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# NEW EXAMPLES OF NON-LOCALLY EMBEDDABLE *CR* STRUCTURES (WITH NO NON-CONSTANT *CR* DISTRIBUTIONS)

by Jean-Pierre ROSAY(*)

Several years ago, L. Nirenberg gave a famous example of a strictly pseudoconvex, non-locally embeddable, *CR* structure on a manifold of dimension 3, ([7], [8] see also [4]).

This phenomenon of non-embeddability is important enough in order that it may be worthwhile to give new examples which may shed a new light and seem to present some improvement. In the previous examples it seems that only the existence of $C^1$ *CR* function could be investiged. We can just as well handle the case of *CR* distributions. There may be more than just a theoretical interest to it, since, for example, $L^\infty$ functions arise quite naturally as weak limits.

Our example has the possible additional feature that it can be adapted so that the *CR* stucture that we construct can be extended to a complex structure on the concave side of the manifold. This relates to a theorem by D. Catlin [14], and an example (in dimension 5, with degenerate Levi form) by D. Hill, who conjectured that such one sided extensions to the concave side always exist, and asked explicitly the question for Nirenberg's example [2]. Full details will not be provided, but some indications (that we hope to be sufficient) are given in a remark at the end of the paper. It seems also that our examples complete some examples by Jacobowitz [3], concerning the canonical bundle.

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In part I, we describe a simple "basic construction" from which everything follows easily. This is the main part of the paper. In II, we get immediately an example of a strictly pseudo convex *CR* structure on a manifold of dimension 3 (arbitrarily close to any given structure, as in [4]), for which there is a point so that every *CR* distribution defined in a connected neighborhood of this point, must be constant.

We get also examples of structures with only one "independant solution" (see further, for more details).

In III, we give a new approach to results by Jacobowitz and Trèves [4]. Jacobowitz and Trèves were the first to show that the phenomenon of non-embeddability occurs too in higher dimension for stuctures whose Levi form is non-degenerate and has only one positive eigenvalue. Then all *CR* functions, or distributions, must be smooth, if the *CR* structure is smooth ([12] page 493- , or [10]). In III we give examples of non-embeddable *CR* structure on manifolds of dimension $2m-1$, whose Levi form has one positive and $(m-2)$ negative eigenvalue (arbitrarily close to any given embedded stucture), with locally, no non-constant *CR* functions. Or with only $k$ "independent solutions" ($k$ arbitrary $k \in \{0, \dots, m\}$) : $z_1, \dots, z_k$ which satisfy $dz_1 \wedge \dots \wedge dz_k \neq 0$ and so that every *CR* function is a function of $(z_1, \dots, z_k)$. By *CR* structure we will always mean *CR* structure of hypersurface type, and we will consider only smooth ($\mathcal{C}^\infty$) structure.

Let us remind the reader that it is still an open question whether any strictly pseudoconvex *CR* structure of dimension 5 can be locally embedded (for dimension $> 7$, this is known to be true [1], [5], [13]). Our example sheds some light on the difficulty to build a counterexample, from the point of view of function theory.

## I. Basic construction

### 1. Lupacciolu's theorem.

Although very simple, our construction relies entirely on a very nice result by G. Lupacciolu [6]. We are using only a very particular case, but is seems much more pleasant to state the result in some generality. A different and simple proof of Lupacciolu's theorem is given in the appendix in [9].

"*Let $\Omega$ be bounded domain in $\mathbf{C}^n$ and $K$ a polynomially convex compact set in $b\Omega$ (the boundary of $\Omega$). If $b\Omega$ is smooth and strictly* *pseudoconvex off $K$, then every $CR$ distribution defined on $b\Omega - K$ has a holomorphic extension to $\Omega$ (i.e. in some sense $K$ is negligible).*"

Since every *CR* distribution on $b\Omega - K$ has a local holomorphic extension inside to $\Omega$ (due to strict pseudoconvexity), the case of *CR* distributions reduces easily to the case of smooth *CR* functions.

## 2. Construction of a *CR* structure.

We start with $S$ a smooth strictly convex hypersurface defined in $\Omega$, an open neighborhood of $O$ in $\mathbf{C}^2$. We assume that $O \in S$ and that $\operatorname{Re} z_2 \geq 0$ on $S$ (therefore $> 0$ on $S - \{0\}$). We fix $\beta > 0$, small enough in order that $S \cap \{\operatorname{Re} z_2 \leq \beta\}$ is a compact set in $\Omega$, and that the line segment from $(0,0)$ to $(0,\beta)$ does not intersect $S$ except at $(0,0)$. For $0 < \alpha < \beta$ we set :

$$
\begin{aligned}
\Sigma &= S \cap \{\operatorname{Re} z_2 < \beta\} \\
\Sigma^- &= S \cap \{\operatorname{Re} z_2 < \alpha\} \\
\Sigma^+ &= S \cap \{\alpha < \operatorname{Re} z_2 < \beta\} \\
S_\alpha &= S \cap \{\operatorname{Re} z_2 = \alpha\}.
\end{aligned}
$$

Then, we consider $z_1^{\#}$ a smooth function defined on $\overline{\Sigma}^-$ with the following properties :

(i) $\overline{\partial}_b\, z_1^{\#}$ vanishes to infinite order along $S_\alpha$.

(ii) $z_1^{\#}$ is sufficiently close to $z_1$ (say in the $\mathcal{C}^2$ topology) in order that the map $\tau :\ (z_1, z_2) \mapsto \left(z_1^{\#}(z_1, z_2), z_2\right)$ maps $\overline{\Sigma}^-$ into a strictly convex surface (diffeomorphically).

(iii) For $| \ y_2 \ |$ small $z_1^{\#}(z_1, \alpha + iy_2) = z_1 + \dfrac{\epsilon}{z_1}$ for some positive $\epsilon$, $\epsilon < \dfrac{d^2}{25}$, where $d$ is the distance from $(0,\alpha)$ to $S$.

(iv) For $\operatorname{Re} z_2 < \dfrac{2\alpha}{3} \quad z_1^{\#}(z_1, z_2) = z_1$.

To construct such a function $z_1^{\#}$ is extremely easy. One can set $z_1^{\#}(z_1, z_2) = z_1 + \dfrac{\epsilon}{z_1}\chi(z_2)$. For $\chi$ one picks a smooth cut-off function, defined in $\mathbf{C}$, which is identifically 1 in a neighborhood of $\alpha$, but has small support around $\alpha$, and such that $\overline{\partial}\chi$ vanishes to infinite order along the real line $\{\operatorname{Re} z = \alpha\}$. Notice, as a crucial fact, that for $z_2$ close to $\alpha$, $z_1 \neq 0$ if $(z_1, z_2) \in S$. Once $\chi$ is chosen, it is enough to take $\epsilon$ small enough. Observe that one can therefore take $z_1^{\#}$ as close as wanted to $z_1$ (in the $C^\infty$ topology).

It follows from (ii) that there is a unique *CR* structure on $\Sigma^-$ such that $z_1^{\#}$ and $z_2$ are *CR* functions. It is the pullback of the structure induced from $\mathbf{C}^2$ on $\tau(\Sigma^-)$ and it is therefore a strictly pseudoconvex structure. It follows from (i) that this structure patches smoothly with the standard *CR* structure along $S_\alpha$. On $\Sigma^+$ we use the standard *CR* structure (induced from $\mathbf{C}^2$).

This finally gives rise to a smooth *CR* structure on $\Sigma$ which is strictly pseudoconvex, and for which we have as independant *CR* functions :

$$\begin{cases} \text{(a) on } \Sigma^-, \ z_1^{\#} \text{ and } z_2 \\ \text{(b) on } \Sigma^+, \ z_1 \text{ and } z_2. \end{cases}$$

As a last piece of notation, let $\Sigma' = \Sigma \cap \{\operatorname{Re} z_2 > \frac{\alpha}{2}\}$. For some applications we will need to consider functions defined only on $\Sigma'$. The only idea in our construction is in the following claim.

CLAIM — *It follows from (iii) that every CR function defined on $\Sigma$ or $\Sigma'$, (equipped with the above CR structure) is a function of $z_2$ only. The same holds for CR distributions. In addition, any CR distribution on $\Sigma$ (or $\Sigma'$), that is constant on $\Sigma^+$, is constant.*

## 3. Proof of the claim.

**3.1.** For simplicity let us first treat the case of continuous *CR* function on $\Sigma$ (or $\Sigma'$). The restriction of $f$ to $\Sigma^+$ is a *CR* function (for the usual structure). By 1, it extends to a holomorphic function on the open set bounded by $\Sigma$ and the hyperplanes $\{\operatorname{Re} x_2 = \alpha \text{ or } \beta\}$.

Let $\gamma$ be the curve in the $z_1$ plane constituted by the points $z_1$ such that $(z_1, \alpha) \in S$. We conclude from what precedes that the function $z_1 \mapsto f(z_1, \alpha)$, defined on $\gamma$, extends holomorphically inside the convex curve $\gamma$, and is therefore the uniform limit on $\gamma$ of a sequence of polynomials in $z_1$.

The same holds, for $\mid y_2 \mid$ small, on the corresponding curve

$\{(z_1 \in \mathbf{C},\ (z_1, \alpha + iy_2) \in S\})$, for the function $z_1 \mapsto f(z_1, \alpha + iy_2)$.

Together with $\tau(\Sigma^-)$, the hyperplane $\{\operatorname{Re} z_2 = \alpha\}$ (or the hyperplanes $\{\operatorname{Re} z_2 = \alpha$ or $\frac{\alpha}{2}\})$ bounds a domain to which we can again apply Lupacciolu's theorem, since *CR* functions on $\Sigma^-$ for our structure are usual *CR* functions on $\tau(\Sigma^-)$.

The conclusion is that on the curve $\gamma$ the function $z_1 \mapsto f(z_1, \alpha)$ is a uniform limit of polynomials in $z_1 + \frac{\epsilon}{z_1}$ $(= z_1^{\#})$. And , again, the same holds, for $\mid y_2 \mid$ small, on the corresponding curves. Notice that 0 is inside $\gamma$.

It now follows from the above facts and from the Lemma below (which we immediately state for distributions), that for $\mid y_2 \mid$ small $f(z_1, \alpha + iy_2)$ does not depend on $z_1$. So the holomorphic extensions of $f$, from $\Sigma^+$ and $\tau(\Sigma^-)$ respectively, do not depend on $z_1$. Hence $f$ depends only on $z_2$, as claimed. And it is also clear that if $f$ is constant on $\Sigma^+$ then $f$ is constant.

It remains to state and prove the following Lemma.

**3.2.** Lemma. — *Let $\gamma$ be a smooth closed curve in $\mathbf{C}$, so that $O \notin \gamma$ and $O$ does not belong to the unbounded component of $\mathbf{C} - \gamma$. Let $d$ be the distance from 0 to $\gamma$ (or any smaller positive number). Let $\epsilon \in \mathbf{C}$, $0 <\mid \epsilon \mid< \frac{d^2}{25}$. Then no non-constant distribution on $\gamma$ can be both the limit in the sence of distributions of :*

$$\begin{cases} \text{(a) a sequence of polynomials in } z \\ \text{(b) a sequence of polynomials in } z + \dfrac{\epsilon}{z}. \end{cases}$$

*Proof of the Lemma.* — We can assume $\epsilon$ to be real and positive. Let $\tau_1$ be the mapping defined on $\mathbf{C} - \{0\}$ by $\tau_1(z) = z + \frac{\epsilon}{z}$. If $z \in \gamma \mid \tau_1(z) \mid \geq d - \frac{\epsilon}{d} > \frac{d}{2}$ and $\tau$ is $1-1$ on $\gamma$. Indeed $\tau(z) = \tau(z')$ implies $z = z'$ or $z' = -\frac{\epsilon}{z}$, but $\mid zz' \mid \geq d^2$ for $z$ and $z' \in \gamma$. Set $\Gamma_\epsilon = \{z \in \mathbf{C},\ \mid z \mid = \frac{\sqrt{\epsilon}}{2}\}$. $\tau$ maps $\Gamma_\epsilon$ to $\Gamma'_\epsilon$ the ellipse with vertices at $\pm\left(2\sqrt{\epsilon} + \frac{\sqrt{\epsilon}}{2}\right)$ and $\pm\left(2\sqrt{\epsilon} - \frac{\sqrt{\epsilon}}{2}\right) i$, which lies inside the curve $\tau_1(\gamma)$, since $\frac{5\sqrt{\epsilon}}{2} < \frac{d}{2}$. But a crucial fact is that as a map from $\Gamma_\epsilon$ to $\Gamma'_\epsilon$, $\tau_1$ reverses orientation. If $f$ is a distribution on $\gamma$ which is on $\gamma$ the limit of a sequence of polynomials, $f$ extends holomorphically inside $\gamma$. Set $g = (\tau_1)_* f (i.e.\ f \circ \tau_1^{-1}$ if $f$ is a function) on $\tau_1(\gamma)$. If $f$ is a limit of polynomials in $z + \frac{\epsilon}{z}$, $g$ extends holomorphically inside $\tau_1(\gamma)$. Denote by $\tilde{f}$ and $\tilde{g}$ the above holomorphic extensions. By unique analytic continuation, we have $\tilde{g} \circ \tau_1 = \tilde{f}$ on the region of the plane delimited by $\gamma$ and $\Gamma_\epsilon$ (which is indeed mapped, by $\tau_1$, inside $\tau_1(\gamma)$).

If $\tilde{g}$ were not constant, by adding a constant we could assume that $\tilde{g}$ does not vanish on $\Gamma'_\epsilon$ but vanishes at some point inside $\Gamma'_\epsilon$. Since $\tau_1$ reverses orientation (from $\Gamma_\epsilon$ to $\Gamma'_\epsilon$), the variation of the argument of $\tilde{f}$ along $\Gamma_\epsilon$ would be $2\pi k$ for some $k < 0$, a contradiction.

*Remark.* — The bound $| \epsilon | \leq \frac{d^2}{25}$ should not be taken seriously. In fact if $\gamma$ is a circle it is easy to see that the proof above gives the result for every $\epsilon \neq 0$. In case of functions (instead of distributions) one clearly does not need the curve to be smooth.

**3.3.** *The distributional case.* — The basic fact to be used is the following : if $\gamma$ is a curve in a *CR* manifold, which is not "complex tangential" (i.e. $\dot{\gamma}$ does not belong to the span of the Cauchy Riemann vector fields and their conjugates) then the restriction of any *CR* distribution to the curve $\gamma$ makes sense as a distribution ( and in the space of distribution varies smoothly as the curve varies smoothly). This does not require the *CR* structure to be embedded. On the contrary, it comes from a much more general theory, see [11] Proposition 5.2, page 39. To extend the proof given in 3.1 is then easy. To avoid any difficulty on $\Sigma^\pm$, one can take advantage of the fact the structure on $\Sigma^+$ is embedded, the one on $\Sigma^-$ is embeddable (by $\tau$). Then any *CR* distribution has local holomorphic extensions to the convex side.

## 4. Remarks.

**4.1.** For the next paragraph it will be useful to indicate how to reformulate our claim, to lead obviously to an "invariant" formulation :

Let $N = (0, 1)$. This is a normal vector to $S$ at the point $p, p = 0$, pointing towards the convex side of $S$. Then $\{\operatorname{Re} z_2 = \alpha\} = \{\operatorname{Re}\langle (z - p) \cdot N \rangle = \alpha\}$. And the conclusion has been that *CR* functions were depending only on $\langle z \cdot N \rangle$.

**4.2.** It is not hard to see that the canonical bundle of the structure constructed in 2, cannot have on $\Sigma$ (or even $\Sigma'$) a non-zero closed section. For definitions, and interesting related results, see [3]. Indeed such a section could be written $h(z_1, z_2) dz_1 \wedge dz_2$ on $\Sigma^+$ and $k(z_1, z_2) dz_1^{\#} \wedge dz_2$ on $\Sigma^-$, where $h$ and $k$ should be *CR* (for our structure) on $\Sigma^\pm$. On $\operatorname{Re} z_2 = \alpha$ one must then have $h(z_1, z_2) = \left(1 - \frac{\epsilon}{z_1^2}\, \chi(z_2)\right) k\,(z_1, z_2)$. Let $\gamma$ be as in 3, and let us use again the notations in the proof of the Lemma. For $z_2 = \alpha$, fixed, the function $z_1 \mapsto h(z_1, \alpha)$ extends holomorphically inside $\gamma$, while the function $z_1 \mapsto k\left(\tau_1^{-1}(z_1), \alpha\right)$ defined on $\tau_1(\gamma)$ extends holomorphically inside the curve $\tau_1(\gamma)$. Denote these holomorphic extensions, by $\tilde{k}$ and $\tilde{h}$. By unique analytic continuation one must have :

$$\frac{z_1^2 \tilde{h}(z_1)}{z_1^2 - \epsilon} = \tilde{k} \circ \tau_1(z_1), \text{ for } z_1$$

inside $\gamma$, $\mid z_1 \mid > \frac{\sqrt{\epsilon}}{2}$. The singularities of the left-hand side, at $z_1 = \pm\sqrt{\epsilon}$, are therefore removable. By the Lemma, $\frac{z_1^2 \tilde{h}}{z_1^2 - \epsilon}$ must be constant, which can happen only if $\tilde{h}$ is identically 0. Again, the same must happen for $\operatorname{Re} z_2 = \alpha$, $z_2$ close to $\alpha$, on the corresponding curves. And one gets that the canonical bundle has only the zero section, on $\Sigma$.

## II. Examples

**1.** We consider a strictly convex surface $S$ and $\beta > 0$ as in I.2. Let $S_j = \left(\lambda_j, \frac{\beta}{2^j}\right)$ be a sequence of points in $S$ (converging to 0). Denote by $N_j$ the unit normal to $S$ at $S_j$, pointing towards the convex side. Set $\Gamma_j = \{z \in s,\ \operatorname{Re}\langle (z - s_j) \cdot N_j \rangle \leq \alpha_j\}$, where the $\alpha_j' s$ are positive numbers, chosen small enough in order that on $\Gamma_j$ $\frac{\beta}{2^j} - \frac{\beta}{2^{j+2}} < \operatorname{Re} z_2 < \frac{\beta}{2^j} + \frac{\beta}{2^{j+2}}$. The convex hulls of the $\Gamma_j$ are disjoint. On each $\Gamma_j$ make a small pertubation of the *CR* structure induced from $\mathbf{C}^2$, following the basic construction with 0 replaced by $\zeta_j$, so that every *CR* function on $\Gamma_j$ depends only on $\langle z \cdot N_j \rangle$. One gets a *CR* structure on $S$, arbitrarily close to the original one, for which :

PROPOSITION 1. — *If $f$ is any CR distribution defined on a connected neighborhood of 0 in $S$ (for the CR structure just defined), then $f$ must be constant.*

*Proof.* — Set $\lambda = \frac{\beta}{2^r} + \frac{\beta}{2^{r+2}}$ for $r \geq 1$, $r$ large enough in order that $f$ be defined on $S \cap \{\operatorname{Re} z_2 < \lambda\}$. Notice that our choice of $\lambda$ makes sure that the hyperplane $\{\operatorname{Re} z_2 = \lambda\}$ does not intersect any $\Gamma_j$. Let $\omega$ be the open set bounded by $S$ and the hyperplane $\{\operatorname{Re} z_2 = \lambda\}$. Set $\widehat{\Gamma}_j$ to be the convex hull of $\Gamma_j$. Due to Lupacciolu's theorem, $f$ extends holomorphically to $\omega - \bigcup_j \widehat{\Gamma}_j$. Indeed one can show that $b\omega \cap \left[\left(\bigcup_j \widehat{\Gamma}_j\right) \cup \{\operatorname{Re} z_2 = \lambda\}\right]$ is polynomially convex, or more easily one can take care of each $\Gamma_j$, separately. And our structure is just the usual one off the $\Gamma_j$'s. Due to what we have seen before, for infinitely many indices $j$ (therefore for at least two distinct), this holomorphic extension must depend only on $\langle z \cdot N_j \rangle$. So it must be constant. Then, as seen in I.2, $f$ must be constant on $S \cap \{\operatorname{Re} z_2 < \lambda\}$ (by the last part of the claim). If one desires, it is not hard to prove that in fact $f$ must be constant on the whole connected neighborhood on which it is defined, by reasoning as in I.

**2.** Going back to the construction given in I, for $j = 1, 2, \ldots$ set $\alpha_j = \frac{\beta}{2^j}$. For each $\alpha_j$, we can do a small pertubation of the standard *CR* structure exactly as described in I (just take $\alpha = \alpha_j$). This pertubation takes place in this region $\left\{\frac{2\alpha_j}{3} < \operatorname{Re} z_2 < \alpha_j\right\}$. One ends with a *CR* stucture on $S$, arbitrarily close to the standard one. From the claim in I (this why we introduced $\Sigma'$), we get immediately that, for this *CR* structure :

PROPOSITION 2. — *$z_2$ is a CR function, and every CR function or distribution, defined in a connected neighborhood of 0 in $S$, depends only on $z_2$.*

**3.** *Remarks.* — A variant of the construction done in 1 will be presented in III. Somewhat "closer" to the construction in 2, this construction will lead more easily to a *CR* structure which extends to a complex structure on the concave side.

Every *CR* structure on a 3-dimensional manifold can be approximated by a real analytic one (since there is no integrability condition in dimension 3), and therefore a locally embeddable one. So we can obtain examples of non-embeddable *CR* structure with no non-constant *CR* distribution arbitrarily close to any given *CR* structure.

From I.4.2, one sees that in our examples the canonical bundles has no non-zero section in any connected neighborhood of 0.

## III. Higher dimensions

**1.** Let us make some comments on the situation. To carry out our construction in higher dimensions, with extra variables $z_3, \ldots, z_m$, we meet the following problem. We would like to set again $z_1^{\#} = z_1 + \frac{\epsilon}{z_1}\chi$, where $\chi$ is a cut off function which must be 0 when $z_1 = 0$.

And we wish to keep $\overline{\partial}_b z_1^{\#} = 0$ on $\{\operatorname{Re} z_2 = \alpha\}$. This is clearly impossible if, as in general, $\chi$ has to depend effectively on the extra variables $z_3, \ldots, z_m$. Things however can be worked out in case of concavity (in the extra variables). Another way to see an obstruction is the following : if for $z_1 = \alpha$ the trace of $S$ on $\{z_2 = \alpha\}$ were a bounded domain in $\mathbf{C}^{n-1}$, the matching of the *CR* structures of $\Sigma^{\pm}$ along $\{\operatorname{Re} z_2 = \alpha\}$ would require that the (lower dimensional) *CR* structures induced on $S \cap \{z_2 = \alpha\}$ coincide. Then Hartogs extension phenomenon would go precisely in the opposite direction to the Lemma given in I, by imposing some matching of the *CR* functions on $\Sigma^{\pm}$.

**2.** In the remaining part of this paper we will consider $S$ an hypersurface defined in a neighborhood of 0 in $\mathbf{C}^m$, $0 \in S$. And we assume that $S$ is defined by

$$\operatorname{Re} z_2 = |\, z_1 \,|^2 + |\, z_2 \,|^2 - \sum_{j=3}^{n} |\, z_j \,|^2 + \mathcal{O}(|\, z \,|^3).$$

This covers (after holomorphic change of variables) the case of hypersurfaces with non-degenerate Levi form with only one positive eigenvalue. If we fix $\alpha > 0$ (small) the key fact is that, on $S \cap \{z_2 = \alpha\}$, $z_1$ does not vanish. This allows to adapt *immediately* the construction done in II.2 (treating $z_3, \ldots, z_m$ as mere parameters on which $\chi$ would not depend) to give an example of a non-embeddable *CR* structure arbitrarily close to the standard *CR* structure induced from $\mathbf{C}^m$ on $S$, with following properties. The functions $z_2, \ldots, z_m$ are *CR* functions. Every *CR* function $f$

depends only on $(z_2, \ldots, z_m)$, not on $z_1$, and therefore, being $CR$, satisfies $df \wedge dz_2 \wedge \ldots \wedge dz_m = 0$.

To work with several sets of coordinates allows one to strengthen the conclusion. This is what is just sketched in the next paragraph.

**3.** Let $k \geq 2$. We want to get, for a perturbed $CR$ structure on $S$, that $z_{k+1}, \ldots, z_m$ are $CR$ functions and that, in a neighborhood of 0, any $CR$ function can depend only on $z_{k+1}, \ldots, z_m$ (in particular is constant if $k \geq m$). Here, there is a change of notations with respect to the introduction.

All the perturbations which follow will be made small enough to keep the signature of the Levi form, and therefore the smoothness of $CR$ functions. We pick "small" disjoint open neighborhoods $V_j$ of the complex hyperplanes $\{z_2 = 2^{-j}\}$, for $j$ large. Set $S_j = S \cap V_j$ and $S' = S - \left(\bigcup_j V_j\right)$. On a neighborhood of $S'$ in $S$ we keep the standard structure for which, in a neighborhood of 0, the Levi form has one positive eigenvalue and $(m-2)$ negative eigenvalues. We assume, as one can, that $S'$ is connected, and locally connected, and that each $S_j$ is connected. On each $S_j$ we modify the structure in such a way that :

(i) Every $CR$ function on $S_j$ satisfies

$$df \wedge dz_2^{(j)} \wedge \ldots \wedge dz_k^{(j)} \wedge dz_{k+1} \wedge \ldots \wedge dz_m = 0,$$

where each $z_p^{(j)}$ is some linear combination (depending on $j$) of the coordinates functions $z_1, \ldots, z_k$, very close in the $C^\infty$ topology to $z_p$ $(p = 2, \ldots, k)$.

(ii) Every $CR$ function $f$ on $S_j$, which on some non empty open subset of $S_j$ satisfies the equation $df \wedge dz_{k+1} \wedge \ldots \wedge dz_m = 0$, satisfies the same equation on $S_j$.

We assume the $z_2^{(j)}$ to be closed enough to $z_2$ so that for $\Omega$ a fixed neighborhood of 0, in which the construction will be carried, $\{z_2^{(j)} = 2^{-j}\} \cap \Omega \subset V_j$.

To achieve (i) one proceeds as already done several times. Ones modifies the standard structure in a neighborhood of the complex hyperplane $\{z_2^{(j)} = 2^{-j}\}$, by patching along the real hyperplane $\{\operatorname{Re} z_2^{(j)} = 2^{-j}\}$ the standard structure with the structure defined by prescribing the functions

$$z_1^{\#} = z_1 + \frac{\epsilon}{z_1}\chi(z_2^{(j)}), z_2^{(j)}, \ldots, z_k^{(j)}, z_{k+1}, \ldots, z_m$$

to be *CR* [one can apply Lupacciolu's theorem "slicewise" for $z_3^{(j)}, \dots, z_k^{(j)}$, $z_{k+1}, \dots, z_m$ fixed and small]. The function $\chi$ is a smooth cut off function with small support around $2^{-j}$ and $\overline{\partial}$ flat on the real line $\{\operatorname{Re} z = 2^{-j}\}$, $\epsilon$ is a small positive number.

Let us turn to (ii). For $\delta > 0$, small enough the set $S_j - \{z \in \mathbf{C}^n, \operatorname{Re} z_2^{(j)} = 2^{-j}, \mid \operatorname{Im} z_2^{(j)} \mid < \delta\}$ is connected. If the support of $\chi$ has been chosen small enough (that for simplicity of the proof we will assume) our *CR* structure is locally embeddable at each point of this set $\Big($by $z \mapsto (\zeta_1, \dots, \zeta_k, z_{k+1}, \dots, z_m)$, for $\zeta_p = z_p$ or $z_p^{(j)}$ $p = 2, \dots, k$ and $\zeta_1 = z_1$ or $z_1^{\#}\Big)$. Then (ii) follows immediately from the fact that a holomorphic function defined on a connected open set cannot vanish on a nonempty open subset of an hypersurface without being identically 0, since on $S_j$ the functions $f, z_{k+1}, \dots, z_m$ are *CR* functions.

The same argument shows the following. Let $\mathcal{O}$ be an open neighborhood of $O$ in $S$, assume $\mathcal{O} \cap S'$ to be connected. Let $f$ be a *CR* function defined on $\mathcal{O}$. For large $j$, $S_j \subset \mathcal{O}$. From (i) we have

$$df \wedge dz_2^{(j)} \wedge \dots \wedge dz_k^{(j)} \wedge dz_{k+1} \wedge \dots \wedge dz_m = 0, \text{ on } S_j$$

but, on a neighborhood of $S'$ $z_2^{(j)}, \dots, z_k^{(j)}$, $z_{k+1}, \dots, z_m$ are *CR* functions and we then get that

$$df \wedge dz_2^{(j)} \wedge \dots \wedge dz_k^{(j)} \wedge dz_{k+1} \wedge \dots \wedge dz_m = 0, \text{ on } S'.$$

We do not immediately get this on $S_\ell$ for $\ell \neq j$, since on $S_\ell$ the $z_p^{(j)}$ are not *CR*, for our structure.

By using, for different indices $j$, different linear combinaisons $z_2^{(j)}, \dots, z_k^{(j)}$ of the functions $z_1, z_2, \dots, z_k$, (it is crucial that $z_1$ appears!) we can get that on a neighborhood of $\mathcal{O} \cap S'$ : $df \wedge dz_{k+1} \wedge \dots \wedge dz_m = 0$.

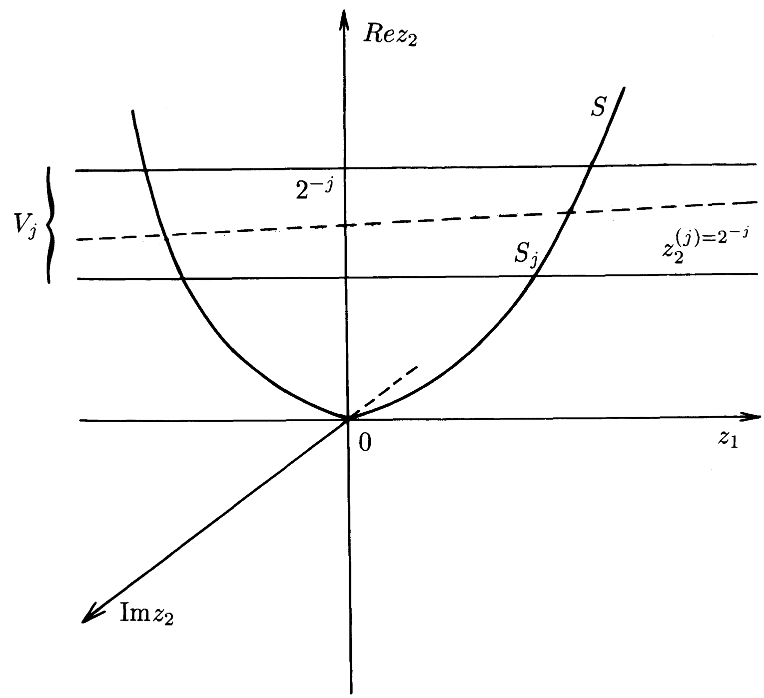
Then according to (ii) $df \wedge dz_{k+1} \wedge \dots \wedge dz_m = 0$ on $S$ and this yields the desired conclusion.

**5.** *Remark.* — We wish to present some remarks about the possibility to do our construction in such a way that the *CR* structure extends to one side of $S$ to a complex structure.

The perturbation of the *CR* structure that we did on $S_j$ leads to a corresponding perturbation on $V_j - \{z_1 = 0\}$. While one keeps the usual complex structure on one side of the real hyperplane $\{\operatorname{Re} z_2^{(j)} =$

$2^{-j}\}$, one imposes on the other side the functions $z_1^{\#}$ (same definition), $z_2^{(j)}, \ldots, z_k^{(j)}, z_{k+1}, \ldots, z_m$ to be holomorphic to define the complex structure.

On a neighborhood of the boundary of $V_j$, this is still (in a fixed neighborhood of 0) the standard structure. Finally, this gives an extension of the *CR* structure to a complex structure on the side of $S$ towards which $(0, -1, 0, \ldots, 0)$ is pointing out (but not on the other side). The following figure, for $(z_3, \ldots, z_m) = 0$, may help :




For $\mathrm{Re}\, z_2 < 0$, the complex structure to be considered is just the usual one.

*Added in proof :* Hill's conjecture has been recently proved by D. Catlin (communication at the AMS summer Institute at Santa Cruz, July 89).

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