ANNALES DE L'INSTITUT FOURIER

PATRICK BONCKAERT

Conjugacy of normally tangent diffeomorphisms : a tool for treating moduli of stability

Annales de l'institut Fourier, tome 40, nº 1 (1990), p. 213-236

<http://www.numdam.org/item?id=AIF_1990__40_1_213_0>

Ann. Inst. Fourier, Grenoble
40, 1 (1990), 213-236

CONJUGACY OF NORMALLY TANGENT DIFFEOMORPHISMS : A TOOL FOR TREATING MODULI OF STABILITY

by P. BONCKAERT(1)

1. Introduction.

Let M be a Riemannian manifold and $f, f' : M \to M$ two C^1 diffeomorphisms leaving a submanifold $V \subset M$ invariant. We say that f and f' are *normally tangent* at V if $f|V = f'|V$ and $Nf = Nf'$, where N denotes the derivative in the normal direction at V. In particular, f is normally tangent to its normal linear part $N(f)$, that is, if we identify a neighbourhood of the zero section in the normal bundle N of V in M with a neighbourhood of V in M by exponentiating (see e.g. Spivak [12]), $N(f)$ is the unique map $N \to N$ which covers $f|V : V \to V$ such that (i) $N(f)$ is linear in each fiber of $N \to V$ and (ii) $TN(f) = Tf|N$ along V. Note that we do not ask that $Tf = Tf'$ along V, so the "shear" terms may be different. A similar definition can be given for vector fields. We want to give sufficient conditions such that f and f' are conjugate near V; moreover we will require some extra properties of the conjugating homeomorphism; these extra properties are useful in the treatment of moduli of stability. Moduli of stability appear in various problems in dynamical systems; for detailed descriptions we refer to f.i. [9], [10], [11], [16]; it is not our aim to define nor treat them in this paper.

---

(1)The author wishes to thank the IMPA, Rio de Janeiro, Brasil, for the hospitality during part of the preparation of this paper; partially supported by NFWO Belgium.
Key-words: Conjugacy – Diffeomorphism – Modulus of stability.
A.M.S. Classification: 58F14.

Let us just describe them informally. Moduli are usually due to non-transversal connections between two saddle type behaviors. For example, let P and Q be two normally hyperbolic invariant manifolds for f and f' such that $W_f^s(Q)$ and $W_f^u(P)$ resp. $W_{f'}^s(Q)$ and $W_{f'}^u(P)$ have a non-transversal intersection. A typical attempt to conjugate f and f' is first to construct a conjugating homeomorphism near P and then trying to extend it to a neighbourhood of Q. In general this last step is impossible, unless f and f' satisfy rigid (i.e. necessary) spectral conditions, cfr. the cited references. This type of rigidity is called a modulus of stability. We give, in section 2, a simple example to indicate how this extra property of the conjugating homeomorphism can be used to overcome the extension problem just mentionned.

Normally tangent diffeomorphisms or vector fields appear, for example, when we blow up a singularity of two vector fields with the same first non-vanishing jet (see [2]), or in the study of dynamical systems with boundary (see [7]), or in the presence of symmetry.

The extra property, mentionned above, is roughly the following.

Suppose, for simplicity, that $M = V \times N$ with N some normed space.

Let $h = (h_v, h_r) : V \times N \to V \times N$ be a homeomorphism defined near $V \times \{0\}$ conjugating f and f'; then we want to have the following estimates (d is some metric on V) :

$$\text{(1)} \qquad \begin{cases} d(h_v(v,r),v) &= O(|r|^\alpha) \\ |h_r(v,r) - r| &= O(|r|^{1+\alpha}) \end{cases}$$

for some $\alpha \in]0,1[$. Remark that (1) implies that $\dfrac{\partial h_r}{\partial r}(v,0)$ exists and is equal to the identity. If we don't require extra condition (1), then classical results on this subject can be found in [5].

2. An example.

Let $L : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n \times \mathbf{R}$ be a linear map of the form $L(v,r) = (L_s v, \mu r)$ with $|L_s| < 1 < \mu$. Suppose that h is a homeomorphism such that $h \circ L = L \circ h$ on a domain W of the form $V \times]-\epsilon, \epsilon[$, where $V \subset \mathbf{R}^n \backslash \{0\}$ is a fundamental domain for L_s in the sense that there exists a disk D in $\mathbf{R}^n$ containing 0 such that $V = D \backslash L_s(D)$.

By "saturation", this conjugacy extends to (at least) $(D\backslash\{0\}) \times]-\epsilon,\epsilon[$ since for every (v,r) in $(D\backslash\{0\}) \times]-\epsilon,\epsilon[$ there exists a (unique) $N \in \mathbb{N}$ such that $L^{-N}(v,r) \in V \times]-\epsilon,\epsilon[$; put $h(v,r) = L^N \circ h \circ L^{-N}(v,r)$. In general, h does not necessarily extend to $\{0\} \times]-\epsilon,\epsilon[$. However, if h satisfies estimates like in (1) on W, it extends to the identity on $\{0\} \times]-\epsilon,\epsilon[$. In fact, we claim that for any sequence (v_i,r_i) converging to $(0,r)$ we have that $h(v_i,r_i)$ converges to $(0,r)$. Let $N_i \in \mathbb{N}$ be so that $L^{-N_i}(v_i,r_i) \in W$. Since

$$\begin{aligned} h(v_i,r_i) &= L^{N_i} \circ h \circ L^{-N_i}(v_i,r_i) \\ &= L^{N_i}(h(L_s^{-N_i}v_i, \mu^{-N_i}r_i)) \\ &= L^{N_i}(L_s^{-N_i}v_i + O(|\mu^{-N_i}r_i|^\alpha), \mu^{-N_i}r_i + O(|\mu^{-N_i}r_i|^{1+\alpha})) \\ &= (v_i + L^{N_i}\mu^{-N_i\alpha}O(|r_i|^\alpha), r_i + \frac{O(|r_i|^{1+\alpha})}{\mu^{N_i\alpha}}) \end{aligned}$$

we get that $h(v_i,r_i) \to (0,r)$.

3. Conjugacy near an invariant manifold.

If $v_0 \in V$ is a fixed point of f and if the codimension of V in M is one, then in [2] one obtained the following result near v_0 :

THEOREM 1 [2]. — *Let f, f' be C^2 diffeomorphisms on $\mathbb{R}^n \times \mathbb{R}$ with a fixed point in $(0,0)$, leaving $\mathbb{R}^n \times \{0\}$ invariant. Write $f = (f_v, f_r) \in \mathbb{R}^n \times \mathbb{R}$ and similarly for f'. Denote (v,r) for the coordinates on $\mathbb{R}^n \times \mathbb{R}$. Suppose that $f|\mathbb{R}^n \times \{0\} = f'|\mathbb{R}^n \times \{0\}$ and*

$$\frac{\partial f_r}{\partial r}|\mathbb{R}^n \times \{0\} = \frac{\partial f'_r}{\partial r}|\mathbb{R}^n \times \{0\}.$$

If $\left|\frac{\partial f_r}{\partial r}(0,0)\right| \neq 1$ then there exists a neighbourhood U of 0 and a homeomorphism h conjugating f and f' on U, i.e. $h \circ f = f' \circ h$, and writing $h = (h_v, h_r)$ there exists an $\alpha > 0$ such that on U :

$$\begin{cases} |h_v(v,r) - v| & \leq |r|^\alpha \\ |h_r(v,r) - r| & \leq |r|^{1+\alpha}. \end{cases} \qquad \square$$

It is our aim to replace $(0,0)$ by an invariant manifold V. This is not always possible in general, even for example if we ask that the normal derivative Nf is a hyperbolic (pure) contraction i.e. if $\sup_{x\in V} |Nf_x| < 1$. A counterexample for this, even without asking extra conditions (1), was given by S. van Strien, [13].

THEOREM 2 [13]. — *Let $M = S^3 = \mathbb{R}^3 \cup \{\infty\}$ and consider $V = S^2$ as a submanifold of M. There exists a C^∞ diffeomorphism $f: M \to M$ leaving V invariant such that :*

(i) $\sup_{x\in V} |Nf_x| < 1$ *for some Riemannian structure on M*

(ii) f *is not C^0 conjugate to $N(f)$.*

A similar example exists for flows.

We need a few preliminaries :

DEFINITION 1. — *A Riemannian manifold V is said to have a* strictly positive radius of injectivity *ρ if every point of it has a normal chart of radius at least ρ, in other words, if for all $v \in V$ the exponential map $\exp_v : \{w \in T_vV : |w| < \rho\} \to V$ is a diffeomorphism onto its image. We say that this image is a normal chart of radius ρ.*

(Question : give a "natural" sufficient condition for this if V is not compact.)

PROPERTY 1 (See [1], [2]). — *Let V, ρ, v, be as above. Let $|\cdot|_v$ be the norm on T_vV and d the Riemannian metric on V. Then for all $\sigma < \rho$ there exists a $C_\sigma > 0$ such that for all $w_1, w_2 \in T_vV$ with $|w_1| < \sigma$ and $|w_2| < \sigma$, denoting $v_1 = \exp_v(w_1)$ and $v_2 = \exp_v(w_2)$:*

$$(1 - C_\sigma)d(v_1, v_2) \le |w_1 - w_2| \le (1 + C_\sigma)d(v_1, v_2)$$

and $C_\sigma \to 0$ if $\sigma \to 0$.

DEFINITION 2. — *If V and W are Riemannian manifolds, V having strictly positive radius of injectivity, then the set*

$$\pounds = \bigcup_{\substack{x\in V\\ y\in W}} L(T_xV, T_yW)$$

can be given a uniform structure (see [6]) as follows. A base B for the uniformity is given by subsets U_ϵ of $\pounds\times\pounds$ consisting of pairs $(T, S) \in \pounds\times$

$\mathcal{L}$ such that (i) $T \in L(T_xV, T_yW), S \in L(T_{x'}V, T_{y'}W)$; (ii) x and x' lie in a normal chart ϕ of radius ϵ; (iii) y and y' lie in a normal chart ψ of radius ϵ; (iv) and

$$|D\psi_y \circ T \circ D(\phi^{-1})_{\phi(x)} - D\psi_{y'} \circ S \circ D(\phi^{-1})_{\phi(x')}| < \epsilon.$$

If $f : V \to W$ is a differentiable map, we consider the map $Df : V \to \mathcal{L}$: $x \to Df_x$. So we say that Df is uniformly continuous for the uniform structure defined above if for each $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x, x' \in V : d(x, x') < \delta$ implies $(Df_x, Df_{x'}) \in U_\epsilon$. Here d is the distance on V.

DEFINITION 3. — *If V is a metric space and $W \subset V$ then a* uniform neighbourhood *of W in V is a set U of the form*

$$U = \{v \in V : d(v, W) < \epsilon\}$$

for some $\epsilon > 0$.

Notation. — If $A : E \to E$ is an invertible continuous linear map between normed spaces we denote $m(A) = |A^{-1}|^{-1}; m(A)$ is called the minimum norm of A.

Let V be a manifold and let $f, f' : V \times \mathbf{R} \to V \times \mathbf{R}$ be two diffeomorphisms, normally tangent at $V \times \{0\}$. Suppose that the radial behavior (in the $\mathbf{R}$-direction) is expansive, and that the tangential behavior (in V-direction) is "less expansive" (a precise statement follows). Then we have the following result.

THEOREM 3. — *Let V be a manifold and let $f, f' : V \times \mathbf{R} \to V \times \mathbf{R}$ be a two C^2 diffeomorphisms leaving $V \times \{0\}$ invariant. Write $f = (f_v, f_r) \in V \times \mathbf{R}$ and similarly for f'. Denote (v, r) for the coordinates on $V \times \mathbf{R}$. Suppose that $f|V \times \{0\} = f'|V \times \{0\}$ and*

$$\frac{\partial f_r}{\partial r}|V \times \{0\} = \frac{\partial f'_r}{\partial r}|V \times \{0\}.$$

Let $\alpha \in]0, 1[$ be given. Suppose that there exists a Riemannian structure on V such that V has a strictly positive radius of injectivity and such that, denoting

$$A = \inf_{v \in V} m(\frac{\partial f_r}{\partial r}(v, 0)) \text{ and } B = \sup_{v \in V} |\frac{\partial f_v}{\partial v}(v, 0)|,$$

one has :

$$A > 1 \text{ and } B < A^{\alpha}. \tag{2}$$

Suppose furthermore that df and $d(\frac{\partial f_r}{\partial r})$ *are uniformly continuous and bounded along* $V \times \{0\}$. *Let finally* $\beta \in]0, \alpha[$.

Then there exists a homeomorphism h conjugating f and f′ on a uniform neighbourhood (depending on α *and* β*) of V and satisfying there the following inequalities, writing* $h = (h_v, h_r) \in V \times \mathbf{R}$:

(i) $d(h_v(v,r), r) \leq |r|^{\alpha}$,

(ii) $|h_r(v,r) - r| \leq |r|^{1+\beta}$;

d is the Riemannian metric on V.

Proof. — Since we only claim a conjugacy on a uniform neighbourhood of V we may modify f and f' outside such a uniform neighbourhood as follows.

Let $\varphi : \mathbf{R} \to [0,1]$ be a fixed C^{∞} "bump" function such that

(i) $\varphi(t) = 1$ on the neighbourhood of 0,

(ii) $\varphi(t) = 0$ for $t \notin [-1,1]$.

Let $\tau : \mathbf{R} \to [-1/2, 1/2]$ be a fixed C^{∞} function with the following properties :

(i) $\tau(t) = t$ for $t \in [-1/3, 1/3]$,

(ii) $\tau(t) = 0$ for $t \notin [-1,1]$,

(iii) $|\tau'(t)| \leq 1$ for all $t \in \mathbf{R}$.

We denote, for $\epsilon > 0$ and $r \in \mathbf{R}$:

$$\varphi_{\varepsilon}(r) = \varphi(\frac{|r|}{\varepsilon}) \text{ and } \tau_{\varepsilon}(r) = \varepsilon.\tau(\frac{|r|}{\varepsilon})\frac{r}{|r|}.$$

Instead of f and f' we will consider, for $\varepsilon > 0$, the maps f_{ε} resp. f'_{ε} defined as follows (we only give the definition for f_{ε} since f'_{ε} is treated similarly). Let $f_{\varepsilon} = (f_{\varepsilon,v}, f_{\varepsilon,r})$ with

$$\begin{cases} f_{\varepsilon,v}(v,r) &= f_v(v, \tau_{\varepsilon}(r)) \\ f_{\varepsilon,r}(v,r) &= \frac{\partial f_r}{\partial r}(v,0).r + \varphi_{\varepsilon}(r)(f_r(v,r) - \frac{\partial f_r}{\partial r}(v,0).r). \end{cases}$$

One immediately checks that on a uniform neighbourhood of $V \times \{0\}$ f and f_ε coincide. We want to have control on the derivatives of f_ε. The proofs of the next two lemmas are lengthly but straightforward. For the second one we use results in [4, chapter 2, section 1].

LEMMA 1. — *There exists $M > 0$ such that for all $\sigma > 0$ there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0]$ and for all $(v, r) \in V \times \mathbf{R}$ we have that $f_{\varepsilon,v}(v, r)$ lies in the same normal chart as $f_v(v, 0)$ and in this chart the following estimates hold :*

(i) $\left|\dfrac{\partial f_{\varepsilon,v}}{\partial v}(v,r) - \dfrac{\partial f_v}{\partial v}(v,0)\right| < \sigma$;

(ii) $\left|\dfrac{\partial f_{\varepsilon,v}}{\partial v}(v,r)\right| \leq M$;

(iii) $\left|\dfrac{\partial f_{\varepsilon,r}}{\partial v}\right| \leq M|r|$;

(iv) $\left|\dfrac{\partial f_{\varepsilon,r}}{\partial r}(v,r) - \dfrac{\partial f_{\varepsilon,r}}{\partial r}(v,0)\right| < \sigma$;

(v) $\left|f_{\varepsilon,r}(v,r) - \dfrac{\partial f_r}{\partial r}(v,0).r\right| < M|r|^2$;

(vi) $\left|(f_\varepsilon^{-1})_r(v,r) - \dfrac{\partial (f^{-1})_r}{\partial r}(v,0).r\right| < M|r|^2$ *if f_ε is invertible;*

(vii) $\left|\dfrac{\partial (f_\varepsilon^{-1})_v}{\partial r}(v,r)\right| < M$;

(viii) $\left|\dfrac{\partial^2 f_{\varepsilon,r}}{\partial v \partial r}(v,r)\right| < M$;

(ix) $\left|\dfrac{\partial^2 f_{\varepsilon,r}}{\partial r^2}(v,r)\right| < M$;

(x) *let $\pi_v : V \times \mathbf{R} \to V$ be the projection and $\widetilde{f} = f|V \times \{0\}$, then* $\left|\dfrac{\partial}{\partial r}(\widetilde{f} \circ \pi_v \circ f_\varepsilon^{-1})\right| < M$. □

LEMMA 2. — *If $\varepsilon > 0$ is small then f_ε is a diffeomorphism.* □

From now on we drop the index ε and we assume that f and f' satisfy the properties and estimates of the lemmas above.

LEMMA 3. — *If $\varepsilon > 0$ is small in the construction above then there exists a unique mapping $h = (h_v, h_r) : V \times \mathbf{R} \to V \times \mathbf{R}$ such that*

(i) $h \circ f = f' \circ h$,

(ii) *h is continuous,*

(iii) $d(h_v(v,r), v) \le |r|^\alpha$ *for* $|r| \le \varepsilon$ *and*

(iv) $|h_r(v,r) - r| \le |r|^{1+\beta}$ *for* $|r| \le \varepsilon$.

Proof. — The proof will include two sublemmas. Let us introduce some notations.

For a map $h = (h_v, h_r) : V \times \mathbf{R} \to V \times \mathbf{R}$ we define for each $\varepsilon > 0$

$$D_\varepsilon(h, Id) = \sup_{v \in V, |r| \le \varepsilon} \left\{ \frac{d(h_v(v,r), v)}{|r|^\alpha}, \frac{|h_r(v,r) - r|}{|r|^{1+\beta}} \right\}$$

where Id is the identity of $V \times \mathbf{R}$. We put

$$E_\varepsilon = \{h : V \times \mathbf{R} \to V \times \mathbf{R} \text{ is continuous and } D_\varepsilon(h, Id) < \infty\}$$

and for $h, h' \in E_\varepsilon$:

$$D_\varepsilon(h, h') = \sup_{v \in V, |r| \le \varepsilon} \left\{ \frac{d(h_v(v,r), h'_v(v,r))}{|r|^\alpha}, \frac{|h_r(v,r) - h'_r(v,r)|}{|r|^{1+\beta}} \right\}.$$

Then $(E_\varepsilon, D_\varepsilon)$ is a complete pseudometric space. For $h \in E_\varepsilon$ we define $Ph : V \times \mathbf{R} \to V \times \mathbf{R}$ by

$$(Ph) = f' \circ h \circ f^{-1}.$$

SUBLEMMA 1. — *Let $E_\varepsilon(1) = \{h \in E_\varepsilon : D_\varepsilon(h, Id) \le 1\}$. If $\varepsilon > 0$ is small then $P(E_\varepsilon(1)) \subset E_\varepsilon(1)$.*

Proof. — Let $h \in E_\varepsilon(1)$. Let π_v, π_r denote the projections from $V \times \mathbf{R}$ onto V resp. $\mathbf{R}$.

Denote $\tilde{f} = f | V \times \{0\}$ and write $f^{-1}(v,r) = (v', r')$.

a) One has, using Lemma 1

$$\begin{aligned}
\frac{d((Ph)_v(v,r),v)}{|r|^\alpha} &\leq \frac{d(h'_v \circ h(v',r'),\widetilde{f}\circ h_v(v',r'))}{|r|^\alpha} + \frac{d(\widetilde{f}\circ h_v(v',r'),\widetilde{f}(v'))}{|r|^\alpha} \\
&\quad + \frac{d(\widetilde{f}(v'),v)}{|r|^\alpha} \\
&\leq \frac{M|h_r(v',r')|}{|r|^\alpha} + \frac{B+\sigma}{(A+\sigma)^\alpha} + \frac{M|r|}{|r|^\alpha} \\
&= \frac{B+\sigma}{(A+\sigma)^\alpha} + O(|r|^{1-\alpha}) < 1
\end{aligned}$$

if σ, ε are sufficiently small and if $|r| \leq \varepsilon$.

b) Also, using the assumption on β,

$$\begin{aligned}
\frac{|(Ph)_r(v,r) - r|}{|r|^{1+\beta}} &\leq \frac{|f'_r \circ h(v',r') - \frac{\partial f_r}{\partial r}(h_v(v',r'),0)\cdot h_r(v',r')|}{|r|^{1+\beta}} \\
&+ \frac{|[\frac{\partial f_r}{\partial r}(h_v(v',r'),0) - \frac{\partial f_r}{\partial r}(v',r')]\cdot h_r(v',r')|}{|r|^{1+\beta}} \\
&+ \frac{|\frac{\partial f_r}{\partial r}(v',r')\cdot[h_r(v',r')-r']|}{|r|^{1+\beta}} + \frac{|\frac{\partial f_r}{\partial r}(v',r')\cdot r' - r|}{|r|^{1+\beta}} \\
&\leq \frac{M|h_r(v',r')|^2}{|r|^{1+\beta}} + \frac{[M\cdot d(h_v(v',r'),v') + M|r'|]O(|r|^{1+\beta})}{|r|^{1+\beta}} \\
&+ \frac{|\frac{\partial f_r}{\partial r}(v',0) + O(|r|)\cdot|r'|^{1+\beta}}{|r|^{1+\beta}} \\
&+ \frac{|(\frac{\partial f_r}{\partial r}(v',0) + O(|r|))\cdot(\frac{\partial (f^{-1})_r}{\partial r}(v,0) + O(|r|))\cdot r - r|}{|r|^{1+\beta}} \\
&\leq O(|r|^{1-\beta}) + O(|r|^\alpha) \\
&+ |\frac{\partial f_r}{\partial r}(v',0) + O(|r|)|\cdot\left(\frac{1}{m(\frac{\partial f_r}{\partial r}(v',0))} + O(|r|)\right)^{1+\beta} + O(|r|^{1-\beta}) \\
&< 1
\end{aligned}$$

if ε is sufficiently small and if $|r| \leq \varepsilon$. This proves the sublemma. $\square$

SUBLEMMA 2. — *If $\varepsilon > 0$ is small then $P : E_\varepsilon(1) \to E_\varepsilon(1)$ is a contraction.*

Proof. — Let $h, h' \in E_\varepsilon(1)$ and write $f^{-1}(v,r) = (v',r')$.

a) First of all

$$\frac{d((Ph)_v(v,r),(Ph')_v(v,r))}{|r|^\alpha}$$
$$\leq \frac{(B+\sigma+O(|r|))\cdot D_\varepsilon(h,h')|r'|^\alpha + M\cdot D_\varepsilon(h,h')|r'|^{1+\beta}}{|r|^\alpha}$$
$$\leq \left[\frac{B+\sigma}{(A+\sigma)^\alpha} + O(|r|^{1+\beta-\alpha})\right]\cdot D_\varepsilon(h,h').$$

b) Secondly

$$\frac{|(Ph)_r(v,r)-(Ph')_r(v,r)|}{|r|^{1+\beta}}$$
$$\leq \frac{M|r|\cdot d(h_v(v',r'),h'_v(v',r')) + |\frac{\partial f_r}{\partial r}(v',0)+O(|r|)|\cdot|h_r(v',r')-h'_r(v',r')|}{|r|^{1+\beta}}$$
$$\leq O(|r|^{\alpha-\beta})\cdot D_\varepsilon(h,h') + |\frac{\partial f_r}{\partial r}(v',0)$$
$$+O(|r|)|\cdot\left(\frac{1}{m(\frac{\partial fr}{\partial r}(v',0))}+O(|r|)\right)^{1+\beta}\cdot D_\varepsilon(h,h').$$

So if $\varepsilon > 0$ is small we get a contraction.

Continuation of the proof of Lemma 3. From the foregoing sublemma it follows that there exists a $h \in E_\varepsilon(1)$ such that $D_\varepsilon(Ph,h) = 0$, and for all $h' \in E_\varepsilon(1)$: if $D_\varepsilon(Ph',h') = 0$ then $D_\varepsilon(h,h') = 0$. Let us analyse this a bit more in detail. The condition $D_\varepsilon(Ph,h) = 0$ means : for all $(v,r) \in V \times \mathbb{R}$ with $|r| \leq \varepsilon$ we have $Ph(v,r) = (v,r)$. As can be checked form the definition of P this implies

$$h \circ f|f^{-1}(V\times B(0,\varepsilon)) = f' \circ h|f^{-1}(V\times B(0,\varepsilon)).$$

But now we can extend this conjugacy to $V \times \mathbf{R}$ by saturating by means of f and f'.

This extension is unique. □

LEMMA 4. — *If we take $\overline{\alpha} > \alpha$ and $\overline{\beta} > \beta$ sufficiently close to α resp. β, and if we apply Lemma 3 to $\overline{\alpha}, \overline{\beta}$ then the obtained mapping $\overline{h}$ is a homeomorphism.*

Proof. — We apply Lemma 3 to $\overline{\alpha}$ and $\overline{\beta}$ instead of α, β. This is possible if $\overline{\alpha}$ and $\overline{\beta}$ are close to α resp. β. We obtain an $\overline{\varepsilon}$ close to ε, say $\varepsilon/2 \leq \overline{\varepsilon} \leq \varepsilon$, and we obtain a unique mapping $\overline{h} = (\overline{h}_v, \overline{h}_r) : V \times \mathbf{R} \to V \times \mathbf{R}$ such that

(i) $\overline{h} \circ f = f' \circ \overline{h}$;

(ii) $\overline{h}$ is continuous;

(iii) $d(\overline{h}_v(v,r), v) \leq |r|^{\overline{\alpha}}$ for $|r| \leq \overline{\varepsilon}$;

(iv) $|h_r(v,r) - r| \leq |r|^{1+\overline{\beta}}$ for $|r| \leq \overline{\varepsilon}$.

But by interchanging the role of f and f' we also infer the existence of a unique mapping $\widetilde{h}$ such that $\widetilde{h} \circ f' = f \circ \widetilde{h}$ and with similar properties as $\overline{h}$. Combining these two we can write

$$\begin{aligned} \widetilde{h} \circ \overline{h} \circ f &= \widetilde{h} \circ f' \circ \overline{h} \\ &= f \circ \widetilde{h} \circ \overline{h}. \end{aligned}$$

Write $\approx{h} = \widetilde{h} \circ \overline{h}$. Then a straightforward estimation shows that $\overset{\approx}{h}$ satisfies all the properties of Lemma 3 (using the original α and β); necessarily $\overset{\approx}{h}$ = Identity. Hence $\overline{h}$ has an inverse, namely $\widetilde{h} \cdot / \cdot$.

Theorem 3 is proved. □

THEOREM 4. — *In Theorem 3 we may replace "$V \times \mathbf{R}$" by "$V \times E$", where E is a Hilbert space, provided that we add the following assumption: denote*

$$A = \inf_{v \in V} m\left(\frac{\partial f_r}{\partial r}(v,0)\right)$$

and

$$\beta_0 = \sup_{v_1 \in V_1} \frac{\log \left|\frac{\partial f_r}{\partial r}(v_1,0,0)\right|}{\log m\left(\frac{\partial f_r}{\partial r}(v_1,0,0)\right)} - 1$$

then assume $\beta_0 < \alpha$ and take $\beta \in]\beta_0, \alpha[$.

The *Proof* only differs from the preceding ones in the treatment of the r-direction in Sublemmas 1 and 2. We choose $\beta \in]\beta_0, \alpha[$ and remark that this yields

$$\sup_{v_1 \in V_1} \frac{|\frac{\partial f_r}{\partial r}(v_1, 0, 0)|}{(m(\frac{\partial f_r}{\partial r}(v_1, 0, 0))^{1+\beta}} < 1.$$

Now it is straightforward to check that the desired estimates can be redone.

Remark. — Recent techniques of S. van Strien [15] in the case $V =$ one point and $E = \mathbb{R}^n$ give an indication that the extra assumption might be superfluous.

4. Application.

DEFINITION. — *Let V be a topological space and $f : V \to V$ a homeomorphism. We say that a set $A \subset V$ is a uniform ω-set if for any neighbourhood U of A there exists $P = P(U) \in \mathbb{N}$ such that for all $y \in V$:*

$$\# \{n | f^n(y) \notin U\} \leq P.$$

Similar definition for a uniform α-set.

If M is a manifold and $f : M \to M$ is a diffeomorphism leaving a submanifold $V \subset M$ invariant we write $Vf = Tf|TV$, and Nf denotes the normal derivative along V.

THEOREM 5. — *Let M be a manifold, $f : M \to M$ a diffeomorphism leaving the relatively compact codimension 1-submanifold V invariant. Assume that V has a trivializable normal bundle. Let us write $\widetilde{f} = f|V$. Suppose that there exists a uniform ω-set $A \subset V$ for $\widetilde{f}$ such that, for some metric on TM :*

$$\sup_{x \in A} |Vf_x| < \inf_{x \in A} m(Nf_x),$$
$$\inf_{x \in A} m(Nf_x) > 1.$$

Then there exists $N \in \mathbb{N}$ and $\alpha \in]0, 1[$ such that for all $n \geq N$: f^n satisfies the assumptions of Theorem 3.

Proof. — Denote

$$\begin{aligned} a &= \sup_{x \in A} |Vf_x| \\ b &= \inf_{x \in A} m(Nf_x) > 1 \end{aligned}$$

and fix $\eta, a', b' \in \mathbb{R}^+$ with $a < a' < b' < b$. Choose for all $x \in A$ a neighbourhood U_x of x in V such that for all $y \in U_x$:

$$|Vf_y| < a' < b' < m(Nf_y).$$

Put $U = \bigcup_{x \in A} U_x$. As U is a neighbourhood of A we know that there exists $P \in \mathbb{N}$ such that for all $y \in V$:

$$\#\ \{n \in \mathbb{N} | f^n(y) \notin U\} \leq P.$$

Now let $y \in V$ and $n \in \mathbb{N}$, $n \geq P$. Denote $I = \inf\{m(Nf_x) : x \in V\}$ and $S = \sup\{|Vf_x| : x \in V\}$. We can write

$$V(f^n)_y = \prod_{i=0}^{n-1} Vf_{f^i(y)}$$

so $|V(f^n)_y| \leq I^P \cdot (a')^{n-p}$ and $m(N(f^n)_y) \geq S^P(b')^{n-P}$.

We can find constants $M > 1$ and $C, D > 0$ with $C > 1$ and $N \in \mathbb{N}$ such that for all $n \geq N$:

$$m(N(f^n)_y) \geq C \geq MD \geq M|V(f^n)_y|.$$

Then $\alpha \in]0, 1[$ works in order to have the conditions of Theorem 3 for f^n.

□

5. Conjugacy near an invariant submanifold of V.

If we want to replace $\{(0,0)\}$ in Theorem 1 by an invariant submanifold V_1 of V towards which we allow contractive as well as expansive behavior, then things become more complicated.

Let us describe the situation in the next theorem, without being too precise for the moment. Let $V_1 \subset V$ be an f-invariant submanifold on which f is "almost" not expansive, that is $|T(f|V_1)| \leq 1 + \eta$ for a "very small" $\eta > 0$. The manifold V_1 will play the role of $\{(0,0)\}$ in Theorem

1. We do not ask that V_1 is compact, because we also want to cover cases like the following. Suppose f.i. that V_1 is a compact f-invariant manifold on which $|T(f|V_1)| \leq 1 + \eta$, and suppose that there exists an invariant manifold $\widetilde{V}_1$ with $V_1 \subset \widetilde{V}_1 \subset V$ such that the normal bundle N_1 of V_1 in $\widetilde{V}_1$ is contracted by Tf, i.e. $|N_1(f)| < 1$.

Then on a neigbourhood of V_1 in $\widetilde{V}_1$ we still have that the norm of the derivative is $\leq 1 + \eta$. We want to let this neighbourhood take over the role of V_1. So we allow open manifolds. Hence we have to impose extra conditions on f and its derivatives such as uniform continuity, boundedness etc. This is not too restrictive, since the applications we have in mind concern neighbourhoods of compact manifolds. By presenting the theorem in this way we avoid unnecessary repetitions of arguments in the proof of it. We will assume that the Riemannian manifold V_1 has a strictly positive radius of injectivity for the exponential mapping. If we want to apply the theorem to neighbourhoods of compact manifolds this is no problem if we make a decent rescaling using diffeomorphisms mapping $]-\varepsilon, \varepsilon[$ to $\mathbf{R}$ (ε small). See further on for the details.

THEOREM 6. — *Let (V_1, d_1) be a Riemannian manifold having strictly positive radius of injectivity, let $V = V_1 \times \mathbf{R}$ and $M = V \times \mathbf{R} = V_1 \times \mathbf{R} \times \mathbf{R}$. Let $f, f' : M \to M$ be two C^2 diffeomorphisms leaving $V \times \{0\}$ and $V_1 \times \{(0,0)\}$ invariant. Write $f = (f_1, f_2, f_r) \in V_1 \times \mathbf{R} \times \mathbf{R}$ and $f_v = (f_1, f_2)$; similarly for f'. We use variables $(v_1, v_2, r) \in V_1 \times \mathbf{R} \times \mathbf{R} = M$ and $v = (v_1, v_2) \in V_1 \times \mathbf{R} = V$.*

Suppose that

$$f|_{V \times \{0\}} = f'|_{V \times \{0\}}$$

and

$$\frac{\partial f_r}{\partial r}|_{V \times \{0\}} = \frac{\partial f'_r}{\partial r}|_{V \times \{0\}}$$

(i.e. : f and f' are normally tangent at V). Suppose furthermore that there exist constants $a, a', a'' \in \mathbf{R}$ and $\eta > 0$ such that for all $v_1 \in V_1$:

$$1 < a' \leq |\frac{\partial f_r}{\partial r}(v_1, 0, 0)| \leq a''$$

$$|\frac{\partial f_1}{\partial v_1}(v_1, 0, 0)| \leq 1 + \eta$$

$$1 < a < \inf_{v_1 \in V_1} |\frac{\partial f_2}{\partial v_2}(v_1, 0, 0)|;$$

denote $\alpha_0 = \log a / \log a''$*; we assume that* $\eta < (a')^{\alpha_0} - 1$*;*
finally we assume that f *is uniformly continuous along* $V_1 \times \{(0,0)\}$*, and that* $Df, \dfrac{\partial^2 f_2}{\partial v_2^2}$ *and* $D(\dfrac{\partial f_r}{\partial r})$ *are bounded and uniformly continuous on* $V_1 \times \{(0,0)\}$.

Then there exists an $\varepsilon > 0$ *and a homeomorphism* $h : V_1 \times]-\varepsilon, \varepsilon[^2 \to V_1 \times \mathbf{R}^2$ *conjugating* f *and* f' *on* $(V_1 \times]-\varepsilon,\varepsilon[^2) \cap f^{-1}(V_1 \times]-\varepsilon,\varepsilon[^2)$ *i.e.* $h \circ f = f' \circ h$*. Moreover* h *satisfies the following inequalities : write* $h = (h_1, h_2, h_r)$ *with respect to the product* $V_1 \times \mathbf{R} \times \mathbf{R}$*; there exists an* $\alpha \in]0,1[$ *such that for all* $(v_1, v_2, r) \in V_1 \times]-\varepsilon, \varepsilon[^2$

$$\begin{aligned} &d_1(h_1(v_1,v_2,r),v_1) \le |r|^\alpha \\ &|h_2(v_1,v_2,r) - v_2| \le |r|^\alpha \\ &|h_r(v_1,v_2,r) - r| \le |r|^{1+\alpha}. \end{aligned}$$

Proof. — a) *An "almost" diagonalization of* $T(f|V \times \{0\})$ *along* V_1.

For shortness sake let us denote $\widetilde{f} = f|V \times \{0\} = f'|V \times \{0\}$.

We consider V_1 as a submanifold of V and look at its normal bundle.

Let $T_{V_1}V$ be the restriction of the tangent bundle TV to V_1. We would like to have a C^2 splitting

$$T_{V_1}V = TV_1 \oplus N$$

such that for $v_1 \in V_1$ the tangent map $D\widetilde{f}_{v_1},: T_{v_1}V \to T_{f(v_1)}V$ has, with respect to this splitting, a matrix of the form

$$Df_{v_1} = \begin{pmatrix} \dfrac{\partial f_1}{\partial v_1}(v_1,0,0) & B(v_1) \\ 0 & \dfrac{\partial f_2}{\partial v_2}(v_1,0,0) \end{pmatrix}$$

with $|B(v_1)| < \sigma$, where σ is any given "small" number. We will need this in the estimates further on in the proof. First of all remark that the zero entry is a trivial consequence of the invariance of V_1 for f; also the diagonal elements are independent of the N in the splitting. So the point is to find a decent representative N for the normal bundle $T_{V_1}V/TV_1$ of V_1

in V. From [5] or [14] and from the assumption that for all $v_1, v_1' \in V_1$:

$$|\frac{\partial f_1}{\partial v_1}(v_1, 0, 0)| \leq 1 + \eta < a \leq |\frac{\partial f_2}{\partial v_2}(v_1', 0, 0)|$$

it follows that $T_{V_1}V$ has a continuous $D\widetilde{f}$-invariant splitting

$$T_{V_1}V = TV_1 \oplus N^*.$$

First of all, like in [5], [14], fix some C^2 splitting

$$T_{V_1}V : TV_1 \oplus N^0$$

(not necessarily being invariant for $D\widetilde{f}$). Each linear bundle map $U : N^0 \to TV_1$ defines splitting by putting, for $v_1 \in V_1$:

$$T_{v_1}V = T_{v_1}V_1 \oplus (I + U_{v_1})(N^0_{v_1})$$

where I denotes the identity. Let U^* denote the continuous linear bundle map obtained in [5], [14] corresponding to the splitting $T_{V_1}V = TV_1 \oplus N^*$, that is : $N^* = (I + U^*)N^0$. From their iteration process it follows that, given $\sigma' > 0$, there exists a C^2 linear bundle map $U : N^0 \to TV_1$ with

$$\sup_{v_1 \in V_1} |U_{v_1} - U^*_{v_1}| < \sigma'.$$

We will choose σ' in a moment. The splitting corresponding to U need not be invariant. Let us look at the decomposition of $D\widetilde{f}$ with respect to this splitting, which we denote

$$T_{V_1}V = TV_1 \oplus N.$$

So let $n_0 \in N^0_{v_1}$ and consider

$$n = U_{v_1}n_0 + n_0 \in N_{v_1}.$$

We must show that

$$|B(v_1)(n)| \leq \sigma|n|.$$

Put

$$n^* = U^*_{v_1}n_0 + n_0;$$

we can decompose $D\widetilde{f}_{v_1}(n^*)$ by writing

$$D\widetilde{f}_{v_1}(n^*) = U^*_{f(v_1)}(n_1) + n_1$$

for some $n_1 \in N^0_{f(v_1)}$. As V_1 is invariant for f, the N^0-component of $D\widetilde{f}_{v_1}(n)$ is equal to n_1. So with respect to the splitting

$$T_{V_1}V = TV_1 \oplus N$$

we have

$$B(v_1)(n) = D\widetilde{f}_{v_1}(n_1) - (n_1 + U_{f(v_1)}(n_1));$$

hence, for some $M > 0$:

$$\begin{aligned}
|B(v_1)| &\leq |D\widetilde{f}_{v_1}(n) - D\widetilde{f}_{v_1}(n^*)| + |D\widetilde{f}_{v_1}(n^*) - (n_1 + U_{f(v_1)}(n_1))| \\
&\leq M|n - n^*| + |n_1 + U^*_{f(v_1)}(n_1) - (n_1 + U_{f(v_1)}(n_1))| \\
&\leq M|U_{v_1}(n_0) - U^*_{v_1}(n_0)| + |U^*_{f(v_1)} - U_{f(v_1)}||n_1| \\
&\leq M\sigma'|n_0| + \sigma'|\frac{\partial f_2}{\partial v_2}(v_1, 0, 0)||n_0| \\
&\leq 2M\sigma'|n_0|
\end{aligned}$$

and this is smaller than $\sigma|n|$ provided σ' is small enough. So from now on we assume that $|B(v_1)| < \sigma$ for all $v_1 \in V_1$.

In the usual way we can identify a small neighbourhood of V_1 in V with a neighbourhood of the zero section in N : see for example [5], [8], [12]. This defines a coordinate system $(v_1, v_2) \in V_1 \times \mathbb{R}$ in the neighbourhood of $V_1 \times \{0\}$ in $V_1 \times \mathbb{R}$ of the form $V_1 \times]-\varepsilon, \varepsilon[$ for some $\varepsilon > 0$. In this coordinate system all the assumptions of the theorem remain valid since the exponential mapping preserves the distance to V_1; moreover in this coordinate system we have

$$|\frac{\partial f_1}{\partial v_2}(v_1, 0, 0)| < \sigma.$$

b) *A "bumping off" construction for f and f'.*

Since we only claim a conjugacy on a uniform neighbourhood of $V_1 \times \{(0,0)\}$ in M we may modify f and f' outside such a uniform neighbourhood as follows. We may assume that V_1 is connected. Fix a number A with

$$A > \inf_{v_1 \in V_1} |\frac{\partial f_2}{\partial v_2}(v_1, 0, 0)| > a;$$

denote

$$K = \sup_{v_1 \in V_1} |\frac{\partial f_2}{\partial v_2}(v_1, 0, 0) - A|;$$

let $\phi : \mathbf{R} \to [0,1]$ be a fixed C^∞ bump function such that

(i) $\phi(t) = 1$ on a neighbourhood of 0;

(ii) $\phi(t) = 0$ for $\notin [-1,1]$

(iii) $\forall\ t \in \mathbf{R} : |t\phi'(t)| < \dfrac{A-a}{2K}$ if $K > 0$, (such a function can be found since $\int_0^1 \dfrac{A-a}{2Kt} dt = \infty$).

Fix moreover a C^∞ function $\tau : \mathbf{R} \to [-\frac{1}{2}, \frac{1}{2}]$ with the following properties :

(i) $\tau(t) = t$ for $t \in [-\frac{1}{3}, \frac{1}{3}]$;

(ii) $\tau(t) = 0$ for $t \notin [-1,1]$;

(iii) $\forall t \in \mathbf{R} : |\tau'(t)| \leq 1$.

We denote, for $\varepsilon > 0, v_2 \in \mathbf{R}$ and $r \in \mathbf{R}$:

$$\begin{aligned}
\tau_\varepsilon(v_2) &= \varepsilon\tau(\frac{|v_2|}{\varepsilon})\frac{v_2}{|v_2|} : \\
\tau_\varepsilon(r) &= \varepsilon\tau(\frac{|r|}{\varepsilon})\frac{r}{|r|}; \\
\phi_\varepsilon(v_2) &= \phi(\frac{|v_2|}{\varepsilon}); \\
\phi_2(r) &= \phi(\frac{|r|}{\varepsilon}).
\end{aligned}$$

Instead of f and f' we will consider, for $\varepsilon > 0$, the maps f_ε resp. f'_ε defined as follows (we only give the definition for f_ε since f'_ε is treated similarly). We put

$$\begin{aligned}
f_{\varepsilon,1}(v_1, v_2, r) &= f_1(v_1, \tau_\varepsilon(v_2), \tau_\varepsilon(r)) \\
f_{\varepsilon,2}(v_1, v_2, r) &= Av_2 + \phi_\varepsilon(r)\phi_\varepsilon(v_2)(f_2(v_1, v_2, 0) - Av_2) \\
&\quad +\phi_\varepsilon(r)(f_2(v_1, \tau_\varepsilon(v_2), r) - f_2(v_1, \tau_\varepsilon(v_2), 0)) \\
f_{\varepsilon,r}(v_1, v_2, r) &= \frac{\partial f_r}{\partial r}(v_1, \tau_\varepsilon(v_2), 0) \cdot r + \phi_\varepsilon(r)(f_r(v_1, \tau_\varepsilon(v_2), r) \\
&\quad -\frac{\partial f_r}{\partial r}(v_1, \tau_\varepsilon(v_2), 0) \cdot (r),
\end{aligned}$$

if $\dfrac{\partial f_2}{\partial v_2}(v_1, 0, 0) > 0$; in the other case we replace A by $-A$; let us treat the first case, since the second case goes similarly. One immediately checks that f and f_ε coincide on a uniform neighbourhood of $V_1 \times \{(0,0)\}$. Also:

$$|r| \geq \varepsilon \Rightarrow f_{\varepsilon,2}(v_1, v_2, r) = Av_2$$

and

$$f_{\varepsilon,r}(v_1, v_2, r) = \frac{\partial f_r}{\partial r}(v_1, \tau_\varepsilon(v_2), 0) \cdot r.$$

c) *Construction of the conjugacy.*

From now on we drop the index ε and we assume that f and f' satisfy the estimates of parts a) and b). Let α_0 be as in the theorem. If $\alpha < \alpha_0$ then $(a'')^\alpha / a < 1$. Fix $\alpha < \alpha_0$ with the property that $\eta < (a')^\alpha - 1$, and fix $\beta \in]0, \alpha[$. The major step in the proof is the following.

LEMMA 5. — *Let f, f' be like in parts a) and b). Fix $\tau > a''$. If $\varepsilon > 0$ is small then there exists a unique mapping $h = (h_v, h_r) : V \times \mathbb{R} \to V \times \mathbb{R}$ such that*

(i) $h \circ f = f' \circ h$;

(ii) *h is continuous;*

(iii) *write $h_v = (h_1, h_2)$ with respect to the product $V_1 \times \mathbb{R}$; then $h_2(v, r) = v_2$ if $|r| \geq \tau^2\varepsilon$;*

(iv) $d_1(h_1(v_1, v_2, r), v_1) \leq |r|^\alpha$ *for* $|r| \leq \tau^3\varepsilon$;

(v) $|h_2(v_1, v_2, r) - v_2| \leq |r|^\alpha$ *for* $|r| \leq \tau^3\varepsilon$;

(vi) $|h_r(v_1, v_2, r) - r| \leq |r|^{1+\beta}$ *for* $|r| \leq \tau^3\varepsilon$.

The *Proof* of this lemma and of the rest of the theorem is now to a large extent similar to the proof of Theorem 3, and is hence omitted. Let us just mention that the operator P is to be taken as follows :

$$\begin{cases} (Ph)_1 &= f \circ h \circ f^{-1} \\ (Ph)_2 &= \begin{cases} (f')_2^{-1} \circ h \circ f(v, r) & \text{if } |f_r(v, r)| \leq \tau^3\varepsilon \\ v_2 & \text{if } |f_r(v, r)| > \tau^2\varepsilon \end{cases} \\ (Ph)_r &= f'_r \circ h \circ f^{-1} \end{cases}$$

THEOREM 7. — *In Theorem 6 we may replace "$V = V_1 \times \mathbb{R}$" by "$V = V_1 \times \mathbb{R}^p$", $p > 1$, provided that we add the following assumption :*

there exists an invertible linear map $A : \mathbf{R}^p \to \mathbf{R}^p$ *such that*

$$\inf_{v_1 \in V_1, \lambda \in [0,1]} m(A + \lambda(\frac{\partial f_2}{\partial v_2}(v_1, 0, 0) - A)) > a. \tag{3}$$

The *Proof* is almost a copy of the preceding one. Let us indicate where the extra assumption is used. In order to let the estimates in work, we made a cut off construction for the operator P in the v_2-direction. For this purpose we need the fact that, for large r, $f_2(v_1, v_2, r)$ is independent of v_1 and r. This was achieved by the "bumping off" construction in part b) of the proof. Now in this construction we use the extra assumption for higher dimensions. Note that, for $p = 1$, assumption (3) is always satisfied if V_1 is connected. In general it means that "$\frac{\partial f_2}{\partial v_2}(v_1, 0, 0)$ may not rotate too much if v_1 varies".

6. Application.

First a new preliminaries. According to [8], Chapter VII. par. 3, any Hilbert bundle which is trivializable as a vector bundle is Hilbert isomorphic to a trivial Hilbert bundle (i.e. with a constant inner product on each fiber); let us explain this in more detail. Let $\pi : E \to V$ be a vector bundle over the base space V, isomorphic to $V \times H$, where $H, \langle .,. \rangle$ is a Hilbert space and such that for each $x \in V$ there is an inner product $\langle .,. \rangle_x$ on H. There exists a continuous linear map $A_x : H \to H$ such that

$$\langle v, w \rangle_x = \langle A_x v, w \rangle$$

for all $v, w \in H$. This A_x is positive definite and symmetric. Hence we can consider a square root $B = A^{1/2}$ of it, and one has

$$\langle A_x v, w \rangle = \langle B_x, v, B_x w \rangle$$

for all $v, w \in H$. If we take

$$B : V \times H \to V \times H : (x, v) \to (x, B_x v)$$

then B is a Hilbert bundle isomorphism between the original Hilbert bundle and a trivial Hilbert bundle. It is clear that B has the same differentiability as the metric on E.

Let M be a smooth manifold, $V \subset M$ a C^2 submanifold of codimension one and $V_1 \subset V$ a compact C^2 submanifold. Let $f, f' : M \to M$ be

C^2 diffeomorphisms leaving V and V_1 invariant. Suppose that f and f' are normally tangent at V. Let N be the normal bundle of V in M. Suppose that $\forall\ x \in V_1 : (Nf)_x : N_x \to N_{f(x)}$ is a hyperbolic (pure) expansion for a some Riemannian on N. (For a hyperbolic (pure) contraction one should consider $f^{-1}, (f')^{-1}$.) Suppose that N is trivializable on a neighbourhood A of V_1 in V, that is : there exists a vector bundle isomorphism $\psi = (\psi_v, \psi_r) : N|A \to A \times E$ covering the identity (see [8]).

Let $T_{V_1}V$ be the restriction of the tangent bundle of V to V_1. Suppose that $T_{V_1}V$ has a trivializable C^2 Tf-invariant splitting

$$T_{V_1}V = N_1^u \oplus TV_1 \oplus N_1^s.$$

This means that the splitting of the normal bundle of V_1 in V, i.e. $N_1^u \oplus N_1^s$, is isomorphic in the sense of vector bundle morphisms (see [8]) to $V_1 \times V_u \times V_s$ for some fixed vector spaces V_u and V_s. For any $x \in V_1$ put

$$\begin{aligned} V_{1,x}f &= Tf_x|T_xV_1 \\ N_{1,x}^u f &= Tf_x|N_{1,x}^u \\ N_{1,x}^s f &= Tf|N_{1,x}^s. \end{aligned}$$

Suppose that there exists a C^2 Riemannian structure on TV such that for all $x \in V_1 : V_{1,x}f$ is an isometry and

$$|N_{1,x}^s f| < 1 < m(N_{1,x}^u f).$$

Let us introduce some more abbreviations :

$$\begin{aligned} a_0 &= \inf_{x \in V_1} m(N_{1,x}^u f) \\ a_0'' &= \sup_{x \in V_1} |Nf_x| \\ \alpha_0 &= \frac{\log a_0}{\log a_0''}, \\ \beta_0 &= \sup_{x \in V_1} \frac{\log |Nf_x}{\log m(Nf_x)} - 1. \end{aligned}$$

If W is a small neighbourhood of V_1 in M we can consider the orthogonal projections $\pi_v : W \to V$ and $\pi_1 : V \to V_1$. Let finally $\psi : W \to N|W \cap V$ denote the natural identification between W and a neighbourhood of the zero section in $N|W \cap V$ and put $\pi_E = \varphi_r \circ \psi : W \to E$.

THEOREM 8. — *Let $M, V, V_1, f, f', a_0, a_0'', \alpha_0$ and β_0 be as above. Let $V_1 \times (V_u \times V_s)$ (considered as a trivial bundle vector bundle) have the metric* *induced by the isomorphism with $N_1^u \oplus N_1^s$, and let $B : V_1 \times V_u \to V_1 \times V_u$ be a Hilbert bundle isomorphism trivializing the metric as described above. Denote $\overline{N}^u = B_*(Tf|N_1^u) : V_1 \times V_u \to V_1 \to V_u$. Assume that there exists a linear map $A : V_u \to V_u$ such that*

$$\inf_{x \in V_1, \lambda \in [0,1]} m(A + \lambda(\overline{N}^u_x - A)) > a_0 \tag{4}$$

and assume that $\beta_0 < \alpha_0$ and $\beta_0 < 1$.

Then there exists a neighbourhood W of V_1 in M and a homeomorphism $h : W \to h(W) \subset M$ conjugating f and f' on $W \cap f^{-1}(W)$ and having the following properties with respect to some Riemannian on M and V. Let d_M, d_V resp. d_1 be the metrics on M, V resp. V_1. There exists $\alpha, \beta \in]0,1[$ and $K > 0$ such that for all $w \in W$:

$$d_1(\pi_1 \circ \pi_V(h(w)), \pi_1 \circ \pi_V(w)) \leq K d_M(w, \pi_v(w))^\alpha$$
$$d_v(\pi_V(h(w), \pi_V(w)) \leq K d_M(w, \pi_V(w))^\alpha$$
$$d_M(\pi_E(h(w)), \pi_E(w)) \leq K d_M(w, \pi_V(w))^{1+\beta}.$$

Proof. — Up to a C^2 change of coordinates we can assume that $f, f' : V_1 \times V_s \times V_u \times E \to V_1 \times V_s \times V_u \times E$ with $V = V_1 \times V_s \times V_u \times \{0\}$ and $V_1 \times \{(0,0,0)\}$ invariant. Consider $N = V \times E = M$ as the trivial normal bundle of V in M and consider $N_1 = V_1 \times V_s \times V_u = V$ as the trivial normal bundle of V_1 in V. Let g be the metric on M for which we have the normal hyperbolicity conditions as stated in the theorem.

As we have explained we can find a C^2 vector bundle isomorphism $\phi : N \to N$ such that $\phi_* g$ is constant, that is, on V_s, V_u and E have a fixed inner product. Let us prove the theorem for $\phi_* f$ and $\phi_* f'$ and let us denote these diffeomorphisms again f and f'. According to [5] there exists invariant manifolds for $f|V$ near V_1, more precisely, there exist unique C^2 invariant manifolds W^s and W^u tangent at V_1 to $V_1 \times V_s$ resp. to $V_1 \times V_u$. Let $V_s(\varepsilon)$ denote the ball of radius ε in V_s.

Up to C^2 change of coordinates ("straightening out the invariant manifold") we may, and do, assume that the stable manifold W^s is locally equal to $V_1 \times V_s(\varepsilon)$. Let us write $f = (f_1, f_s, f_u, f_r)$ with respect to the product $V_1 \times V_s \times V_u \times E$, and similarly for f'. Let $a'' \geq a_0''$ be close to a_0''. If ε is small enough then we can find $a' > 1$ such that for all

$(v_1, v_s) \in V_1 \times V_s(\varepsilon)$ we have

$$1 < a' \leq m(\frac{\partial f_r}{\partial r}(v_1, v_s, 0, 0)) \leq |\frac{\partial f_r}{\partial r}(v_1, v_s, 0, 0)| \leq a''.$$

In the same way, if $\eta > 0$ is a small number then, since V_1 is invariant,

$$|\frac{\partial(f_1, f_s)}{\partial(v_1, v_s)}(v_1, v_s, 0, 0)| \leq 1 + \eta,$$

and if $a \leq a_0$ is close to a_0 then

$$1 < a \leq m(\frac{\partial f_u}{\partial v_u}(v_1, v_s, 0, 0))$$

for all $(v_1, v_s) \in V_1 \times V_s(\varepsilon)$. We want to replace $V_s(\varepsilon)$ by V_s. For that purpose we use a diffeomorphism $\tau : \mathbf{R} \to]-\varepsilon, \varepsilon[$ satisfying the following properties:

(i) $\forall\, t \in [\frac{-\varepsilon}{2}, \frac{\varepsilon}{2}] : \tau(t) = t,$

(ii) $\forall\, t \in \mathbf{R} : 0 < \tau'(t) \leq 1,$

and we put

$$T : V_1 \times V_s(\varepsilon) \times V_u \times E \to V_1 \times V_s \times V_u \times E : (v_1, x, y, r) \to (v_1, \tau^{-1}(|x|)\frac{x}{|x|}, y, r).$$

Now it suffices to consider $T_* f$ and $T_* f'$ instead of f resp. f'. Obviously, the estimates on the partial derivatives for f on $V_1 \times V_s(\varepsilon) \times \{(0,0)\}$ hold for $T_* f$ on $V_1 \times V_s \times \{(0,0)\}$. So from now on we assume that $W^s = V_1 \times V_s$ and we denote $T_* f$ and $T_* f'$ again f resp. f'.

W^s will play the role of V_1 in the Theorem 7 and V_u will take the role of $\mathbf{R}$. By construction V_1 is invariant. □

Remark. — Precisely as in section 3 it should be noted that (i) if the codimension of V in M is one then $\beta_0 = 0$, yielding a simplification of the assumptions and that (ii) if the codimension of V_1 in V is equal to one condition (4) is always satisfied.

BIBLIOGRAPHY

[1] M. BERGER, P. GAUDUCHON and E. MAZET, Le Spectre d'une Variété Riemannienne, Lecture Notes in Mathematics 194, Springer-Verlag, Berlin-Heidelberg-New York (1971).

[2] P. BONCKAERT, F. DUMORTIER and S. STRIEN, Singularities of Vector fields on $\mathbf{R}^3$ determined by their first nonvanishing jet, Ergodic Theory and Dynamical Systems, 9 (1989) 281-308.

[3] H. BRAUNER, Differentialgeometrie, Viewig, Braunschweig, 1981.

[4] M.W. HIRSCH, Differential Topology, Springer, New York, 1976.

[5] M.W. HIRSCH, C. PUGH and M. SHUB, Invariant Manifolds, Lecture Notes in Math., 583, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[6] J.L. KELLEY, General Topology, Van Nostrand, New York, 1955.

[7] R. LABARCA and M.J. PACIFICO, Morse-Smale vector fields on 4-manifolds with boundary, Dynamical Systems and bifurcation theory, Pitman series 160, Longman, 1987.

[8] S. LANG, Differential Manifolds, Addison-Wesley, Reading Massachusetts, 1972.

[9] W. de MELO, Moduli of Stability of two-dimensional diffeomorphisms, Topology 19 (1980), 9-21.

[10] W. de MELO and F. DUMORTIER, A type of moduli for saddle connections of planar diffeomorphisms, J. of Diff. Eq. 75 (1988), 88-102.

[11] J. PALIS, A differentiable invariant of topological conjugacies and moduli of stability, Astérisque, 51 (1978), 335-346.

[12] M. SPIVAK, Differential Geometry, Volume I, Publish or Peris Inc., Berkeley, 1979.

[13] S. van STRIEN, Normal hyperbolicity and linearisability, Invent. Math., 87 (1987), 377-384.

[14] S. van STRIEN, Linearisation along invariant manifolds and determination of degenerate singularies of vector fields in $\mathbf{R}^3$, Delft Progr. Rep., 12 (1988), 107-124.

[15] S. van STRIEN, Smooth linearization of hyperbolic fixed points without resonace conditions, preprint.

[16] S. van STRIEN and G. TAVARES DA SANTOS, Moduli of stability for germs of homogeneous vector fields on $\mathbf{R}^3$, J. of Diff. Eq., 69 (1987), 63-84.

Manuscrit reçu le 11 décembre 1989.

Patrick BONCKAERT,
Limburgs Universitair Centrum
B-3610 Diepenbeek (Belgique).