1} \wedge dd^c \max \{\log |z_1|, \log \varepsilon\}$.

CorOLLARY 4.2. — *If w, u are locally bounded* psh *functions on $\Omega \subset \mathbb{C}^n$, then*

$$\lim_{\varepsilon \to 0} \int \psi w \, dd^c \max \{\log |z|, \log \varepsilon\} \wedge (dd^c u)^{n-1}$$

$$= 2\pi \int \psi(0, z') w(0, z')(dd^c u(0, z'))^{n-1},$$

for every continuous function ψ with compact support in Ω.

With the aid of the slices, one can make sense of boundary integrals which occur in integration by parts formulas involving $\log |z|$. For example, we formally have

$$\int w(z) \, dd^c \max \{\log |z_1|, \log \varepsilon\} \wedge (dd^c u)^{n-1}$$

$$= \int_{|z_1| = \varepsilon} w(z) \, d^c \log |z_1| \wedge (dd^c u(z_1, z'))^{n-1}.$$

Write $z_1 = \varepsilon e^{i\theta}$ so that $d^c \log |z_1| = d\theta$, and the last integral can be written as

$$\int_{-\pi}^{\pi} w(\varepsilon e^{i\theta}, z') \langle T, z_1, \varepsilon e^{i\theta} \rangle \, d\theta$$

where $T = (dd^c u)^{n-1}$.

COROLLARY 4.3. — *If ψ is any test function, then*

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(\varepsilon e^{i\theta}, z') w(\varepsilon e^{i\theta}, z')(dd_z^{c'} u(\varepsilon e^{i\theta}, z'))^{n-1} \, d\theta$$

$$= \int_{\mathbb{C}^{n-1}} \psi(0, z') w(0, z')(dd^c u(0, z'))^{n-1}$$

for u,w locally bounded psh *functions.*

We can also consider the slightly more general case when $v = \log |f|$ for f a holomorphic function on $\Omega \subset \mathbb{C}^n$. All the results are local, so at regular points of the manifold $M = \{f = 0\}$, we have just to make an analytic change of variable to see that

$$\langle w(dd^c u)^{n-1}, f, 0 \rangle = w|_M (dd^c u)^{n-1} = w \, dd^c \log |f| \wedge (dd^c u)^{n-1}.$$

5. The Integral Formula.

In this section, we study psh functions which have logarithmic singularities. We will prove a local version of the integral formula from which the version stated in (1.2) of the introduction can be easily deduced. The idea here is quite direct. For functions with a logarithmic singularity on $z_1 = 0$ e.g.

$$u(z) = \log \frac{1}{|z_1|} + \tilde{u}(z), \, \tilde{u}(z) = O(1),$$

we want to prove the (formally obvious) formula,

$$d[(u \, d^c v - v \, d^c u) \wedge (dd^c w)^{n-1}] = (u \, dd^c v - v \, dd^c u) \wedge (dd^c w)^{n-1}$$
$$- 2\pi \chi_{\{z_1 = 0\}} (\tilde{u}(0, z') - \tilde{v}(0, z'))(dd^c \tilde{w}(0, z'))^{n-1}.$$

The « δ-function » term arises because $dd^c \log \dfrac{1}{|z_1|}$ is concentrated on $z_1 = 0$. The precise version of this result is given in Lemma 5.2.

Let Ω be an open set in $\mathbb{C}^n$ with $\Omega \cap M \neq \varnothing$, where

$$M = \{z_1 = 0\}.$$

Consider the classes $\mathscr{L}_\pm(\Omega, M)$ of all locally bounded psh functions u on $\Omega \backslash M$ such that

$$u(z) = \mp \log |z_1| + O(1), \qquad z_1 \to 0$$

uniformly on compact subsets of Ω. For $u \in \mathscr{L}_{\pm}(\Omega, M)$, let

(5.1)
$$\tilde{u}(z) = u(z) \pm \log |z_1|$$

so that $\tilde{u}(z)$ is locally bounded on $\Omega \backslash M$ near points of M. The function $\tilde{u}$ is psh on $\Omega \backslash M$ and hence has a unique psh extension to Ω given by

$$\tilde{u}(0, z') = \limsup_{z_1 \to 0, \, \zeta \to z'} \tilde{u}(z_1, \zeta).$$

For $u, v, w_2, \ldots, w_n \in \mathscr{L}_{\pm}(\Omega, M)$ the current

(5.2)
$$\theta = (u \, dd^c v - v \, dd^c u) \wedge dd^c w_2 \wedge \cdots \wedge dd^c w_n$$

is a well-defined (n, n) current on $\Omega \backslash M$, since u, v, w_j are locally bounded on $\Omega \backslash M$. The trivial extension by 0 of θ to all of Ω is a current of locally finite mass on Ω, since the formula (5.1) substituted into (5.2) gives

(5.3)
$$\theta = \pm \log |z_1| \, dd^c(\tilde{u} - \tilde{v}) \wedge dd^c \tilde{w}_2 \wedge \cdots \wedge dd^c \tilde{w}_n$$
$$+ (\tilde{u} \, dd^c \tilde{v} - \tilde{v} \, dd^c \tilde{u}) \wedge dd^c \tilde{w}_2 \wedge \cdots \wedge dd^c \tilde{w}_n$$

which has locally finite mass on Ω by Theorem 2.1. Hereafter, when we write the current on the right hand side of (5.2), we will always mean the current θ of (5.3). Throughout the rest of this section, we will assume

(5.4)
$$T = dd^c w_2 \wedge \cdots \wedge dd^c w_n, \, w_j \in \mathscr{L}_+(\Omega, M), j = 2, \ldots, n.$$

LEMMA 5.1. − *Let* $u, v \in \mathscr{L}_+(\Omega, M)$ *or* $u, v \in \mathscr{L}_-(\Omega, M)$. *Then there is a* $(2n-1)$ *current,* $PV[(u \, d^c v - v \, d^c v) \wedge T]$ *defined on* Ω *by the formula*

$$(PV[(u \, d^c v - v \, d^c v) \wedge T], \psi) = \lim_{\varepsilon \to 0} \int_{|z_1| > \varepsilon} \psi \wedge (u \, d^c v - v \, d^c v) \wedge T$$

for every test 1-form, ψ.

Proof. − We give the proof only for the case $w_2 = \cdots = w_n = w$. The proof in the general case is exactly the same. It also follows from this special case by applying it to functions of the form $w = t_2 w_2 + \cdots + t_n w_n$, $t_j \geqslant 0$, $\sum t_j = 1$, and then equating coefficients of the resulting polynomial in $(t_2, \ldots, t_n)$.

We will give the proof for the case $u, v \in \mathscr{L}_+(\Omega, M)$. The proof in the other case is similar. Write $\ell(z_1) = \log |z_1|$, and recall that

$\tilde{u} = u + \ell$ is a locally bounded psh function on Ω. Then

$$u\, d^c v - v\, d^c u = (\tilde{u} - \tilde{v}) d^c(-\ell) - (-\ell) d^c(\tilde{u} - \tilde{v}) + \tilde{u}\, d^c \tilde{v} - \tilde{v}\, d^c \tilde{u}.$$

Thus,

$$(5.5) \quad \int_{|z_1| > \varepsilon} \psi \wedge (u\, d^c v - v\, d^c u) \wedge (dd^c w)^{n-1}$$

$$= \int_{|z_1| > \varepsilon} \psi \wedge (\tilde{v} - \tilde{u})\, d^c \ell \wedge (dd^c \tilde{w})^{n-1}$$

$$+ \int_{|z_1| > \varepsilon} \ell \psi \wedge d^c(\tilde{u} - \tilde{v}) \wedge (dd^c \tilde{w})^{n-1}$$

$$+ \int_{|z_1| > \varepsilon} \psi \wedge (\tilde{u}\, d^c \tilde{v} - \tilde{v}\, d^c \tilde{u}) \wedge (dd^c \tilde{w})^{n-1}.$$

The current $(\tilde{u}\, d^c \tilde{v} - \tilde{v}\, d^c \tilde{u}) \wedge (dd^c \tilde{w})^{n-1}$ is a $(2n-1)$ current of finite mass on Ω with 0 mass on the polar set M, since $\tilde{u}, \tilde{v}, \tilde{w}$ are bounded psh functions (see e.g. [BT 1], Theorem 6.9). Thus, the limit of the last term as $\varepsilon \to 0$ is

$$\int_\Omega \psi \wedge (\tilde{u}\, d^c \tilde{v} - \tilde{v}\, d^c \tilde{u}) \wedge (dd^c \tilde{w})^{n-1}.$$

The limit of the next to last term also exists and is equal to

$$\int_\Omega \ell \psi \wedge d^c(\tilde{u} - \tilde{v}) \wedge (dd^c \tilde{w})^{n-1} = \int_{\Omega \setminus M} \ell \psi \wedge d^c(\tilde{u} - \tilde{v}) \wedge (dd^c w)^{n-1}$$

$$= \int_{\Omega \setminus M} \ell \psi \wedge d^c(u - v) \wedge (dd^c w)^{n-1}$$

by Theorem 2.1 and the dominated convergence theorem. Thus, the most singular term is the first one on the right hand side of (5.5). To study it write the 1-form ψ in the form

$$\psi = \sum_{j=1}^{2n} \psi_j\, dx_j.$$

Since the (1,1) parts of $dx_j \wedge d^c \ell$ and $d\ell \wedge d^c x_j$ are the same, this term is equal to

$$-\sum_{j=1}^{2n} \int_{|z_1| > \varepsilon} \psi_j(\tilde{u} - \tilde{v})\, d\ell \wedge d^c x_j \wedge (dd^c \tilde{w})^{n-1}.$$

However, we claim that

$$(5.6) \quad \int_{|z_1|>\varepsilon} \psi_j(\tilde{u}-\tilde{v}) \, d\ell \wedge d^c x_j \wedge (dd^c \tilde{w})^{n-1}$$

$$= \int_{|z_1|>\varepsilon} \ell \, d(\psi_j(\tilde{u}-\tilde{v})) \wedge d^c x_j \wedge (dd^c \tilde{w})^{n-1}$$

$$+ \log \varepsilon \int_{|z_1|>\varepsilon} d(\psi_j(\tilde{u}-\tilde{v})) \wedge d^c x_j \wedge (dd^c \tilde{w})^{n-1}.$$

If $\tilde{u}, \tilde{v}, \tilde{w}$ are smooth, then (5.6) follows from an integration by parts and Stokes' theorem. In the general case, approximate $\tilde{u}, \tilde{v}, \tilde{w}$ by decreasing limits of smooth psh functions. We then get the identity for all except possibly the countably many ε for which the measures in the integrand put positive mass on $\{|z_1| = \varepsilon\}$. However, all three terms in (5.6) are right continuous functions of ε. Hence they are equal for all $\varepsilon > 0$.

By Theorem 2.1, the first term on the right hand side of (5.6) converges to the integral over $\{z_1 \neq 0\}$, as $\varepsilon \to 0$. The second term converges to zero, by Corollary 2.5. Thus, the left hand side of (5.6) has limit

$$\sum_{j=1}^{2n} \int_{z_1 \neq 0} \ell \, d(\psi_j(\tilde{u}-\tilde{v})) \wedge d^c x_j \wedge (dd^c \tilde{w})^{n-1}$$

as $\varepsilon \to 0$, which proves the first assertion of Lemma 5.1.

LEMMA 5.2. $-$ *Under the hypotheses and notation of Lemma* 5.1, *we have*

$$dPV[(u \, d^c v - v \, d^c u) \wedge dd^c w_2 \wedge \cdots \wedge dd^c w_n] = \theta$$
$$\mp 2\pi \chi_{\{z_1 = 0\}}[\tilde{u}(0,z') - \tilde{v}(0,z')] \wedge dd^c \tilde{w}_2(0,z') \wedge \cdots \wedge dd^c \tilde{w}_n(0,z')$$

where θ *is given in* (5.2), $z' = (z_2, \ldots, z_n)$. *The* $-$ *sign is chosen when* $u, v \in \mathscr{L}_+(\Omega, M)$, *and the* $+$ *sign is chosen when* $u, v \in \mathscr{L}_-(\Omega, M)$.

Proof. $-$ We give the proof for $u, v \in \mathscr{L}_+(\Omega, M)$, $w_2 = \cdots = w_n = w$. The assertion of the Lemma is that

$$\lim_{\varepsilon \to 0} - \int_{|z_1|>\varepsilon} d\psi \wedge (u \, d^c v - v \, d^c u) \wedge (dd^c w)^{n-1}$$

$$= \int_{\Omega} \psi(u \, dd^c v - v \, dd^c u)) \wedge (dd^c w)^{n-1}$$

$$- \int_{\Omega \cap M} \psi(0,z')(\tilde{u}(0,z') - \tilde{v}(0,z')) \wedge (dd^c \tilde{w}(0,z'))^{n-1}$$

$$= \int_\Omega \psi(\tilde{u}\, dd^c\tilde{v} - \tilde{v}\, dd^c\tilde{u}) \wedge (dd^c\tilde{w})^{n-1}$$

$$+ \int_\Omega \ell\, dd^c(\tilde{u} - \tilde{v}) \wedge (dd^c\tilde{w})^{n-1}$$

$$- \int_{\Omega \cap M} \psi(0,z')(\tilde{u}(0,z') - \tilde{v}(0,z')) \wedge (dd^c\tilde{w}(0,z'))^{n-1}.$$

By the argument given in the proof of Lemma 5.1, the limit is equal to

$$(5.7) \quad - \int_\Omega \ell(\tilde{u} - \tilde{v})\, dd^c\psi (dd^c\tilde{w})^{n-1}$$

$$- 2 \int_\Omega \ell\, d\psi \wedge d(\tilde{u} - \tilde{v}) \wedge (dd^c\tilde{w})^{n-1}$$

$$- \int_\Omega d\psi \wedge (\tilde{u}\, d^c\tilde{v} - \tilde{v}\, d^c u) \wedge (dd^c\tilde{w})^{n-1}.$$

However, if we consider the functions

$$\ell_\varepsilon(z_1) = \max (\log |z_1|, \log \varepsilon) = \max (\ell, \log \varepsilon)$$
$$u_\varepsilon(z) = \tilde{u}(z) - \ell_\varepsilon(z)$$
$$v_\varepsilon(z) = \tilde{v}(z) - \ell_\varepsilon(z)$$

then we can also study the limit, $\lim_{\varepsilon \to 0} I(\varepsilon)$ where

$$(5.8) \qquad I(\varepsilon) = \int \psi(u_\varepsilon\, dd^c v_\varepsilon - v_\varepsilon\, dd^c u_\varepsilon) \wedge (dd^c w)^{n-1}.$$

Since

$$u_\varepsilon\, dd^c v_\varepsilon - v_\varepsilon\, dd^c u_\varepsilon = (\tilde{u} - \tilde{v})\, dd^c(-\ell_\varepsilon) + \ell_\varepsilon\, dd^c(\tilde{u} - \tilde{v}) + \tilde{u}\, dd^c\tilde{v} - \tilde{v}\, dd^c\tilde{u},$$

and $dd^c w = dd^c\tilde{w}$ on $\Omega \backslash M$, we have from Corollary 4.2, Theorem 2.1, and the dominated convergence theorem,

$$(5.9) \quad \lim_{\varepsilon \to 0} I(\varepsilon) = + \int \psi\ell\, dd^c(\tilde{u} - \tilde{v}) \wedge (dd^c\tilde{w})^{n-1}$$

$$+ \int \psi(\tilde{u}\, dd^c\tilde{v} - v\, dd^c\tilde{u}) \wedge (dd^c\tilde{w})^{n-1}$$

$$- 2\pi \int_{\Omega \cap M} \psi(0,z')(\tilde{u}(0,z') - \tilde{v}(0,z')) \wedge (dd^c\tilde{w})^{n-1}.$$

On the other hand,

$$
\begin{aligned}
I(\varepsilon) &= -\int d\psi \wedge (u_\varepsilon d^c v_\varepsilon - v_\varepsilon d^c u_\varepsilon) \wedge (dd^c \tilde{w})^{n-1} \\
&= +\int d\psi \wedge ((\tilde{u}-\tilde{v})d^c \ell_\varepsilon - \ell_\varepsilon d^c(\tilde{u}-\tilde{v})) \wedge (dd^c \tilde{w})^{n-1} \\
&\quad -\int d\psi \wedge (\tilde{u}d^c\tilde{v} - \tilde{v}d^c\tilde{u}) \wedge (dd^c \tilde{w})^{n-1} \\
&= +\int (\tilde{u}-\tilde{v})d\ell_\varepsilon \wedge d^c\psi \wedge (dd^c \tilde{w})^{n-1} \\
&\quad +\int \ell_\varepsilon d\psi \wedge d^c(\tilde{u}-\tilde{v}) \wedge (dd^c \tilde{w})^{n-1} \\
&\quad -\int d\psi \wedge (\tilde{u}d^c\tilde{v} - \tilde{v}d^c\tilde{u}) \wedge (dd^c \tilde{w})^{n-1}.
\end{aligned}
$$

Integrate by parts in the first integral to obtain

$$
-\int \ell_\varepsilon(\tilde{u}-\tilde{v}) dd^c\psi \wedge (dd^c \tilde{w})^{n-1} - \int \ell_\varepsilon d\psi \wedge d^c(\tilde{u}-\tilde{v}) \wedge (dd^c \tilde{w})^{n-1}.
$$

Thus, we have

$$
\begin{aligned}
I(\varepsilon) = &-\int \ell_\varepsilon(\tilde{u}-\tilde{v}) dd^c\psi \wedge (dd^c \tilde{w})^{n-1} \\
&-2\int \ell_\varepsilon d\psi \wedge d^c(\tilde{u}-\tilde{v}) \wedge (dd^c \tilde{w})^{n-1} \\
&-\int d\psi \wedge (\tilde{u}\,d^c\tilde{v} - \tilde{v}\,d^c\tilde{u}) \wedge (dd^c \tilde{w})^{n-1}.
\end{aligned}
$$

As before, it follows that $I(\varepsilon)$ converges to the expression in (5.7) as $\varepsilon \to 0$. This completes the proof.

Remark 5.3. — Note that, in general, the term $u\,d^c v \wedge (dd^c w)^{n-1}$ is not a current representable by integration on Ω, because the singularity of $u\,d^c v$ is too strong when $u, v \in \mathscr{L}_+(\Omega, M)$. It is because of the cancellation between the terms $u\,d^c v$ and $v\,d^c u$ that the principal value exists.

Remark 5.4. — We can consider the more general case of the classes $\mathscr{L}_\pm(\Omega, M, f)$ where f is a holomorphic function on Ω, $M = \{f=0\}$ is a complex submanifold of codimension 1, and $u = \mp \log |f| + O(1)$ as $z \to M$. The only change is that the current on M is replaced by the residual mass of $(\tilde{u}-\tilde{v}) \wedge (dd^c \tilde{w})^{n-1}$ on M with respect to the psh function $\log |f|$.

Proof of the integral formula. — We now give the proof of the integral formula for functions in the class $\mathscr{L}_+$, defined in (1.1). Suppose $u, v, w_2, \ldots, w_n \in \mathscr{L}_+$. Let $T = dd^c w_2 \wedge \cdots \wedge dd^c w_n$, and let $\tilde{T}$ denote the current which is the slice of T on the hyperplane at infinity; i.e.

$$\tilde{T} = (dd^c \rho^*_{w_2} + \Omega) \wedge \cdots \wedge (dd^c \rho^*_{w_n} + \Omega)$$

$$= dd^c \tilde{w}_2(0,t) \wedge \cdots \wedge dd^c \tilde{w}_n(0,t),$$

where we are using the notation of Section 3.

THEOREM 5.5. — $\displaystyle\int_{\mathbb{C}^n} (u\, dd^c v - v\, dd^c u) \wedge T = 2\pi \int_{\mathbb{P}^{n-1}} (\rho^*_u - \rho^*_v) \wedge \tilde{T}.$

Proof. — We can assume $w_2 = \cdots = w_n = w$. Consider $u, v, w \in \mathscr{L}_+$ as functions on $\mathbb{P}^n$, the n-dimensional projective space with $\mathbb{C}^n$ imbedded in $\mathbb{P}^n$ in the usual way, i.e. $(z_1, \ldots, z_n)$ corresponds to the homogeneous coordinates $[z_1, \ldots, z_n, 1]$. We claim that

$$(5.10) \quad \lim_{R \to \infty} - \int_{|z| < R} \psi \wedge (u\, d^c v - v\, d^c u) \wedge (dd^c w)^{n-1} = S(\psi)$$

defines a current on $\mathbb{P}^n$ and

$$dS = \theta - U$$

where $\theta = (u\, dd^c v - v\, dd^c u) \wedge (dd^c w)^n$, on $\mathbb{C}^n$, extended by 0 to the hyperplane at ∞, U is a current with support on the hyperplane at ∞, and U is given by

$$2\pi(\rho^*_u - \rho^*_v) \wedge (\Omega + dd^c \rho^*_w)^{n-1}.$$

To prove this, it is enough to prove the formula for test forms with support in the coordinate patches

$$\mathscr{U}_i = \{[z_1, \ldots, z_{n+1}] \in \mathbb{P}^n : z_i \neq 0\}, \quad 1 \leqslant i \leqslant n + 1.$$

Without loss of generality, take $i = n + 1$ and use coordinates

$$z_n = 1/s, \quad z_j = t_j/s, \quad 2 \leqslant j \leqslant n.$$

Then $u \in \mathscr{L}_+$ implies

$$u\left[\frac{1}{s}(t_2, \ldots, t_n, 1)\right] = -\log|s| + O(1)$$

so $\mathscr{L}_+ \subset \mathscr{L}_+(\mathscr{U}_n, M)$, where $M = \{s=0\} = \mathscr{U}_n \cap \{$hyperplane at $\infty\}$. Thus the limit in (5.10) exists by Lemma 5.1 and so S is a current on $\mathbb{P}^n$. The assertion about dS follows from Lemma 5.2, provided we note also that

$$\tilde{u}(s,t) = \tilde{u}\left[\frac{1}{s}(t_1, \ldots, t_{n-1})\right] + \log |s|,$$

and, in the notation of Section 3,

$$\rho_u^*(t) = \tilde{u}(0,t) - \frac{1}{2}\log(1+|t|^2)$$

so that

$$(\tilde{u}(0,t) - \tilde{v}(0,t))(dd^c\tilde{w})^{n-1} = (\rho_u^*(t) - \rho_v^*(t)) \wedge (\Omega + dd^c\rho_w^*(t))^{n-1}$$

$\left(\text{where } \Omega = dd^c \frac{1}{2}\log(1+|t|^2)\right)$. This completes the proof.

6. Applications to the Robin function.

Recall from Section 1 the definition of the classes $\mathscr{L}$, $\mathscr{L}_+$ and the Robin function ρ_u associated to $u \in \mathscr{L}_\rho$. In this section, we show how the Green formula can be used to establish several properties of this function. Recall from Section 3, the functions $\tilde{u} = \tilde{u}_1 \in \mathscr{L}_+$, the current $\omega(u)$ on $\mathbb{P}^n$, and the local coordinate t on $\mathbb{C}^{n-1} \subset \mathbb{P}^{n-1}$ given in (3.1). It is convenient to introduce the notations

$$\Omega(v) = dd^c\tilde{v}(0,t) = dd^c\rho_v^* + \Omega$$

$$\Omega = dd^c \frac{1}{2}\log(1+|t|^2)$$

for these currents on $\mathbb{P}^{n-1}$, and

$$(6.2) \qquad M(v,u_1,\ldots,u_n) = \int_{\mathbb{C}^n} v \, dd^c u_1 \wedge \cdots \wedge dd^c u_n$$

$$= \int_{\mathbb{P}^n} v \, dd^c u_1 \wedge \cdots \wedge dd^c u_n.$$

In this notation, the integral formula, Theorem 5.5, can be viewed as computing the commutator of the first two slots in the multilinear form M,

$$(6.3) \quad M(v, u_1, \ldots, u_n) = M(u_1, v, u_2, \ldots, u_n)$$

$$+ 2\pi \int_{\mathbb{P}^{n-1}} (\rho_v^* - \rho_{u_1}^*)\Omega(u_2) \wedge \cdots \wedge \Omega(u_n)$$

$$= M(u_1, v, u_2, \ldots, u_n)$$

$$+ 2\pi \int_{\mathbb{P}^{n-1}} [\tilde{v}(0,t) - \tilde{u}_1(0,t)] \, dd^c\tilde{u}_2(0,t) \wedge \cdots \wedge dd^c\tilde{u}_n(0,t).$$

Our first application is to an important inequality for functions $u, v \in \mathscr{L}_+$.

THEOREM 6.1. $-$ *If $u, v \in \mathscr{L}_+$ and $u \geqslant v$, then*

$$\int_{\mathbb{C}^n} u(dd^c v)^n \leqslant \int_{\mathbb{C}^n} v(dd^c u)^n + \sum_{j=0}^{n-1} 2\pi \int_{\mathbb{P}^{n-1}} (\rho_u^* - \rho_v^*)\Omega(v)^{n-1-j} \wedge \Omega(u)^j.$$

Proof. $-$ For $u, v \in \mathscr{L}_+$ we have the identity

$$(6.4) \quad u(dd^c v)^n - v(dd^c u)^n = (u \, dd^c v - v \, dd^c u) + \sum_{j=0}^{n-1} (dd^c v)^{n-1-j} \wedge (dd^c u)^j$$

$$+ (v-u) \sum_{j=1}^{n-1} (dd^c u)^j \wedge (dd^c v)^{n-j}$$

or equivalently,

$$(6.5) \quad M(u, v, \ldots, v) - M(v, u, \ldots, u)$$

$$= \sum_{j=0}^{n-1} M(u, v, \underbrace{v, \ldots, v}_{n-1-j}, \underbrace{u, \ldots, u}_{j}) - M(v, u, \underbrace{v, \ldots, v}_{n-1-j}, \underbrace{u, \ldots, u}_{j})$$

$$+ \sum_{j=1}^{n-1} M(v-u, \underbrace{u, \ldots, u}_{j}, \underbrace{v, \ldots, v}_{n-j}).$$

Since $u \geqslant v$, the last term on the right hand side of (6.4) or (6.5) is negative. The first term is the commutator of the first two slots of M. So from (6.3), the inequality of Theorem 6.1 follows.

As a corollary of Theorem 6.1, we have the basic inequality of [T].

COROLLARY 6.2. $-$ If $u \in \mathscr{L}_+$, $u \geqslant 0$ for $|z| \leqslant 1$, and $(dd^c u)^n$ is supported in $|z| \leqslant 1$, then

$$\int_{|z|=1} u(z) \, d\sigma(z) \leqslant n \sup_{\alpha \in P^{n-1}} \rho_u(\alpha)$$

($d\sigma(z) =$ normalized surface area measure on $|z|=1$, $\int_{|z|=1} d\sigma(z) = 1$).

Proof. $-$ Since $u \geqslant 0$ on $|z| \leqslant 1$ and $(dd^c u)^n = 0$ in $|z| > 1$, we have from the maximum principle for the Monge-Ampere operator that

$$(1+\varepsilon)u(z) \geqslant \log^+ |z|, \qquad 1 \leqslant |z| \leqslant R$$

because the estimate holds for $|z| = 1$ and $|z| = R \gg 1$. Letting $R \to +\infty$ and $\varepsilon \to 0$ then gives $u(z) \geqslant \log^+ |z| = v(z)$ for all z. We claim this choice in Theorem 6.1 gives the corollary. Because, we clearly have $\int_{\mathbb{C}^n} v(dd^c u)^n = 0$, $\rho_v \equiv 0$, $\Omega(v) = \Omega$, so the right hand side of the inequality of Theorem 6.1 is the sum over $0 \leqslant j \leqslant n-1$ of the terms

$$(6.6) \quad 2\pi \int_{\mathbb{P}^{n-1}} \rho_u^* \Omega^{n-1-j} \wedge \Omega(u)^j \leqslant 2\pi \rho_u \int_{\mathbb{P}^{n-1}} \Omega^{n-1-j} \wedge \Omega(u)^j.$$

Now, for any choice of $u_1, \ldots, u_n \in \mathscr{L}_+(\mathbb{C}^n)$, we have

$$\int_{\mathbb{C}^n} dd^c u_1 \wedge \cdots \wedge dd^c u_n = \int_{\mathbb{P}^n} \omega(u_1) \wedge \cdots \wedge \omega(u_n) = (2\pi)^n.$$

So, the last integral in (6.6) is $(2\pi)^{n-1}$. Also, $\int_{\mathbb{C}^n} (dd^c v)^n = (2\pi)^n$, and the inequality follows.

We next study the convergence of the multilinear form $M(v_1, u_1, \ldots, u_n)$ on $\mathscr{L}_+$. It follows from known convergence theorems that M is continuous under decreasing limits (e.g. [D 1] Chapter 2), combined with the localization procedure of Section 3). However, M is not continuous under increasing limits. Here we will show, in Proposition 6.4 and Theorem 6.6 that this lack of continuity in M is measured by the Robin functions ρ_u^*. The first step is to note that M is continuous in the first slot in a strong way.

LEMMA 6.3. − *Suppose v_j, v and u_i, $u_{i,j} \in \mathscr{L}_+$ are monotone increasing sequences in j and*

$$\lim_{j \to \infty} v_j(z) = v(z), \quad \lim_{j \to \infty} u_{i,j}(z) = u_i(z),$$

for almost all $z \in \mathbb{C}^n$. Then for any $u \in \mathscr{L}_+$

$$\lim_{j \to \infty} M(u - v_j, u_{1,j}, \ldots, u_{n,j}) = M(u - v, u_1, \ldots, u_n).$$

Proof. − We can write the integral defining M as an integral over $\mathbb{P}^n$. Then choose a partition of unity subordinate to the coordinate patches $\mathscr{U}_0, \ldots, \mathscr{U}_n$ discussed in Section 3. On each patch, say on $\mathscr{U}_1$, we have in terms of local coordinates in the patch, say in $\mathscr{U}_1$

$$\int_{\mathscr{U}_1} \chi(\tilde{u} - \tilde{v}_j) \, dd^c \tilde{u}_{1,j} \wedge \cdots \wedge dd^c \tilde{u}_{n,j}$$

where χ is a smooth function with compact support in $\mathscr{U}_1$. This integral converges to $\displaystyle\int_{\mathscr{U}_1} \chi(\tilde{u} - \tilde{v}) \, dd^c \tilde{u}_1 \wedge \cdots \wedge dd^c \tilde{u}_n$ by the continuity of the Monge-Ampère operator on monotone limits of bounded psh functions.

Combining this lemma with the Green formula yields the next Proposition.

PROPOSITION 6.4. − *Suppose u_j, $u \in \mathscr{L}_+$ and u_j increases to u for almost all $z \in \mathbb{C}^n$. Then for $0 \leqslant k \leqslant n$*

$$\lim_{j \to +\infty} \int_{\mathbb{C}^n} u(dd^c u_j)^{n-k} \wedge (dd^c u)^k = \int_{\mathbb{C}^n} u(dd^c u)^n$$

$$+ \sum_{\ell=0}^{k-1} \int_{\mathbb{C}^{n-1}} (\tilde{u}(0,t) - w(t))(dd^c w(t))^{n-1-k-\ell} \wedge (dd^c \tilde{u}(0,t))^{k+\ell}$$

where $w(t) \in \mathscr{L}_+(\mathbb{C}^{n-1})$ is the usc regularization of $\displaystyle\lim_{j \to +\infty} \tilde{u}_j(0,t)$.

Proof. − From the Green formula, or, equivalently, the identity (6.3), and the symmetry of $M(v, w_1, \ldots, w_n)$ in $w_1, \ldots, w_n$, we have

$$(6.7) \quad \int_{\mathbb{C}^n} u(dd^c u_j)^{n-k} \wedge (dd^c u)^k = \int_{\mathbb{C}^n} u(dd^c u)^n$$

$$+ \sum_{\ell=1}^{k} \int_{\mathbb{C}^n} (u_j - u)(dd^c u_j)^{n-k-\ell} \wedge (dd^c u)^{k+\ell}$$

$$\sum_{\ell=0}^{k-1} \int_{\mathbb{C}^{n-1}} (\tilde{u}(0,t) - \tilde{u}_j(0,t))(dd^c \tilde{u}_j(0,t))^{n-1-k-\ell} \wedge (dd^c \tilde{u}(0,t))^{k+\ell}.$$

When $j \to +\infty$, Lemma 6.3 implies that the second term on the right hand side of (6.7) tends to zero and the third term converges to the corresponding integrals with $\tilde{u}_j(0,t)$ replaced by $w(t)$, as asserted.

We also need the following potential theoretic lemma (see e.g. [Car], Theorem 1, p. 15 for the case $n = 1$).

LEMMA 6.5. — *Suppose* $u \in \mathscr{L}$, $v \in \mathscr{L}_+$ *and* $u \leqslant v$ *for* $(dd^c v)^n$-*almost all points in the support of* $(dd^c v)^n$. *Then* $u \leqslant v$ *on* $\mathbb{C}^n$.

Proof. — It is no loss of generality to suppose that

$$(6.8) \qquad v(z) \geqslant \frac{1}{2} \log (2 + |z|^2).$$

Suppose by way of contradiction that $u(z_0) > v(z_0)$ for some $z_0 \in \mathbb{C}^n$. Select $\varepsilon > 0$, $\delta > 0$ such that $\delta < \varepsilon/2$ and the set

$$S = S_{\varepsilon, \delta} = \left\{ z \in \mathbb{C}^n : u(z) + \frac{\delta}{2} \log (2 + |z|^2) > (1 + \varepsilon) v(z) \right\}$$

contains z_0. Then S has positive Lebesgue measure, and S is bounded, since $u \in \mathscr{L}$, $v \in \mathscr{L}_+$ and $\delta < \varepsilon$. Thus, by the comparison inequality, Theorem 4.2 of [BT 1], we have

$$(6.9) \qquad 0 < \int_S \left\{ dd^c \left[u(z) + \frac{\delta}{2} \log (2 + |z|^2) \right] \right\}^n \leqslant (1 + \varepsilon)^n \int_S (dd^c v)^n.$$

However, for almost all points in $S \cap \mathrm{spt}\,(dd^c v)^n$, we have

$$(1 + \varepsilon) v(z) \leqslant u(z) + \frac{\delta}{2} \log (2 + |z|^2) \leqslant v(z) + \frac{\delta}{2} \log (1 + |z|^2),$$

or $v(z) + \frac{1}{4} \log (2 + |z|^2)$ (since $\delta < \varepsilon/2$), contrary to the normalization (6.8). Thus, the right hand side of (6.9) vanishes, which is a contradiction. Therefore, $u \leqslant v$ on $\mathbb{C}^n$.

We can now give good conditions which relate the continuity of ρ_u^* and M under increasing limits.

THEOREM 6.6. — *Let* u_j, $u \in \mathscr{L}_+$ *and suppose* u_j *increases to* u *for almost all* $z \in \mathbb{C}^n$. *Then the following are equivalent :*

(i) $\rho_{u_j}^*$ *increases to* ρ_u^* *for almost all points in* $\mathbb{P}^{n-1}$;

(ii) *for all* $0 \leqslant k \leqslant n$ *and all* $v \in \mathscr{L}_+$,

$$\lim_{j \to \infty} \int (v \pm u_j)(dd^c u_j)^{n-k} \wedge (dd^c u)^k = \int_{\mathbb{C}^n} (v \pm u)(dd^c u)^n;$$

(iii) *for all* $0 \leqslant k \leqslant n$ *and each* $\varepsilon > 0$, *there exists* $R > 0$ *such that*

$$\int_{|z| > R} \log (1 + |z|)(dd^c u_j)^{n-k} \wedge (dd^c u)^k < \varepsilon;$$

(iv) *there exists* $v \in \mathscr{L}_+$ *such that* $\displaystyle\lim_{j \to +\infty} \int_{\mathbb{C}^n} v(dd^c u_j)^n = \int_{\mathbb{C}^n} v(dd^c u)^n$

(v) $\displaystyle\lim_{j \to +\infty} \int_{\mathbb{C}^n} u(dd^c u_j)^n = \int_{\mathbb{C}^n} u(dd^c u)^n.$

Proof. — (i) $\Rightarrow$ (ii). Because of Lemma 6.3 it is enough to prove

$$\lim_{j \to +\infty} \int_{\mathbb{C}^n} u(dd^c u_j)^{n-k} \wedge (dd^c u)^k = \int_{\mathbb{C}^n} u(dd^c u)^n.$$

However, since $\rho_u^*(t) = \tilde{u}(0,t) - \dfrac{1}{2} \log (1 + |t|^2)$, it follows from $\rho_{u_j}^* \nearrow \rho_u^*$ that in Proposition 6.4, $w(t) = \tilde{u}(0,t)$, so (ii) follows.

(ii) $\Rightarrow$ (iii). This is a standard fact from real analysis. Write $\mu_j = (dd^c u_j)^{n-k} \wedge (dd^c u)^k$ and $\mu = (dd^c u)^n$ so that, with $f(z) = \log (1 + |z|)$, we have $f d\mu_j \to f d\mu$ weakly as Borel measures on $\mathbb{C}^n$. If χ_R is a continuous function with $0 \leqslant \chi_R \leqslant 1$, $\chi_R = 0$ for $|z| \geqslant R$ and $\chi_R = 1$ for $|z| < R - 1$, then

$$(6.10) \qquad \int \chi_R f d\mu_j = \int f d(\mu_j - \mu) + \int \chi_R f d\mu + \int (1 - \chi_R) f d(\mu_j - \mu).$$

If $\varepsilon > 0$ is given, choose R_0 so large that the second term on the right hand side of (6.10) does not exceed $\varepsilon/3$. Then choose $J = J(R_0)$ so large that the other two terms on the right hand side of (6.10) do not exceed $\varepsilon/3$ for $j > J$. Finally, choose $R_1 > R_0$ so that $\int \chi_R f d\mu_j < \varepsilon/3$ for the finitely many $j = 1, 2, \ldots, J$, with $R > R_1$. Then (iii) holds for all $R > R_1$.

(iii) $\Rightarrow$ (iv) With the notation of the previous paragraph,

$$\limsup_{j \to -\infty} \left| \int v \, d(\mu_j - \mu) \right| \leqslant \limsup_{j \to +\infty} \left| \int (1 - \chi_R) v \, d(\mu_j - \mu) \right| + \sup_j \left| \int \chi_R v \, d\mu_j \right|.$$

If R is large, then the last term is small because of (iii), while the lim sup of the other term is zero, since $u\mu_j \to u\mu$ weakly as Borel measures on $\mathbb{C}^n$. Thus, (iv) is proved.

(iv) $\Rightarrow$ (v) We have, with $v \in \mathcal{L}_-$ as in (iv)

$$\int_{\mathbb{C}^n} u(dd^c u_j)^n = \int_{\mathbb{C}^n} v(dd^c u_j)^n + \int_{\mathbb{C}^n} (u - v)(dd^c u_j)^n.$$

The limit of the last term is $\int_{\mathbb{C}^n} (u - v)(dd^c u)^n$, by Lemma 6.3 (take $v_j \equiv v$) so by hypothesis (iv), the limit is $\int_{\mathbb{C}^n} u(dd_c u)_n$.

(v) $\Rightarrow$ (i). By Proposition 6.4, we have

$$\lim_{j \to -\infty} \int_{\mathbb{C}^n} u(dd^c u)^n = \int_{\mathbb{C}^n} u(dd^c u)^n$$

$$+ \sum_{\ell=0}^{n-1} \int_{\mathbb{C}^{n-1}} [\tilde{u}(0,t) - w(t)][dd^c w(t)]^{n-1-\ell} \wedge [dd^c \tilde{u}(0, t)]^\ell.$$

Thus, since $w(t) = \left[\lim_{j \to \infty} \tilde{u}_j(0;t) \right]^* \leqslant \tilde{u}(0,t)$, under hypothesis (iv) every term in the sum must vanish. In particular, the term with $\ell = 0$ vanishes, so $\tilde{u}(0,t) \leqslant w(t)$ for $[dd^c w(t)]^{n-1}$ almost all t on $\mathbb{C}^{n-1}$. Because $w(t) \in \mathcal{L}_+(\mathbb{C}^{n-1})$, we conclude from Lemma 6.5 that $\tilde{u}(0,t) \leqslant w(t)$. That is $w(t) = \tilde{u}(0,t)$, or, equivalently, $\rho_{u_j}^*$ increases to ρ_u^* for almost all points in $\mathbb{P}^{n-1}$. This completes the proof.

There are two immediate corollaries of Theorem 6.6.

COROLLARY 6.7. — *Suppose $u_j, u \in \mathcal{L}_+$, that $u_j(z)$ increases to $u(z)$ for almost all $z \in \mathbb{C}^n$, and that there exists a compact set $K \subset \mathbb{C}^n$ such that $(dd^c u_j)^n$ is supported in K for all $j = 1, 2, \ldots$. Then $\rho_{u_j}^*(z)$ increases to $\rho_u^*(z)$ quasi-everywhere in $\mathbb{P}^{n-1}$; i.e. except for z in a polar subset of $\mathbb{P}^{n-1}$.*

Proof. – It is already known that $u(dd^c u_j)^n$ converges weakly to $u(dd^c u)^n$ as Borel measures on $\mathbb{C}^n$ (see e.g. [BT 1], Theorem 7.4). Thus, if all the measures are supported in K, then (iv) of Theorem 6.6 holds, so condition (i) does also.

We also obtain the following result of Kolodziej (see Section 1 for the definition of the capacity C).

COROLLARY 6.8. – *If $K_1 \supset K_2 \supset \cdots$ are compact sets in $\mathbb{C}^n$ and if*

$$K = \bigcap_{j=1}^{\infty} K_j, \ \text{then}$$

$$\lim_{j \to \infty} C(K_j) = C(K).$$

Proof. – This results directly from Corollary 6.7; (the case when $\sup \rho^*_{u_K} = +\infty$ follows from Corollary 6.2).

We next study the sense in which the Robin function ρ^*_u determines u. In particular, for K a compact subset of $\mathbb{C}^n$, let

$$L_K(z) = \sup \{v(z) : v \in \mathscr{L}, v(z) \leqslant 0 \text{ for } z \in K\}$$

denote the extremal function of K and L_K^* its uppersemicontinuous regularization. Then either K is pluripolar ($L_K^* \equiv +\infty$) or else $L_k^* \in \mathscr{L}_+$ [Si 1]. Let ρ_K^* denote the Robin function associated to L_K^*. We want to decide if ρ_K^* determines the compact set K.

Now, this is clearly false in general. If one translates K, then ρ_K^* doesn't change. Also, if K is changed by a pluripolar set, then ρ_K^* is unchanged. And, if

$$\hat{K} = \text{polynomially convex hull of } K$$

then $L_{\hat{K}}^* = L_K^*$, so $\rho_{\hat{K}}^* = \rho_K^*$. However, we do have the following comparison.

THEOREM 6.9. – *Let K_1, K_2 be nonpolar compact sets in $\mathbb{C}^n$ with $K_1 \subset K_2$. If $\rho^*_{K_1} \equiv \rho^*_{K_2}$, then $L^*_{K_1} \equiv L^*_{K_2}$. Further, there is a polar set E such that $\hat{K}_1 = \hat{K}_2 \backslash E$.*

Proof. – It is no loss of generality to assume $K_i = \hat{K}_i$. If we show $L^*_{K_1} = L^*_{K_2}$, then $K_1 = K_2 \backslash E$, since $\{z \in \mathbb{C}^n : L_K^*(z) = 0\}$ is equal to $\hat{K}_1$, up to a pluripolar set. We have $L^*_{K_1} \leqslant L^*_{K_2}$ and $L^*_{K_2}(z) = 0$ for $z \in K_2$,

except possibly for z in a pluripolar set; in particular, a set of $(dd^c L^*_{K_2})^n$-measure zero. Thus we have

$$0 \leqslant \int_{\mathbb{C}^n} L^*_{K_1}(dd^c L^*_{K_2})^n = \int_{\mathbb{C}^n} L^*_{K_1}(dd^c L^*_{K_2})^n - \int_{\mathbb{C}^n} L^*_{K_2}(dd^c L^*_{K_1})^n.$$

By Theorem 6.1, we see that the last expression is $\leqslant 0$. Hence $L^*_{K_1} = 0 = L^*_{K_2}$ almost everywhere with respect to the measure $(dd^c L^*_{K_2})^n$. So, by Lemma 6.5, $L^*_{K_1} \leqslant L^*_{K_2}$, which implies the two functions are equal. This completes the proof.

7. Applications to polar sets.

In this section, we show that the class

$$\mathscr{L}_\rho = \{ u \in \mathscr{L} : \rho^*_u \not\equiv -\infty \}$$

is rich enough to deal with (pluri-) polar sets in $\mathbb{C}^n$. Recall that a subset E of $\mathbb{C}^n$ is a polar set if there exists a psh function U on $\mathbb{C}^n$ such that U is not identically $-\infty$, and $E \subset \{U = -\infty\}$. It was shown by Josefson [J] that the concepts of « locally » and « globally » polar sets coincide. Siciak [Si 1] showed that if E is polar, then it is $\mathscr{L}$-polar; that is, $E \subset \{u = -\infty\}$ for some $u \in \mathscr{L}$, a psh function of logarithmic growth. Here we obtain a more precise result.

THEOREM 7.1. — *If $E \subset \mathbb{C}^n$ is polar, then there exists $u \in \mathscr{L}_\rho$ with $E \subset \{u = -\infty\}$.*

Note that, in general, it is not possible to take u such that $\rho_u(\alpha) > -\infty$ for all $\alpha \in \mathbb{P}^{n-1}$. For instance, if E contains a nonpolar portion of a complex line α, then $\rho_u(\alpha) = -\infty$ for this $\alpha \in \mathbb{P}^{n-1}$. Siciak [Si 3] has also proved that Theorem 7.1 is a consequence of the theorem of Kolodziej when E is an F_σ-set (see Corollary 6.8).

Theorem 7.1 and its proof also give other information about polar sets. A polar set E in $\mathbb{C}^n$ is *complete* if it has the form $E = \{U = -\infty\}$ for some U psh on $\mathbb{C}^n$.

THEOREM 7.2. — *If E is a complete polar set in $\mathbb{C}^n$, then there exists $u \in \mathscr{L}_\rho$ such that $E = \{u = -\infty\}$.*

With the added hypothesis that E is an F_σ-set, it was shown by Souhail [Sou] and Zeriahi [Ze] that $E = \{u = -\infty\}$ for some $u \in \mathcal{L}$.

For polar sets which are not complete, we obtain information about how they propagate. Given E, a polar set in $\mathbb{C}^n$, define

$$E^* = \cap \{U = -\infty\}$$

where the intersection over all psh functions U on $\mathbb{C}^n$ with $E \subset \{U = -\infty\}$. It is evident that $E^{**} = E^*$.

In general, E^* can be dense in $\mathbb{C}^n$, even when E is compact, and E^* need not be a polar set. When the set E^* is an F_σ and a G_δ it was shown by Zeriahi [Ze] that it is a complete polar set. It also follows from our results that the propagation of E is restricted.

THEOREM 7.3. — *If H is a smooth algebraic hypersurface in $\mathbb{P}^n$ and if E is a polar set disjoint from H, then H is not a subset of E^*.*

Before proving these results, we record some simple equivalences for determining when $u \in \mathcal{L}_\rho$.

LEMMA 7.4. — *Let $u \in \mathcal{L}$. Then the following are equivalent :*

(i) $u \in \mathcal{L}_\rho$; i.e. $\rho_u^* \not\equiv -\infty$.

(ii) $\inf\limits_r \int_{\alpha = 1} u(r\alpha)\, d\sigma(\alpha) - \log r > -\infty$, *where $d\sigma(\alpha)$ denotes surface area on $|\alpha| = 1$, normalized so that* $\int_{|\alpha| = 1} d\sigma(\alpha) = 1$.

(iii) $\lim\limits_{r \to \infty} \int_{\alpha = 1} |u(r\alpha) - \log r| d\sigma(\alpha) < +\infty$.

(iv) $\int_{\alpha = 1} u(r\alpha)\, d\sigma(\alpha) - \log r$ *decreases to* $(2\pi)^{-n+1} \int_{\mathbb{P}^{n-1}} \rho_u^* \Omega^{n-1}$, *as* $r \to +\infty$.

Proof. — These are easy facts from potential theory. When $n = 1$, one dimensional potential theory applied to the subharmonic function $u_\alpha(\zeta) = u(\zeta\alpha)$, $\zeta \in \mathbb{C}$, $\alpha \in \mathbb{C}^n$, $|\alpha| = 1$, shows that

$$\lim_{r \to \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} u_\alpha(re^{i\theta})\, d\theta - \log r = \rho_\alpha(u) > -\infty$$

if and only if

$$(7.1) \qquad u_\alpha(\zeta) = \rho_\alpha + \frac{1}{2\pi} \int_{t \in \mathbb{C}} \log |\zeta - t| \, \Delta u_\alpha(t), \quad \zeta \in \mathbb{C}$$

where Δu_α is the positive Borel measure on $\mathbb{C}$ of total mass 2π given by the Laplacian of u_α. In particular, from this representation it follows that

$$(7.2) \qquad \rho_u(\alpha) = \rho_\alpha.$$

For the proof of the Lemma, (ii) $\Leftrightarrow$ (iii) follows directly from Jensen's formula. The limit in (iv) always exists, because $r \to \int_{|\alpha|=1} u(r\alpha) \, d\sigma(\alpha)$ is a convex function of $\log r$. In fact, it is a decreasing limit, which is equal to $(2\pi)^{-(n-1)} \int_{\mathbb{P}^{n-1}} \rho_u^* \Omega^{n-1}$ by the monotone convergence theorem and (7.2), because

$$\int_{|\alpha|=1} u(r\alpha) \, d\sigma(\alpha) - \log r = \int_{|\alpha|=1} \frac{1}{2\pi} \int_{-\pi}^{\pi} (u(re^{i\theta}\alpha) \, d\theta \, d\sigma(\alpha).$$

We omit the details.

Proof of Theorem 7.1. — Suppose $E \subset \mathbb{C}^n$ is pluripolar. By Siciak's theorem, there exists $\psi \in \mathscr{L}$ such that $E \subset \{\psi = -\infty\}$ and $\psi(z) \leqslant \log^+ |z|$. Consider the psh functions

$$\psi_{\varepsilon,j}(z) = \max \{\varepsilon\psi(z) + (1-\varepsilon)\log^+ |z|, -2^{-j}\}$$

and the balayage of $\psi_{\varepsilon,j}$ from the ball $|z| \leqslant j$,

$$w_{\varepsilon,j}(z) = \sup \{u(z) : u \in \mathscr{L}, u(z) \leqslant \psi_{\varepsilon,j}(z) \text{ for } |z| \leqslant j\}.$$

Let $w_{\varepsilon,j}^*$ denote the uppersemicontinuous regularization of $w_{\varepsilon,j}$. Then $w_{\varepsilon,j}^* \in \mathscr{L}_+$ and has the following properties:

    (i) $w_{\varepsilon,j}^*$ is increasing as $\varepsilon \searrow 0$;

    (ii) $\varepsilon\psi(z) + (1-\varepsilon)\log^+ |z| \leqslant w_{\varepsilon,j}^* \leqslant \log^+ |z|$;

    (iii) $w_{\varepsilon,j}^*(z) = w_{\varepsilon,j}(z)$ if $|z| \leqslant j$;

    (iv) $(dd^c w_{\varepsilon,j}^*)^n$ is supported in $|z| \leqslant j$.

Assertions (i), (ii), (iii) are clear from the definition of $w_{\varepsilon,j}^*$, while (iv)

follows from the method of Section 9 of [BT 1] (see e.g. Corollaries 9.3, 9.4, p. 32).

We therefore have that $w_{\varepsilon,j}^*(z) \nearrow \log^+ |z|$ for almost all $z \in \mathbb{C}^n$ as $\varepsilon \searrow 0$. Because of (iv) and Corollary 6.7, it follows that $\rho_{\varepsilon,j}^* \nearrow 0$ at almost all points of $\mathbb{P}^{n-1}$ and, by the bounded convergence theorem,

$$\lim_{\varepsilon \to 0} \int_{\mathbb{P}^{n-1}} (-\rho_{\varepsilon,j}^*)\Omega^{n-1} = 0.$$

Choose $\varepsilon = \varepsilon_j \searrow 0$ such that

$$\int_{\mathbb{P}^{n-1}} (-\rho_{\varepsilon,j}^*)\Omega^{n-1} = \int_{\mathbb{P}^{n-1}} |\rho_{\varepsilon,j}^*|\Omega^{n-1} < 1.$$

Let $w_j = w_{\varepsilon,j}^*$ for $\varepsilon = \varepsilon_j$ and define

$$w(z) = \sum_{j=1}^{\infty} 2^{-j}w_j(z).$$

Then (from (ii)), w is psh and $w \in \mathscr{L}$. If $|z| < j$ and $\psi(z) = -\infty$, then $w_j(z) = \psi_{\varepsilon_j,j}(z) = -2^j$, by (iii). Hence, $w(z) = -\infty$ on the set $\psi(z) = -\infty$. Finally, we have $w \in \mathscr{L}_\rho$ by (ii) of Lemma 7.4, because by (iv) of that Lemma,

$$\int_{\alpha=1} w(R\alpha)\,d\sigma(\alpha) - \log R = \sum_{j=1}^{\infty} 2^{-j} \int_{|\alpha|=1} (w(R\alpha) - \log R)\,d\sigma(\alpha)$$

$$\geq \sum_{j=1}^{\infty} 2^{-j}(2\pi)^{-n+1} \int_{\mathbb{P}^{n-1}} \rho_j^*\Omega^{n-1} > -(2\pi)^{-n+1}.$$

This completes the proof.

COROLLARY 7.5. — *If E is a complete $\mathscr{L}$-polar set then E is a complete $\mathscr{L}_p$-polar set.*

Proof. — By (ii) of the proof of Theorem 7.1, the function $w(z)$ constructed there has the property,

$$w(z) \geq \left(\sum_{j=1}^{\infty} 2^{-j}\varepsilon_j \right)\psi(z).$$

Hence, if E is a complete $\mathscr{L}$-polar set, i.e. $E = \{\psi = -\infty\}$, then $E = \{w = -\infty\}$ is also a complete $\mathscr{L}_p$-polar set.

PROOF OF THEOREM 7.2. — We have $E = \{U = -\infty\}$ for some U psh on $\mathbb{C}^n$. By Corollary 7.5, it suffices to construct $\psi \in \mathcal{L}$ such that $E = \{\psi = -\infty\}$. Choose $R_j \to +\infty$ so fast that $\sum_{j=1}^{\infty} 2^{-j} \log R_j = +\infty$ and set

$$K_j = \{z \in \mathbb{C}^n : |z| \leqslant R_j, \ U(z) \geqslant -j\}.$$

Choose c_j so large that

(7.3) $U(z) < c_j$ for $|z| \leqslant R_j$

and then choose ε_j so small that

(7.4) $\varepsilon_j(U(z) - c_j) \geqslant -1,$ $z \in K_j$

(7.5) $\displaystyle \varepsilon_j \int_{|z| < R_j} (c_j - U) < 1.$

Define

$$\psi_j(z) = \begin{cases} \max\{\varepsilon_j(U(z) - c_j), \log^+|z| - \log R_j\}, & |z| \leqslant R_j \\ \log|z| - \log R_j & |z| > R_j. \end{cases}$$

Then ψ_j is psh and $\psi_j < 0$ for $|z| < R_j$. Further,

$$\int_{|z| < R_j} |\psi_j| \leqslant \varepsilon_j \int_{|z| < R_j} (c_j - U(z)) < 1.$$

Thus

$$\psi(z) = \sum_{j=1}^{\infty} 2^{-j} \psi_j(z)$$

is psh, since the sum is locally convergent in L^1 and since, on any compact set, the partial sums of the series are eventually decreasing. Note that if $U(z) = -\infty$ and $|z| < R_j$, then $\psi_j(z) = \log^+|z| - \log R_j$ so that $\psi(z) = -\infty$, since $\sum 2^{-j} \log R_j = +\infty$. Also, $\psi(z) > -\infty$ if $z \notin E$, since if $U(z) \geqslant -J$, $|z| \leqslant R_J$, then

$$\psi(z) = \sum_{j=1}^{\infty} 2^{-j} \psi_j(z) \geqslant \psi(z) = \sum_{j=1}^{J-1} 2^{-j} \psi_j(z) + \sum_{j=J}^{+\infty} 2^{-j} > -\infty.$$

It remains to show that $\psi \in \mathcal{L}$. But, $\psi_j(z) \leqslant 0$ if $|z| \leqslant R_j$. And, if $|z| > R_j$; then $\psi_j(z) = \log|z| - \log R_j \leqslant \log^+|z|$, so $\psi(z) \leqslant \log^+|z|$. This completes the proof.

Proof of Theorem 7.3. — We will only give the proof in case H is an affine hyperplane. The general case results from applying the integral formula with the hyperplane replaced by the hypersurface. We can choose coordinates $(s,t) = (s,t_2,\ldots,t_n)$ on $\mathbb{C}^n$ so that the hyperplane is $\{s=0\}$. Then the analytic change of coordinate of Section 3, $z_1 = 1/s$, $(z_2,\ldots,z_n) = t/s$ maps $\{s=0\}$ into the hyperplane at infinity in $\mathbb{P}^n$ and E into a pluripolar set $E_1 \subset \mathbb{C}^n$. Then choose $u \in \mathscr{L}_\rho$, $u = -\infty$ on E_1. The function $\tilde{u}(s,t) = u(1/s, t/s) + \log|s|$ is $-\infty$ on E and $\tilde{u}(0,t) = \rho_u^*(t) + \dfrac{1}{2}\log(1+|t|^2)$ is not $\equiv -\infty$. Thus, $\{s = 0\}$ is not a subset of E^*.

BIBLIOGRAPHY

[AT] H. ALEXANDER and B. A. TAYLOR, Comparison of two capacities in $\mathbb{C}^n$, Math. Z., 186 (1984), 407-417.

[BT 1] E. BEDFORD and B. A. TAYLOR, A new capacity for plurisubharmonic functions, Acta Math., 149 (1982), 1-40.

[BT 2] E. BEDFORD and B. A. TAYLOR, Fine topology, Silov boundary, and $(dd^c)^n$, J. Funct. Anal., 72 (1987), 225-251.

[Car] L. CARLESON, Selected Problems in Exceptional Sets, Van Nostrand Math. Studies # 13, D. Van Nostrand (1968).

[Ceg] U. CEGRELL, An estimate of the complex Monge-Ampere operator. Analytic functions. Proceedings, Blazejwko 1982. Lecture Notes in Math.,1039, pp. 84-87. Berlin, Heidelberg, New York, Springer, 1983.

[D 1] J.-P. DEMAILLY, Mesure de Monge-Ampère et caractérisation des variétés algébriques affines, mémoire (nouvelle série) n° 19, Soc. Math. de France, 1985.

[D 2] J.-P. DEMAILLY, Mesures de Monge-Ampère et mesures pluriharmoniques, Math. Z., 194 (1987), 519-564.

[F] H. FEDERER, Geometric Measure Theory, Springer-Verlag, New York, 1969.

[J] B. JOSEFSON, On the equivalence between locally polar and globally polar sets for plurisubharmonic functions on $\mathbb{C}^n$, Ark. Math., 16 (1978), 109-115.

[Km] M. KLIMEK, Extremal plurisubharmonic functions and invariant pseudo-distances, Bull. Soc. Math. France, 113 (1985), 123-142.

[K 1] S. KOLODZIEJ, Logarithmic capacity in $\mathbb{C}^n$, to appear in Ann. Pol. Math.

[K 2] S. KOLODZIEJ, On capacities associated to the Siciak extremal function (preprint).

[Lp 1] L. LEMPERT, La métrique de Kobayashi et la représentation des domaines sur la boule, Bull. Soc. Math. France, 109 (1981), 427-474.

[Lp 2] L. LEMPERT, Solving the degenerate Monge-Ampere equation with one concentrated singularity, Math. Ann., 263 (1983), 515-532.

[L] N. LEVENBERG, Capacities in several complex variables, Dissertation, The University of Michigan, 1984.

[Sa] A. SADULLAEV, Plurisubharmonic measures and capacities on complex manifolds, Russian Math. Surveys, 36 (1981), 61-119.

[Si 1] J. SICIAK, Extremal plurisubharmonic functions in $\mathbb{C}^n$, Ann. Polon. Math., 39 (1981).

[Si 2] J. SICIAK, Extremal plurisubharmonic functions and capacities in $\mathbb{C}^n$, Sophia University, Tokyo, 1982.

[Si 3] J. SICIAK, On logarithmic capacities and pluripolar sets in $\mathbb{C}^n$ (preprint).

[So] S. SOUHAIL, Relations entre différentes notions d'ensembles pluripolaires complets dans $\mathbb{C}^n$, Thèse, L'Université Paul Sabatier de Toulouse, 1987.

[T] B. A. TAYLOR, An estimate for an extremal plurisubharmonic function on $\mathbb{C}^n$, Séminaire P. Lelong, P. Dolbeault, H. Skoda, 1982-1983, Lectures Notes in Math., 1028 (1983), 318-328.

[Z] V. P. ZAHARJUTA, Transfinite diameter, Čebyšev constants, and capacity for compacta in $\mathbb{C}^n$, Math. USSR Sbornik, 25 (1975), 350-364.

[Ze] A. ZERIAHI, Ensembles pluripolaires exceptionnels pour croissance partielle des fonctions holomorphes, preprint.

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