ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 39, n° 1 (1989), p. 155-192
<http://www.numdam.org/item?id=AIF_1989__39_1_155_0>

Ann. Inst. Fourier, Grenoble
39, 1 (1989), 155-192

ASYMPTOTIC BEHAVIOR OF SCATTERING AMPLITUDES IN SEMI-CLASSICAL AND LOW ENERGY LIMITS

by D. ROBERT and H. TAMURA

0. Introduction.

In the present paper we study the asymptotic behavior of scattering amplitudes of Schrödinger operators is semi-classical and low energy limits.

Consider the Schrödinger operator $H(h) = -(1/2)h^2\Delta + V, 0 < h \leq 1$, in the n-dimensional space $R^n_x, n \geq 2$. We assume the potential $V(x)$ to satisfy the following condition.

Assumption $(V)_\rho$. — $V(x)$ is a real C^∞-smooth function and satisfies

$$|\partial^\alpha_x V(x)| \leq C_\alpha < x >^{-\rho-|\alpha|}, \quad \rho > 1,$$

for any multi-index α, where $< x >= (1 + |x|^2)^{1/2}$.

Under the above assumption, $H(h)$ admits a unique self-adjoint realization in $L^2 = L^2(R^n_x)$ with domain $D(A) = H^2(R^n_x), H^2(R^n_x)$ being the Sobolev space of order 2. We denote by the same notation $H(h)$ this realization. Assumption $(V)_\rho$ also enables us to define the scattering matrix $S(\lambda; h)$ with energy $\lambda > 0$ as a unitary operator acting on $L^2(S^{n-1}), S^{n-1}$ being the $(n-1)$-dimensional unit sphere. The scattering matrix $S(\lambda; h)$ takes the form

$$S(\lambda; h) = \mathrm{Id} - (2\pi i)T(\lambda; h)$$

with T called T-matrix. We know ([2], [9]) that $T(\lambda; h)$ is an integral operator and the kernel $T(\theta, \omega; \lambda, h)$ is smooth in (θ, ω), $\theta \neq \omega$. Then

Key-words : Schrödinger operators – Scattering theroy – Scattering amplitudes – Semi-classical and low energy limits.

A.M.S Classification : 35P20

the scattering amplitude $f(\omega \to \theta; \lambda, h)$ with the initial direction ω and the final one θ is defined by

$$(0.1) \qquad f(\omega \to \theta; \lambda, h) = c(\lambda, h) T(\theta, \omega; \lambda, h)$$

with

$$c(\lambda, h) = -2\pi(2\lambda)^{-(n-1)/4}(2\pi h)^{(n-1)/2} \exp(-i(n-3)(\pi/4)).$$

If $V(x)$ satisfies $(V)_\rho$ with $\rho > (n+1)/2$, then the scattering amplitude is usually defined through the asymptotics as $|x| \to \infty$ of the outgoing eigenfunction of $H(h)$. We can easily see ([5], [15]) that $f(\omega \to \theta; \lambda, h)$ is related to the kernel $T(\theta, \omega; \lambda, h)$ through relation (0.1). The quantity $|f(\omega \to \theta; \lambda, h)|^2$ is called the differential cross section, which is observable through actual physical experiments and is one of the most fundamental quantities in scattering theory. One aim of this paper is to study the asymptotic behavior of $f(\omega \to \theta; \lambda, h)$ in the semi-classical limit $h \to 0$.

We first formulate the obtained result precisely and give a brief comment on some results related to it. The precise formulation requires many notations, assumptions and definitions.

We begin by reviewing briefly the classical particle scattering. For details, see the book [19]. Assume $(V)_\rho$ with $\rho > 1$. The classical phase trajectory is defined as a solution to the Hamilton system

$$(0.2) \qquad (d/dt)q = p, \quad (d/dt)p = -\nabla_x V(q).$$

If $|q(t)| \to \infty$ as $t \to \pm\infty$, then the particle behaves like a free particle as $t \to \pm\infty$. Thus there exist $(r_\pm, v_\pm) \in R^n \times (R^n \backslash 0)$ such that

$$\lim_{t \to \pm\infty} (|q(t) - v_\pm t - r_\pm| + |p(t) - v_\pm|) = 0.$$

The mapping

$$S_{\mathrm{cl}} : (r_-, v_-) \to (r_+, v_+)$$

is called the scattering transformation in classical mechanics.

We now fix the initial direction $\omega \in S^{n-1}$ arbitrarily. For notational brevity, we take $(0, \dots, 0, 1)$ as the direction ω. We denote by Λ_ω the hyperplane (impact plane) orthogonal to ω and write the coordinates in Λ_ω as $z = (z_1, \dots, z_{n-1})$. We also use the notation $\widehat{z} = (z,0) = (z_1, \dots, z_{n-1}, 0)$ so that $\widehat{z}$ is considered as a position vector in R^n. For given $z \in \Lambda_\omega$, we consider the phase trajectory $\{q_\infty(t; z, \lambda), p_\infty(t; z, \lambda)\}$ asymptotic to $\{\sqrt{2\lambda}\omega t + \widehat{z}, \sqrt{2\lambda}\omega)$ as $t \to -\infty$;

$$
\begin{aligned}
&\lim_{t \to -\infty} |q_\infty(t; z, \lambda) - \sqrt{2\lambda}\omega t - \widehat{z}| = 0, \\
&\lim_{t \to -\infty} |p_\infty(t; z, \lambda) - \sqrt{2\lambda}\omega| = 0
\end{aligned}
\tag{0.3}
$$

in the C^∞-topology for the impact parameter z. Such a trajectory is uniquely defined as the solution to (0.2) with initial condition (0.3) at $t = -\infty$. We assume that energy $\lambda > 0$ under consideration is non-trapping in the following sense.

Non-trapping condition. — Let $\{q(t; y, \eta), p(t; y, \eta)\}$ be the solution to (0.2) with initial state (y, η) at $t = 0$. We say that energy $\lambda > 0$ is non-trapping, if for any $R \gg 1$ large enough, there exists $T = T(R)$ such that $|q(t; y, \eta)| > R$ for $|t| > T$ when $|y| < R$ and $\lambda = (1/2)|\eta|^2 + V(y)$.

The non-trapping condition means that all particles with energy λ go to infinity as $t \to \pm\infty$. Let $\{q_\infty(t; z, \lambda), p_\infty(t; z, \lambda)\}$ be as above. Assume that λ is in non-trapping energy range. Then, by definition, it follows that

$$
\lim_{t \to +\infty} |q_\infty(t; z, \lambda)| = \infty
\tag{0.4}
$$

and hence there exist $\xi_\infty(z; \lambda)$ and $r_\infty(z; \lambda)$ smooth in $z \in \Lambda_\omega$ such that

$$
\begin{aligned}
&\lim_{t \to +\infty} |q_\infty(t; z, \lambda) - \sqrt{2\lambda}\xi_\infty(z; \lambda)t - r_\infty(z; \lambda)| = 0, \\
&\lim_{t \to +\infty} |p_\infty(t; z, \lambda) - \sqrt{2\lambda}\xi_\infty(z; \lambda)| = 0
\end{aligned}
\tag{0.5}
$$

in the C^∞-topology for z. Thus the classical scattering transformation S_{cl} maps $(\widehat{z}, \sqrt{2\lambda}\omega)$ into $(r_\infty(z; \lambda), \sqrt{2\lambda}\xi_\infty(z; \lambda))$. By conservation of energy, $\xi_\infty(z; \lambda)$ takes values in S^{n-1}; $\xi_\infty : \Lambda_\omega \to S^{n-1}$.

Recall the notation $z = (z_1, \dots, z_{n-1})$ (coordinates in Λ_ω). We define the angular density $\widehat{\sigma}(z; \lambda)$ by

$$
\widehat{\sigma}(z; \lambda) = |\det\,(\xi_\infty, (\partial/\partial z_1)\xi_\infty, \dots, (\partial/\partial z_{n-1})\xi_\infty)|
\tag{0.6}
$$

for the phase trajectory $\{q_\infty(t; z, \lambda), p_\infty(t; z, \lambda)\}$ with properties (0.3) and (0.5).

Regular condition. — We say that final direction $\theta \in S^{n-1}, \theta \neq \omega$, is regular for initial direction ω, if for all $z \in \Lambda_\omega$ with $\xi_\infty(z;\lambda) = \theta$,

$$\widehat{\sigma}(z;\lambda) \neq 0. \tag{0.7}$$

Remark. — The regular condition means that the system $\{(\partial/\partial z_1)\xi_\infty, \dots, (\partial/\partial z_{n-1})\xi_\infty\}$ can be taken as a base of the tangent vector space at θ on S^{n-1}.

Assume that final direction θ is regular. By the implicit function theorem, this implies that many particles are not concentrated near the direction θ. Let $\widetilde{\theta} \in S^{n-1}$ be in a small neighborhood of θ. Then there exists only a finite number of $w_j = w_j(\widetilde{\theta};\lambda) \in \Lambda_\omega, 1 \leq j \leq \ell(= \ell(\theta;\lambda))$, such that $\xi_\infty(w_j,\lambda) = \widetilde{\theta}$.

We are now in a position to formulate the first main theorem.

THEOREM 1. — *Fix the initial direction $\omega \in S^{n-1}$ as $\omega = (0,\dots,0,1)$ and keep the above notations. Assume : (i)$(V)_\rho$ with $\rho > 1$; (ii) energy $\lambda > 0$ is in a non-trapping energy range; (iii) final direction $\theta \in S^{n-1}, \theta \neq \omega$, is regular for ω. Let $w_j = w_j(\theta;\lambda) \in \Lambda_\omega, 1 \leq j \leq \ell$, be such that $\xi_\infty(w_j;\lambda) = \theta$. Then the scattering amplitude $f(\omega \to \theta;\lambda,h)$ obeys the following asymptotic formula as $h \to 0$:*

$$f(\omega \to \theta;\lambda,h) = \sum_{j=1}^{\ell} \widehat{\sigma}(w_j;\lambda)^{-1/2} \exp(ih^{-1}S_j - i\mu_j\pi/2) + O(h), \tag{0.8}$$

where

$$S_j = \int_{-\infty}^{\infty} \{|p_\infty(t;w_j,\lambda)|^2/2 - V(q_\infty(t;w_j,\lambda)) - \lambda\}dt - < r_\infty(w_j;\lambda), \sqrt{2\lambda}\theta >, \tag{0.9}$$

$<,>$ being the scalar product in R^n_x, and μ_j is the path index (Keller-Maslov-Morse index) of the phase trajectory $\{q_\infty(t;z,\lambda), p_\infty(;z,\lambda)\}$ with $z = w_j$ on the Lagrangian manifold

$$\{(x,\xi) \in R^n_x \times R^n_\xi \ : \ x = q_\infty(t;z,\lambda), \xi = p_\infty(t;z,\lambda), z \in \Lambda_\omega, t \in R^1\}.$$

Remarks. — (i) Under the same assumptions as in Theorem 1, we can prove that $f(\omega \to \theta;\lambda,h)$ admits an asymptotic expansion in h. (ii) The Lagrangian functions $L_0(q,\dot{q})$ and $L(q,\dot{q})$ associated with the free and interacting systems are defined as $L_0 = |\dot{q}|^2/2$ and $L = |\dot{q}|^2/2 - V(q)$, respectively. Thus the integral in (0.9) may be considered as the difference of actions along the trajectories $\{q_\infty(t; z, \lambda), p_\infty(t; z, \lambda)\}$ and $\{\sqrt{2\lambda}\omega t + \widehat{z}, \sqrt{2\lambda}\omega\}$ with $z = w_j$, which are asymptotic to each other as $t \to -\infty$.

We shall explain the relation between $|f(\omega \to \theta; \lambda, h)|^2$ and the differential cross section $\sigma_{\mathrm{cl}}(\theta; \lambda)$ in classical mechanics in the semi-classical limit $h \to 0$. For initial direction $\omega, \sigma_{\mathrm{cl}}(\theta; \lambda)$ is defined by

$$\sigma_{\mathrm{cl}}(\theta; \lambda) = \sum_{\theta = \xi_\infty(z; \lambda)} \widehat{\sigma}(z; \lambda)^{-1},$$

where the sum is taken over $z \in \Lambda_\omega$ satisfying the relation $\xi_\infty(z; \lambda) = \theta$. Fix a final direction θ_0. Assume that θ_0 is regular and that $\ell = 1$ for this fixed θ_0. Then it follows from the above theorem that $|f(\omega \to \theta_0; \lambda, h)|^2 \to \sigma_{\mathrm{cl}}(\theta_0; \lambda)$ as $h \to 0$. As is expected, this implies that the differential cross section $|f(\omega \to \theta_0; \lambda, h)|^2$ in quantum mechanics is convergent to the classical one $\sigma_{\mathrm{cl}}(\theta_0; \lambda)$ in the limit $h \to 0$. In general case $\ell > 1$, we have the convergence

$$\int_U |f(\omega \to \theta; \lambda, h)|^2 d\theta \to \int_U \sigma_{\mathrm{cl}}(\theta; \lambda) d\theta \tag{0.10}$$

for a small neighborhood U of θ_0, if $\nabla_\theta(S_j - S_k) \neq 0, 1 \leq j < k \leq \ell$, for $\theta \in U$. Unfortunately we do not know whether or not this assumption is really satisfied.

We shall make a brief comment on the results related to the above theorem. The semi-classical asymptotics for scattering amplitudes has been studied by Protas [13] and Vainberg [16] for a class of finite range potentials under the assumption that $\lambda > \sup V(x), V \in C_0^\infty(R_x^n)$, is in a non-trapping energy range. The case of non-compact support has been recently studied by Yajima [18] under assumption $(V)_\rho$ with $\rho > \max\,(1, (n-1)/2)$. Assuming only (0.4) and (0.7) for (λ, θ), he has proved the formula (0.8) with the L^2 error estimate, when $f(\omega \to \theta; \lambda, h)$ is considered as a function of (λ, θ) and hence the convergence (0.10) follows when averaged over energy λ satisfying (0.4). The proof is based on the Enss-Simon formula defining scattering amplitudes in the framework of time-dependent scattering theory ([6]). It should be noted that the strong non-trapping condition is not assumed for energy λ. However it seems to be difficult to prove (0.8) for fixed energy without assuming the non-trapping condition. Roughly speaking, the difficulty comes from the uncertainty principle for the time-energy variables and from the effects of resonances. To prove (0.8) for fixed energy, we require an information on the behavior as $t \to \pm\infty$ of the propagator $\exp(-ih^{-1}tH(h))$. In particular, we have to estimate the local decay as $t \to \pm\infty$ uniformly in h. The non-trapping condition enables us to obtain such an estimate.

The other aim of this paper is to study the asymptotic behavior of scattering amplitudes in the low energy limit for a class of slowly decreasing potentials to which the Born approximation method cannot be directly applied. The argument used in the proof of Theorem 1 applies to such a low energy case.

Let $H = -(1/2)\Delta + V$, where the potential $V(x)$ is assumed to satisfy $(V)_\rho$ with $1 < \rho < 2$. Consider the Schrödinger equation

$$(-1/2)\Delta + V(x) - \lambda)\psi = 0.$$

We now make a change of variables $x \to y = \lambda^{1/\rho}x$ to transform this equation into

$$(-(1/2)\lambda^{2/\rho-1}\Delta + \lambda^{-1}V(\lambda^{-1/\rho}y) - 1)\psi = 0.$$

Define the Hamiltonian $H_\lambda(h)$ as $H_\lambda(h) = -(1/2)h^2\Delta + V_\lambda$ with $h = \lambda^\gamma, \gamma = 1/\rho - 1/2 > 0$, where $V_\lambda = \lambda^{-1}V(\lambda^{-1/\rho}x)$. We denote by $f_\mu(\omega \to \theta; H)$ and $f_\mu(\omega \to \theta; H_\lambda(h)$, the scattering amplitudes at energy $\mu > 0$ of the Hamiltonians H and $H_\lambda(h)$, respectively. Then we can easily prove that

$$f_\lambda(\omega \to \theta; H) = \lambda^{-(n-1)/2\rho} f_1(\omega \to \theta; H_\lambda(h)) \tag{0.11}$$

with h as above. This relation follows from the general representation formula for scattering matrices (Theorem 7.2, [1]).

Thus the argument used in the semi-classical case is applied to study the low energy behavior as $\lambda \to 0$ of $f_\lambda(\omega \to \theta; H)$. We will prove that $f_\lambda(\omega \to \theta; H)$ behaves like $f_\lambda \sim \lambda^{-(n-1)/2\rho}$ as $\lambda \to 0$ for a class of repulsive potentials behaving like

$$V(x) = \Phi(x/|x|)|x|^{-\rho} + o(|x|^{-\rho}); |x| \to +\infty,$$

with $\Phi \in C^\infty(S^{n-1}), \Phi > 0$. The precise formulation will be given in section 6, together with some comments on the related results.

We conclude this section by making some comments on the notations accepted in the present paper. (1) $<,>$ denotes the scalar product in

R^n_x. (2) For given self-adjoint operator A, we denote by $R(\zeta; A)$ the resolvent of A; $R(\zeta; A) = (A-\zeta)^{-1}, \mathrm{Im}\zeta \neq 0$. (3) We denote by $|\ |_0$ and $(,)$ the L^2 norm and scalar product in $L^2 = L^2(R^n_x)$, respectively. (4) We define the weighted L^2 space $L^2_\alpha = L^2_\alpha(R^n_x)$ with the norm $|\ |_\alpha$ by $L^2_\alpha = \{f : |f|_\alpha = | < x >^\alpha f|_0 < \infty\}$. (5) We denote by $\|\ \|_{\alpha,\beta}$ the operator norm when considered as an operator from L^2_α into L^2_β.

1. Representation for scattering amplitudes.

The proof of Theorem 1 is done by applying the stationary phase method to the representation formula for $T(\theta, \omega; \lambda, h)$ obtained by [9]. The aim of this section is to formulate such a formula.

Let $H_0(h) = -(1/2)h^2\Delta$. We denote by $\psi_0(x; \lambda, \omega, h)$ the generalized eigenfunction of $H_0(h)$;

$$\psi_0(x; \lambda, \omega, h) = \exp(ih^{-1}\sqrt{2\lambda} < x, \omega >).$$

By the principle of limiting absorption, we can define the operator $R(\lambda \pm i0; H(h) : L^2_\gamma \to L^2_{-\gamma}, \gamma > 1/2$, by

$$R(\lambda \pm i0; H(h)) = s - \lim_{\kappa \downarrow 0} R(\lambda \pm i\kappa; H(h)) \text{ in } L^2_{-\gamma}.$$

If $V(x)$ satisfies $(V)_\rho$ with $\rho > (n+1)/2$, then the outgoing eigenfunction $\psi_+(|x|\tilde{x}; \lambda, \omega, h), x = |x|\tilde{x}$, of $H(h)$ is given by $\psi_+ = \psi_0 - R(\lambda + i0; H(h))V\psi_0$ and the kernel $T(\theta, \omega; \lambda, h)$ is expressed as

$$T(\theta, \omega; \lambda, h) = c_0(\lambda, h)^2(V\psi_+(\cdot; \lambda, \omega, h), \psi_0(\cdot; \lambda, \theta, h))$$

with

$$c_0(\lambda, h) = (2\pi h)^{-n/2}(2\lambda)^{(n-2)/4}. \tag{1.1}$$

On the other hand, the scattering amplitude $f(\omega \to \theta; \lambda, h)$ is defined through the asymptotic behavior as $|x| \to \infty$ of ψ_+ along the direction θ;

$$\psi_+ = \psi_0 + f(\omega \to \theta; \lambda, h)|x|^{-(n-1)/2}\exp(ih^{-1}\sqrt{2\lambda}|x|)(1 + o(1))$$

and we can easily see that $f(\omega \to \theta; \lambda, h)$ defined in this way satisfies the relation (0.1).However, in the general case $\rho > 1$, we cannot necessarily define $T(\theta, \omega; \lambda, h)$ as above. Thus, the first step toward the proof of

Theorem 1 is to establish the nice representation formula for $T(\theta,\omega;\lambda,h)$ which is well defined even in the case $1<\rho\leq(n+1)/2$. Such a nice formula has been already obtained by [9].

1.1. The precise formulation requires many notations and definitions. We begin by introducing a class of symbols.

DEFINITION. — *For given $\Omega\subset R^n_x\times R^n_\xi$, we denote by $A_m(\Omega)$ the set of all $a(x,\xi),(x,\xi)\in\Omega$, such that*

$$|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)|\leq C_{\alpha\beta L}<x>^{m-|\alpha|}<\xi>^{-L}$$

for any $L\gg 1$. If, in particular, $\Omega=R^n_x\times R^n_\xi$, then we write A_m for $A_m(\Omega)$.

We say that a family of $a(x,\xi;\varepsilon)$ with parameter ε belongs to $A_m(\Omega)$ uniformly in ε, if the constants $C_{\alpha\beta L}$ above can be taken independently of ε. Most of symbols we use in later application have compact support in ξ and hence are of class $A_m(\Omega)$.

We write $\widetilde{x}=x/|x|$ and $\widetilde{\xi}=\xi/|\xi|$. For given triplet $(R,d,\sigma),R\gg 1$, $d>1,-1<\sigma<1$, we introduce the notation

$$\Gamma_\pm(R,d,\sigma)=\{(x,\xi)\ :\ |x|>R,d^{-1}<|\xi|<d,<\widetilde{x},\widetilde{\xi}>\ \gtrless\sigma\}.$$

We now fix d_0 and σ_0, σ_0 being taken close enough to 1.

Then according to the result (Proposition 2.4) of [8], we can construct a real C^∞-smooth function $\phi_\pm(x,\xi)$ with the following properties : (i) $\phi_\pm(x,\xi)$ solves the eikonal equation

$$(1/2)|\nabla_x\phi_\pm(x,\xi)|^2+V(x)=(1/2)|\xi|^2$$

in $\Gamma_\pm(R_0,d_0,\pm\sigma_0)$ for some $R_0\gg 1$; (ii) $\phi_\pm(x,\xi)-<x,\xi>$ belongs to A_0; (iii) For all $(x,\xi)\in R^n_x\times R^n_\xi$

$$|(\partial^2/\partial x_j\partial\xi_k)(\phi_\pm(x,\xi)-\delta_{jk}(|\leq\varepsilon(R_0)<1/2,$$

δ_{jk} being the Kronecker notation, where $\varepsilon(R_0)$ can be made as small as we desire by taking R_0 large enough. Throughout the entire discussion, we fix the triplet (R_0,d_0,σ_0) with the meaning ascribed above.

Let $a(x,\xi)\in A_m$ and $\phi_\pm(x,\xi)$ be as above. Then we define the Fourier integral operator $I_h(a;\phi_\pm):S(R^n_x)\to S(R^n_x)$ by :

$$(I_h(a;\phi_\pm)f)(x) = (2\pi h)^{-n}\iint \exp(ih^{-1}(\phi_\pm(x,\xi) - < y,\xi >))a(x,\xi)f(y)dyd\xi,$$

where the integration with no domain attached is taken over the whole space.

1.2. We can define $F_0(\lambda,h) : L^2_\gamma \to L^2(S^{n-1}), \gamma > 1/2$, by

$$(F_0(\lambda,h)f)(\omega) = c_0(\lambda,h)\int \overline{\psi}_0(x;\lambda,\omega,h)f(x)dx,$$

$c_0(\lambda,h)$ being defined by (1.1). Let $W_\pm(h)$ be the wave operator defined by

$$W_\pm(h) = s - \lim_{t\to\pm\infty} \exp(ih^{-1}tH(h))\exp(-ih^{-1}tH_0(h)).$$

We fix d, $1 < d < d_0$, and σ, $0 < \sigma < \sigma_0$, arbitrarily. Assume that $f \in L^2_\gamma, \gamma > 1/2$, has energy support in $(d^{-2}/2, d^2/2)$; $F_0(\lambda,h)f = 0$ if $\lambda \geq d^2/2$ or $\lambda \leq d^{-2}/2$. Following the argument in [8] (see also [15]), we construct an approximate representation for

$$\exp(-ih^{-1}tH(h))W_\pm(h)f, \quad t \gtrless 0,$$

in the form

$$I_h(c_\pm;\phi_\pm)\exp(-ih^{-1}tH_0(h))f.$$

The symbol $c_\pm = c_\pm(x,\xi;h)$ is determined to satisfy

$$\exp(-ih^{-1}\phi_\pm)(-(1/2)h^2\Delta + V(x) - (1/2)|\xi|^2)\exp(ih^{-1}\phi_\pm)c_\pm \sim 0.$$

We formally set

$$c_\pm(x,\xi;h) \sim \sum_{j=0}^{\infty} c_{\pm j}(x,\xi)h^j$$

and determine $c_{\pm j}$ inductively by solving the transport equation

$$< \nabla_x\phi_\pm, \nabla_x c_{\pm 0} > +(1/2)(\Delta_x\phi_\pm)c_{\pm 0} = 0$$

(1.2)

$$< \nabla_x\phi_\pm, \nabla_x c_{\pm j} > +(1/2)(\Delta_x\phi_\pm)c_{\pm j} = (i/2)\Delta_x c_{\pm j-1}, \quad j \geq 1,$$

with the condition at infinity

(1.3) $$c_{\pm 0} \to 1, \quad c_{\pm j} \to 0, \quad j \geq 1, \text{ as } |x| \to \infty.$$

Since $\nabla_x\phi_\pm = \xi + O(|x|^{-1})$ as $|x| \to \infty$, we can solve (1.2) with (1.3) in $\Gamma_\pm(2R_0, d, \pm\sigma)$ by the standard characteristic curve method.

The solutions $c_{\pm j}(x,\xi)$ are represented as follows. We consider the + case only. Assume that $(x,\xi) \in \Gamma_+(2R_0,d,-\sigma)$. Consider the characteristic curve $r_+(t;x,\xi), t \geq 0$;

$$(d/dt)r_+ = \nabla_x\phi_+(r_+,\xi), r_+(0,;x,\xi) = x.$$

Set

$$F_+(t;x,\xi) = (1/2)\int_t^\infty (\Delta_x\phi_+)(r_+(\tau;x,\xi),\xi)d\tau.$$

Then

$$c_{+0} = \exp(F_+(0;x,\xi))$$

and

$$c_{+j} = c_{+0}(x,\xi)\int_0^\infty -(i/2)(\Delta_x c_{+j-1})r_+(\tau;x,\xi),\xi)\exp(-F_+(\tau;x,\xi))d\tau.$$

If $(x,\xi) \in \Gamma_+(2R_0,d,-\sigma)$, then $r_+(t;x,\xi), t \geq 0$ satisfies

$$|r_+(t;x,\xi| \geq C(1+t+|x|)$$

and

$$|\partial_x^\alpha\partial_\xi^\beta r_+(t;x,\xi)| \leq C_{\alpha\beta} < x >^{1-|\alpha|}, |\alpha| \geq 1.$$

This proves that $c_{+j}(x,\xi)$ belongs to $A_{-j}(\Gamma_+(2R_0,d,-\sigma))$. In a similar way, we can also prove that $c_{-j}(x,\xi)$ belongs to $A_{-j}(\Gamma_-(2R_0,d,+\sigma))$.

We can extend $c_{\pm j}$ obtained in this way to the whole space $R^n_x \times R^n_\xi$ so that : (c.0) $c_{\pm j} \in A_{-j}$; (c.1) supp $c_{\pm j} \subset \Gamma_\pm(R_0,d_0,\pm\sigma_0)$; (c.2) $c_{\pm j}$ solves (1.2) with (1.3) in $\Gamma_\pm(2R_0,d,\pm\sigma)$; (c.3)$c_{\pm j}$ solves (1.2) in $\Gamma_\pm(2R_0,d_0,\pm\sigma)$. This is possible, because equation (1.2) is linear, but the condition $c_{\pm 0} \to 1$ is not necessarily satisfied for ξ with $d_0^{-1} < |\xi| < d^{-1}$ or $d < |\xi| < d_0$.

We now fix an integer N large enough (e.g. $N = 100n$), and set

$$c_\pm(h) = c_\pm(x,\xi;h) = \sum_{j=0}^N c_{\pm j}(x,\xi)h^j.$$

We define $J_{\pm c}(h)$ by

$$J_{\pm c}(h) = I_h(c_\pm(h);\phi_\pm). \tag{1.4}$$

Then we can prove that

$$W_\pm(h)f = \lim_{t\to\pm\infty} \exp(ih^{-1}tH(h))J_{\pm c}(h)\exp(-ih^{-1}tH_0(h))f$$

for $f \in L^2_\gamma, \gamma > 1/2$, with energy support in $(d^{-2}/2, d^2/2)$.

1.3. We take d_j and $\sigma_j, 1 \le j \le 4$, as follows : $1 < d_4 < d_3 < d_2 < d_1 < d_0$ and $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < \sigma_0$. For the region $\Gamma_\pm(3R_0, d_1, \pm\sigma_1)$ we define $J_{\pm a}(h)$ in the same way as $J_{\pm c}(h)$;

$$J_{\pm a}(h) = I_h(a_\pm(h); \phi_\pm),$$

where the symbol

$$a_\pm(h) = a_\pm(x, \xi; h) = \sum_{j=0}^{N} a_{\pm j}(x, \xi) h^j$$

has the properties $(c.0), (c.1)$ for $\Gamma_\pm(3R_0, d_1, \mp\sigma_1), (c.2)$ for $\Gamma_\pm(4R_0, d_2, \mp\sigma_2)$ and $(c.3)$ for $\Gamma_\pm(4R_0, d_1, \mp\sigma_2)$. We further define $K_{\pm a}(h)$ by

$$K_{\pm a}(h) = H(h) J_{\pm a}(h) - J_{\pm a}(h) H_0(h).$$

This operator can be written as

$$k_{\pm a}(h) = I_h(k_{\pm a}(h); \phi_\pm)$$

with the symbol $k_{\pm a}(h) = k_{\pm a}(x, \xi; h)$ given by

$$\exp(-ih^{-1}\phi_\pm(-(1/2)h^2\Delta + V - (1/2)|\xi|^2)\exp(ih^{-1}\phi_\pm)a_\pm(h). \tag{1.5}$$

By definition, it follows that $k_{\pm a}(h)$ has the following properties : $(k.0)$ $h^{-1}k_{\pm a}(h) \in A_{-1}$ uniformly in h; $(k.1)$ supp $k_{\pm a} \subset \Gamma_\pm(3R_0, d_1, \mp\sigma_1)$; $(k.2)$ $h^{-(N+2)}k_{\pm a}(h) \in A_{-(N+2)}(\Gamma_\pm(4R_0, d_1, \mp\sigma_2))$ uniformly in h.

Similarly we define

$$J_{\pm b}(h) = I_h(b_\pm(h); \phi_\pm)$$

for the region $\Gamma_\pm(5R_0, d_3, \pm\sigma_4) \subset \Gamma_\pm(3R_0, d_1, \mp\sigma_1)$ (be careful for the notations), where the symbol

$$b_\pm(h) = b_\pm(x, \xi; h) = \sum_{j=0}^{N} b_{\pm j}(x, \xi) h^j$$

has the properties $(c.0), (c.1)$ for $\Gamma_\pm(5R_0, d_3, \pm\sigma_4), (c.2)$ for $\Gamma_\pm(6R_0, d_4, \pm\sigma_3)$ and $(c.3)$ for $\Gamma_\pm(6R_0, d_3, \pm\sigma_3)$. We also define

$$K_{\pm b}(h) = H(h) J_{\pm b}(h) - J_{\pm b}(h) H_0(h) = I_h(k_{\pm b}(h); \phi_\pm).$$

The symbol $k_{\pm b}(h) = k_{\pm b}(x, \xi; h)$ satisfies $(k.0), (k.1)$ for $\Gamma_{\pm}(5R_0, d_3, \pm\sigma_4)$ and $(k.2)$ for $\Gamma_{\pm}(6R_0, d_3, \pm\sigma_3)$.

1.4. We now formulate the Isozaki-Kitada representation formula for $T(\lambda; h)$ (cf. Theorem 3.3, [9]).

PROPOSITION **1.1.** — *Let the notations be as above. Let energy λ be fixed in the interval $(d_4^{-2}/2, d_4^2/2)$. Then :*

$$T(\lambda, h) = T_{+1}(\lambda, h) + T_{-1}(\lambda, h) - T_2(\lambda, h),$$

where

$$T_{\pm 1} = F_0(\lambda, h) J_{+a}^*(h) K_{\pm b}(h) F_0^*(\lambda, h)$$

$$T_2 = F_0(\lambda, h) K_{+a}^*(h) R(\lambda + i0; H(h))(K_{+b}(h) + K_{-b}(h)) F_0^*(\lambda, h).$$

We now denote by $T_{\pm 1}(\theta, \omega; \lambda, h)$ and $T_2(\theta, \omega; \lambda, h)$ the kernel of $T_{\pm 1}(\lambda, h)$ and $T_2(\lambda, h)$, respectively. It is easy to prove that $T_{\pm 1}(\theta, \omega, \lambda, h) = O(h^\infty)$ for $\theta \neq \omega$. In fact, the kernel $T_{\pm 1}(\theta, \omega; \lambda, h)$ is represented as

$$c_0(\lambda, h)^2 \int \exp(ih^{-1}\psi_{\pm}(x, \theta, \omega)) k_{\pm b}(x, \sqrt{2\lambda}\omega; h) \overline{a}_+(x, \sqrt{2\lambda}\theta; h) dx,$$

where $\psi_{\pm} = \phi_{\pm}(x, \sqrt{2\lambda}\omega) - \phi_+(x, \sqrt{2\lambda}\theta)$. If $\theta \neq \omega$, then $|\nabla_x \psi_{\pm}| \geq C > 0$, as is easily seen, and hence the above relation follows immediately by partial integration. Thus, by definition (0.1), we have

$$f(\omega \to \theta; \lambda, h) = -c(\lambda, h) T_2(\theta, \omega; \lambda, h) + O(h^\infty). \tag{1.6}$$

2. Resolvent estimates.

In this section we analyze the kernel $T_2(\theta, \omega; \lambda, h)$ by making use of microlocal resolvent estimates. We keep the same notations as in section 1 and assume, without loss of generality, that $\lambda \in (d_4^{-2}/2, d_4^2/2)$. For $R \gg 1$, let $\chi(x; R)$, $0 \leq \chi \leq 1$, be a smooth cut-off function such that $\chi = 1$ for $|x| < R$ and $\chi = 0$ for $|x| > R + 1$. We set

$$\chi_a(x) = \chi(x; 20R_0), \ \chi_b(x) = \chi(x; 10R_0).$$

The aim of this section is to prove the following lemma.

LEMME 2.1. — *Assume that λ is in a non-trapping energy range. Let $\gamma > n/2$ be fixed close enough to $n/2$. Then :*

(i) $\|K_{+a}^*(h)R(\lambda + i0; H(h))K_{+b}(h)\|_{-\gamma,\gamma} = O(h^{N/2})$

(ii) $\|K_{+a}^*(h)R(\lambda + i0; H(h))(1 - \chi_b)K_{-b}(h)\|_{-\gamma,\gamma} = O(h^{N/2})$

(iii) $\|((1 - \chi_a)K_{+a}(h))^* R(\lambda + i0; H(h))\chi_b K_{-b}(h)\|_{-\gamma,\gamma} = O(h^{N/2})$.

Remark. — The above lemma will be intuitively clear. Let

$$\Sigma_+(R, \sigma; \xi) = \{x \in R^n_x \ : \ |x| > R, < \widetilde{x}, \widetilde{\xi} >> \sigma\}$$

for $\xi \neq 0$. Then estimate (i) follows from the fact that outgoing particles starting from $\Sigma_+(5R_0, \sigma_4, \xi)$ ($\supset$ support in x of $k_{+b}(h)$) with momentum $\nabla_x\phi_+ \sim \xi$ never pass over $\Sigma_+(3R_0, -\sigma_1; \xi)\backslash\Sigma_+(4R_0, -\sigma_2; \xi)$ where the symbol $k_{+a}(h)$ of $K_{+a}(h)$ has only a weak estimate by property $(k.2)$. Estimate (ii) will be also clear from the same consideration as above. Estimate (iii)follows from the fact that scattered particles pass over $\Sigma_+(20R_0, -\sigma_1; \xi)\backslash\Sigma_+(20R_0, -\sigma_2; \xi)$ with momentum different from $\nabla_x\phi_+ \sim \xi$.

Now, we shall accept the above lemma as proved. If we set

$$e_{+a}(x; h) = \chi_a(x)k_{+a}(x, \sqrt{2\lambda}\theta; h)$$
$$e_{-b}(x; h) = \chi_b(x)k_{-b}(x, \sqrt{2\lambda}\omega; h),$$

and if we write ϕ_+ and ϕ_- for $\phi_+(x, \sqrt{2\lambda}\theta)$ and $\phi_-(x, \sqrt{2\lambda}\omega)$, respectively, then it follows from Lemma 2.1 that

$$T_2(\theta, \omega; \lambda, h) = c_0(\kappa, h)^2(w_{-b}, e_{+a}\exp(ih^{-1}\phi_+)) + O(h^{N/3})$$

with

$$w_{-b}(x; h) = R(\lambda + i0; H(h))e_{-b}\exp(ih^{-1}\phi_-).$$

Recall the definition of $k_{-b}(h)$ ((1.5)). Then a simple calculation proves that

$$w_{-b} = f_{-b}\exp(ih^{-1}\phi_-) + R(\lambda + i0; H(h))g_{-b}\exp(ih^{-1}\phi_-),$$

where

$$f_{-b}(x; h) = \chi_b(x)b_-(x, \sqrt{2\lambda}\omega; h)$$

$$g_{-b}(x; h) = \exp(ih^{-1}\phi_-)[\chi_b, H_0(h)]b_-(x, \sqrt{2\lambda}\omega; h)\exp(ih^{-1}\phi_-), \tag{2.1}$$

$[,]$ being the commutator notation. The same argument as in the previous section shows that

$$(f_{-b}\exp(ih^{-1}\phi_-), e_{+a}\exp(ih^{-1}\phi_+)) = O(h^\infty)$$

and hence

$$T_2(\theta,\omega;\lambda,h) = c_0(\lambda,h)^2(g_{-b}\exp(ih^{-1}\phi_-), w_{+a}) + O(h^{N/3}),$$

where

$$w_{+a}(x;h) = R(\lambda - i0; H(h))e_{+a}\exp(ih^{-1}\phi_+).$$

Thus, as an immediate consequence of Lemma 2.1, we obtain the following result by repeating the above argument and by making use of relation (1.6).

COROLLARY. — *Define $G_0(\theta,\omega;\lambda,h)$ by*

$$G_0 = (R(\lambda + i0; H(h))g_{-b}\exp(ih^{-1}\phi_-), g_{+a}\exp(ih^{-1}\phi_+)) \tag{2.2}$$

with

$$g_{+a}(x;h) = \exp(-ih^{-1}\phi_+)[\chi_a, H_0(h)]a_+(x,\sqrt{2\lambda}\theta;h)\exp(ih^{-1}\phi_+). \tag{2.3}$$

Then

$$f(\omega\to\theta;\lambda,h) = c_1(\lambda,h)G_0(\theta,\omega;\lambda,h) + O(h^{N/3}) \tag{2.4}$$

with

$$c_1(\lambda,h) = (2\pi)(2\lambda)^{(n-3)/4}(2\pi h)^{-(n+1)/2}\exp(-i(n-3)(\pi/4)).$$

The proof of Theorem 1 is now reduced to the study on the asymptotic behavior as $h\to 0$ of $G_0(\theta,\omega;\lambda,h)$.

2.1. We prepare two lemmas to prove Lemma 2.1.

LEMMA 2.2. — *If λ is in a non-trapping energy range, then*

$$\|R(\lambda \pm i0; H(h))\|_{\alpha,-\alpha} = O(h^{-1}),\ \alpha > 1/2.$$

The above lemma has been already proved in [15] by making use of outgoing and incoming parametrices for $\exp(-ih^{-1}tH(h))$ constructed by [8] and also Gérard and Martinez [7] have recently given a short proof based on the Mourre estimate. The non-trapping condition is essentially used to prove this lemma only.

LEMMA 2.3. — *Assume that* λ *is in a non-trapping energy range. Let* $\omega_{\pm}(x,\xi) \in A_0$ *has support in* $\Gamma_{\pm}(R,d,\sigma_{\pm})$ *for* $R > 2R_0$. *Then :*

(i) $$\|R(\lambda \pm i0; H(h))\omega_{\pm}(x, hD_x)\|_{-\alpha+\delta,-\alpha} = O(h^{-1}),\ \delta > 1$$

for any $\alpha > 1/2$.

(ii) *If* $\sigma_{+} > \sigma_{-}$, *then*

$$\|\omega_{\mp}(x, hD_x)R(\lambda \pm i0; H(h))\omega_{\pm}(x, hD_x)\|_{-\alpha,\alpha} = O(h^{\infty})$$

for any $\alpha \gg 1$.

(iii) *If* $\omega(x,\xi) \in A_0$ *has support in* $|x| < (9/10)R$, *then*

$$\|\omega(x, hD_x)R(\lambda \pm i0; H(h))\omega_{\pm}(x, hD_x)\|_{-\alpha,\alpha} = O(h^{\infty})$$

for any $\alpha \gg 1$.

The microlocal resolvent estimates as in Lemma 2.3 have been studied in the case $h = 1$ by [10]. (In this case, the non-trapping condition is not necessary.) The proof of the lemma is essentially based on the same idea as in [10] (see also [17]).

2.2. We first prove Lemma 2.1, accepting Lemma 2.3 as proved.

Proof of Lemma 2.1. — We begin by the following general remark. Let $a_{\pm}(x,\xi) \in A_0$ be supported in $\Gamma_{\pm}(R,d,\sigma), R > R_0$, and let $\omega_{\pm}(x,\xi) \in A_0$ vanish in $\Gamma_{\pm}(\widehat{R},\widehat{d},\widehat{\sigma}_{\pm}) \supset \Gamma_{\pm}(R,d,\sigma)$, $\quad R > \widehat{R} > R_0$, $\quad \widehat{d} > d$, $\quad \widehat{\sigma}_{-} > \sigma > \widehat{\sigma}_{+}$. If R_0 is chosen large enough, then

$$\omega_{\pm}(x, hD_x)I_h(a_{\pm}; \phi_{\pm}) \ :\ L^2_{-\alpha} \to L^2\alpha$$

is bounded with bound $O(h^{\infty})$ for any $\alpha \gg 1$. This follows from the standard calculus of Fourier integral and pseudodifferential operators.

(i) By properties (k.1) and (k.2), the symbol $k_{+a}(h)$ of $K_{+a}(h)$ has support in $\Gamma_{+}(3R_0, d_1, -\sigma_1)$ and satisfies $k_{+a}(h) = O(h^N < x >^{-N})$ in $\Gamma_{+}(4R_0, d_1, -\sigma_2)$. On the other hand, the symbol $k_{+b}(h)$ of $K_{+b}(h)$ has support in $\Gamma_{+}(5R_0, d_3, \sigma_4)$. We note that

$$\Gamma_{+}(4R_0, d_1, -\sigma_1)\backslash\Gamma_{+}(4R_0, d_1, -\sigma_2) \subset \Gamma_{-}(4R_0, d_1, -\sigma)$$

for any $\sigma, 0 > -\sigma > -\sigma_2$. Hence, by the above remark, estimate (i) follows from Lemma 2.3.

(ii) Estimate (ii) is verified in the same way as (i). We give only a sketch. The symbol $(1-\chi_b)k_{-b}(h)$ has support in $\Gamma_-(10R_0, d_3, -\sigma_4)$ and is of order $O(h^N < x >^{-N})$ in $\Gamma_-(10R_0, d_2, -\sigma_3)$. Since

$$\Gamma_-(10R_0, d_3, -\sigma_4)\backslash\Gamma_-(10R_0, d_3, -\sigma_3) \subset \Gamma_+(10R_0, d_3, -\sigma)$$

for any $\sigma, -\sigma_3 > -\sigma > -\sigma_2$, we can prove (ii) by making use of the same argument as used to prove (i).

(iii) By taking the adjoint, it suffices to show that

$$\|(\chi_b K_{-b}(h))^* R(\lambda - i0; H(h))(1-\chi_a)K_{+a}(h)\|_{-\gamma,\gamma} = O(h^{N/3}).$$

This also follows from Lemmas 2.2 and 2.3 by making use of the same argument as above. We omit the detailed proof. □

2.3 *Proof of Lemma 2.3.* — We may assume that $1 < d < d_0$ and $\sigma_+ > -\sigma_0$. Take the triplets $(\widehat{R}_j, \widehat{d}_j, \widehat{\sigma}_j)$, $1 \leq j \leq 2$, as follows : $R > \widehat{R}_2 > \widehat{R}_1 > 2R_0$, $d_0 > \widehat{d}_1 > \widehat{d}_2 > d$ and $\sigma_+ > \widehat{\sigma}_2 > \widehat{\sigma}_1 > -\sigma_0$. For brevity, we further assume that $\alpha < N/2 - 1$ for N fixed arbitrarily in section 1.

(i) We consider the + case only. Let the symbol $c_{+0}(x,\xi)$ be as in subsection 1.2. Recall that $c_{+0} = 1 + O(|x|^{-1})$ as $|x| \to \infty$ in $\Gamma_+(2R_0, \widehat{d}_1, \widehat{\sigma}_1)$. Hence, by the calculus of Fourier integral and pseudodifferential operators, we can find a symbol $e_N(h) = e_N(x,\xi;h) \in A_0$ (uniformly in h) with support in $\Gamma + (\widehat{R}_2, \widehat{d}_2, \widehat{\sigma}_2)$ such that

$$J_{+c}(h)(I_h(e_N(h);\phi_+))^* = \omega_+(x, hD_x) - h^N \omega_N(x, hD_x; h),$$

where $J_{+c}(h)$ is defined by (1.4) and $\omega_N(x,\xi;h)$ belongs to A_{-N} uniformly in h. By the same argument as in subsection 1.2, we can construct an approximate representation for

$$U(t;h) = \exp(-ih^{-1}tH(h))\omega_+(x, hD_x), \quad t \geq 0.$$

We define $U_N(t;h)$ and $R_N(t;h), t \geq 0$, as follows :

$$U_N = J_{+c}(h)\exp(-ih^{-1}tH_0(h))(I_h(e_N(h);\phi_+))^*$$
$$R_N = K_{+c}(h)\exp(-ih^{-1}tH_0(h))(I_h(e_N(h);\phi_+))^*,$$

where

$$K_{+c}(h) = H(h)J_{+c}(h) - J_{+c}(h)H_0(h) = I_h(k_{+c}(h);\phi_+)$$

and the symbol $k_{+c}(h) = k_{+c}(x,\xi;h)$ satisfies $(k.0),(k.1)$ for $\Gamma_+(R_0,d_0,-\sigma_0)$ and $(k.2)$ for $\Gamma_+(2R_0,d_0,\widehat{\sigma}_1)$. Then it follows that

$$ih(\partial/\partial t)U_N = H(h)U_N + R_N.$$

Hence the Duhamel principle yields

$$U(t;h) = U_N + h^N \exp(-ih^{-1}tH(h))\omega_N(x,hD_x;h) + G_N(t;h), \tag{2.5}$$

where

$$G_N = ih^{-1}\int_0^t \exp(-ih^{-1}(t-s)H(h))R_N(s;h)ds.$$

We assert that :

$$\|U_N(;h)\|_{-\alpha+\delta,-\alpha} = O(<t>^{-\delta}), \delta > 1, \tag{2.6}$$

$$\|R_N(t;h)\|_{-\alpha+\delta,\alpha} = O(h^N <t>^{-\delta}). \tag{2.7}$$

Then estimate (i) follows from Lemma 2.2 and the assertions above. In fact, by (2.5), we have

$$R(\lambda+i0;H(h))\omega_+(x,hD_x) = \sum_{k=1}^{3} Q_k(\lambda,h),$$

where

$$Q_1 = ih^{-1}\int_0^\infty \exp(ih^{-1}t\lambda)U_N(t;h)dt$$
$$Q_2 = h^N R(\lambda+i0;H(h))\omega_N(x,hD_x;h)$$
$$Q_3 = ih^{-1}\int_0^\infty R(\lambda+i0;H(h))\exp(ih^{-1}s\lambda)R_N(s;h)ds.$$

This relation proves (i) immediately.

We now prove (2.6) and (2.7). We may write $(U_N(t;h)f)(x)$ as

$$(2\pi h)^{-n}\iint \exp(ih^{-1}\psi(t,x,\xi,y))c_+(x,\xi;h)\overline{e}_N(y,\xi;h)f(y)dyd\xi, \tag{2.8}$$

where $\psi = \phi_+(x,\xi) - \phi_+(y,\xi) - (1/2)t|\xi|^2$. If $(y,\xi) \in$ supp e_N, then $<\widetilde{y},\widetilde{\xi}> > \widehat{\sigma}_2$ and hence

$$|\nabla_\xi(\phi_+(y,\xi) + (1/2)t|\xi|^2)| \geq C(1+|y|+t), \quad t \geq 0.$$

Hence, by repeated use of partial integration, we can write $(U_N(t;h)f)(x)$ as

$$(U_N(t;h)f(x) = (2\pi h)^{-n} \iint \exp(ih^{-1}\psi)a(t,x,\xi,y)f(y)dyd\xi,$$

where $|a| \le C_\alpha < x >^\alpha < y >^{-\alpha+\delta} < t >^{-\delta}$. This proves (2.6).

To prove (2.7), we again represent $(R_N(t;h)f)(x)$ in a form similar to (2.8). Assume that $(x,\xi) \in$ supp $k_{+c}(h)$. If $(x,\xi) \in \Gamma_+(2R_0,d_0,\widehat{\sigma}_1)$, then $k_{+c}(h) = O(h^N < x >^{-N})$ by property $(k.2)$ and if $(x,\xi) \notin \Gamma_+(2R_0,d_0,\widehat{\sigma}_1)$ and $(y,\xi) \in$ supp e_N (and hence $< \widetilde{y}, \widetilde{\xi} >> \widehat{\sigma}_2$), then

$$|\nabla_\xi \psi| \ge C(1 + |x| + |y| + t), \quad t \ge 0,$$

by the choice of $\widehat{\sigma}_1, \widehat{\sigma}_2 > \widehat{\sigma}_1$. Thus the same argument as above proves (2.7).

(ii) Estimate (ii) is proved in the same way as (i). We give only a sketch. We may assume that $\sigma_+ > \widehat{\sigma}_2 > \widehat{\sigma}_1 > \sigma_-$. Then we can show by the same argument as used to prove (2.6) that

$$\|\omega_-(x,hD_x)U_N(t;h)\|_{-\beta,\beta} = O(h^\infty < t >^{-\infty})$$

for any $\beta \gg 1$. Since $\omega_N \in A_{-N}$, it follows from estimate (i) that

$$\|\omega_-(x,hD_x)R(\lambda + i0; H(h))\omega_N(x,hD_x;h\|_{-\beta,\beta} = O(h^{-1})$$

for any $\beta, \beta < N/2 - 1$. Similarly it follows from (i) and Lemma (2.2) that

$$\|\omega_-(x,hD_x)R(\lambda + i0; H(h))R_N(t;h)\|_{-(\beta-\delta),\beta-\delta} = O(h^N < t >^{-\delta})$$

for any $\beta < N/2$. Thus estimate (ii) is proved by combining the three estimates above.

(iii) Estimate (iii) is also proved in the same way as (i) and (ii). We omit the detailed proof. □

3. Preliminary step by short time parametrices.

In this section we analyze $G_0 = G_0(\theta,\omega;\lambda,h)$ defined by (2.2) by making use of short time parametrices for $\exp(-ih^{-1}tH(h))$.

Let $\{q(t;y,\eta), p(t;y,\eta)\}$ be the solution to the Hamilton system (0.2) with initial state (y,η) at $t=0$. We denote by F^t the canonical mapping

$$F^t \ : \ (y,\eta) \to (q(t;y,\eta), p(t;y,\eta)).$$

The analysis is based on the following proposition.

PROPOSITION 3.1. — *Let $\omega(x,\xi) \in A_0$ be of compact support. Assume that $\omega_t(x,\xi) \in A_0$ vanishes in a small neighborhood of*

$$\{(x,\xi) \ : \ (x,\xi) = F^t(y,\eta), (y,\eta) \in \ \mathrm{supp}\ \omega\}.$$

Then

$$\|\omega_t(x,hD_x)\exp(-ih^{-1}tH(h))\omega(x,hD_x)\|_{-\alpha,\alpha} = O(h^\infty)$$

for any $\alpha \gg 1$, where the order relation is uniform in t when t ranges over a compact interval in R^1.

The above proposition corresponds to the famous Egorov theorem on the propagation of singularities for hyperbolic equations. For a proof, see [14] (Chap. IV).

3.1. We keep the same notations as in sections 1 and 2, and we write again ϕ_+ and ϕ_- for $\phi_+(x,\sqrt{2\lambda}\theta)$ and $\phi_-(x,\sqrt{2\lambda}\omega)$, respectively.

Assume that $\lambda \in (d_4^{-2}/2, d_4^2/2)$ is a non-trapping energy range and that $y \in \ \mathrm{supp}\ g_{-b} \subset \{y \ : \ 10R_0 < |y| < 10R_0+1\}$. By the non-trapping condition, there exists $T_0 \gg 1$ such that

$$\{(x,\xi) \ : \ x = q(T_0;y,\nabla_x\phi_-), \xi = p(T_0;y,\nabla_x\phi_-)\} \subset \Gamma_+(30R_0,d_4,0)$$

with $\nabla_x\phi_- = \nabla_x\phi_-(y,\sqrt{2\lambda}\omega)$. Recall that $g_{+a}(x;h)$ is supported in $\{x \ : \ 20R_0 < |x| < 20R_0+1\}$. We now use the relation

$$\begin{aligned} R(\lambda+i0;H(h)) = ih^{-1}\int_0^T &\exp(ih^{-1}t\lambda)\exp(-ih^{-1}tH(h))dt \\ &+\exp(ih^{-1}T\lambda)R(\lambda+i0;H(h))\exp(-ih^{-1}TH(h)) \end{aligned}$$

for $T \geq 0$. Then it follows from Proposition 3.1 and (iii) of Lemma 2.3 that

$$(R(\lambda+i0;H(h))\exp(-ih^{-1}T_0H(h))g_{-b}\exp(ih^{-1}\phi_-), g_{+a}\exp(ih^{-1}\phi_+)) = O(h^\infty).$$

Thus we obtain

$$G_0 = ih^{-1}\int_0^{T_0}\exp(ih^{-1}t\lambda)F(t;\theta,\omega,h)dt + O(h^\infty),$$

where

$$F = (\exp(-ih^{-1}tH(h))g_{-b}\exp(ih^{-1}\phi_-), g_{+a}\exp(ih^{-1}\phi_+)).$$

3.2. We further analyze $G_0(\theta, \omega; \lambda, h)$. We recall the notations in Introduction. Let $\{q_\infty t; z, \lambda), p_\infty(t; z, \lambda)\}$ be the phase trajectory with properties (0.3) and (0.5). Let $w_j = w_j(\theta; \lambda) \in \Lambda_\omega, 1 \le j \le \ell$, be as in Theorem 1; $\xi_\infty(w_j; \lambda) = \theta$. Define

$$Z_j = \{z \in \Lambda_\omega \ : \ |z - w_j| < \varepsilon\}, \quad 1 \le j \le \ell,$$

for $\varepsilon > 0$ small enough (and hence $Z_j \cap Z_k = \emptyset, j \ne k$), and

$$Y_j = \{y \in \text{ supp } g_{-b} \ : \ y = q_\infty(s; w_j, \lambda), s < 0\}.$$

Then there exist S_0 and $S_1, S_1 \gg S_0 \gg 1$, such that

$$Y_j \subset \Pi_{-j} = \{y \ : \ y = q_\infty(x; z, \lambda), z \in Z_j, -S_1 < s < -S_0\}.$$

Let $\pi_{-j} \in C_0^\infty(\Pi_{-j}), 0 \le \pi_{-j} \le 1$, be such that $\pi_{-j} = 1$ on Y_j.

Recall that $\phi_- = \phi_-(x, \sqrt{2\lambda}\omega)$ solves the eikonal equation $(1/2)|\nabla_x\phi_-|^2 + V(x) = \lambda$ in $\Sigma_- = \{x \ : \ |x| > R_0, < \widetilde{x}, \omega >< \sigma_0\}$ and $\nabla_x\phi_-$ behaves like $\nabla_x\phi_- = \sqrt{2\lambda}\omega + O(|x|^{-1})$ as $|x| \to \infty$ in Σ_-.

Thus it follows from the Hamilton-Jacobi theory that $\nabla_x\phi_-(y, \sqrt{2\lambda}\omega) = p_\infty(s; z, \lambda)$ for $y = q_\infty(s; z, \lambda) \in \Sigma_-$ and hence

$$q(t; y, \nabla_x\phi_-) = q_\infty(t + s; z, \lambda) \tag{3.1}$$

for y as above. Therefore, if $y \notin \Pi_{-j}$, then the classical particle starting from y with momentum $\nabla_x\phi_- = \nabla_x\phi_-(y, \sqrt{2\lambda}\omega)$ passes over the support of g_{+a} with momentum different from $\nabla_x\phi_+ \sim \sqrt{2\lambda}\theta$. Thus, by Proposition 3.1, we have

$$G_0 = ih^{-1}\sum_{j=1}^{\ell}\int_0^{T_0}\exp(ih^{-1}t\lambda)F_{-j}(t; \theta, \omega, \lambda, h)dt + O(h^\infty),$$

where

$$F_{-j} = (\exp(-ih^{-1}tH(h))\pi_{-j}g_{-b}\exp(ih^{-1}\phi_-), g_{+a}\exp(ih^{-1}\phi_+)).$$

We further define

$$X_j = \{x \in \text{ supp } g_{+a} \ : \ x = q(t; y, \nabla_x\phi_-), y \in Y_j\}.$$

Then there exists $T_1 \gg 1, T_1 < T_0$, such that

$$X_j \subset \Pi_{+j} = \{x : x = q(t; y, \nabla_x \phi_-), y \in \Pi_{-j}, T_1 < t < T_0\}.$$

Let $\pi_{+j} \in C_0^\infty(\Pi_{+j}), 0 \leq \pi_{+j} \leq 1$, be such that $\pi_{+j} = 1$ on X_j. We again use Proposition 3.1 to obtain

$$G_0 = ih^{-1} \sum_{j=1}^{\ell} \int_{T_1}^{T_0} \exp(ih^{-1}t\lambda) F_j(t; \theta, \omega, \lambda, h) dt + O(h^\infty), \tag{3.2}$$

where

(3.3)

$$F_j = (\exp(-ih^{-1}tH(h))\pi_{-j}g_{-b} \exp(ih^{-1}\phi_-), \pi_{+j}g_{+a} \exp(ih^{-1}\phi_+)).$$

3.3. We now construct an approximate representation for $\psi_j(t, x; h) = \exp(-ih^{-1}tH(h))\pi_{-j}g_{-b} \exp(ih^{-1}\phi_-), T_1 < t < T_0$, with $x \in \Pi_{+j}$ with the aid of the Maslov theory ([12]).

LEMMA 3.2. — *The point $x = q(t; y, \nabla_x \phi_-) \in \Pi_{+j}$ with $y \in \Pi_{-j}$ is non-focal in the Maslov sense;*

$$D(t, y) = \det\ (\partial q(t; y, \nabla_x \phi_-)/\partial y) \neq 0, \quad T_1 < t < T_0.$$

Proof. — By (3.1), the determinant under consideration is written as

$$D(t, y) = D_\infty(t + s, z) D_\infty(s, z)^{-1}, \tag{3.4}$$

where

$$D_\infty(x, z) = \det\ (\partial q_\infty(s; z, \lambda)/\partial(s, z)). \tag{3.5}$$

By (0.3), $D_\infty(s, z) \neq 0$ for $s, -S_1 < s < -S_0$. By (0.5),

$$q_\infty(t + s; z, \lambda) \sim \sqrt{2\lambda}\xi_\infty(z; \lambda)(t + s) + r_\infty(z; \lambda)$$

as $t \to \infty$. We may assume that $t + s \gg 1$. Hence $D_0(t + s, z) \sim (t + s)^{n-1}(2\lambda)^{n/2}\widehat{\sigma}(z; \lambda)$, $\widehat{\sigma}(z, \lambda)$ being defined by (0.6). By assumption, $\widehat{\sigma}(z; \lambda) \neq 0$ for $z \in Z_j$. This proves the lemma. □

We now recall the representation (2.1) for g_{-b}. We can write g_{-b} as

$$g_{-b}(x;h) = ihg_{0b}(x) + h^2 g_b(x;h),$$

where

$$g_{0b} = < \nabla_x \phi_-, \nabla \chi_b > b_{-0}(x, \sqrt{2\lambda}\omega) \tag{3.6}$$

and $g_b \in C_0^\infty(R_x^n)$. Similarly

$$g_{+a}(x;h) = ihg_{0a}(x) + h^2 g_a(x;h),$$

where

$$g_{0a} = < \nabla_x \phi_+, \nabla \chi_a > a_{+0}(x, \sqrt{2\lambda}\theta). \tag{3.7}$$

Lemma 3.2 enables us to construct an approximate representation for $\psi_j(t,x;h)$, $T_1 \le t \le T_0$, in the asymptotic form

$$\exp(ih^{-1}S_j(t,y) - i\mu_j\pi/2)|D(t,y)|^{-1/2}\sum_{k=1}^{\infty} h^k \psi_{jk}(t,y) \tag{3.8}$$

for $x = q(t;y,\nabla_x\phi_-) \in \Pi_{+j}$ with $y \in \Pi_{-j}$, where $S_j(t,y)$ is the action along the trajectory joining the points x and y;

$$S_j = \phi_-(y, \sqrt{2\lambda}\omega) + \int_0^t (|p(\tau;y,\nabla_x\phi_-)|^2/2 - V(q(\tau;y,\nabla_x\phi_-))d\tau \tag{3.9}$$

and μ_j is the path index of the above trajectory, while ψ_{jk} is smooth in $t, T_1 < t < T_0$, and $y \in \Pi_{+j}$, and

$$\psi_{j1}(t,y) = i\pi_{-j}(y)g_{0b}(y),$$

g_{0b} being defined by (3.6).

Remark. — We should note that the above path index μ_j coincides with the one defined for the trajectory $\{q_\infty(t;w_j,\lambda), p_\infty(t;w_j,\lambda)\}$ in Theorem 1.

We insert the above approximate representation into the integral (3.3) and make a change of variables $x \to y$ with $x = q(t;y,\nabla_x\phi_-)$. This is possible by Lemma 3.2. For brevity, we consider only the leading term $L_j(t;\theta,\omega,\lambda,h), 1 \le j \le \ell$, given by

$$L_j = h^2 \int \exp(ih^{-1}\psi_j(t,y) - i\mu_j\pi/2)M_j(t,y)|D(t,y)|^{1/2}dy,$$

where

$$\psi_j = S_j(t, y) - \phi_+(q(t; y, \nabla_x \phi_-), \sqrt{2\lambda}\theta)$$

$$M_j = \pi_{-j}(y) g_{0b}(y) \pi_{+j}(q(t; y, \nabla_x \phi_-)) g_{0a}(q(t; y, \nabla_x \phi_-)).$$

Thus the proof of Theorem 1 is reduced to the study on the asymptotic behavior as $h \to 0$ of the integral L_j.

4. Stationary phase method.

In this section we complete the proof of Theorem 1. We apply the stationary phase method to the integral

$$N_j(\theta, \omega; \lambda, h) = \int_{T_1}^{T_0} \exp(ih^{-1} t\lambda) L_j(t, \theta, \omega, \lambda, h) dt, \ 1 \le j \le \ell,$$

and prove that

$$N_j = c_2(\lambda, h) \exp(ih^{-1} S_j - i\mu_j \pi/2) \widehat{\sigma}(w_j; \lambda)^{-1/2} (1 + O(h)) \tag{4.1}$$

with

$$c_2(\lambda, h) = -(2\lambda)^{-(n-3)/4} (2\pi h)^{(n-1)/2} h^2 \exp(i(n-1)(\pi/4)).$$

(See Theorem 1 for the notation S_j.) This, together with (3.2) and (2.4), yields the desired asymptotic formula for $f(\omega \to \theta; \lambda, h)$.

We make a change of variables $y \to (x, z)$ with $y = q_\infty(s; z, \lambda) \in \Pi_{-j}, z \in Z_j, -S_1 < s < -S_0$. Then, by (3.1) and (3.4), N_j is represented as

$$N_j = h^2 \int_{T_1}^{T_0} \int_{-S_1}^{-S_0} \exp(ih^{-1} t\lambda - i\mu_j \pi/2) I_j(t, s; \theta, \omega, \lambda, h) ds dt,$$

where

$$I_j = \int \exp(ih^{-1} \Phi_j(t, s, z)) f_j(t, s, z) |D_\infty(t+s, z)|^{1/2} |D_\infty(s, z)|^{1/2} dz$$

$$\Phi_j = S_j(t, q_\infty(s; z, \lambda)) - \phi_+(q_\infty(t+s; z, \lambda), \sqrt{2\lambda}\theta)$$

$$f_j = \pi_{-j}(q_\infty(s; z, \lambda)) g_{0b}(q_\infty(s; z, \lambda)) \pi_{+j}(q_\infty(t+s; z, \lambda)) g_{0a}(q_\infty(t+s; z, \lambda)).$$

We may assume f_j to have support in Z_j with respect to the z variables.

We apply the stationary phase method to study the asymptotic behavior as $h \to 0$ of the integral I_j. As is easily seen from the proof of Lemma 4.5 below, $z = w_j$ is the only stationary point of $\Phi_j(t,s,z),(t,s)$ being fixed.

4.1. This subsection is devoted to the preliminary step for applying the stationary phase method to the integral I_j.

We denote by $\{q_+(t;x,\xi),p_+(t;x,\xi)\}$ the solution to the Hamilton system (0.2) with $q_+(0;x,\xi) = x$ and $p_+(\infty;x,\xi) = \lim_{t\uparrow\infty} p_+(t;x,\xi) = \xi$. If $(x,\xi) \in \Gamma_+(R_0,d_0,-\sigma_0)$, then such a trajectory exists uniquely and $q_+(t;x,\xi)$ behaves like

$$\lim_{t\to\infty} |q_+(t;x,\xi) - \xi t - a_\infty(x,\xi)| = 0$$

in the C^∞-topology, where

$$a_\infty(x,\xi) = x + \int_0^\infty (p_+(\tau;x,\xi) - \xi)d\tau. \tag{4.2}$$

Since

$$\nabla_x\phi_+(q_+(t;x,\xi),\xi) = p_+(t;x,\xi), \tag{4.3}$$

we can represent $\phi_+ = \phi_+(x,\xi)$ as
(4.4)

$$\phi_+ =< a_\infty(x,\xi),\xi > - \int_0^\infty (|p_+(\tau;x,\xi)|^2/2 - V(q_+(\tau,x,\xi)) - |\xi|^2/2)d\tau$$

for $(x,\xi) \in \Gamma_+(R_0,d_0-\sigma_0)$. Similarly we have

$$\phi_-(y,\sqrt{2\lambda}\omega = 2s\lambda + \int_{-\infty}^s (|p_\infty(\tau;z,\lambda)|^2/2 - V(q_\infty(\tau;z,\lambda)) - \lambda)d\tau \tag{4.5}$$

for $y = q_\infty(s;z,\lambda) \in \Pi_{-j}$, because

$$\lim_{\tau\to-\infty} < q_\infty(\tau+s;z,\lambda) - \sqrt{2\lambda}\tau\omega, \sqrt{2\lambda}\omega >= 2s\lambda.$$

LEMMA 4.1. — *$\nabla_\xi\phi_+(x,\xi) = a_\infty(x,\xi)$ in $\Gamma_+(R_0,d_0,-\sigma_0)$.*

Proof. — Set $f_k(x,\xi) = (\partial/\partial\xi_k)\phi_+(x,\xi), 1 \le k \le n$. Then f_k obeys the equation $< \nabla_x\phi_+, \nabla_x f_k > -\xi_k = 0$ with the condition

$f_k(x,\xi) - x_k \to 0$ as $|x| \to \infty$. We can solve this equation by making use of relation (4.3) and the solution f_k is represented in the form (4.2). This proves the lemma. □

LEMMA 4.2. — *Denote by $A(x,\xi), (x,\xi) \in \Gamma_+(R_0, d_0, -\sigma_0)$, the $n\times n$ matrix $A(x,\xi) = (\partial^2\phi_+(x,\xi)/\partial x_\alpha \partial \xi_\beta)_{1\le\alpha,\beta\le n}$. Then*

$$\det\, A(x,\xi) = \exp(\int_0^\infty (\Delta_x\phi_+)(q_+(\tau;x,\xi),\xi)d\tau).$$

Proof. — By (4.3),

$$(d/dt)q_+(t;x,\xi) = \nabla_x\phi_+(q_+(t;s,\xi),\xi)$$

with $q_+(0;x,\xi) = x$. Hence, by the Liouville theorem,

$$\det\, (\partial q_+(t;x,\xi)/\partial x) = \exp(\int_0^t (\Delta_x\phi_+)(q_+(\tau;x,\xi),\xi)d\tau).$$

This, together with Lemma 4.1, proves the lemma by letting $t\to\infty$. □

LEMMA 4.3. — *Let $D_\infty(s,z)$ be defined by (3.5). Then*

$$D_\infty(s,z) = (2\lambda)^{1/2}\exp(\int_{-\infty}^s (\Delta_x\phi_-)(q_\infty(\tau;z,\lambda),\sqrt{2\lambda}\omega)d\tau).$$

Proof. — Since

$$(d/dt)q_\infty(t;z,\lambda) = \nabla_x\phi_-(q_\infty(t;z,\lambda),\sqrt{2\lambda}\omega)$$

and since $q_\infty(t;z,\lambda)$ behaves like $q_\infty \sim \sqrt{2\lambda}\omega t + \hat{z}$ as $t\to -\infty$, the Liouville theorem again proves the lemma. □

LEMMA 4.4. — *Let g_{0a} and g_{0b} be as defined by (3.7) and (3.6), respectively. Then :*

(i) $g_{0a}(x) = \chi_{0a}(t,s)\exp(2^{-1}\int_{t+s}^\infty (\Delta_x\phi_+)(q_\infty(\tau;w_j,\lambda),\sqrt{2\lambda}\theta)d\tau)$

with $x = q_\infty(t+s; w_j, \lambda) \in \Pi_{+j}$, where

$$\chi_{0a} = < \nabla_x \phi_+(x, \sqrt{2\lambda}\theta), \nabla_x \chi_a(x) >, \quad x = q_\infty(t+s; w_j, \lambda). \tag{4.6}$$

(ii) $g_{0b}(y) = \chi_{0b}(s) \exp(-2^{-1} \int_{-\infty}^{s} (\Delta_x \phi_-)(q_\infty(\tau; w_j, \lambda), \sqrt{2\lambda}\omega) d\tau)$

with $y = q_\infty(s; w_j, \lambda) \in \Pi_{-j}$, where

$$\chi_{0b} = < \nabla_x \phi_-(y, \sqrt{2\lambda}\omega), \nabla_x \chi_b(y) >, \quad y = q_\infty(s; w_j, \lambda). \tag{4.7}$$

Proof. — (i) Recall that $a_{+0} = a_{+0}(x, \xi)$ satisfies the transport equation

$$< \nabla_x \phi_+(x, \xi), \nabla_x a_{+0} > + (1/2)(\Delta_x \phi_+)(x, \xi) a_{+0} = 0$$

with the condition $a_{+0} \to, |x| \to \infty$, in $\Gamma_+(4R_0, d_2, -\sigma_2)$. Hence the solution a_{+0} is written as

$$a_{+0}(x, \xi) = \exp(2^{-1} \int_0^\infty (\Delta_x \phi_+)(q_+(\tau; x, \xi), \xi) d\tau).$$

For $x = q_\infty(t+s; w_j, \lambda)$, we have

$$q_+(\tau; x, \sqrt{2\lambda}\theta) = q_\infty(\tau + t + s; w_j, \lambda) \tag{4.8}$$

$$p_+(\tau; x, \sqrt{2\lambda}\theta) = p_\infty(\tau + t + s; w_j, \lambda). \tag{4.9}$$

Thus (4.8) proves (i).

(ii) (ii) is proved in a similar way. We omit the detailed proof.

4.2. We now calculate the determinant and signature of the $(n-1) \times (n-1)$ matrix $\partial^2 \Phi_j / \partial z^2 = (\partial^2 \Phi_j / \partial z_\alpha \partial z_\beta)_{1 \le \alpha, \beta \le n-1}$ at the stationary point $z = w_j$.

LEMMA 4.5. — *At the stationary point $z = w_j$:*

(i) $\mathrm{sgn}(\partial^2 \Phi_j / \partial z^2) = n - 1$.

(ii) $|\det (\partial^2 \Phi_j / \partial z^2)| = (2\lambda)^{(n-2)/2} \widehat{\sigma}(w_j; \lambda) D_\infty(t+s, w_j) E_0(t, s)$,

where $\widehat{\sigma}(z; \lambda)$ is defined by (0.6) and

$$\begin{aligned} E_0 &= \det A(q_\infty(t+s; w_j, \lambda), \sqrt{2\lambda}\theta) \\ &= \exp(\int_{t+s}^\infty (\Delta_x \phi_+)(q_\infty(\tau; w_j, \lambda), \sqrt{2\lambda}\theta) d\tau). \end{aligned}$$

Proof. — We first note that by Lemma 4.2 and (4.8), det $A(x, \sqrt{2\lambda}\theta)$ with $x = q_\infty(t+s; w_j, \lambda)$ is represented as above.

The $S_j = S_j(t, q_\infty(s; z, \lambda))$, S_j being defined by (3.9), is the action along the trajectory joining the points $x = q_\infty(t+s; z, \lambda)$ and $y = q_\infty(s; z, \lambda)$. Therefore, by the Hamilton-Jacobi theory,

$$\partial S_j/\partial z_\alpha = < p_\infty(t+s; z, \lambda), \partial q_\infty(t+s; z, \lambda)/\partial z_\alpha >, \quad 1 \leq \alpha \leq n-1.$$

Since $p_\infty(t+s; z, \lambda)$ behaves like $p_\infty \sim \sqrt{2\lambda}\xi_\infty(z; \lambda)$ as $t \to \infty$, we can write $p_\infty = p_\infty(t+s; z, \lambda)$ as

$$p_\infty = p_+(t; q_\infty(s; z, \lambda), \sqrt{2\lambda}\xi_\infty(z; \lambda)) = \nabla_x \phi_+(q_\infty(t+s; z, \lambda), \sqrt{2\lambda}\xi_\infty(z; \lambda)).$$

Thus, $\partial \Phi_j/\partial z_\alpha$ is represented as

$$\partial \Phi_j/\partial z_\alpha = < \nabla_x \phi_+(q_\infty, \sqrt{2\lambda}\xi_\infty(z; \lambda)) - \nabla_x \phi_+(q_\infty, \sqrt{2\lambda}\theta), \partial q_\infty/\partial z_\alpha >$$

with $q_\infty = q_\infty(t+s; z, \lambda)$ and hence

$$\partial^2 \Phi_j/\partial z_\alpha \partial z_\beta = \sqrt{2\lambda} < A(q_\infty, \sqrt{2\lambda}\theta)\partial \xi_\infty(w_j; \lambda)/\partial z_\beta, \partial q_\infty/\partial z_\alpha >$$

at the stationary point $z = w_j$. We may assume that $t + s \gg 1$. Then $A(q_\infty, \sqrt{2\lambda}\theta) \sim \mathrm{Id}(n \times n$ identity matrix) and

$$\partial q_\infty/\partial z_\alpha \sim (2\lambda)^{1/2}(t+s)\partial \xi_\infty(w_j; \lambda)/\partial z_\alpha.$$

This proves (i) at once.

To prove (ii), we use Lemma 4.1 to obtain

$$\nabla_\xi \phi_+(q_\infty(t+s; w_j, \lambda), \sqrt{2\lambda}\theta) = \sqrt{2\lambda}\xi_\infty(w_j; \lambda)(t+s) + r_\infty(w_j; \lambda)$$

and hence

$$\xi_\infty(w_j; \lambda) = (2\lambda)^{-1/2}\ {}^t\!A(q_\infty, \sqrt{2\lambda}\theta)\partial q_\infty/\partial s.$$

Since $< \xi_\infty, \xi_\infty >= 1$ and $< \xi_\infty, \partial \xi_\infty/\partial z_\alpha >= 0$, we can easily prove the relation

$$\begin{bmatrix} 1, & * \\ 0, & \partial^2 \Phi_j/\partial z^2 \end{bmatrix} = {}^t\Sigma(w_j; \lambda) \cdot {}^t\!A(q_\infty, \sqrt{2\lambda}\theta) \cdot \partial q_\infty/\partial(s, z),$$

where

$$\Sigma(z; \lambda) = ((2\lambda)^{-1/2}\xi_\infty, (2\lambda)^{1/2}\partial \xi_\infty/\partial z_1, \ldots, (2\lambda)^{1/2}\partial \xi_\infty/\partial z_{n-1}).$$

Thus (ii) follows immediately from the above relation. □

LEMMA 4.6. — *Let $S_j, 1 \leq j \leq \ell$, be as in Theorem 1. Then*

$$\Phi_j(t,s,w_j) = S_j - t\lambda.$$

Proof. — We recall the representation (3.9) for $S_j(t,y)$.

We have $q(\tau; y, \nabla_x(h) i_-) = q_\infty(\tau + s; w_j, \lambda)$ and $p(\tau; y, \nabla_x \phi_-) = p_\infty(\tau + s; w_j, \lambda)$ for $y = q_\infty(s; w_j, \lambda)$. Hence, by (4.5),

$$S_j(t,y) = \lambda(t+2s) + \int_{-\infty}^{t+s} (|p_\infty(\tau; w_j, \lambda)|^2/2 - V(q_\infty(\tau; w_j, \lambda)) - \lambda) d\tau$$

for y as above. On the other hand, we have

$$a_\infty(x, \sqrt{2\lambda}\theta) = \sqrt{2\lambda}\theta(t+s) + r_\infty(w_j; \lambda)$$

for $x = q_\infty(t+s; w_j, \lambda)$. Hence, by (4.4), (4.8) and (4.9), $\phi_+(x, \sqrt{2\lambda}\theta)$ is represented as

$$\begin{aligned} &2\lambda(t+s) + < r_\infty(w_j;\lambda), \sqrt{2\lambda}\theta > \\ &\quad - \int_{t+s}^{\infty} (|p_\infty(\tau; w_j, \lambda)|^2/2 - V(q_\infty(\tau; w_j, \lambda)) - \lambda) d\tau \end{aligned}$$

for x as above. The two expressions above prove the lemma. □

4.3. We now apply the stationary phase method to the integral $I_j = I_j(t,s;\theta,\omega,\lambda,h)$. Since $\pi_{-j}(y) = 1$ for $y = q_\infty(s; w_j, \lambda) \in \Pi_{-j}$ and $\pi_{+j}(x) = 1$ for $x = q_\infty(t+s; w_j, \lambda) \in \Pi_{+j}$, we combine Lemmas 4.3 $\sim$ 4.6 to obtain that

$$I_j = c_3(\lambda, h) \exp(ih^{-1}(S_j - t\lambda)) \chi_{0b}(s) \chi_{0a}(t,s)(1 + O(h))$$

with

$$c_3(\lambda, h) = (2\lambda)^{-(n-3)/4}(2\pi h)^{(n-1)/2} \exp(i(n-1)(\pi/4)),$$

where χ_{0a} and χ_{0b} are defined by (4.6) and (4.7), respectively. As is easily seen,

$$\chi_{0a}(t,s) = (d/dt)\chi_a(q_\infty(t+s; w_j, \lambda))$$
$$\chi_{0b}(s) = (d/ds)\chi_b(q_\infty(s; w_j, \lambda)).$$

Hence

$$\int_{T_1}^{T_0} \int_{-S_1}^{-S_0} \chi_{0b}(s) \chi_{0a}(t,s) ds dt = -1.$$

Thus we obtain (4.1) and the proof of Theorem 1 is now complete.

5. Resolvent estimate at low energies.

In the previous sections, the semi-classical resolvent estimate (Lemma 2.2) has played an important role in proving the asymptotic formula (0.8). In this section, we will prove such a resolvent estimate at low energies which is also important in studying the asymptotic behavior of scattering amplitudes in the low energy limit.

Consider the Schrödinger operator $H = -(1/2)\Delta + V$ in $R^n_x, n \geq 3$, where the potential $V(x)$ is assumed, in addition to $(V)_\rho$ with $1 < \rho < 2$, to satisfy the following conditions :

(V.0) $V(x)$ is repulsive, $V \geq 0$;

(V.1) $-(\partial/\partial r)V = -|x|^{-1} < x, \nabla_x V > \geq C|x|^{-\rho-1}$ for $|x| > R \gg 1$.

The aim of this section is to prove the following

THEOREM 5.1. — *Assume that the space dimension $n \geq 3$ and that $V(x)$ satisfies $(V)_\rho$ with $1 < \rho < 2$, (V.0) and (V.1). Then*

$$\|R(\lambda \pm i0; H)\|_{\alpha,-\alpha} = O(\lambda^{-1}), \quad \lambda \to 0,$$

for any $\alpha > 1/2$.

As stated in Introduction, we make a change of variables $x \to y = \lambda^{1/\rho}x$ and consider the Hamiltonian $H_\lambda(h)$ defined by

$$H_\lambda(h) = -(1/2)h^2\Delta + V_\lambda \tag{5.1}$$

with $h = \lambda^\gamma, \gamma = 1/\rho - 1/2 > 0$, where $V_\lambda(x) = \lambda^{-1}V(\lambda^{-1/\rho}x)$.

As an application of Theorem 5.1, we obtain the following corollary which corresponds to the semi-classical resolvent estimate (Lemma 2.2).

COROLLARY 5.2. — *Assume the same assumptions as in Theorem 5.1 Then*

$$\|R(1 \pm i0; H_\lambda(h))\|_{\alpha,-\alpha} = O(\lambda^{-2\alpha/\rho}) = O(h^{-2\alpha/\gamma\rho})$$

with $h = \lambda^\gamma, \gamma = 1/\rho - 1/2 > 0$, for any $\alpha > 1/2$.

Proof. — We may assume that $\alpha > 1/2$ is close enough to $1/2$. Let $u_\lambda(x) = (R(1 + i0; H_\lambda(h))f)(x)$ with $f \in L^2_\alpha$. Then $u_\lambda(x)$ can be represented as $u_\lambda(x) = (R(\lambda + i0; H)g_\lambda)(\lambda^{-1/\rho}x)$ with $g_\lambda(x) = \lambda f(\lambda^{1/\rho}x)$. Since

$< \lambda^{1/\rho}x >^{-2\alpha} \leq \lambda^{-2\alpha/\rho} < x >^{-2\alpha}$ and $< \lambda^{-1/\rho}x >^{2\alpha} \leq \lambda^{-2\alpha/\rho} < x >^{2\alpha}$, it follows from Theorem 5.1 that $|u_\lambda|_{-\alpha} \leq C\lambda^{-2\alpha/\rho}|f|_\alpha$. This proves the corollary. □

5.1. Theorem 5.1 is proved through a series of elementary lemmas. Throughout the discussion below, the assumptions in Theorem 5.1 are assumed to be satisfied.

LEMMA 5.3. — *If $u \in L^2_{-3/2}$ and satisfies $-(1/2)\Delta u + Vu = 0$ in the distribution sense, then $u = 0$.*

Proof. — The proof uses the assumption $n \geq 3$. Set $\psi(r) = (1+r^2)^{-1/2}, r = |x|$. Let $\phi \in C_0^\infty(R_x^n), 0 \leq \phi \leq 1$, be a smooth cut-off function such that $\phi = 1$ for $|x| \leq 1$ and $\phi = 0$ for $|x| \geq 2$. We further set $\psi_R(x) = \phi(x/R)\psi(r)$. We multiply the equation $-(1/2)\Delta u + Vu = 0$ by $\psi_R \bar{u}$. Then, by partial integration,

$$\int 2^{-1}\psi_R|\nabla u|^2 dx + \int (\psi_R V - 4^{-1}\Delta\psi_R)|u|^2 dx = 0.$$

Since $V \geq 0$ by assumption (V.0) and since $\Delta\psi_R = O(|x|^{-3})$ as $|x| \to \infty$, we can let $R \to \infty$ to obtain

$$\int 2^{-1}\psi|\nabla u|^2 dx + \int (\psi V - 4^{-1}\Delta\psi)|u|^2 dx = 0.$$

By direct calculation,

$$\Delta\psi = (1+r^2)^{-5/2}((3-n)r^2 - n) < 0.$$

Thus the lemma follows at once. □

LEMMA 5.4. — *Let $\chi_R = \chi_R(x)$ be the characteristic function of $B_R = \{x : |x| < R\}, R \gg 1$. Then*

$$\|\chi_R R(\lambda \pm i0; H)\|_{\sigma,0} = O(1), \quad \lambda \to 0,$$

for $\sigma = (1+\rho)/2$.

Proof. — We consider the + case only. The proof is done by contradiction. Deny the statement. Then there exist sequences

$\{\lambda_j\}_{j=1}^{\infty}, \{\kappa_j\}_{j=1}^{\infty}, \kappa_j > 0$, and $\{f_j\}_{j=1}^{\infty}, f_j \in L^2_\sigma$, such that : (i) $\lambda_j \to \lambda_0$; (ii) $\kappa_j \to 0$; (iii) $f_j \to 0$ strongly in L^2_σ; (iv)

$$|\chi_R u_j|_0 = |\chi_R R(\lambda_j + i\kappa_j; H) f_j|_0 = 1.$$

By elliptic estimate, $\{u_j\}_{j=1}^{\infty}$ forms a precompact set in $L^2(B_R)$. Thus we may assume that u_j converges to some $u_0, u_0 \neq 0$, with $|\chi_R u_0|_0 = 1$. We further assume that the limit $\lambda_0 = 0$. If $\lambda_0 \neq 0$, then we can prove that $u_0 = 0$, following the argument used in proving the principle of limiting absorption ([1]). This contradicts $u_0 \neq 0$. In the case $\lambda_0 = 0$ also, we can prove that $u_0 = 0$.

Let $\{\chi^{(\alpha)}\}_{1\le\alpha\le n}$ be a real C^∞-smooth vector field of the form $\chi^{(\alpha)} = g(r)x_\alpha/r$, $r = |x|$, where $g(r)$ has the following properties : (g.1) $g(r)/r > 0$; (g.2) $g'(r) > 0$; (g.3) $g(r) = 1 - r^{-\delta}, 0 < \delta \ll 1$, for $r > R \gg 1$. Since

$$\chi^{(\alpha)}_\beta = \partial\chi^{(\alpha)}/\partial x_\beta = g'\widetilde{x}_\alpha\widetilde{x}_\beta + (g/r)(\delta_{\alpha\beta} - \widetilde{x}_\alpha\widetilde{x}_\beta), \quad \widetilde{x}_\alpha = x_\alpha/r,$$

it follows from properties (g.1) $\sim$ (g.3) that the $n \times n$ matrix $Q(x) = \{\chi^{(\alpha)}_\beta\}_{1\le\alpha,\beta\le n}$ is positive definite and

$$Q(x) \ge C_\delta < x >^{-(1+\delta)} \ge C_\delta < x >^{-2\sigma} . \tag{5.2}$$

For notational brevity, we use the summation convention and write $u_\alpha = (\partial/\partial x_\alpha)u$ and $u_{\alpha\beta} = (\partial^2/\partial x_\alpha\partial x_\beta)u$, etc.

We now multiply the equation

$$-(1/2)\Delta u_j + Vu_j - (\lambda_j + i\kappa_j)u_j = f_j$$

by $\chi^{(\alpha)}\overline{u}_{j\alpha} + 2^{-1}\chi^{(\alpha)}_\alpha\overline{u}_j$. Then, by partial integration, we obtain the following relation :

$$\begin{aligned}\int \{2^{-1} \text{ Re } \chi^{(\alpha)}_\beta u_{j\alpha}\overline{u}_{j\beta} - 2^{-1}\chi^{(\alpha)}V_\alpha|u_j|^2 - 8^{-1}\chi^{(\alpha)}_{\alpha\beta\beta}|u_j|^2\}dx \\ = \text{ Re } (f_j, \chi^{(\alpha)}u_{j\alpha} + 2^{-1}\chi^{(\alpha)}_\alpha u_j) - \kappa_j \text{Im } (u_j, \chi^{(\alpha)}u_{j\alpha}).\end{aligned}$$

Since

$$\kappa_j|u_j|_0^2 = |\text{Im}(f_j, u_j)| \le |f_j|_\sigma |u_j|_{-\sigma'}$$

we have by elliptic estimate that the right side is estimated from above by

$$C|f_j|_\sigma(|f_j|_\sigma + |\nabla u_j|_{-\sigma} + |u_j|_{-\sigma}).$$

On the other hand, by making use of (V.1) and (5.2), we see that the left side is estimated from below by

$$C\{|\nabla u_j|^2_{-(1+\delta)/2} + |u_j|^2_{-\sigma} - |\chi_R u_j|^2_0\}.$$

Thus we obtain

$$|\nabla u_j|^2_{-(1+\delta)/2} + |u_j|^2_{-\sigma} \leq C(|f_j|^2_\sigma + |\chi_R u_j|^2_0). \tag{5.3}$$

Hence, $\{u_j\}_{j=1}^\infty$ forms a bounded set in $L^2_{-\sigma}$ and $u_j \to u_0$ weakly in $L^2_{-\sigma}$. The limit u_0 satisfies $-(1/2)\Delta u_0 + Vu_0 = 0$. By Lemma 5.3, it follows that $u_0 = 0$. This contradicts $u_0 \neq 0$ and the proof is complete. □

LEMMA 5.5. — *Let $\sigma = (1+\rho)/2$ be as above. Then :* (i) $\|\nabla R(\lambda \pm i0; H)\|_{\sigma,-\alpha} = O(1)$ *for any* $\alpha > 1/2$. (ii) $\|R(\lambda \pm i0; H)\|_{\sigma,-\sigma} = O(1)$.

Proof. — The lemma is an immediate consequence of Lemma 5.4 and (5.3). □

5.2. *Proof of theorem 5.1.* — Let $u_{\pm\lambda} = R(\lambda \pm i0; H)f$ with $f \in L^2_\sigma, \sigma = (1+\rho)/2$. We know that $u_{\pm\lambda} \in L^2_{-\alpha}$ for any $\alpha > 1/2$. We multiply the equation $-(1/2)\Delta u_{\pm\lambda} + Vu_{\pm\lambda} - \lambda u_{\pm\lambda} = f$ by $\psi \overline{u}_{\pm\lambda}, \psi =< x >^{-2\alpha}$. Then we have by partial integration that

$$\int 2^{-1}\psi|\nabla u_{\pm\lambda}|^2 dx + \int (\psi V - 4^{-1}\Delta\psi)|u_{\pm\lambda}|^2 dx = \text{ Re } (f + \lambda u_{\pm\lambda}, \psi u_{\pm\lambda}).$$

By Lemma 5.5, $|\nabla u_{\pm\lambda}|_{-\alpha} + |u_{\pm\lambda}|_{-\sigma} \leq C|f|_\sigma$ for any $\alpha > 1/2$ and hence $\lambda|u_{\pm\lambda}|^2_{-\alpha} \leq C|f|^2_\sigma$. This proves that $\|R(\lambda \pm i0; H)\|_{\sigma,-\alpha} = O(\lambda^{-1/2})$ and also $\|R(\lambda \pm i0; H)\|_{\alpha,-\sigma} = O(\lambda^{-1/2})$. The theorem is proved by repeating the above argument for $f \in L^2_\alpha, \alpha > 1/2$. □

6. Asymptotics at low energies.

In this section we study the asymptotic behavior of scattering amplitudes in the low energy limit. Consider the Schrödinger operator $H = -(1/2)\Delta + V$ in $R^n_x, n \geq 3$, where the potential $V(x)$ is assumed to satisfy $(V)_\rho$ with $1 < \rho < 2$. Let $H_\lambda(h)$ be defined by (5.1) with

$h = \lambda^\gamma, \gamma = 1/\rho - 1/2 > 0$. Denote by $f_\mu(\omega \to \theta; H)$ and $f_\mu(\omega \to \theta; H_\lambda(h))$ the scattering amplitudes with energy $\mu > 0$ for the Hamiltonians H and $H_\lambda(h)$, respectively. As stated in Introduction, $f_\lambda(\omega \to \theta; H)$ is related to $f_1(\omega \to \theta; H_\lambda(h))$ through relation (0.11);

$$f_\lambda(\omega \to \theta; H) = \lambda^{-(n-1)/2\rho} f_1(\omega \to \theta; H_\lambda(h)).$$

Thus the problem is reduced to the study on the asymptotic behavior as $h \to 0$ of $f_\mu(\omega \to \theta; H_\lambda(h))$ with fixed energy $\mu = 1$.

6.1. We require many assumptions to formulate the obtained result precisely. Roughly speaking, the asymptotics as $\lambda \to 0$ of $f_\lambda(\omega \to \theta; H)$ is determined by the asymptotic behavior as $|x| \to \infty$ of $V(x)$ in the case of slowly decreasing potentials. We consider the following class of potentials with homogeneous property at infinity.

Assumption (A). — (A.0) $V(x)$ satisfies $(V)_\rho$ with $1 < \rho < 2$.

(A.1) $V(x)$ is repulsive; $V \geq 0$.

(A.2) There exists $\Phi \in C^\infty(S^{(n-1)}), \Phi > 0$ (strictly), such that

$$|\partial_x^\alpha(V(x) - \Phi(x/|x|)|x|^{-\rho}) = o(|x|^{-\rho-|\alpha|}), \quad |\alpha| \leq 2, \text{ as } |x| \to \infty.$$

We fix again the initial direction $\omega \in S^{n-1}$ as $\omega = (0, \dots, 0, 1)$ and use the notation Λ_ω with the meaning ascribed in Introduction. Define $V_0(x)$ by $V_0(x) = \Phi(x/|x|)|x|^{-\rho}$ with Φ as in (A.2) and denote by $\{q_{0\infty}(t; z), p_{0\infty}(t; z)\}, z \in \Lambda_\omega$, the phase trajectory satisfying (0.3) with energy $\lambda = 1$, which is defined as a solution to the Hamilton system (0.2) with $V = V_0$. Similarly we denote by $\{q_{\lambda\infty}(t; z), p_{\lambda\infty}(t; z)\}$ the phase trajectory associated with the potential $V_\lambda = \lambda^{-1}V(\lambda^{-1/\rho}x)$. In general, $V_0(x)$ has a singularity at the origin and also $V_\lambda(x)|_{x=0} \to \infty$ as $\lambda \to 0$. However, in the repulsion case which we consider here, classical particles never pass over a neighborhood of the origin (classically forbidden region).

Assumption (B). — $|q_{0\infty}(t; z)| \to \infty$ as $t \to \infty$ for $z \in \Lambda_\omega$.

It follows from (A.2) that as $\lambda \to 0$

$$\partial_x^\alpha V_\lambda(x) = \partial_x^\alpha V_0(x) + o(1)|x|^{-\rho-|\alpha|}, \quad |\alpha| \leq 2, \tag{6.1}$$

uniformly in $x, |x| > c > 0$. Hence, under assumption (B), we can easily prove that $|q_{\lambda\infty}(t; z)| \to \infty$ as $t \to \infty$ for $\lambda, 0 < \lambda \ll 1$. Assumption

(B) also enables us to define the angular densities $\widehat{\sigma}(z;V_0)$ and $\widehat{\sigma}(z;V_\lambda)$ by (0.6) with energy $\lambda = 1$ for the trajectories $\{q_{0\infty}(t;z), p_{0\infty}(t;z)\}$ and $\{q_{\lambda\infty}(t;z), p_{\lambda\infty}(t;z)\}$, respectively. For final direction $\theta \in S^{n-1}$, we make the following assumption.

Assumption (C). — $\widehat{\sigma}(z;V_0) \neq 0$ for all $z \in \Lambda_\omega$ such that

$$\lim_{t\uparrow\infty} |p_{0\infty}(t;z) - \sqrt{2}\theta| = 0.$$

LEMMA 6.1. — *$\widehat{\sigma}(z;V_\lambda) \to \widehat{\sigma}(z;V_0)$ as $\lambda \to 0$ uniformly in z.*

We further proceed with the argument, accepting the above lemma as proved. Under assumption (C), it follows from Lemma 6.1 that there exists only a finite number of $v_j = v_j(\theta;\lambda) \in \Lambda_\omega, 1 \leq j \leq \ell, \ell$ being independent of $\lambda, 0 < \lambda \ll 1$, such that $\lim_{t\uparrow\infty} |p_{\lambda\infty}(t;v_j) - \sqrt{2}\theta| = 0$.

6.2. We are now in a position to formulate the second main theorem.

THEOREM 2. — *Assume that the space dimension $n \geq 3$ and that $(A), (B)$ and (C) are satisfied. Let the notations be as above. Then the scattering amplitude $f_\lambda(\omega \to \theta; H)$ obeys the following asymptotic formula as $\lambda \to 0$:*

$$f_\lambda(\omega \to \theta; H) = \lambda^{-(n-1)/2\rho} \sum_{j=1}^{\ell} \widehat{\sigma}(v_j;V_\lambda)^{-1/2}$$
$$\exp(i\lambda^{-\gamma} S_j - i\mu_j\pi/2)(1 + O(\lambda^\gamma))$$

with $\gamma = 1/\rho - 1/2 > 0$, where μ_j is the path index of the phase trajectory $\{q_{\lambda\infty}(t;v_j), p_{\lambda\infty}(t;v_j)\}$ with $v_j = v_j(\theta;\lambda)$ as above and

$$S_j = \int_{-\infty}^{\infty} \{|p_{\lambda\infty}(t;v_j)|^2/2 - V(q_{\lambda\infty}(t;v_j)) - 1\}dt - < r_{\lambda\infty}(v_j), \sqrt{2}\theta >$$

with $r_{\lambda\infty}(v_j) = \lim_{t\uparrow\infty}(q_{\lambda\infty}(t;v_j) - \sqrt{2}\theta t)$.

Remarks. — (i) The path index μ_j is independent of $\lambda, 0 < \lambda \ll 1$. (ii) The quantity S_j can be described in terms of the phase trajectory associated with the original Hamiltonian $H = -(1/2)\Delta + V$. Let $\{q_\infty(t;z,\lambda), p_\infty(t;z,\lambda)\}$ be the phase trajectory defined as the solution with property (0.3) to the Hamilton system(0.2). Then a simple calculation yields that

$$q_{\lambda\infty}(t;z) = \lambda^{1/\rho} q_\infty(\lambda^{-(1/\rho+1/2)}t; \lambda^{-1/\rho}z, \lambda)$$
$$p_{\lambda\infty}(t;z) = \lambda^{-1/2} p_\infty(\lambda^{-(1/\rho+1/2)}t; \lambda^{-1/\rho}z, \lambda).$$

Thus we have

$$\lambda^{-\gamma} S_j = \int_{-\infty}^{\infty} \{|p_\infty(t; u_j, \lambda)|^2/2 - V(q_\infty(t; u_j, \lambda)) - \lambda\} dt - < r_\infty(u_j; \lambda), \sqrt{2\lambda}\theta > \text{ with } u_j = \lambda^{-1/\rho} v_j(\theta; \lambda).$$

Proof. — As stated above, in the case of repulsion, any new difficulty does not occur from the fact that $V_\lambda(x)|_{x=0} \to \infty$ as $\lambda \to 0$. The proof is done by applying to $f_1(\omega \to \theta; H_\lambda(h))$ the same arguments as used to prove the semi-classical asymptotic formula (0.8). In particular, the resolvent estimate (Corollary 5.2) enables us to follow the same arguments as in sections 2 and 3. We omit the detailed proof. □

6.3. *Proof of Lemma 6.1.* — Let $\xi_{0\infty}(z) = \lim_{t\uparrow\infty} p_{0\infty}(t;z)/\sqrt{2}$ and $\xi_{\lambda\infty}(z) = \lim_{t\uparrow\infty} p_{\lambda\infty}(t;z)/\sqrt{2}$. To prove the lemma, it suffices to show that

$$|\partial_z^\alpha(\xi_{\lambda\infty}(z) - \xi_{0\infty}(z))| \to 0, \quad |\alpha| \le 1, \text{ as } \lambda \to 0, \tag{6.2}$$

uniformly in $z \in \Lambda_\omega$.

In the proof, we denote by $\varepsilon(\lambda)$ a quantity of order $o(1)$ as $\lambda \to 0$. Assume z to be fixed. By (6.1), we can easily prove that

$$|q_{\lambda\infty}(t;z) - q_{0\infty}(t;z)| + |p_{\lambda\infty}(t;z) - p_{0\infty}(t;z)| \le \varepsilon_T(\lambda)$$

for $t, -\infty < t < T$, $T \gg 1$ being fixed arbitrarily. Define the mapping Q from $C([T,\infty); R^n)$ into itself by

$$(Qq)(t) = q_{\lambda\infty}(T;z) + p_{\lambda\infty}(T;z)(t-T) - \int_T^t \int_T^s (\nabla_x V_\lambda) q(\tau)) d\tau ds.$$

Then the $q_{\lambda\infty}(t;z), t > T$, is obtained as the fixed point of the mapping Q. We now introduce the norm $|\cdot|_\infty$ in $C([T,\infty); R^n)$ by $|q|_\infty = \sup_{t>T} |t|^{-1} |q(t)|$ and define the subset D_T as

$$D_T = \{q \in C([T,\infty); R^n) : |q - q_{0\infty}(\cdot; z)|_\infty \le \varepsilon(\lambda)\}$$

with $\varepsilon(\lambda)$ to be determined below. If T is large enough, then we can choose $\varepsilon(\lambda)$ in such a way that $Q : D_T \to D_T$ is a contraction mapping. Thus we have

$$|q_{\lambda\infty}(t;z) - q_{0\infty}(t;z)| \le \varepsilon(\lambda)(1+|t|).$$

Since

$$\sqrt{2}(\xi_{\lambda\infty}(z) - \xi_{0\infty}(z)) = \int_{-\infty}^{\infty} \{(\nabla_x V_0)(q_{0\infty}(\tau;z)) - (\nabla_x V_\lambda)(q_{\lambda\infty}(\tau;z))\}d\tau,$$

this, together with (6.1), proves (6.2) with $|\alpha| = 0$. A similar argument applies to the case $|\alpha| = 1$ and also it is easy to see that the convergence is uniform in z. This proves the lemma. □

6.4. We end the section by making a brief comment on the results related to Theorem 2. For brevity, we confine ourselves to the case $n = 3$. Many works have been done on the asymptotic behavior of scattering amplitudes at low energies. See, for example, [3], [4] and references quoted there. Roughly speaking, in the case of rapidly decreasing potentials ($V = O(|x|^{-\rho}), \rho > 2$), the behavior is determined by the Born approximation and it strongly depends on the fact whether the Hamiltonian H under consideration has a zero energy resonance and bound state or not. In the case of repulsion we consider here, H does not have zero energy resonances and bound states. On the other hand, little attention has been paid to the case of slowly decreasing potentials. In [11], Kvitsinskii has dealt with the special case $V(x) = a|x|^{-\rho}, 1 < \rho < 2$, including the case of attraction, $a < 0$. Our theorem may be considered as a slight generalization to the case without spherical symmetry, although the restrictive smoothness and repulsion conditions are assumed. In the case of attraction, we have to take into account not only the possibility of zero energy resonances and bound states but also the fact that classical particles pass over a neighborhood of the origin, where $\lambda^{-1}V(\lambda^{-1/\rho}x) \to \infty$ as $\lambda \to 0$. These facts produce additional difficulties. Furthermore, it is reported in [11] that the backward scattering amplitude $f_\lambda(\omega \to -\omega; H)$ has the different type of singularity at zero energy; $f_\lambda \sim \lambda^{-\beta}, \beta = (6-\rho)/4\rho$, for $V = a|x|^{-\rho}$, $a < 0$, with $1 < \rho < 2$ (Glory effect, [20]). In our terminology, this implies that the final direction $-\omega$ is not regular. Thus it will be interesting to extend Theorem 2 to a wide class of potentials without spherical symmetry.

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Manuscrit reçu le 7 mars 1988,
révisé le 26 septembre 1988.

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