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JUAN ELIAS

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A NOTE ON THE ONE-DIMENSIONAL SYSTEMS OF FORMAL EQUATIONS

by Juan ELIAS

To Joan

0. Introduction.

Let $(X,0)$ be an algebroid singularity defined by the ideal $I \subset \mathbf{k}[[X_1,...,X_N]]$. J. Nash in [N] proposed to study $(X,0)$ using the set of arcs A_X, i.e. the set of $\alpha_. \in \mathbf{k}[[T]]^N$ such that $\alpha_.(0) = 0$, and $f(\alpha_.) = 0$ for all $f \in I$. Let A_X^n be the set of n-th truncations of A_X: $\gamma_. \in \mathbf{k}[[T]]^N$ belongs to A_X^n if and only if $\deg(\gamma_i) \leq n$ for all $i = 1,...,N$ and there exists $\alpha \in A_X$ such that $\alpha_. - \gamma_. \in (T)^{n+1}\mathbf{k}[[T]]^N$. Denote by $\pi_n : A_X^n \to A_X^{n-1}$ the truncation map $\pi_n((\sum_{j=0}^n \gamma_j^i T^j)_{i=1,...,N}) = (\sum_{j=0}^{n-1} \gamma_j^i T^j)_{i=1,...,N}$, so we have a projective system of sets $\{A_X^n, \pi_n\}_{n\geq 0}$ and a isomorphism of sets $A_X \cong lim_{\leftarrow} A_X^n$. Hence a way to study A_X is look into A_X^n. In the complex case from the existence of a non-singular model of $(X,0)$ J. Nash deduces that A_X^n is constructible for all n (see [N],[Le]), on the other hand J.C. Tougeron (see [Le]) proves that A_X^n is constructible from the formal version of the approximation theorem of M. Artin ([A]) due to J. Wavrik ([W1]). In particular from this result one can deduce that there exists a numerical function $\beta : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ such that: $\gamma_. \in A_X^n$ if and only if there exists $\tilde{\gamma}_. \in \mathbf{k}[[T]]^N$ such that $f(\tilde{\gamma}_.) \in (T)^{\beta(n)}\mathbf{k}[[T]]^N$ for all $f \in I$ and $\gamma_. - \tilde{\gamma}_. \in (T)^{n+1}$.

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As far as we know only in a few cases we have an explicit determination of β: first case is due to J. Wavrik for X reduced plane curve taking non-singular arcs ([W2]), the second one is due to M. Lejeune for hypersurface singularities taking general arcs ([Le]).

In this paper we determine the function β in the case of one-dimensional singularities X, taking non-singular arcs, in terms of the Milnor number associated to X_{red}. See [La] for other results on β.

The paper is divided in two sections, in the first we give some preliminaires results about contact between curves. In the second one we define the numerical function β and we prove the main result of this paper (Theorem 2.1).

Throughout this paper R will be the power series ring $\mathbf{k}[[X_1, ..., X_N]]$, where $\mathbf{k}$ is an infinite field. We denote by M the maximal ideal of R.

A curve of $(\mathbf{k}^N, 0) = \text{Spec}(R)$ is a one-dimensional, Cohen-Macaulay closed subcheme X of $(\mathbf{k}^N, 0)$, i.e. $X = \text{Spec}(R/I)$ where $I = I(X)$ is a perfect height N-1 ideal of R; we put $\mathcal{O}_{\mathcal{X}} = R/I$. A branch is an integral curve.

2. Contact of curves.

If X is a reduced curve of $(\mathbf{k}^N, 0)$ then we denote by $\delta(X)$ the dimension over $\mathbf{k}$ of the quotient $\tilde{\mathcal{O}_{\mathcal{X}}}/\mathcal{O}_{\mathcal{X}}$ where $\tilde{\mathcal{O}_{\mathcal{X}}}$ is the integral closure of $\mathcal{O}_{\mathcal{X}}$. If r is the number of branches of X then we define the Milnor number of X by $\mu(X) = 2\delta - r + 1$.

Let X be a reduced curve and let Q be an infinitely near point of X, see [ECh], [VdW]. It is known that there exists a unique sequence $\{Q\}_{i=0,...,s}$ of infinitely near points of X such that $Q_0 = 0, ..., Q_s = Q$, and that Q_{i+1} belongs to the first neighbourhood of Q_i for $i = 0, ..., s-1$. We denote by (X, Q) the union of the irreducible components through Q of the proper transform of X by the composition of the blowing-up centered at Q_i for $i = 0, ..., s-1$. We denote by $p_{(X,Q)}(T) = e(X, Q)T - \rho(X, Q)$ the Hilbert polynomial of the local ring $\mathcal{O}_{(X,Q)}$.

For the readers convenience we will summarize some properties of $e(X, Q)$ and $\rho(X, Q)$ that we will use in the paper:
(1) $e(X, Q) - 1 \leq \rho(X, Q)$, ([M] Proposition 12.14),
(2) $e(X, Q) = 1$ if and only if $\rho(X, Q) = 0$, ([M] Proposition 12.16),

(3) $e(X,Q) = 2$ if and only if $\rho(X,Q) = 1$, ([M] Proposition 12.17),
(4) $\dim_{\mathbf{k}}(R/I + M^n) = p_{(X,Q)}(n)$ for all $n \geq e(X,Q) - 1$, ([K] Theorem 6, or [M] Proposition 12.11).

Let $T(X)$ be the set of infinitely near point Q of X such that its multiplicity $e(X,Q)$ is greater than one. From [Ca] we obtain that

$$\delta(X) = \sum_{Q \in T(X)} \rho(X,Q).$$

Let X, Y be curves of $(\mathbf{k}^N, 0)$, without components in common, we denote by $(X.Y)$ the number $dim_{\mathbf{k}}(R/I(X) + I(Y))$ ([H]).

Let Z_1 be a branch, for every reduced curve Z_2 , such that Z_1 is not a component of Z_2, we define $f(Z_1, Z_2)$ as the number of non-singular points shared by Z_1 and Z_2.

PROPOSITION 1.1. — *If Z_1 is a non-singular branch then*

$$(Z_1.Z_2) \leq \mu(Z_2) + f(Z_1, Z_2) + 1.$$

Proof. — From [C] and [M], Proposition 12.16, we deduce

$$(Z_1.Z_2) \leq \sum_{Q \in K} (\rho(Z_1 \cup Z_2, Q) - \rho(Z_2, Q))$$

where K is the set of infinitely near points shared by Z_1 and Z_2.

Since (Z_i, Q) is a curve of $(\mathbf{k}^N, Q) \cong (\mathbf{k}^N, 0)$, we put $(Z_i, Q) = \mathrm{Spec}(R/I_{i,Q})$ for $i = 1, 2$. Consider the projection

$$\frac{R}{(I_{1,Q} \cap I_{2,Q}) + M^n} \to \frac{R}{I_{2,Q} + M^n}$$

for all $n \geq e(Z_2, Q)$; from this and [K], Corollary 6, we get

$$(e(Z_2, Q) + 1)n - \rho(Z_1 \cup Z_2, Q) \geq e(Z_2, Q)n - \rho(Z_2, Q).$$

Therefore $\rho(Z_1 \cup Z_2, Q) - \rho(Z_2, Q) \leq e(Z_2, Q)$, and hence

$$(Z_1.Z_2) \leq \sum_{Q \in K} e(Z_2, Q) \ . \tag{1}$$

Assume that Z_2 is singular. Since $e(Z_2, 0) \leq \rho(Z_2, 0) + 1$, [M] Proposition 12.14, and $r \leq e(Z_2, 0)$ we deduce

$$e(Z_2, 0) \leq (2\rho(Z_2, 0) + 1 - r) + 1 \ . \tag{2}$$

Let K^* be the set of points belonging to K such that $e(Z_2,0) \geq 2$. From [M], Proposition 12.17, we obtain that for all $Q \in K^*$

$$e(Z_2,Q) \leq 2\rho(Z_2,Q) \ . \tag{3}$$

By (2) and (3) we get

$$\sum_{Q\in K^*} e(Z_2,Q) \leq \left(2 \sum_{Q\in K^*} \rho(Z_2,Q) + 1 - r\right) + 1,$$

since $\rho(Z_2,Q) = 0$ if and only if $e(Z_2,Q) = 1$ we have

$$\sum_{Q\in K^*} e(Z_2,Q) \leq \mu(Z_2) + 1 \ .$$

Recall that $e(Z_2,Q) = 1$ for $Q \in K - K^*$, from (1) we obtain the claim.

PROPOSITION 1.2. — *Let $Z_i = \mathrm{Spec}(R/I_i)$ be curves, $i = 1,2$. Assume that Z_1 is non-singular and $I_2 + M^{\mu(Z_2)+n+1} \subset I_1 + M^{\mu(Z_2)+n+1}$. Then we have*

$$n \leq f(Z_1,Z_2).$$

Proof. — From the hypothesis we deduce that

$$I_1 + I_2 \subset I_1 + I_2 + M^{\mu(Z_2)+n+1} = I_1 + M^{\mu(Z_2)+n+1},$$

so that

$$\mu(Z_2) + n + 1 \leq \dim_{\mathbf{k}}(R/I_1 + M^{\mu(Z_2)+n+1}) \leq (Z_1.Z_2).$$

The claim follows from Proposition 1.1.

COROLLARY 1.3. — *If $n \geq 2$ then there exists a non-singular branch Y of Z_2 such that $n \leq f(Z_1,Y)$.*

Proof. — By Proposition 1.2 we get $f(Z_1,Z_2) \geq n \geq 2$, so there exists a branch Y of Z_2 such that Z_1 and Y share n non-singular infinitely near points. Since a non-singular branch and a singular branch cannot share two non-singular near points, we get that Y is non-singular.

The following result is well known :

PROPOSITION 1.4. — *Let Z_1, Z_2 be non-singular branches, for all n the following inequalities are equivalent :*
(1) $(Z_1.Z_2) \geq n$,

(2) Z_1 *and* Z_2 *share* n *infinitely near points,*
(3) *for all parametrization of* Z_1*:*

$$Z_1 : \begin{cases} X_1 = & t \\ X_i = & X_i(t) \text{ for all } i = 2, ..., N, \end{cases}$$

there exists a parametrization of Z_2*:*

$$Z_2 : \begin{cases} X_1 = & t \\ X_i = & \tilde{X}_i(t) \text{ for all } i = 2, ..., N, \end{cases}$$

such that

$$X_i(t) - \tilde{X}_i(t) \equiv 0 \text{ modulo } (t)^n,$$

for all $i = 2, ..., N$.

2. The function β.

DEFINITION. — *We say that a system of formal equations* $\{F = 0\} = \{F_1 = 0, ..., F_s = 0\}$, $F_i \in R$, *is one-dimensional if and only if* $(F) = (F_1, ..., F_s)$ *is a height* $N - 1$ *ideal of* R. *We denote by* $\mathcal{F}$ *the set of one-dimensional systems of formal equations.*

Let $\{F = 0\}$ be a one-dimensional system of formal equations, we define the curve $Z_F = \operatorname{Spec}(R/\operatorname{rad}(F))$, and the numbers $\mu(\{F = 0\}) = \mu(Z_F)$ and $m(\{F = 0\}) = \operatorname{Min}\{n \in \mathbf{N} \mid \operatorname{rad}((F))^n \subset (F)\}$.

DEFINITION. — *Let* $\beta : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ *be the numerical function:*

$$\beta(n, \{F = 0\}) = m(\{F = 0\})(2\mu(\{F = 0\}) + n + 1).$$

THEOREM 2.1. — *Given a one-dimensional system of formal equations* $\{F = 0\}$, *and a non-negative integer* $n \geq 0$ *if* Z_F *is singular and* $n \geq 1$ *if* Z_F *is non-singular. Let* $X_i(X_1, ..., X_r) \in \mathbf{k}[[X_1, ..., X_r]]$, $1 \leq r \leq N$, $i = r + 1, ..., N$ *be a set of formal power series such that for every* $G \in (F)$ *:*

$$G(X_1, ..., X_r, X_{r+1}(X_1, ..., X_r), ..., X_N(X_1, ..., X_r)) \equiv 0 \\ \text{modulo } (X_1, ..., X_r)^{\beta(n, \{F=0\})} .$$

Then there exist $\tilde{X}_i(X_1, ..., X_r) \in \mathbf{k}[[X_1, ..., X_r]]$, $i = r+1, ..., N$, *such that:*

(1) $G(X_1,...,X_r,\tilde{X}_{r+1},...,\tilde{X}_N)=0$ *for all* $G\in(F)$,
(2) $X_i(X_1,...,X_r)-\tilde{X}_i(X_1,...,X_r)\equiv 0$ *modulo* $(X_1,...,X_r)^n$ *for all* $i=r+1,...,N$.

Proof. — First of all we will prove that $r=1$. From now on we put $\mu(\{F=0\})=\mu(Z_F)=\mu$, $\rho(Z_F,0)=\rho$ and $e(Z_F,0)=e$.

Let J be the ideal of R generated by $X_i-X_i(X_1,...,X_r)$ for $i=r+1,...,N$. Notice that J is the kernel of the map $\varphi: R\to \mathbf{k}[[X_1,...,X_r]]$ defined by

$$\varphi(G)=G(X_1,...,X_r,X_{r+1}(X_1,...,X_r),...,X_N(X_1,...,X_r))\ .$$

From the hypothesis we deduce that

$$(F)\subset J \text{ modulo } (X_1,...,X_r)^{\beta(n,\{F=0\})},$$

so

$$\text{rad}((F))\subset J \text{ modulo } (X_1,...,X_r)^{2\mu+1+n}\ . \tag{1}$$

Recall [C] that

$$\delta(Z_F)=\sum_{Q\in T(Z_F)}\rho(Z_F,Q),$$

by [M], Proposition 12.14, we obtain that $\delta(Z_F)+1\geq e$; from this we deduce $\mu\geq\delta(Z_F)$, so $\mu\geq\rho$.

From [M], Proposition 12.11, we get

$$\dim_{\mathbf{k}}\left(\frac{R}{\text{rad}((F))+M^{2\mu+n+1}}\right)=e(2\mu+n+1)-\rho.$$

Since $\text{Spec}(R/J)$ is non-singular, from (1) we have

$$e(2\mu+n+1)-\rho\geq\binom{2\mu+n+r}{r}\ .$$

Assume that $r\geq 2$, then $(2\mu+n+1)(e-(\mu+1)-n/2)\geq\rho$. Since $\mu\geq\rho\geq e-1$ ([M], Proposition 12.14) we obtain: $\rho\leq(2\mu+n+1)(-n/2)$. If Z_F is singular then we get $\rho\leq 0$, but from [M], Propositions 12.14 and 12.17, we have that $\rho\geq 1$, so r=1. If Z_F is non-singular we get that $\rho<0$, since ρ is a non-negative integer ([M], Propositions 12.14) we deduce $r=1$.

Consider the non-singular branch Z_1 which admits the parametrization:

$$Z_1:\begin{cases}X_1= & t\\ X_i= & X_i(t) \text{ for all } i=2,...,N\ .\end{cases}$$

Notice that the series $H_i = X_i - X_i X_1)$, $i = 2, ..., N$, form a system of generators of the ideal I_1 defining the curve Z_1. If $G \in \text{rad}(F)$ then

$$G(X_1, X_2(X_1), ..., X_N(X_1)) \equiv 0 \text{ modulo } (X_1)^{\mu+1+n},$$

thus

$$\text{rad}((F)) \subset I_1 \text{ modulo } (X_1)^{\mu+1+n}.$$

From Propositions 1.2,1.3 and 1.4 we deduce the claim.

Remark. — (1) From the proof of the theorem it is easy to prove that for the systems of formal equations with $r = 1$ one can take

$$\beta(n, \{F = 0\}) = m(\{F = 0\})(\mu(\{F = 0\}) + n + 1).$$

(2) If we consider reduced systems of formal equations, i.e. $\text{rad}((F)) = (F)$, then we have

$$\beta(n, \{F = 0\}) = 2\mu(\{F = 0\}) + n + 1.$$

Notice that the number $2\mu(\{F = 0\}) + 1$ has the following property ([E]): the analytic type of Z_F is determined by any of its truncations: $(Z_F)_n = \text{Spec}(R/(F) + M^n)$ for all $n \geq 2\mu(\{F = 0\}) + 1$.

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Juan ELIAS,
Universitat de Barcelona
Facultat de Matematiques
Departament d'Algebra i Geometria
Gran Via 585
08007 BARCELONA
(Espagne).