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GALOIS MODULE STRUCTURE OF THE RINGS OF INTEGERS IN WILDLY RAMIFIED EXTENSIONS

by Stephen M.J. WILSON

1. Introduction.

The purpose of this paper is the generalisation to wildly ramified extensions of Taylor's result [T] on the Galois module structure of the ring of integers in a tamely ramified extension of algebraic number fields. Our main results may be loosely stated as follows.

THEOREM. — *Let N and N' be sums of Galois algebras with group Γ over algebraic number fields. Suppose that N and N' have the same dimension over $\mathbf{Q}$ and that they are identical at their wildly ramified primes. Then (writing $\mathcal{O}_N$ for the maximal order in N)*

$$\mathcal{O}_N \oplus \mathcal{O}_N \oplus \mathbf{Z}\Gamma \cong_{\mathbf{Z}\Gamma} \mathcal{O}_{N'} \oplus \mathcal{O}_{N'} \oplus \mathbf{Z}\Gamma.$$

In many cases $\mathcal{O}_N \cong_{\mathbf{Z}\Gamma} \mathcal{O}_{N'}$.

A precise statement of these results is given in §6 (6.1, 6.2 and 6.3) where we give the details of the rôle played by the root numbers of N and N' at the symplectic characters of Γ in determining the relationship between the $\mathbf{Z}\Gamma$-modules $\mathcal{O}_N$ and $\mathcal{O}_{N'}$.

Our theorem includes as a special case the theorem of Taylor referred to above. (Take N/K to be any tame Galois extension and N' to be the

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Galois algebra induced from the trivial extension K/K. In this case the $\mathbf{Z}\Gamma$ factors may be cancelled.) This would not be so if the theorem applied simply to field extensions (instead of to Galois algebras). We make the extra effort to establish our result for sums of (rather than single) Galois algebras so that in future work it may be possible to restrict attention to extensions with only one wildly ramified prime.

We use here many of the results (of Fröhlich, Martinet, Taylor *et al.*) which Taylor deploys in the proof of his theorem. Fröhlich's book [F2] is a virtually complete reference to the results that we need.

In [W1] the following adaptation of Martinet's conjecture (see [F2] I 1.18) was proved.

Under the conditions of the above theorem, (and in the case where N and N' are fields) $\mathcal{O}_N\mathcal{M}\oplus\mathcal{M} \cong_{\mathcal{M}} \mathcal{O}_{N'}\mathcal{M}\oplus\mathcal{M}$ where $\mathcal{M}$ is a maximal order containing $\mathbf{Z}\Gamma$. It was also there conjectured that the result is true under the (at least apparently) weaker assumption that $\mathcal{O}_N$ and $\mathcal{O}_{N'}$ belong to the same genus of $\mathbf{Z}\Gamma$-modules. This was subsequently proved by Queyrut [Q2]. We conjecture that our present results are also true under this weaker assumption. Futher discussion of the context and direction of this work may be found in the record [W2] of the author's talk to the Bordeaux Seminar.

In §2 (because we must necessarily deal with non-projective modules) we introduce the *class group* of a genus (of lattices over an order). We place this class group in the Heller exact sequence of a functor which is effectively $\otimes_{\mathbf{Z}}\overline{\mathbf{Q}}$. (Here $\overline{\mathbf{Q}}$ stands for an algebraic closure of $\mathbf{Q}$). We investigate the "relative" group in this sequence and obtain an idèlic formula for it and for the class group. (This idea of employing the relative groups of $\otimes_{\mathbf{Z}}\overline{\mathbf{Q}}$ is due to Queyrut. In [Q1] he investigates the relative group of this functor applied to the category of those lattices (over a given order) which are locally free outside a given set of primes.)

In §3 we introduce a *canonical* "funny" norm and recall the "Hom" notation of Fröhlich. In §4 we set up our resolvent machinery. This is effectively the machinery of [F2] with modifications to allow for our more general context and also our different view of the "funny" norm. In Theorem 4.8 we establish the pivotal connection between our resolvents and elements of the relative groups of §2.

In §5 we introduce Gauss sums and root numbers in a manner appropriate to our requirements. We detail and adapt those properties that we need.

Notation and conventions :

(i) 'Module' means *right* module unless otherwise stated.

(ii) A mapping may be written on the right (possibly exponentially) or on the left (when the argument will usually be delimited with parentheses).

(iii) The factors of explicitly composite maps are always written (with or without a small circle, '$\circ$') in the natural order (i.e. as if they were written on the right). Such explicit composites are never written on the left.

(iv) Homomorphisms of the form $g \otimes 1$ inherited by a tensor product from one of its factors (for example, after extension of scalars) are written plainly g. (For a worse example of abuse of notation see before 3.1.)

Most of the work for this paper was done while the author was visiting the University of Bordeaux. He would like to thank the members of the U.E.R. de Mathématiques et d'Informatiques very much for their continued hospitality, help and encouragement.

2. The class group of a genus and its relative group. Idèlic formulae.

The work of this section will accept considerable generalisation (e.g. A need not be semisimple and M need not be finitely generated — cf. [BKW]). We keep, however, to the situation in which we are really interested.

Let A be a finite dimensional semisimple $\mathbf{Q}$-algebra. We write $\mathcal{J}(A)$ for the idèles of A and $\mathcal{U}(\Lambda)$ for the unit idèles with respect to an order Λ in A. Let U be a normal, totally complex, finite extension of $\mathbf{Q}$ which is big enough for A (i.e. $A \otimes_{\mathbf{Q}} U$ is a direct sum of matrix rings over U). If Y is an abelian group we put $\overline{Y}$ for $Y \otimes_{\mathbf{Z}} U$.

Let M be a full $\mathbf{Z}$-lattice in an A-module V. We assume, for convenience, that V_A is faithful. Choose an order Λ of A such that $M\Lambda \subset M$. We put $E = \mathrm{End}_A(V)$ and $\Theta = \mathrm{End}_A(M) \stackrel{\mathrm{def}}{=} \{\alpha \in E \mid M\alpha \subset M\} \cong \mathrm{End}_\Lambda(M)$.

Put $C = Z(A)$. So $C = Z(E)$ since V is faithful. We say that a simple component of C_∞ is *symplectic* if it is the centre of a component of A_∞ (or, equivalently, of E_∞) which is a full matrix ring over $\mathbf{H}$. Put $\mathcal{J}(C)^+$ for the set of those idèles of C whose projection into any symplectic component of C_∞ is positive. Put $C^{\times +} \stackrel{\mathrm{def}}{=} C^\times \cap \mathcal{J}(C)^+$. We write Nrd_A for the reduced norm with respect to A (see §3). We recall (see e.g. [F2] II §1)

THEOREM 2.1. — (i) $\mathrm{Nrd}_A \mathcal{J}(A) = \mathcal{J}(C)^+$;

(ii) $\mathcal{K}_1(A) \overset{\mathrm{Nrd}}{\simeq} C^{\times +}$;

(iii) $C^\times \mathcal{J}(C)^+ = C^\times \mathcal{J}(C)$. □

We put $\mathcal{LS}(M, A)$ for the full subcategory of $\mathrm{Mod}(\Lambda)$ whose objects are those Λ-lattices which are locally isomorphic to $M^{(t)}$ for some t. We have (cf. [CR] § 6.3)

THEOREM 2.2. — *The functor* $\mathrm{Hom}_\Lambda(M, -)$ *gives an equivalence from* $\mathcal{LS}(M, A)$ *to the category* $\mathcal{LS}(\Theta, E^{\mathrm{op}})$ *of finitely generated, locally free left* Θ*-modules.*

We need to describe local isomorphism in terms of idèles. We state the following lemma. It is easy to prove directly but the result can also be imported from the theory of locally free modules ([F1]) via 2.2.

LEMMA 2.3. — *Let* W *and* W' *be isomorphic* A*-modules spanned by* T *and* T' *in* $|\mathcal{LS}(M, A)|$. *Put* $X = \mathrm{End}_A(W)$ *and* $\Xi = \mathrm{End}_A(T)$. *Let* $\left\{ \overset{\beta=}{\beta_p : W_p \to W'_p} \right\}$ *be a local* A*-isomorphism from* W *to* W'. *We write* $T\beta$ *for* $\{t \in W' \mid t \in T_p\beta_p \,\forall p\}$. *Then decale*(i) $T' = T\beta$ *for some* β.

(ii) *If* $W = W'$ *then* $T\beta$ *is a lattice locally isomorphic to* T *if and only if* $\beta \in \mathcal{J}(X)$.

(iii) $T\beta = T \iff \beta \in \mathcal{U}(\Xi)$.

Write $\mathcal{K}_0^{\mathrm{ls}}(M, A)$ for the Grothendieck group of $\mathcal{LS}(M, A)$ (with respect to direct sums) and put $\mathcal{F}ib$ for the fibre category (see [H]) of the functor $\otimes_{\mathbf{Z}} U : \mathcal{LS}(M, A) \to \mathcal{LS}(\overline{M}, \overline{A})$. From [H] 5.2 (working with split exact sequences) we obtain the Heller exact sequence of this functor:

$$\mathcal{K}_1(\Theta) \to \mathcal{K}_1(\overline{E}) \xrightarrow{\partial} \mathcal{K}_0^{\mathrm{ls}}(M, A, \otimes_{\mathbf{Z}} U) \xrightarrow{\delta} \mathcal{K}_0^{\mathrm{ls}}(M, A) \to \mathcal{K}_0^{\mathrm{ls}}(\overline{M}, \overline{A})$$

where $\mathcal{K}_0^{\mathrm{ls}}(M, A, \otimes_{\mathbf{Z}} U) = \mathcal{K}_\#(\mathcal{F}ib)$ is Heller's relative group and we have used 2.2 to obtain the form of the $\mathcal{K}_1$ groups. The last group in the sequence is free of rank 1 and we denote the kernel of the last map – essentially a rank map – by $\mathcal{C}l(M, A)$. We call this group the *class group of the genus of* M. Writing $\mathcal{K}'_1(\Theta)$ for the image of $\mathcal{K}_1(\Theta)$ in $\mathcal{K}_1(\overline{E})$, we may rewrite our sequence

$$(2.4) \qquad 1 \to \mathcal{K}'_1(\Theta) \to \mathcal{K}_1(\overline{E}) \xrightarrow{\partial} \mathcal{K}_0^{\mathrm{ls}}(M, A, \otimes_{\mathbf{Z}} U) \xrightarrow{\delta} \mathcal{C}l(M, A) \to 0 \,.$$

Now the objects of $\mathcal{F}ib$ are triples (T_1, f, T_2) where $T_i \in |\mathcal{LS}(M, A)|$ and (we reverse Heller's notation) $f : \overline{T}_1 \simeq \overline{T}_2$ is an $\overline{A}$-isomorphism. The relative group $\mathcal{K}_0^{\rm ls}(M, A, \otimes_{\mathbf{Z}} U)$ is generated by isomorphism classes of these triples subject to the relations obtained from direct sum and composition of triples. We denote by $[T_1, f, T_2]$ the class of (T_1, f, T_2) in $\mathcal{K}_0^{\rm ls}(M, A, \otimes_{\mathbf{Z}} U)$.

LEMMA 2.5. — *With T, W, W', X, Ξ and β as in 2.3,*

(i) *Every object of $\mathcal{F}ib$ is $(T\beta, f, T)$ for some T and β.*

(ii) *Every element of $\mathcal{K}_0^{\rm ls}(M, A, \otimes_{\mathbf{Z}} U)$ is $[T\beta, f, T]$ for some T and some $\beta \in \mathcal{J}(X)$.*

(iii) *If $f \in \overline{X}^\times$ and $\beta \in \mathcal{J}(X)$ and if* $\mathrm{Nrd}_X(f) = 1 = \mathrm{Nrd}_X(\beta)$ *then* $[T\beta, f, T] = 0$.

Proof. — (i) If $(T_1, f, T_2) \in \mathcal{F}ib$ then $\overline{T}_1 \cong \overline{T}_2$. Hence the T_i have the same rank and must therefore be locally isomorphic and we can apply 2.3(i).

(ii) So every element can be written as an algebraic sum of elements $[T\beta, f, T]$. But we can make all terms positive by the rule $-[T\beta, f, T] = [(T\beta)\beta^{-1}, f^{-1}, T\beta]$ and put them all together with the direct sum. Moreover, choosing an isomorphism $g: W' \to W$, we have $(T\beta, f, T) \cong (T\beta g, g^{-1}f, T)$ and $\beta g \in \mathcal{J}(X)$ by 2.3(ii).

(iii) Replacing T by $T \oplus M^{(t)}$ if necessary (by adding copies of $(M, 1, M)$ to $(T\beta, f, T)$) we can have $\beta \in [\mathcal{J}(X), \mathcal{J}(X)] \subset \mathcal{U}(\Xi)[X^\times, X^\times]$ (by Wang's Theorem and strong approximation — see [F1]). So $\beta = ug$ with $u \in \mathcal{U}(\Xi)$ and $g \in [X^\times, X^\times]$. Then $[T\beta, f, T] \overset{2.3(\mathrm{iii})}{=} [Tg, f, T] = [T, gf, T] = \partial[\overline{W}, gf] = 0$ by 2.1(ii). □

Put $\partial_{\mathcal{J}} = \partial_{\mathcal{J}(C)}$ for the canonical projection of $\mathcal{J}(\overline{C})$ onto its quotient mod $\mathrm{Nrd}_E \mathcal{U}(\Theta)$.

THEOREM 2.6. — (i) *The map* $\tilde{\lambda}: (T\beta, f, T) \mapsto \partial_{\mathcal{J}}(\mathrm{Nrd}_X(\beta f))$ *induces an isomorphism*

$$\lambda: \mathcal{K}_0^{\rm ls}(M, A, \otimes_{\mathbf{Z}} U) \xrightarrow{\sim} \frac{\overline{C}^\times \mathcal{J}(C)}{\mathrm{Nrd}_E(\mathcal{U}(\Theta))} \subset \frac{\mathcal{J}(\overline{C})}{\mathrm{Nrd}_E(\mathcal{U}(\Theta))}.$$

(ii) *There is an isomorphism of exact sequences*

$$\begin{array}{ccccccccc}
1 \to & \mathcal{K}_1'(\Theta) & \to & \mathcal{K}_1(\overline{E}) & \xrightarrow{\partial} & \mathcal{K}_0^{\rm ls}(M, A, \otimes_{\mathbf{Z}} U) & \xrightarrow{\delta} & \mathcal{C}l(M, A) & \to 0 \\
& \| & & \downarrow \wr\, \mathrm{Nrd} & & \downarrow \wr\, \lambda & & \downarrow \wr\, \lambda_{\mathcal{C}l} & \\
1 \to & \mathcal{K}_1'(\Theta) & \xrightarrow{\mathrm{Nrd}} & \overline{C}^\times & \to & \dfrac{\overline{C}^\times \mathcal{J}(C)}{\mathrm{Nrd}_E(\mathcal{U}(\Theta))} & \to & \dfrac{\overline{C}^\times \mathcal{J}(C)}{\overline{C}^\times \mathrm{Nrd}_E(\mathcal{U}(\Theta))} & \to 0 .
\end{array}$$

(iii) *Let A' be a semisimple subalgebra of A such that U is big enough for A' also. Then the diagram*

$$\begin{array}{ccc} \mathcal{K}_0^{\mathrm{ls}}(M, A, \otimes_{\mathbf{Z}} U) & \xrightarrow{\lambda} & \dfrac{\mathcal{J}(\overline{C})}{\mathrm{Nrd}_E(\mathcal{U}(\Theta))} \\ \Big\downarrow \mathrm{res} & & \Big\downarrow \mathrm{res} \\ \mathcal{K}_0^{\mathrm{ls}}(M, A', \otimes_{\mathbf{Z}} U) & \xrightarrow{\lambda'} & \dfrac{\mathcal{J}(\overline{C}')}{\mathrm{Nrd}_{E'}(\mathcal{U}(\Theta'))} \end{array}$$

commutes. Here the dash distinguishes the objects defined with reference to A' as opposed to A and the right hand map is induced by the restriction maps from $\mathcal{K}_1((\overline{A})_p) = (\overline{C})_p^\times$ to $\mathcal{K}_1((\overline{A}')_p) = (\overline{C}')_p^\times$ for each prime p of $\mathbf{Q}$.

Proof. — (i) Firstly, $\tilde{\lambda}$ is constant on isomorphism classes:

Let $(g', g): (T_1\beta_1, f_1, T_1) \rightsquigarrow (T_2\beta_2, f_2, T_2)$ be an isomorphism in $\mathcal{F}ib$. Put Ξ_i for $\mathrm{End}_A(T_i)$, an order in $X_i \stackrel{\mathrm{def}}{=} \mathrm{End}_A(T_i\mathbf{Q})$.

Now $f_1 g = g' f_2$ and $T_1 g \beta_2 = T_2 \beta_2 = T\beta_1 g'$. So $g\beta_2 = u\beta_1 g'$ for some u in $\mathcal{U}(\Xi_1)$. Hence $g\beta_2 f_2 g^{-1} = u\beta_1 f_1$ and now $\partial_{\mathcal{J}}(\mathrm{Nrd}_{X_2}(\beta_2 f_2)) = \partial_{\mathcal{J}}(\mathrm{Nrd}_{X_1}(\beta_1 f_1))$. (Since T_1 is locally isomorphic to $M^{(t)}$, Ξ_1 is locally isomorphic to $\mathrm{Mat}_t(\Theta)$ and so $\mathrm{Nrd}_{X_1}(u)$ lies in $\mathrm{Nrd}_E(\mathcal{U}(\mathrm{Mat}_t(\Theta))) = \mathrm{Nrd}_E(\mathcal{U}(\Theta))$).

Secondly, it is easily verified that $\tilde{\lambda}$ is multiplicative with respect to direct sum and composition of triples. So $\tilde{\lambda}$ factors through $\mathcal{K}_0^{\mathrm{ls}}(M, A, \otimes_{\mathbf{Z}} U)$.

Thirdly (and using the notation of 2.5), with $W' = W$, $f \in \overline{X}^\times$ and $\beta \in \mathcal{J}(X)$ may be arbitrarily chosen. So by 2.1, the image of λ is as claimed.

Finally, if $x \in \ker(\lambda)$ then we can take $x = [T\beta, f, T]$ as in 2.3(ii). So $\mathrm{Nrd}_X(\beta f)$lies in $\mathrm{Nrd}_X(\mathcal{U}(\Xi))$. Thus, pre-multiplying β by an element of $\mathcal{U}(\Xi)$ (and hence not changing x), we have $\mathrm{Nrd}_X(\beta f) = 1$. Then $\mathrm{Nrd}_X(\beta)^{-1} = \mathrm{Nrd}_X(f)$ and so this common value lies in $\mathrm{Nrd}_X(\mathcal{J}(X)) \cap \mathrm{Nrd}(\overline{X}^\times) = \mathrm{Nrd}_X(X^\times)$ (Hasse-Schilling). So we may choose $g \in X^\times$ such that $\mathrm{Nrd}(\beta g) = \mathrm{Nrd}(g^{-1} f) = 1$. Hence $x = [T\beta g, g^{-1} f, T] = 0$ by 2.5(iii).

(ii) and (iii) now follow easily from the description of λ. □

Let $cls = cls_M = cls_{M,A}$ denote the epimorphism $\partial_{\mathcal{J}} \circ \lambda^{-1} \circ \delta$ of $\overline{C}^\times \mathcal{J}(C)$ onto $Cl(M, A)$. We have easily the

COROLLARY 2.7. — *(i) For $i = 1$ and 2, let M_i and M_i' be full lattices in some faithful A-module. Suppose that M_i' lies in the genus of M_i*

and that $[M_i] - [M_i'] = cls_{M_i}(u_i)$ *in* $Cl(M_i, A)$. *Then* $[M_1 \oplus M_2] - [M_1' \oplus M_2'] = cls(u_1 u_2)$ *in* $Cl(M_1 \oplus M_2, A)$.

(ii) *If* M *is a local direct summand of* M' *then* $cls_{M'}(\mathrm{Nrd}_E(\mathcal{U}(\Theta))) = \{0\}$.

There is a particularly useful description of λ on triples containing an ideal of Λ.

COROLLARY 2.8. — *Let* I *be a (right) ideal of* Λ *and* T *a* Λ*-module. Suppose that both* I *and* T *are locally isomorphic to* M. *For all rational places* p *choose* a_p *in* T *so that* $a_p I_p = T_p$. *Put* $a = \{a_p\}$ *and let* $f: \overline{T} \to \overline{A}$ *be an* $\overline{A}$*-isomorphism. Then* $\lambda([T, f, I]) = \partial_{\mathcal{J}}(\mathrm{Nrd}_E(f(a)))$.

Proof. — View a as a local isomorphism from I to N. □

Let K be a sum of fields embeddable in U and let n be the dimension of K over $\mathbf{Q}$. Put $A_K = A \otimes_{\mathbf{Q}} K$ and so also $C_K (\stackrel{\mathrm{def}}{=} Z(A_K)) = C \otimes_{\mathbf{Q}} K$. Suppose that V (the ambient A-module of the lattice M) is an A_K-module which obtains its A-module structure by restriction. We put $\Theta_K \stackrel{\mathrm{def}}{=} \mathrm{End}_{A_K}(M)$ and $E_K \stackrel{\mathrm{def}}{=} \mathrm{End}_{A_K}(V)$. Now $\overline{K}$ is canonically isomorphic to $U^{(n)}$ (see 3.1). So $\overline{A_K} = \overline{A}^{(n)}$, $\overline{C_K} = \overline{C}^{(n)}$ and the following result is now immediate from 2.4(iii) (cf. 3.2(ii)).

THEOREM 2.9. — *The following diagram commutes*

$$\begin{array}{ccc} \mathcal{K}_0^{\mathrm{ls}}(M, A_K, \otimes_{\mathbf{Z}} U) & \to & \dfrac{\mathcal{J}(\overline{C_K})}{\mathrm{Nrd}_{E_K}\mathcal{U}(\Theta_K)} \\ \Big\downarrow \mathrm{res}_A^{A_K} & & \Big\downarrow \mathcal{N}_{K/\mathbf{Q}} \\ \mathcal{K}_0^{\mathrm{ls}}(M, A, \otimes_{\mathbf{Z}} U) & \to & \dfrac{\mathcal{J}(\overline{C})}{\mathrm{Nrd}_E \mathcal{U}(\Theta)} \end{array}$$

where the right hand map is induced by the map from $\mathcal{J}(\overline{C})^{(n)}$ *to* $\mathcal{J}(\overline{C})$ *which multiplies together the* n *components.*

3. Reduced norms and the 'Hom' language.

Let F be a field of characteristic zero and let $\overline{F}$ be a finite normal extension of F which is big enough for our purposes. Let L be an extension of F lying in $\bar{F}$. We put $\Omega_L = \mathrm{Gal}(\bar{F}/L)$. Let $\hat{F}$ be a commutative F-algebra containing $\bar{F}$ and to which the action of Ω_F extends.

Let A be a semisimple F-algebra and put $\hat{A}$ for $A \otimes_F \hat{F}$. We denote by Nrd_A the reduced norm maps (Ω_F-homomorphisms) from $\hat{A}^\times$ and from $\mathcal{K}_1(\hat{A})$ to $Z(\hat{A})^\times$. (Recall that, if $\bar{F}$ is big enough, $\hat{A}$ is a direct sum of matrix rings over $\hat{F}$ and Nrd_A is the direct sum (product) of the determinants on these matrix rings.)

Let K be a direct sum of finite extensions $K_1, \ldots, K_r$ of F. We assume that $\bar{F}$ contains a copy of each K_j. Put $n = \dim_F(K)$ and let Φ be the set of n non-zero F-embeddings $\phi\colon K \to \bar{F}$. Abusing notation, for $\phi \in \Phi$, we also denote by ϕ the map from $K \otimes_F \hat{A}$ to $\hat{A}$ by $k \otimes a \mapsto k^\phi a$. For any module M, we regard the elements of $\mathrm{Map}(\Phi, M)$ as sets $\{m_\phi\}$ indexed by Φ. Then Φ gives a canonical isomorphism

$$(3.1) \qquad (\ )^\Phi\colon K \otimes_F \hat{A} \to \mathrm{Map}(\Phi, \hat{A}) \quad \text{by} \quad b \mapsto \{b^\phi\}\ .$$

We identify $K \otimes_F \hat{A}$ with $\mathrm{Map}(\Phi, \hat{A})$ in this way. We have a norm map $\mathcal{N}_{K/F} = \mathcal{N}(K/F, \hat{A})$ from $(K \otimes_F \hat{A})^\times$ to $\hat{A}^\times$ by $\{a_\phi\} \mapsto \prod_\phi a_\phi$. One easily obtains

LEMMA 3.2. — (i) *If A is commutative then $\mathcal{N}(K/F, \hat{A})$ is an Ω_F-homomorphism.*

(ii) *The following diagram commutes*

$$\begin{array}{ccc} \mathcal{K}_1(K \otimes_F \hat{A}) & \xrightarrow{\mathrm{Nrd}_A} & Z(K \otimes_F \hat{A})^\times \\ \big\downarrow \mathrm{res} & & \big\downarrow \mathcal{N}(K/F, Z(\hat{A})) \\ \mathcal{K}_1(\hat{A}) & \xrightarrow{\mathrm{Nrd}_A} & Z(\hat{A})^\times . \end{array}$$

□

Let Γ be a finite group (and assume that $\bar{F}$ is big enough for Γ). We write $R_\Gamma = R_{\bar{F},\Gamma}$ for the character ring of Γ over $\bar{F}$. We recall the following lemma (cf. [F2] II §1).

LEMMA 3.3. — *There is an isomorphism of Ω_F-modules :*

$$\mathrm{Hom}_{\mathbf{Z}}(R_\Gamma, \hat{F}^\times) \xrightarrow{\sim} Z(\hat{F}\Gamma)^\times \ \ \textit{by}\ \ f \mapsto \sum_{\chi\ \mathrm{irr.}} f(\chi) e_\chi\ .$$

Here e_χ denotes the idempotent of $\bar{F}\Gamma$ corresponding to the irreducible character χ. (We tend to identify these groups. Where necessary we denote the inverse of this isomorphism by 'hom_Γ'.)

Let χ be a character of Γ over $\overline{F}$ and let ρ_χ be a matrix representation with character χ. We extend ρ_χ to $\hat{F}\Gamma$ and obtain a homomorphism $\det_\chi$ from $\hat{F}\Gamma^\times$ to $\hat{F}^\times$ by $\alpha \mapsto \det(\rho_\chi(\alpha))$. We note that $\det_\chi$ restricts to an abelian character of Γ.

We denote by $\det_\Gamma$ the Ω_F-homomorphism from $\hat{F}\Gamma^\times$ to $\mathrm{Hom}(R_\Gamma, \hat{F}^\times)$ such that $\det_\Gamma(\alpha): \chi \mapsto \det_\chi \alpha$ for all characters χ. It follows more or less immediately from the definitions that $\mathrm{Nrd}_{F\Gamma} = \det_\Gamma$ on $\hat{F}\Gamma^\times$ given the identification of 3.3. More precisely,

(3.4)
$$\mathrm{Nrd}_{F\Gamma} \circ \mathrm{hom}_\Gamma = \det_\Gamma \ .$$

Let Δ be a subgroup of Γ. We denote by $(\mathrm{res}^\Gamma_\Delta)^*$ the map — effectively an induction map — from $\mathrm{Hom}(R_\Delta, \hat{F}^\times)$ to $\mathrm{Hom}(R_\Gamma, \hat{F}^\times)$ which is composition with $\mathrm{res}^\Gamma_\Delta : R_\Gamma \to R_\Delta$. We recall one of the most important properties of the 'Hom' notation.

LEMMA 3.5. — *(i) If α lies in $\hat{F}\Delta^\times$ then* $\mathrm{res}^\Gamma_\Delta \circ \det_\Delta(\alpha) = \det_\Gamma(\alpha)$. *(These are both mappings from R_Γ to $\hat{F}^\times$. To put it a different way,* $\det_\Delta \circ (\mathrm{res}^\Gamma_\Delta)^* = \det_\Gamma$.*)*

(ii) *The following diagram commutes.*

$$\begin{array}{ccccc}
\mathcal{K}_1(\hat{F}\Delta) & \xrightarrow{\mathrm{Nrd}_{F\Delta}} & Z(\hat{F}\Delta)^\times & \xrightarrow[\sim]{\mathrm{hom}_\Delta} & \mathrm{Hom}(R_\Delta, \hat{F}^\times) \\
\downarrow {\scriptstyle \mathrm{ind}^\Gamma_\Delta} & & & & \downarrow {\scriptstyle (\mathrm{res}^\Gamma_\Delta)^*} \\
\mathcal{K}_1(\hat{F}\Gamma) & \xrightarrow{\mathrm{Nrd}_{F\Gamma}} & Z(\hat{F}\Gamma)^\times & \xrightarrow[\sim]{\mathrm{hom}_\Gamma} & \mathrm{Hom}(R_\Gamma, \hat{F}^\times).
\end{array}$$

Proof. — (i) The equation asserts that, for a character χ of Γ, $\det_\chi(\alpha)$ is the same whether we regard χ as a character of Γ or as restricted to Δ. Since the representation ρ_χ that we chose above restricts nicely to a suitable representation of Δ, this is entirely obvious.

(ii) Let $x \in \mathcal{K}_1(\hat{F}\Delta)$. Choose $\alpha \in \hat{F}\Delta^\times$ to represent x. Then

$$x\,\mathrm{Nrd}_{F\Delta} \circ \mathrm{hom}_\Delta \circ (\mathrm{res}^\Gamma_\Delta)^* = \alpha\,\mathrm{Nrd}_{F\Delta} \circ \mathrm{hom}_\Delta \circ (\mathrm{res}^\Gamma_\Delta)^* \overset{3.4}{=} \alpha \det_\Delta \circ (\mathrm{res}^\Gamma_\Delta)^*$$
$$\overset{(i)}{=} \alpha \det_\Gamma \overset{3.4}{=} \alpha\,\mathrm{Nrd}_{F\Gamma} \circ \mathrm{hom}_\Gamma = x\ \mathrm{ind}^\Gamma_\Delta \circ \mathrm{Nrd}_{F\Gamma} \circ \mathrm{hom}_\Gamma.$$

□

4. Resolvents and transfer for sums of Galois algebras.

We take F, $\overline{F}$, $\hat{F}$, K, Φ, and Γ as in §3. For $j = 1, \ldots, r$, let N_j be a Γ-Galois algebra over K_j and choose a simple factor L_j of N_j. Let N be the direct sum of the N_j. We denote by ρ and ρ_j the implied representations of Γ in $\mathrm{Aut}_K(N)$ and $\mathrm{Aut}_{K_j}(N_j)$. Let Γ_j be the decomposition (stability) group of L_j in Γ and let σ_j be the representation of Γ_j as $\mathrm{Gal}(L_j/K)$.

For each ϕ in Φ let $\tilde{\phi}: N \to \bar{F}$ be an F-homomorphism extending ϕ. Put $\tilde{\Phi}$ for the set of these $\tilde{\phi}$. We describe $\rho = (\rho, N, \tilde{\Phi})$ as a Γ-Galois algebra over K fully embedded in $\bar{F}$ over F.

Our first aim is to describe a "transfer" map from Ω_F to Γ^{ab}. We need this to give the action of Ω_F upon the resolvents that we shall define.

LEMMA 4.1. — *A homomorphism θ from N to $\bar{F}$ which extends $\phi \in \Phi$ induces a homomorphism $\theta^*: \Omega_{K^\phi} \longrightarrow \Gamma$ such that, for $\omega \in \Omega_{K^\phi}$, we have $(\omega^{\theta^*\rho}) \circ \theta = \theta \circ \omega$.*

Proof. — Let L be the (unique) simple factor of N such that $L^\theta \neq \{0\}$. Then θ induces an isomorphism from $\mathrm{Gal}(L^\theta/K^\phi)$ to $\mathrm{Gal}(L/L \cap K)$ and thence to the decomposition group of L in Γ. Thus θ induces a homomorphism as required. □

Take $\omega \in \Omega_F$ and $\phi \in \Phi$. Then $\phi\omega = \psi$ for some $\psi \in \Phi$. Now, since $N^{\tilde{\psi}}$ is normal over K^ψ, $\tilde{\phi}\omega = \tilde{\psi}\omega'$ for some $\omega' \in \Omega_{K^\psi}$. We put $\omega^{\tilde{\phi}^\#} \stackrel{\mathrm{def}}{=} \omega'^{\tilde{\psi}^*}$ (so that $\tilde{\phi} \circ \omega = (\omega^{\tilde{\phi}^\# \rho}) \circ \tilde{\psi}$ and $\tilde{\phi}^\#$ agrees with $\tilde{\phi}^*$ on Ω_{K^ϕ}). We define the transfer map

$$\mathrm{Ver}(K/F, \rho): \Omega_F \to \Gamma^{\mathrm{ab}} \quad \text{by} \quad \omega \mapsto \prod_{\phi \in \Phi} \omega^{\tilde{\phi}^\#}.$$

THEOREM 4.2. — (i) $\mathrm{Ver}(K/F, \rho)$ *is a homomorphism and independent of the choice of* $\tilde{\Phi}$.
Choose $\omega \in \Omega_F$ and a $\phi_j \in \Phi_j$ for each j. Working in Γ^{ab} we have

(ii) $\omega \,\mathrm{Ver}(K/F, \rho) = \prod_{j=1}^{n} \omega \,\mathrm{Ver}(K_j/F, \rho_j)$.

(iii) $\omega \,\mathrm{Ver}(K/F, \rho_j) \stackrel{(a)}{=} \omega \,\mathrm{Ver}(K/F, \sigma_j) \stackrel{(b)}{=} \omega \,\mathrm{Ver}(K^\phi/F) \circ \tilde{\phi}_j^*$ *where* $\mathrm{Ver}(K^\phi/F)$ *is the transfer mapping from* Ω_F *to* $\Omega_{K^\phi}^{\mathrm{ab}}$.

Proof. — (i) This is a matter of calculation observing that the different choices for $\tilde{\phi}$ differ by an element of $\rho(\Gamma)$.

(ii) and (iii)a follow immediately from 4.1 and the definitions. Part (iii)b pis a matter of calculation (see [F2], proof of Theorem 20A). □

We define the *resolvent mapping* (an $\hat{F}\Gamma$-homomorphism)

$$\zeta_\rho : N \otimes_F \hat{F} \to (N \otimes \hat{F})\Gamma \quad \text{by} \quad b \mapsto \sum_{\gamma \in \Gamma} b^\gamma \gamma^{-1}.$$

For a in $N \otimes_F \hat{F}$ we define the *reduced resolvent* $(a \mid \rho)$ of a with respect to ρ to be $\det_\Gamma(\zeta_\rho(a))$ — this is an element of $Z(N \otimes \hat{F}\Gamma)$. Now the canonical isomorphism of 3.1 extends to a homomorphism

$$(4.3) \qquad (\ )^{\tilde{\Phi}} : N \otimes_F \hat{A} \to \mathrm{Map}(\Phi, \hat{A}) \quad \text{by} \quad b \mapsto \{b^{\tilde{\phi}}\}\ .$$

We then define the *norm resolvent* (of a with respect to ρ over F) to be $\mathfrak{R}(a \mid \rho) = \mathfrak{R}(a \mid \rho, \widetilde{\Phi}) \stackrel{\text{def}}{=} \mathcal{N}_{K/F}\left((a \mid \rho)^{\widetilde{\Phi}}\right)$, possibly suppressing the $\widetilde{\Phi}$ if $F = \mathbf{Q}$. Indeed we find

LEMMA 4.4. — *Let $\omega \in \Omega_F$ and $\phi \in \Phi$. Take ψ as before 4.2. Choose $a \in N \otimes_F \hat{F}$.*

(i) $\zeta_\rho(a)^{\tilde{\phi}\omega} = \omega^{\tilde{\phi}\#} \cdot \zeta_\rho(a)^{\tilde{\psi}}$ *and so* $(a \mid \rho)^{\tilde{\phi}\omega} = (a \mid \rho)^{\tilde{\psi}} \cdot \det_\Gamma(\omega^{\tilde{\phi}\#})$.

(ii) *If $\breve{\Phi} = \{\breve{\phi} \mid \phi \in \Phi\}$ is another choice of extensions to N of the embeddings Φ of K then $\mathfrak{R}(a \mid \rho, \breve{\Phi}) = \mathfrak{R}(a \mid \rho, \widetilde{\Phi}) \cdot \det_\Gamma(\delta)$ for some $\delta \in \Gamma$.*

Proof. — (i)

$$\zeta_\rho(a)^{\tilde{\phi}\omega} = \left(\sum_{\gamma \in \Gamma} a^\gamma \gamma^{-1}\right)^{(\omega^{\tilde{\phi}\#})\tilde{\psi}} = \left(\sum_{\gamma \in \Gamma} a^{\gamma(\omega^{\tilde{\phi}\#})} \gamma^{-1}\right)^{\tilde{\psi}}$$

$$= \left(\sum_{\gamma \in \Gamma} a^\gamma \left(\gamma(\omega^{\tilde{\phi}\#})^{-1}\right)^{-1}\right)^{\tilde{\psi}} = \omega^{\tilde{\phi}\#} \cdot \zeta_\rho(a)^{\tilde{\psi}}.$$

(ii) follows immediately from the second part of (i). □

We obtain immediately from 4.4(i)

THEOREM 4.5. — *If $\omega \in \Omega_F$ and $a \in N \otimes_F \hat{F}$ then* $\mathfrak{R}(a \mid \rho, \widetilde{\Phi})^\omega = \mathfrak{R}(a \mid \rho, \widetilde{\Phi}) \cdot \det_\Gamma\left(\omega \mathrm{Ver}(K/F, \rho)\right)$.

Of course the resolvents obtained from ρ are simply related to those obtained from the ρ_i and, indeed, to those obtained from the σ_i.

THEOREM 4.6. — *For $j = 1, \ldots, r$, put $\Phi_j \stackrel{\text{def}}{=} \{\phi \in \Phi \mid K_j^\phi \neq \{0\}$ and $\widetilde{\Phi}_j \stackrel{\text{def}}{=} \{\tilde{\phi} \mid \phi \in \Phi_j\}$ and choose a set $\breve{\Phi}_j = \{\breve{\phi} \mid \phi \in \Phi_j\}$, where $\breve{\phi}$ is an* *embedding of L_j in $\bar{F}$ which extends $\phi\,|_{K_j}$. Choose a_j in N_j for each j and put $a = \sum_j a_j$. We have*

(i) $$\Re(a\,|\,\rho, \widetilde{\Phi}) = \prod_j \Re(a_j\,|\,\rho_j, \widetilde{\Phi}_j)\ .$$

(ii) *If $a_j \in L_j$ then $\Re(a_j\,|\,\rho_j, \widetilde{\Phi}_j) = \mathrm{res}^{\Gamma}_{\Gamma_j}\left[\Re(a_j\,|\,\sigma_j, \check{\Phi}_j)\right]\cdot \det_\Gamma(\delta)$ for some $\delta \in \Gamma$.*

Proof. — (i) This is clear.

(ii) Let Π be the projection from N_j to L_j. Then $\Pi\check{\Phi}_j$ is a set of F-homomorphisms of N_j into $\bar{F}$ extending those of K_j so, by 4.4(ii),

(*) $$\Re(a_j\,|\,\rho_j, \widetilde{\Phi}_j) = \Re(a_j\,|\,\rho_j, \Pi\check{\Phi}_j)\cdot \det_\Gamma(\delta).$$

for some $\delta \in \Gamma$. But if $\gamma \in \Gamma \setminus \Gamma_j$ then $a_j^\gamma \notin L_j$ and $a^{\gamma\Pi} = 0$. So

$$\zeta_{\rho_j}(a_j)^{\Pi\check{\phi}} = \sum_{\gamma\in\Gamma} a_j^{\gamma\Pi\check{\phi}}\gamma^{-1} = \sum_{\gamma\in\Gamma_j} a_j^{\gamma\check{\phi}}\gamma^{-1} = \zeta_{\sigma_j}(a_j)^{\check{\phi}}\ .$$

Whence, by 3.5, $(a_j\,|\,\rho_j)^{\Pi\check{\phi}} = \mathrm{res}^{\Gamma}_{\Gamma_j}\circ\left((a_j\,|\,\sigma_j)^{\check{\phi}}\right)$ and the result follows from (*). □

We recall (cf. [F2] I 3.1) the following result which expresses the fundamental properties of the resolvent mapping.

THEOREM 4.7. — (i) *The map $\zeta_\rho \circ \widetilde{\Phi}$ from $N \otimes_F \hat{F}$ to* $\mathrm{Map}(\Phi, \hat{F}\Gamma)$ *is an $\hat{F}\Gamma$-isomorphism.*

(ii) *The element a generates $N \otimes_F \hat{F}$ over $K \otimes_F \hat{F}\Gamma$ if and only if $\zeta_\rho(a) \in (N \otimes \hat{F}\Gamma)^\times$ (equivalently $(a\,|\,\rho) \in \det_\Gamma(N \otimes \hat{F}\Gamma)^\times$ or, indeed, $\Re(a\,|\,\rho, \widetilde{\Phi}) \in Z(\hat{F}\Gamma)^\times$) .*

Proof. — (i) This follows from the linear independence of the F-embeddings of N in $\bar{F}$ and the equality of the dimensions.

(ii) From (i) we deduce that a generates $N\otimes_F \hat{F}$ if and only if $\zeta_\rho(a)^{\widetilde{\Phi}}$ is a unit in $\mathrm{Map}(\Phi, \hat{F}\Gamma)$. Moreover this holds for any choice of of $\widetilde{\Phi}$. □

We establish in the rest of this section the notations and specifications for the remainder of the paper. We set $F = \mathbf{Q}$ and $\bar{F} = U$ where U is a subfield of $\mathbf{C}$ which is finite and normal over $\mathbf{Q}$ and is big enough for our purposes (we want U to contain a copy of each component of N, the Gauss sums and root numbers of §5 and the $|\Gamma|^{\text{th}}$ roots of 1). We write, as usual, $\mathcal{O}_K$ for the maximal order in K. By a *prime* of K we mean a prime (finite or infinite) of one of the K_j. The symbol $\wp$ will always stand for such a prime. We adopt the reasonable convention that if M is an $\mathcal{O}_K$-module and $\wp$ is a prime of K_j then the completion of M at $\wp$ is $M_\wp \stackrel{\text{def}}{=} M \otimes_{\mathcal{O}_K} (\mathcal{O}_{K_j})_\wp$ (where $(\mathcal{O}_{K_j})_\wp$, the completion of $\mathcal{O}_{K_j}$ at $\wp$, is understood to be $(K_j)_\wp$ if $\wp$ is Archimedean).

We denote by p some prime of $\mathbf{Q}$ and by $\wp$ some prime of K lying over p. For each such $\wp$ we *choose* a prime $\check{\wp}$ of N lying over $\wp$. We put ρ_p and $\rho_\wp$ for the Galois representations of Γ afforded by N_p and $N_\wp$ and we put $\rho_{check\wp}$ for that of the decomposition group $\Gamma(\check{\wp})$ afforded by $N_{\check{\wp}}$.

We choose an ideal M of $\mathcal{O}_K\Gamma$ in the $\mathcal{O}_K\Gamma$-genus of $\mathcal{O}_N$ such that $M_\wp = (\mathcal{O}_K\Gamma)_\wp$ if $N_\wp/K_\wp$ is tame. We choose $a = \{a_\wp\}$ in the adèle ring of $\mathcal{O}_N$ such that $a_\wp M_\wp = (\mathcal{O}_N)_\wp$ for each $\wp$. Furthermore, if $N_\wp/K_\wp$ is tame, we demand — as we may — that $a_\wp \in N_{\check{\wp}}$ and $a_\wp \mathcal{O}_{K_\wp}\Gamma(\check{\wp}) = \mathcal{O}_{N_{\check{\wp}}}$. The link between the present work and that of the §2 is now given in the following theorem.

THEOREM 4.8. — *In* $\mathcal{K}_0^{\mathrm{ls}}(M, \mathbf{Q}\Gamma, \otimes_{\mathbf{Z}} U)$,

$$[\mathcal{O}_N, M, \zeta_\rho \circ \widetilde{\Phi}] = \lambda^{-1}(\partial_{\mathcal{J}}(\Re(a \mid \rho, \widetilde{\Phi}))) \ .$$

Proof. — Apply 2.8 with $\Lambda = \mathcal{O}_K\Gamma$ and then 2.9 with $\Lambda = \mathbf{Z}\Gamma$. □

Let $\mathcal{P}$ denote a prime of U lying over p. Put $\Pi_{\mathcal{P}}$ for the natural projection of U_p onto $U_{\mathcal{P}}$ and put $\Phi(\wp, \mathcal{P})$ for the $\mathbf{Q}_p$-embeddings of $K_\wp$ in $U_{\mathcal{P}}$. Now $K_p = \bigoplus_{\wp|p} K_\wp$ and we have corresponding dissections $\Phi\Pi_p = \bigcup_{\wp|p} \Phi(\wp, \mathcal{P})$ and $\widetilde{\Phi}\Pi_{\mathcal{P}} = \bigcup_{\wp|p} \widetilde{\Phi}(\wp, \mathcal{P})$ where $\widetilde{\Phi}(\wp, \mathcal{P})$ is a choice of embeddings of N in $U_{\mathcal{P}}$ extending the elements of $\Phi(\wp, \mathcal{P})$. We choose a set $\check{\Phi}(\wp, \mathcal{P}) = \{\check{\psi} \mid \psi \in \Phi(\wp, \mathcal{P})\}$ of extensions to $N_{\check{\wp}}$ of the embeddings $\psi: K_\wp \to U_{\mathcal{P}}$.

The triple $(\rho_p, N_p, \widetilde{\Phi}\Pi_{\mathcal{P}})$ is a Γ-Galois algebra over K_p fully embedded in $U_{\mathcal{P}}$ over $\mathbf{Q}_p$. So, by 4.6, we have the following relationships between a globally formed resolvent, the resolvents formed at $\wp$ and the "decomposed" resolvents at $\check{\wp}$.

THEOREM 4.9. — *If* $a_p = \sum_{\wp|p} a_\wp \in N_p$, *then*

(i) $$\Re(a_p \mid \rho_p, \widetilde{\Phi})^{\Pi_{\mathcal{P}}} = \Re(a_p \mid \rho_p, \widetilde{\Phi}\Pi_{\mathcal{P}}) = \prod_{\wp|p} \Re(a_\wp \mid \rho_\wp, \widetilde{\Phi}(\wp, \mathcal{P})).$$

(ii) *If* $a_\wp \in N_{\check{\wp}}$ *then, for some* $\delta \in \Gamma$,

$$\Re(a_\wp \mid \rho_\wp, \widetilde{\Phi}(\wp, \mathcal{P})) = \left[\mathrm{res}^{\Gamma}_{\Gamma(\check{\wp})} \circ \Re(a_\wp \mid \rho_{\check{\wp}}, \check{\Phi}(\wp, \mathcal{P}))\right] \cdot \det{}_\Gamma(\delta). \qquad \square$$

Finally, we assure ourselves of a case where the resolvent is effectively trivial.

THEOREM 4.10. — *If $\wp$ is unramified in N then*

$$(a_\wp \mid \rho_\wp) \in \det_\Gamma(\mathcal{O}_{N_\wp}\Gamma^\times) .$$

Proof. — As $\wp$ is unramified, $a_\wp$ mod $\wp$ generates a Γ-Galois algebra, namely $\mathcal{O}_{N_\wp}$ mod $\wp$. Thus, by 4.7, the resolvent is a unit mod $\wp$ and therefore a unit, as required. □

5. Galois-Gauss sums and root numbers.

We wish to use the local and global Galois-Gauss sums as discussed in [M] and [F]. Our machinery is already complex and the essential properties that we shall need of these objects are all proved elsewhere. For this reason, we do not consider the sums in their "proper" context (that is defined on characters of the Galois group of some universal extension of our base field) but in a manner best suited to our present needs (that is defined, via ρ, on characters of our group Γ and of its decomposition groups).

We adopt the notation developed before Theorem 4.8. So $F = \mathbf{Q}$, $\bar{F} = U$ and N is a Γ-Galois algebra over K. Also, p, $\wp$ and $\breve{\wp}$ are primes of $\mathbf{Q}$, K and N, repectively, such that $\breve{\wp} \mid \wp \mid p$. Let ψ be a character of $\mathrm{Gal}(N_{\breve{\wp}}/K_\wp)$. Then $W(\psi)$ is the local root number associated with ψ and (if $p \neq \infty$) $\tau(\psi)$ is the Galois-Gauss sum.

Recall that $\rho_{\breve{\wp}}$ is the isomorphism, induced by ρ, from the decomposition group $\Gamma(\breve{\wp})$ of $\breve{\wp}$ to $\mathrm{Gal}(N_{\breve{\wp}}/K_\wp)$. We define $\overline{W}(\rho_{\breve{\wp}})$ and (if $p \neq \infty$) $\tau(\rho_{\breve{\wp}})$ in $\mathrm{Hom}(R_{\Gamma(\breve{\wp})}, U^\times)$ to be the maps which send χ to $W(\rho_{\breve{\wp}}^{-1} \circ \bar{\chi})$ and $\tau(\rho_{\breve{\wp}}^{-1} \circ \chi)$ respectively. These maps depend only on $N_{\breve{\wp}}$, $K_\wp$ and $\rho_{\breve{\wp}}$.

Following [F2], we define the corresponding semi-local objects by "induction". Thus $\overline{W}(\rho_\wp) = \mathrm{res}^\Gamma_{\Gamma(\breve{\wp})} \circ \overline{W}(\rho_{\breve{\wp}})$ and $\tau(\rho_\wp) = \mathrm{res}^\Gamma_{\Gamma(\breve{\wp})} \circ \tau(\rho_{\breve{\wp}})$.

Let S be a finite set of finite primes of $\mathbf{Q}$ which contains all those which are either ramified in N or divisors of $|\Gamma|$. It follows from the definitions that if $\wp \mid p \notin S \cup \{\infty\}$ then $\tau(\rho_\wp) = \overline{W}(\rho_\wp) = 1$. So we may define the corresponding global objects as products of the semilocal ones in the following manner.

$$(5.1) \quad \tau(\rho) = \prod_{\wp < \infty} \tau(\rho_\wp), \quad \overline{W}(\rho) = \prod_{\wp} \overline{W}(\rho_\wp), \quad \overline{W}_\infty(\rho) = \prod_{\wp \mid \infty} \overline{W}(\rho_\wp)$$

We define $W^s(\rho)$ to be the homomorphism which agrees with $\overline{W}(\rho)$ on the symplectic irreducible characters and is 1 on the other irreducibles.

Thus the objects we use are mild generalisations of those studied in [F] and [M]. They inherit the following properties.

THEOREM 5.2. — (i) *The maps $\overline{W}(\rho_\wp)$ and (for $\wp < \infty$) $\tau(\rho_\wp)$ in $\mathrm{Hom}(R_\Gamma, U^\times)$ are independent of the choice of $\breve{\wp}$. (So they depend only upon $N_\wp$, $K_\wp$ and $\rho_\wp$.)*

(ii) *If $\wp < \infty$ then $\tau(\rho_\wp) = |\tau(\rho_\wp)|\, \overline{W}(\rho_\wp)$ and $\tau(\rho_\wp)^\omega = \tau(\rho_\wp)\det_\Gamma(\gamma)$ for any $\omega \in \Omega_{\mathbf{Q}}$ (where γ is some element of Γ depending on ω and $\rho_\wp$).*

(iii) *$W^s(\rho)$ agrees with $\overline{W}(\rho)$ on all symplectic characters and takes only the values ± 1.*

(iv) *If $\omega \in \Omega_{\mathbf{Q}}$ then $\tau(\rho)^\omega = \tau(\rho)\det_\Gamma(\omega\mathrm{Ver}(K/F, \rho))$*

(v) *$\overline{W}(\rho)\, |\tau(\rho)| = \tau(\rho)\overline{W}_\infty(\rho)$.*

(The moduli, $|\tau(\rho)|$ and $|\tau(\rho_\wp)|$, are defined pointwise using the norm that U inherits as a subfield of $\mathbf{C}$.)

Proof. — (i) Given our chosen prime $\breve{\wp}$ of N lying over $\wp$, any other has the form $\breve{\wp}^\gamma$ for some γ in Γ. And then γ gives a $K_\wp$-isomorphism $: N_{\breve{\wp}} \to N_{\breve{\wp}^\gamma}$. Also $\Gamma(\breve{\wp}^\gamma) = \Gamma(\breve{\wp})^\gamma$ and $\rho(\breve{\wp}^\gamma) = \rho(\breve{\wp})^\gamma$. So, in an obvious sense, $\tau(\rho_{\breve{\wp}^\gamma}) = \tau(\rho_{\breve{\wp}})^\gamma$ etc. and the result follows.

(ii) These results with $\rho_{\breve{\wp}}$ replacing $\rho_\wp$ may be found in [M] pp. 38–39 (4.1) and p. 42 (5.1). Our results follow on composition with $\mathrm{res}^\Gamma_{\Gamma(\breve{\wp})}$.

(iii), (iv) & (v) We use the notation of §4. So, in particular, σ_j is an isomorphism from Γ_j to $\mathrm{Gal}(L_j/K_j)$. We have immediately from (i) and 5.1 that

$$(5.3) \qquad \tau(\rho) = \prod_{j=1}^{r} \tau(\rho_j) = \prod_{j=1}^{r} \mathrm{res}^\Gamma_{\Gamma_j} \circ \tau(\sigma_j) \quad \text{and similarly for } \overline{W}(\rho).$$

Part (iii) with $\rho = \sigma_j$ is standard ([M] §7). Since restriction sends symplectic characters to symplectic characters, the result for $\rho = \rho_j$, and hence (iii) in general, follows on composition with $\mathrm{res}^\Gamma_{\Gamma_j}$.

Let $\breve{\phi}$ be an embedding of L_j into U then, translating [F2] p.119 Theorem 20B into our terms (and pulling back to Γ_j) we have
$\tau(\sigma_j)^\omega = \tau(\sigma_j).\det_{\Gamma_j}(\omega\mathrm{Ver}(K_j^\phi/\mathbf{Q}) \circ \breve{\phi}^*) = \tau(\sigma_j).\det_{\Gamma_j}(\omega\mathrm{Ver}(K_j/\mathbf{Q}, \sigma_j))$
where we have applied 4.2(iii)b. On composing with $\mathrm{res}^\Gamma_{\Gamma_j}$ and applying 4.2(iii)a and 3.5, we obtain

$$\tau(\rho_j)^\omega = \tau(\rho_j).\det_\Gamma(\omega\mathrm{Ver}(K_j/\mathbf{Q}), \rho_j)$$

and (iv) now follows by 5.3 and 4.2(ii).

Similarly (v) follows from 5.3 and the result for $\rho = \sigma_j$ ([F2] I 5.22 and 5.23). $\square$

We need to introduce two more elements of $Z(U\Gamma)^\times = \mathrm{Hom}(R_\Gamma, U^\times)$ associated with $\rho_\wp$ for finite $\wp$. Firstly, recall ([F2] p.149) the definition of the *non-ramified characteristic*, $y(\rho_{\breve\wp})$, in $\mathrm{Hom}(R_{\Gamma_{\breve\wp}}, U^\times)$. Let $\mathrm{nr}\colon R_{\Gamma(\breve\wp)} \to R_{\Gamma(\breve\wp)}$ be the "non-ramified part" map such that if χ is the character of a representation space V then $\mathrm{nr}(\chi)$ is the character of $V^{\Gamma(\breve\wp)_0}$ where $\Gamma(\breve\wp)_0$ is the inertia group of $\breve\wp$. Let $\mathrm{art}\colon K_{\breve\wp}^\times \to \Gamma(\breve\wp)^{\mathrm{ab}}$ be the Artin map composed with $\rho_{\breve\wp}^{-1}$ and let π be a generator of $\wp_\wp$. Then we define $y(\rho_{\breve\wp}) \stackrel{\mathrm{def}}{=} [\chi \mapsto (-1)^{\deg(\mathrm{nr}(\chi))}\det_{\mathrm{nr}(\chi)}(\mathrm{art}(\pi))]$ and $y(\rho_\wp) \stackrel{\mathrm{def}}{=} \mathrm{res}^\Gamma_{\Gamma(\breve\wp)} \circ y(\rho_{\breve\wp})$.

Secondly, recall ([F2] p.151) what might be called the "tame fudge factor". We choose c to generate $\wp\times\mathrm{Different}(K_\wp/\mathbf{Q}_p)$. We define $z(\rho_{\breve\wp}) \stackrel{\mathrm{def}}{=} [\chi \mapsto \det_\chi(\mathrm{art}(c))]$ and $z(\rho_\wp) \stackrel{\mathrm{def}}{=} \mathrm{res}^\Gamma_{\Gamma_{\breve\wp}} \circ z(\rho_{\breve\wp})$.

We put $y^*(\rho_\wp) = y(\rho_\wp)z(\rho_\wp)^{-1}$ and we put $y^*(\rho)$ for the product of the $y^*(\rho_\wp)$ where $\wp$ ranges over the primes of K which are *tame* in N and divide a prime in S. Again, our $y^*(\rho_\wp)$ is a slight generalization of the local y^* discussed in [F] (p.155). It inherits (cf. the proof of 5.2(ii)) the following property.

LEMMA 5.4. — *$y^*(\rho_\wp)$ lies in $\mathrm{Hom}^+_{\Omega_{\mathbf{Q}}}(R_\Gamma, U^\times)$ ($= Z(\mathbf{Q}\Gamma)^{\times +} = \det(\mathbf{Q}\Gamma^\times)$).*

In the last two theorems of this section we record the important facts about the integrality and positivity relationships between the resolvants and the Galois-Gauss sums. We put $U(p)$ for the maximal extension of $\mathbf{Q}$ in U which is tamely ramified at p. Recall that the $a_\wp$ were chosen (before 4.8) so that $a_\wp(\mathcal{O}_{K_\wp}\Gamma(\breve\wp) = \mathcal{O}_{N_{\breve\wp}}$ if $N_\wp/K_\wp$ is tame.

THEOREM 5.5. — *Let p be a finite prime of $\mathbf{Q}$.*

(i) *If $p \notin S$ then*

(a) $\tau(\rho_\wp) = 1$ *and*

(b) $\mathfrak{R}(a_\wp \mid \rho_\wp, \widetilde\Phi(\wp, \mathcal{P})) \in \det_\Gamma((\mathcal{O}_{U(p)}\Gamma)^\times_{\mathcal{P}})$.

(ii) *If $N_\wp/K_\wp$ is tame then* $[\tau(\rho_\wp)/y^*(\rho_\wp)]^{-1}\mathfrak{R}(a_\wp \mid \rho_\wp, \widetilde\Phi(\wp, \mathcal{P})) \in \det_\Gamma((\mathcal{O}_{U(p)}\Gamma)^\times_{\mathcal{P}})$.

(iii) *If $N_\wp/K_\wp$ is tame and q is an integer prime different from p then*

$$[\tau(\rho_\wp)/y^*(\rho_\wp)]1_{\mathbf{Q}_q} \in \det_\Gamma((\mathcal{O}_{U(q)}\Gamma)^\times_q).$$

Proof. — (i) (a) follows from the definition of τ and (b) is an immediate consequence of 4.10.

(ii) By [F2], Theorem 31, this is true if $\rho_\wp$ is replaced by $\rho_{\breve{\wp}}$ and $\widetilde{\Phi}(\wp,\mathcal{P})$ by $\check{\Phi}(\rho_{\breve{\wp}},\mathcal{P})$. Our result follows on composing with $\mathrm{res}^{\Gamma}_{\Gamma(\breve{\wp})}$ and applying 4.9(ii).

(iii) [F2], Theorem 30 applies in a similar manner. □

Let R^s_Γ be the subgroup of R_Γ generated by the symplectic characters.

THEOREM 5.6. — (i) *Suppose that* $\wp$ *and* $\mathcal{P}$ *are infinite primes of* K *and* U. *Then*
(a) $\Re(a_\wp \mid \rho_\wp, \widetilde{\Phi}(\wp,\mathcal{P}))\overline{W}(\rho_\wp)^{-1}$ *and* (b) $\Re(a \mid \rho, \widetilde{\Phi})_{\mathcal{P}}\, \overline{W}_\infty(\rho)$

are real and positive when restricted to R^s_Γ.

(ii) *If* $\wp$ *is finite then* $\tau(\rho_\wp) \mid_{R^s_\Gamma}$ *is fixed under* $\Omega_{\mathbf{Q}}$. *Hence its values are totally real.*

(iii) *If* $\wp$ *is finite and tame in* N *then* $\tau(\rho_\wp) \mid_{R^s_\Gamma}$ *takes only rational values. In particular* $\mid \tau(\rho_\wp) \mid$ *is totally positive on* R^s_Γ.

(iv) *If* $\wp$ *is tame in* N *then* $\overline{W}(\rho_\wp) \mid_{R^s_\Gamma}$ *is fixed under* $\Omega_{\mathbf{Q}}$ *and takes only the values* ± 1.

Proof. — (i) From [F2], I 5.16 and III Proposition 4.1 the sign of $\Re(a_\wp \mid \rho_{\breve{\wp}}, \check{\Phi}(\breve{\wp},\mathcal{P}))$ agrees with $\overline{W}(\rho_{\breve{\wp}})$ on the symplectic characters and so (a) follows on composing with $\mathrm{res}^{\Gamma}_{\Gamma(\breve{\wp})}$ and applying 4.9(ii), bearing in mind that $\det_\chi(\Gamma) = \{1\}$ if χ is symplectic. Part (b) follows immediately applying 5.1 and 4.9(i).

(ii) Since $\det_\chi(\Gamma) = \{1\}$ if χ is symplectic, this follows immediately from the second part of 5.2(ii).

(iii) Rationality of $\tau(\rho_{\breve{\wp}})\Big|_{R^s_\Gamma}$ is proved in ([F2], Theorem 21) and our result follows on composition with $\mathrm{res}^{\Gamma}_{\Gamma(\breve{\wp})}$.

(iv) This follows, in a similar way, from [F2] Theorem 21 (for $\wp$ finite) and [F2] I 5.16 (for $\wp$ infinite). □

6. The main theorem.

We now suppose that we have a second sum of Γ-Galois algebras N'/K' fully embedded in U over $\mathbf{Q}$. We suppose defined and chosen (subject perhaps to restrictions yet to be specified) all the objects we have defined and chosen for N/K. We distinguish the objects belonging to N'/K' by a ' ''. Thus we have ρ', Φ' and so on.

We continue with the notation developed towards the end of §4. Moreover, if p is a prime of $\mathbf{Q}$ we put $\mathbf{PW}_p = \mathbf{PW}_p(N/K)$ for the set of primes $\wp$ of K dividing p which are wildly ramified in N and put $\mathbf{PT}_p$ for those which are not. We put also $\mathbf{PW}$ and $\mathbf{PT}$ for the unions of the $\mathbf{PW}_p$ and the $\mathbf{PT}_p$ respectively.

We take A of §3 to be $\mathbf{Q}\Gamma$ and M of §3 to be the ideal M of $\mathcal{O}_K\Gamma$ which was chosen for 4.8. So $C = Z(\mathbf{Q}\Gamma) = \mathrm{Hom}_{\Omega_{\mathbf{Q}}}(R_\Gamma, U^\times)$, $\overline{C} = Z(U\Gamma)$ and so on. We recall the map cls from $\overline{C}^\times \mathcal{J}(C)$ to $Cl(M,A)$ which was defined for 2.7 and we write i_∞ for the inclusion of $\overline{C}^\times$ into $\mathcal{J}(\overline{C})$ via the component at infinity.

THEOREM 6.1. — *Suppose that K and K' have the same dimension over $\mathbf{Q}$ and that for each prime p of $\mathbf{Q}$ there is a bijection $\alpha = \alpha_p$ from $\mathbf{PW}_p$ to $\mathbf{PW}'_p$ and for each $\wp$ in $\mathbf{PW}_p$ there is an isomorphism $\alpha_\wp$ of $\mathbf{Q}_p$-algebras from $N_\wp$ to $N_{\alpha(\wp)}$ which respects the action of Γ. Then*

(i) *$\mathcal{O}_N$ and $\mathcal{O}_{N'}$ lie in the same genus of $\mathbf{Z}\Gamma$-modules.*

(ii) *$W^s(\rho)/W^s(\rho')$ lies in $C^\times$ (i.e. it is fixed under $\Omega_{\mathbf{Q}}$).*

(iii) *In $Cl(M,\mathbf{Q}\Gamma)$, $[\mathcal{O}_N] - [\mathcal{O}_{N'}] \equiv cls(i_\infty(W^s(\rho)/W^s(\rho'))$ modulo $cls(\det_\Gamma(\mathcal{U}(\mathbf{Z}\Gamma)))$.*

Proof. — (i) Since $(\mathcal{O}_N)_\wp$ is $\mathbf{Z}_p\Gamma$-free if $\wp \in \mathbf{PT}_p$, this is clear.

(ii) Restricted to the symplectic characters,

$$W^s(\rho)/W^s(\rho') \overset{5.2(\mathrm{iii})}{=\!=} \overline{W}(\rho)/\overline{W}(\rho') \overset{5.1}{=\!=} \prod_{\wp\in\mathbf{PT}} \overline{W}(\rho_\wp)/ \prod_{\wp'\in\mathbf{PT}'} \overline{W}(\rho'_{\wp'})$$

where for the second equality we have been able to cancel the factors belonging to the wild primes using the correspondence α and the context independence result 5.2(i). Now, since $W^s(\rho)$ is 1 outside the symplectic characters, our result follows immediately from the invariance property (5.6(iv)) of the $\overline{W}(\rho_\wp)$. □

The proof of part (iii) of this theorem will occupy most of the rest of this section. We first deduce the two corollaries which are our principal results.

COROLLARY 6.2. — *Without any further assumption*

(i) *In* $Cl(M \oplus \mathbf{Z}\Gamma, \mathbf{Q}\Gamma)$ *and with* $cls = cls_{M\oplus\mathbf{Z}\Gamma}$,
$$[\mathcal{O}_N \oplus \mathbf{Z}\Gamma] - [\mathcal{O}_{N'} \oplus \mathbf{Z}\Gamma] = cls(i_\infty(W^s(\rho)/W^s(\rho'))).$$

(ii) $\mathcal{O}_N \oplus \mathcal{O}_N \oplus \mathbf{Z}\Gamma \cong_{\mathbf{Z}\Gamma} \mathcal{O}_{N'} \oplus \mathcal{O}_{N'} \oplus \mathbf{Z}\Gamma$.

(iii) *If* Γ *has no irreducible symplectic characters then* $\mathcal{O}_N \oplus \mathbf{Z}\Gamma \cong_{\mathbf{Z}\Gamma} \mathcal{O}_{N'} \oplus \mathbf{Z}\Gamma$.

Proof. — This follows using 2.7(i), 5.2(iii) (that $W^s(\rho)^2 = 1$) and that the Eichler condition is satisfied where required. □

COROLLARY 6.3. — *If, in addition,* $\mathbf{PT}_p$ *is non-empty for all primes* p *of* $\mathbf{Q}$*, then*

(i) *In* $Cl(M, \mathbf{Q}\Gamma)$, $[\mathcal{O}_N] - [\mathcal{O}_{N'}] = cls(i_\infty(W^s(\rho)/W^s(\rho')))$.

(ii) $\mathcal{O}_N \oplus \mathcal{O}_N \cong_{\mathbf{Z}\Gamma} \mathcal{O}_{N'} \oplus \mathcal{O}_{N'}$.

(iii) *If* Γ *has no irreducible symplectic characters then* $\mathcal{O}_N \cong_{\mathbf{Z}\Gamma} \mathcal{O}_{N'}$.

Proof. — Under the added condition, M has $\mathbf{Z}\Gamma$ as a local direct summand. (Indeed, it follows that M has $\mathbf{Z}\Gamma$ as a (global) direct summand unless $K = \mathbf{Q}$.) Hence, by 2.7(ii), $\det_\Gamma(\mathcal{U}(\mathbf{Z}\Gamma))$ $(= \mathrm{Nrd}_{\mathbf{Q}\Gamma}(\mathcal{U}(\mathbf{Z}\Gamma))$) is annihilated by cls_M. The result follows.

From now on we assume the hypotheses of 6.1 to be satisfied. Before 4.8 we chose the ideal M of $\mathcal{O}_K\Gamma$ to be locally isomorphic to $\mathcal{O}_N$ and equal to $\mathcal{O}_K\Gamma$ at the tame places. We choose the ideal M' of $\mathcal{O}_{K'}\Gamma$ in the same way except that we specify also that $M'_\wp = M_\wp^{\alpha_\wp}$ at the wild places $\wp$ in $\mathbf{PW}$. (Here we have regarded the local field homomorphism $\alpha_\wp$ as extended to a homomorphism from $(\mathcal{O}_K\Gamma)_\wp$ to $(\mathcal{O}_{K'}\Gamma)_{\alpha(\wp)}$.)

THEOREM6.4. — *In* $Cl(M, \mathbf{Q}\Gamma)$, $[M] - [M'] \in cls(\det_\Gamma(\mathcal{U}(\mathbf{Z}\Gamma)))$.

Proof. — Let I and I' be modules over $\mathcal{O}_K\Gamma$ and $\mathcal{O}_{K'}\Gamma$, respectively. For each p,
$$I_p = T_p(I) \oplus W_p(I) \quad \text{where} \quad W_p(I) = \bigoplus_{\wp\in\mathbf{PW}_p} I_\wp \quad \text{and} \quad T_p(I) = \bigoplus_{\wp\in\mathbf{PT}_p} I_\wp$$
and similarly for I'. Put
$$w_p = \left[\bigoplus_{\wp\in\mathbf{PW}_p} \alpha_\wp\right] : W_p(K\Gamma) \to W'_p(K'\Gamma).$$
By the definition of M', $W_p(M)w_p = W'_p(M')$ and, clearly, $W_p(\mathcal{O}_K\Gamma)w_p = W'_p(\mathcal{O}_{K'}\Gamma)$.

Since $W_p(K\Gamma)$ and $W'_p(K'\Gamma)$ have the same dimension, so do their complements $T_p(K\Gamma)$ and $T'_p(K'\Gamma)$. Hence the free $\mathbf{Z}_p\Gamma$-modules $T_p(M) = T_p(\mathcal{O}_K\Gamma)$ and $T'_p(M') = T'_p(\mathcal{O}_{K'}\Gamma)$ are isomorphic and we choose a $\mathbf{Q}_p\Gamma$-isomorphism t_p from $T_p(K\Gamma)$ to $T'_p(K'\Gamma)$ which performs this isomorphism. Finally, we put $\beta_p = t_p \oplus w_p$. We have then a local $\mathbf{Q}\Gamma$-isomorphism $\beta = \{\beta_p\}$ between $K\Gamma$ and $K'\Gamma$ such that $M\beta = M'$ and $\mathcal{O}_K\Gamma\beta = \mathcal{O}_{K'}\Gamma$.

Now $\mathcal{O}_{K'}\Gamma$ and $\mathcal{O}_K\Gamma$ are isomorphic (free) $\mathbf{Z}\Gamma$-modules. So choose a $\mathbf{Q}\Gamma$-isomorphism f from $K'\Gamma$ to $K\Gamma$ which performs this isomorphism. Then $\mathcal{O}_K\Gamma\beta f = \mathcal{O}_K\Gamma$ and so βf lies in $\mathcal{U}(\mathrm{End}_{\mathbf{Z}\Gamma}(\mathcal{O}_K\Gamma))$ with reduced norm u in $\det_\Gamma(\mathcal{U}(\mathbf{Z}\Gamma))$. On the other hand, since $M'f \cong M'$, we find that $[M] - [M'] = [M] - [M'f] = [M] - [M\beta f] = cls(u)$. □

Recall that we chose the adèle $a = \{a_\wp\}$ before 4.8 so that $aM = \mathcal{O}_N$. We choose a' for N' in the same way except that we can, and do, demand that, for $\wp$ in $\mathbf{PW}$, $a'_{\alpha(\wp)} = a_\wp^{\alpha_\wp}$.

We choose the set S of primes introduced at the beginning of §5 to be big enough for both ρ and ρ'. So S contains all the finite primes of $\mathbf{Q}$ which either divide $|\Gamma|$ or are ramified in N or N'.

Put now $u(a\,|\,\rho) = \mathfrak{R}(a\,|\,\rho)\tau(\rho)^{-1}y^*(\rho)$.

THEOREM 6.5. — (i) *$u(a\,|\,\rho)$ lies in $\mathcal{J}(C)$ (i.e. it is fixed under $\Omega_{\mathbf{Q}}$).*

(ii) *Let q be a finite prime of $\mathbf{Q}$ then* $\left(\dfrac{u(a\,|\,\rho)}{u(a'\,|\,\rho')}\right)_q \in \det_\Gamma(\mathbf{Z}_q\Gamma^\times)$.

(iii) $\left(\dfrac{u(a\,|\,\rho)}{u(a'\,|\,\rho')}\right) \in i_\infty\left(\dfrac{W^s(\rho)}{W^s(\rho')}\right)\det_\Gamma(\mathcal{U}(\mathbf{Z}\Gamma))$.

Proof. — (i) follows from 5.2(iv), 5.4 and 4.5.

(ii) If $q \in S$ and $\mathcal{Q}$ is a prime of U lying over q then

$$\left(\frac{u(a\,|\,\rho)}{u(a'\,|\,\rho')}\right)_{\mathcal{Q}} = \prod_{\wp\in\mathbf{PT}_q}\frac{\mathfrak{R}(a_\wp\,|\,\rho_\wp,\widetilde{\Phi}(\wp,\mathcal{Q}))}{\tau(\rho_\wp)/y^*(\rho_\wp)}\prod_{\substack{\wp\in\mathbf{PT}_p\\ p\in S\setminus\{q\}}}(\tau(\rho_\wp)/y^*(\rho_\wp))^{-1}$$

$$\times\prod_{\wp\in\mathbf{PW}_q}\frac{\mathfrak{R}(a_\wp\,|\,\rho_\wp,\widetilde{\Phi}(\wp,\mathcal{Q}))}{\mathfrak{R}(a'_{\alpha(\wp)}\,|\,\rho'_{\alpha(\wp)},\widetilde{\Phi}'(\alpha(\wp),\mathcal{Q}))}\prod_{\wp\in\mathbf{PW}}\frac{\tau(\rho_\wp)}{\tau(\rho'_{\alpha(\wp)})}$$

$$\times\prod_{\substack{\wp'\in\mathbf{PT}'_p\\ p\in S\setminus\{q\}}}(\tau(\rho'_{\wp'})/y^*(\rho'_{\wp'}))\prod_{\wp'\in\mathbf{PT}'_q}\left(\frac{\mathfrak{R}(a'_{\wp'}\,|\,\rho'_{\wp'},\widetilde{\Phi}'(\wp',\mathcal{Q}))}{\tau(\rho'_{\wp'})/y^*(\rho'_{\wp'})}\right)^{-1}.$$

By 5.5(ii)&(iii), the first, second, fifth and sixth products all lie in $\det_\Gamma((\mathcal{O}_{U(q)}\Gamma)^\times_{\mathcal{Q}})$. By 4.4(ii), each factor in the third product lies in $\det_\Gamma(\Gamma)$. The fourth product is identically 1 by 5.2(i).

Thus, for each $\mathcal{Q}$ dividing $q, (u(a|\rho)/u(a'|\rho'))_{\mathcal{Q}}$ lies in $\det_\Gamma((\mathcal{O}_{U(q)}\Gamma)^\times_{\mathcal{Q}})$. Hence $(u(a\,|\,\rho)/u(a'\,|\,\rho'))_q$ lies in $\det_\Gamma((\mathcal{O}_{U(q)}\Gamma)^\times_q)$ and, moreover, is fixed under $\Omega_{\mathbf{Q}}$. Therefore Taylor's result ([F2] Theorem 10A) tells us that $(u(a\,|\,\rho)/u(a'\,|\,\rho'))_q$ lies in $\det_\Gamma(\mathbf{Z}_q\Gamma^\times)$, as required. If $q \notin S$ then we achieve the same result with rather less effort by applying 5.5(i) in place of 5.5(ii).

(iii) For f and g lying in $\mathrm{Hom}(R_\Gamma, U^\times_\infty)$ we write $f \sim g$ if $(f(\chi)/g(\chi))_{\mathcal{Q}}$ is real and positive for all symplectic characters χ of Γ and all infinite primes $\mathcal{Q}$ of U. It is easily seen (cf. [F2] I 2.2) that, under the identification of 2.1, $\det_\Gamma(\mathcal{U}(\mathbf{Z}\Gamma))_\infty (\overset{3.5}{=} \mathcal{J}(C)^+_\infty) = \mathrm{Hom}^+_{\Omega_{\mathbf{Q}}}(R_\Gamma, U_\infty) \overset{\text{def}}{=} \{f \in \mathrm{Hom}_{\Omega_{\mathbf{Q}}}(R_\Gamma, U^\times_\infty) \,|\, f \sim 1\}$. So by part (i) and 6.1(ii) we need only prove that $u(a\,|\,\rho)_\infty W^s(\rho) \sim u(a'\,|\,\rho')_\infty W^s(\rho')$. But, applying successively 5.2(iii) and 5.4, 5.2(v), 5.6(i)b, 5.6(iii) and the correspondence α, and then reversing the process, we obtain

$$\begin{aligned} u(a\,|\,\rho)_\infty W^s(\rho) &\sim \Re(a\,|\,\rho)_\infty \tau(\rho)^{-1}\overline{W}(\rho) = \Re(a\,|\,\rho)_\infty \overline{W}_\infty(\rho)|\tau(\rho)| \sim |\tau(\rho)| \\ &\sim \prod_{\wp\in\mathbf{PW}} |\tau(\rho_\wp)| = \prod_{\wp\in\mathbf{PW}} |\tau(\rho'_{\alpha(\wp)})| = \prod_{\wp'\in\mathbf{PW}'} |\tau(\rho'_{\wp'})| \\ &\sim u(a'\,|\,\rho')_\infty W^s(\rho'). \end{aligned}$$

Proof of 6.1(iii). — Since $\tau(\rho)$ lies in $\overline{C}^\times$, $\lambda^{-1}(\partial_{\mathcal{J}}(\tau(\rho))) = \partial(\tau(\rho))$. Thus $cls(\tau(\rho)) = \delta(\partial(\tau(\rho))) = 0$. Consequently $cls(u(a\,|\,\rho)) = cls(\Re(a\,|\,\rho)) \overset{4.8}{=} [\mathcal{O}_N] - [M]$. Thus, modulo $cls(\det_\Gamma(\mathcal{U}(\mathbf{Z}\Gamma)))$,

$$\begin{aligned} cls\left(i_\infty\left(\frac{W^s(\rho)}{W^s(\rho')}\right)\right) &\overset{6.5(\text{iii})}{\equiv} cls\left(\frac{u(a\,|\,\rho)}{u(a'\,|\,\rho')}\right) \\ &\equiv [\mathcal{O}_N] - [\mathcal{O}_{N'}] - [M] + [M'] \overset{6.4}{\equiv} [\mathcal{O}_N] - [\mathcal{O}_{N'}]. \end{aligned}$$

□

It has been remarked (cf. [F2] III, first and last paragraphs of §4) that a natural viewpoint is to consider the root numbers at infinity as "Gauss sums" at infinity and that the natural global object is therefore $T(\rho) \overset{\text{def}}{=} \tau(\rho)\overline{W}_\infty(\rho)$ (the right hand side in 5.2(v)). Indeed, if we put $T^s(\rho)$ for the map in $\mathrm{Hom}(R_\Gamma, U^\times)$ which agrees with $T(\rho)$ on the symplectic irreducible characters and is 1 on the other irreducibles, we find that, by 5.6(ii)&(iv), $T^s(\rho)$ is fixed under $\Omega_{\mathbf{Q}}$. Thus $T^s(\rho)$ lies in $\mathrm{Hom}_{\Omega_{\mathbf{Q}}}(R_\Gamma, U^\times)$, that is, $C^\times$. In particular $i_\infty T^s(\rho)$, unlike $i_\infty W^s(\rho)$, clearly lies in the domain of $cls_{\mathbf{Z}\Gamma}$

and so provides a class group invariant, $t(\rho) = cls(i_\infty(T^s(\rho)))$, of N/K in $Cl(\mathcal{O}_N, \mathbf{Q}\Gamma)$. Moreover, the methods of this section are easily adapted to prove the following alternative to 6.1(iii).

THEOREM 6.6. — *Under the assumptions of 6.1 we have, in* $Cl(\mathcal{O}_N, \mathbf{Q}\Gamma)$,

$$[\mathcal{O}_N] - [\mathcal{O}_{N'}] \equiv t(\rho) - t(\rho') \quad \textit{modulo} \quad cls(\det_\Gamma(U(\mathbf{Z}\Gamma))) \ .$$

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