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FAILURE OF AVERAGING ON MULTIPLY CONNECTED DOMAINS

by David E. BARRETT

Let S be a strip $\{z \in \mathbb{C} : \alpha < \operatorname{Im} z < \beta\}$ and let G be the group of automorphisms of S generated by the translation $z \mapsto z + 1$. The following averaging principle is useful in relating function theory on the annulus S/G to function theory on the simply-connected domain S.

THEOREM 1 (averaging principle). — *Let f be a holomorphic function on S which is bounded on $G \cdot K$ for each compact subset K of S. Then there exists a G-invariant holomorphic function $\tilde{f}$ on S such that $\tilde{f}(z)$ belongs to the closed convex hull of the set $f(G \cdot z)$ for each z in S.*

Proof. — Let $f_n(z) = (2n+1)^{-1} \sum_{j=n}^{n} f(z+j)$. Then the family $\{f_n\}_n$ is uniformly bounded on compact subsets of S so that we can extract a subsequence converging uniformly on compact subsets to a limit function $\tilde{f}$ which clearly satisfies the desired convex hull condition. The G-invariance of $\tilde{f}$ follows from the observation that if f is bounded by C_Z on $G \cdot z$ then $|f_n(z+1) - f_n(z)| \leqslant C_Z/n$ so that indeed $\tilde{f}(z+1) = \tilde{f}(z)$. $\square$

We note in passing that if f is in fact bounded on S then the convex hull condition also holds almost everywhere on the boundary of S.

This averaging principle occurs in work of Scheinberg [Sch] and of Stout [St] and is useful in the study of various problems on annuli involving pointwise estimates, including interpolation problems, corona problems, sup norm estimates for solutions of $\bar{\partial}$, and uniform holomorphic approximation problems. The idea in each case is to pull the problem

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back to a G-invariant problem on the simply-connected domain S, to solve the problem on S by general techniques without attention to the group G, and then to use the averaging principle to obtain G-invariant solutions with no loss of constants.

The argument used in the above proof is known to break down for domains of higher connectivity due to the more complicated nature of the corresponding fundamental groups. The purpose of this note is to exhibit a natural method for constructing counter-examples to the averaging principle in the case of higher connectivity. In particular we will prove the following Theorem.

THEOREM 2. — *Let Ω be any open Riemann surface with non-abelian fundamental group and let $G \cong \pi_1(\Omega)$ be a fixed-point free Fuchsian group with $\Omega \cong \Delta/G$, where Δ is the unit disc. Then there exists a function f holomorphic on Δ and bounded on $G \cdot K$ for all compact subsets K of Δ for which there fails to exist a G-invariant function $\tilde{f}$ on Δ such that $\tilde{f}(z)$ belongs to the closed convex hull of the set $f(G \cdot z)$ for each z in Δ.*

Our construction is based on the study of monodromy properties associated to a certain class of real hypersurfaces in $\Omega \times \mathbb{C}$, namely the class of Levi-flat hypersurfaces with circular fibers. (Indeed, the main thrust of this paper is to show that such a hypersurface need not enclose the graph of a holomorphic function.) These objects appear already in a paper of Nevanlinna in 1929 [Ne], and a paper of Adamyan, Arov, and Krein [AAK] (see also [G]) illustrates their intimate relation with the function-theoretic problems mentioned above. More recently they make an appearance in connection with a very interesting new approach to the corona problem and related function-theoretic problems on the disc, for which see the papers of Alexander and Wermer [AW], Slodkowski [Sk1] [Sk2], and Berndtsson and Ransford [BR] as well as other references cited in these papers. (The reader may also wish to consult the related papers of Forstnerič [Fn1] [Fn2].) Finally, a very recent paper of Berndtsson [B] treats this class of hypersurfaces in its own right in the setting of multiply-connected domains. In view of these developments we introduce these objects in Section 1 at slightly more length than is necessary for the purpose at hand. Sections 2 through 5 contain the proof of Theorem 2. Section 6 contains various remarks about Theorem 2 and its proof.

We should mention that for compact Riemann surfaces Ω with (non-empty) boundary there is a substitute result due to Forelli [Fr] (see also [JM], [EM1], [EM2]) which may be formulated as follows.

THEOREM 3. — *Let $p \in \Omega$, let g_p be Green's function for Ω at p, and let $G \cong \pi_1(\Omega)$ be a fixed-point free Fuchsian group with $\Omega \cong \Delta/G$. Then for any bounded holomorphic function f on Δ there exists a G-invariant meromorphic function $\tilde{f}$ on Δ with poles contained in the preimages of the critical points of g_p (counting multiplicities) and bounded near $\partial\Delta\backslash$ (closure of the set of poles) such that $\tilde{f}(z)$ is contained in the convex hull of the set $f(G \cdot z)$ for almost all $z \in b\Omega$.*

Although this result has shown itself to be very useful it is clear that in some sense it is not optimal, since it is weaker than Theorem 1 in the case of an annulus. It seems to the author that it would be good to devote further study to questions relating to the number and placement of poles required in Theorem 3.

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1. Levi-flat hypersurfaces with circular fibers.

Let Ω be an open Riemann surface. We consider domains D in $\Omega \times \mathbb{C}$ with boundary $S \subset \Omega \times \mathbb{C}$ of class C^2 such that for each $z \in \Omega$ the fiber

$$D z \stackrel{\text{def}}{=} \{w \in \mathbb{C} : (w,z) \in D\}$$

is a (non-degenerate) Euclidean disc with center $c(z)$ and radius $r(z)$.

THEOREM 4. — *For D and S as above the following are equivalent :*

(i) *the Levi-form of S vanishes identically ;*

(ii) *the functions r and c satisfy the following system of partial differential equations with respect to any local coordinate z on Ω :*

$$rr_{z\bar{z}} = |r_{\bar{z}}|^2 + |c_{\bar{z}}|^2$$

$$rc_{z\bar{z}} = 2r_z c_{\bar{z}} ;$$

(iii) *for any simply connected open subset U of Ω there are holomorphic functions α, β, γ, and δ on U so that $\alpha\delta - \beta\gamma \equiv 1$ and*

$$D \cap (U \times \mathbb{C}) = \{(z,w) \in U \times \mathbb{C} : \left|\frac{\alpha(z)w+\beta(z)}{\gamma(z)w+\delta(z)}\right| < 1\} ;$$

(iv) *for any simply connected open subset U of Ω there is a holomorphic mapping* $\rho : U \to \operatorname{Aut} \hat{\mathbb{C}}$ *such that* $D_z = \rho(z)^{-1}(\Delta)$, *where* $\hat{\mathbb{C}}$ *denotes the Riemann sphere.*

This result admits a more or less evident generalization to the case of a domain in $\Omega \times \hat{\mathbb{C}}$ with generalized discs as fibers, where generalized discs include not only interiors of circles but also half-planes as well as exteriors of circles in $\hat{\mathbb{C}}$.

For the purposes of this paper it would suffice to take (iii) or (iv) as the defining property of the class of domains under discussion and to check that this property is in fact local in with respect to Ω. With regard to property (iv) note that the quotient space $\operatorname{Aut} \hat{\mathbb{C}}/\operatorname{Aut} \Delta$ parameterizes the space of generalized discs in $\hat{\mathbb{C}}$, so that our domain D may viewed as the graph of a mapping τ from Ω to $\operatorname{Aut} \hat{\mathbb{C}} /\operatorname{Aut} \Delta$. S is thus Levi-flat if and only if there exists a holomorphic lifting of τ to $\operatorname{Aut} \hat{\mathbb{C}}$ over every simply-connected open subset of Ω. This generalizes the notion of a harmonic function in the sense that a mapping u from Ω to $\mathbb{R} \cong \mathbb{C}/i\mathbb{R}$ is harmonic if and only if there exists a holomorphic lifting of u to $\mathbb{C}$ over every simply-connected open subset of Ω. Note also that if $c \equiv 0$ then the equations in (ii) simply state that $\log r$ is harmonic.

Proof of Theorem 4. — (i) $\Leftrightarrow$ (ii) : The function

$$\psi(z,w) \stackrel{\text{def}}{=} |w-c(z)|^2 - r(z)^2$$

serves as defining function for S. The vanishing of the Levi-form for S is equivalent to the vanishing of the Levi determinant

$$\mathscr{L}(\psi) \stackrel{\text{def}}{=} \det \begin{pmatrix} 0 & \psi_z & \psi_w \\ \psi_{\bar z} & \psi_{z\bar z} & \psi_{w\bar z} \\ \psi_{\bar w} & \psi_{z\bar w} & \psi_{w\bar w} \end{pmatrix}$$

when $\psi = 0$. By direct calculation we have

$$\begin{aligned} \mathscr{L}(\psi) &= 2 \operatorname{Re} (\bar w - \bar c) \{ |w-c|^2 c_{z\bar z} - 2rr_z c_{\bar z} \} \\ &\qquad - 2|w-c|^2 (rr_{z\bar z} + |r_{\bar z}|^2 - |c_{\bar z}|^2) - 4r^2 |r_{\bar z}|^2 \\ &\equiv 2r \operatorname{Re} (\bar w - \bar c)(rc_{z\bar z} - 2r_z c_{\bar z}) + 2r^2 (rr_{z\bar z} - |r_{\bar z}|^2 - |c_{\bar z}|^2) \pmod{\psi} \end{aligned}$$

so that $\mathscr{L}(\psi) \equiv 0 \pmod{\psi}$ if and only if (ii) holds.

(ii) $\Leftrightarrow$ (iv) : Let $p : U \to \operatorname{Aut} \hat{\mathbb{C}}$ be a C^2 map such that $D_z = p(z)^{-1}(\Delta)$. Note that the required holomorphic map ρ must be of the form $\rho = q \cdot p$

for some C^2 map $q : U \to \operatorname{Aut} \Delta$. Our goal, then, is to find conditions on p which are necessary and sufficient for the existence of q such that $q \cdot p$ is holomorphic.

We exploit the equivalence of Δ with the upper half-plane to identify $\operatorname{Aut} \Delta$ with the subgroup $PSL(2,\mathbb{R}) = GL(2,\mathbb{R})/\{\lambda \mathbb{1} : \lambda \in \mathbb{R}^*\}$ of $PSL(2,\mathbb{C}) = GL(2,\mathbb{C})/\{\lambda \mathbb{1} : \lambda \in \mathbb{C}^*\} \cong \operatorname{Aut} \hat{\mathbb{C}}$. Since U is simply-connected we may represent p and q by matrix-valued functions on U, which we denote by P and Q.

Claim. – Let a C^2 map $P : U \to Gl(2,\mathbb{C})$ be given. Then in order to find $Q : U \to Gl(2,\mathbb{R})$ such that $Q \cdot P$ is holomorphic it is necessary and sufficient that the real two-form $\omega \overset{\text{def}}{=} \bar{\partial} P \cdot P^{-1} + \partial \bar{P} \cdot \bar{P}^{-1}$ satisfy $d\omega = \omega \wedge \omega$.

Proof of Claim. – If $Q \cdot P$ is holomorphic then

$$0 = \bar{\partial}(Q \cdot P) = \bar{\partial} Q \cdot P + Q \cdot \bar{\partial} P.$$

By conjugation we have also that

$$0 = \partial Q \cdot \bar{P} + Q \cdot \partial \bar{P}$$

so that

$$dQ = (\partial + \bar{\partial}) Q = - Q \cdot \omega$$

and

$$0 = d^2 Q = Q \cdot \omega \wedge \omega - Q \cdot d\omega ;$$

hence

$$d\omega - \omega \wedge \omega = 0.$$

On the other hand, if $d\omega = \omega \wedge \omega$ then the matrix-valued one-form $dg + g \cdot \omega$ on $U \times Gl(2,\mathbb{R})$ satisfies

$$d(dg + g \cdot \omega) = (dg + g \cdot \omega) \wedge \omega$$

so that the Frobenius Integrability Theorem (see [Na]) guarantees the existence of $Q : U \to Gl(2,\mathbb{R})$ satisfying the linear total differential equation

$$dQ + Q \cdot \omega = 0.$$

Let Q be one such solution. Then

$$\bar{\partial} Q = - Q \cdot \bar{\partial} P \cdot P^{-1}$$

so that $\bar{\partial}(Q \cdot P) = 0$. (We note in passing that Q is determined up to left multiplication by a constant matrix.) □

Thus to prove the equivalence of (iv) and (ii) it suffices now to observe that we may choose p to be the family of automorphisms

$$p(z): \ w \mapsto (w - c(z))/r(z)$$

which is represented by the matrix-valued function

$$P(z) = M \cdot \begin{pmatrix} 1 & -c \\ 0 & r \end{pmatrix},$$

where

$$M = \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}$$

is the matrix taking us from the disc to the half-plane. Then

$$\omega = r^{-1} M \cdot \begin{pmatrix} \partial r & -\bar{\partial} c \\ -\partial \bar{c} & \bar{\partial} r \end{pmatrix} \cdot M^{-1}$$

so that the vanishing of

$$d\omega - \omega \wedge \omega$$
$$= r^{-2} M \cdot \begin{pmatrix} -r\partial\bar{\partial}r + \partial r \wedge \bar{\partial} r + \partial \bar{c} \wedge \bar{\partial} c & -r\partial\bar{\partial}c + 2\partial r \wedge \bar{\partial} c \\ r\partial\bar{\partial}\bar{c} - 2\partial\bar{c} \wedge \bar{\partial} r & r\partial\bar{\partial}r - \partial r \wedge \bar{\partial} r - \partial\bar{c} \wedge \bar{\partial} c \end{pmatrix} \cdot M^{-1}$$

is equivalent to condition (ii).

(iii) $\Leftrightarrow$ (iv) : This follows by definition of the complex structure on Aut $\hat{\mathbb{C}}$.

This completes the proof of Theorem 4. □

Note also that the implication (iii) $\Rightarrow$ (i) follows from the observation that if (iii) holds then S is foliated by the complex curves

$$\frac{\alpha(z)w + \beta(z)}{\gamma(z)w + \delta(z)} = e^{i\theta}.$$

2. Strategy of the construction.

Consider a domain D in $\Omega \times \mathbb{C}$ with disc fibers which satisfies the equivalent conditions of Theorem 4. Let $\Gamma(\bar{D})$ denote the set of all holomorphic functions on Ω whose graph is contained in $D \cup S$.

Suppose that $h \in \Gamma(\bar{D})$ and graph $(h) \cap S \neq \varnothing$. Then there exists a neighborhood U of the z-coordinate of any intersection point together with functions α, β, γ, and δ as in condition (iii) so that

$$\left|\frac{\alpha(z)h(z)+\beta(z)}{\gamma(z)h(z)+\delta(z)}\right| \leqslant 1$$

on U and

$$\left|\frac{\alpha(z)h(z)+\beta(z)}{\gamma(z)h(z)+\delta(z)}\right| = 1$$

at an interior point of U. Thus by the maximum modulus principle it follows that

$$\frac{\alpha(z)h(z) + \beta(z)}{\gamma(z)h(z) + \delta(z)}$$

is constant in U so that graph $(h) \cap (U \times \mathbb{C}) \subset S$. A standard connectedness argument now shows that in fact all of graph (h) is contained in S. Thus we may partition $\Gamma(\bar{D})$ into the set $\Gamma_i(\bar{D})$ of *interior sections* g with graph $(h) \subset D$ and the set $\Gamma_b(\bar{D})$ of *boundary sections* h with graph $(h) \subset S$. It will be useful to define the set $\Gamma_e(\bar{D})$ of *exterior sections* to be the set of all meromorphic functions h on Ω with graph $(h) \cap (D \cup S) = \varnothing$; we explicitly allow $h(z) \equiv \infty$ as a member of $\Gamma_e(\bar{D})$.

Let T be the covering map $T: \Delta \to \Delta/G \cong \Omega$ and let $\tilde{D} = (T \times \mathrm{Id})^{-1}(D) \subset \Delta \times \mathbb{C}$. Then $\tilde{D}$ evidently satisfies the equivalent conditions of Theorem 4 so that we may choose $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$, and $\tilde{\delta}$ holomorphic on Δ so that

$$\tilde{D} = \left\{(z,w) \in \Delta \times \mathbb{C} : \left|\frac{\tilde{\alpha}(z)w+\tilde{\beta}(z)}{\tilde{\gamma}(z)w+\tilde{\delta}(z)}\right| < 1\right\}.$$

Setting

$$\frac{\tilde{\alpha}(z)f(z) + \tilde{\beta}(z)}{\tilde{\gamma}(z)f(z) + \tilde{\delta}(z)} = e^{i\theta}$$

we obtain a holomorphic function f belonging to $\Gamma_b(\bar{\tilde{D}})$. In order to prove Theorem 2 we will define f by exactly this procedure for a particular choice of D over Ω. It is clear that any function f satisfying the conditions of Theorem 2 passes down to a function on Ω belonging to $\Gamma(\bar{D})$. Thus to prove Theorem 2 it will suffice to construct D over Ω for which $\Gamma(\bar{D}) = \varnothing$.

3. Monodromy.

Pick $\rho : \Delta \to \operatorname{Aut}\hat{\mathbb{C}}$ holomorphic so that $\tilde{D}_z = \rho(z)^{-1}(\Delta)$ for $z \in \Delta$. Then

$$\rho(g\cdot z)^{-1}(\Delta) = \tilde{D}_{g\cdot z} = \tilde{D}_z = \rho(z)^{-1}(\Delta)$$

for $g \in G$ so that $\rho(g\cdot z)\rho(z)^{-1} \in \operatorname{Aut}\Delta$. Since $\rho(g\cdot z)\rho(z)^{-1}$ depends holomorphically on z and $\operatorname{Aut}\Delta$ is totally real it follows that $m(g) \overset{\text{def}}{=} \rho(g\cdot z)\rho(z)^{-1}$ is independent of z. The mapping $m : G \to \operatorname{Aut}\Delta$, $g \mapsto m(g)$ is called the *monodromy* of S; it is easy to check that m is a homomorphism.

Observe now that we have a covering map

$$\begin{aligned} \Delta \times \Delta &\to D \\ (z,w) &\mapsto (Tz, \rho(z)^{-1}w) \end{aligned}$$

so that D is obtained as the quotient of the polydisc $\Delta \times \Delta$ by the G-action

$$g\cdot(z,w) = (g\cdot z, m(g)\cdot w). \tag{3.1}$$

Moreover, we see that the graphs of interior sections of D pull back to graphs of holomorphic functions $\varphi : \Delta \to \Delta$ satisfying

$$\varphi(g\cdot z) = m(g)\cdot\varphi(z) \quad \text{for} \quad z \in \Delta \quad \text{and} \quad g \in G. \tag{3.2}$$

Similarly, boundary sections pull back to holomorphic mappings φ from Δ to the unit circle again satisfying (3.2); since φ is necessarily constant these correspond simply to the common fixed points of the image of m on the unit circle. Finally, exterior sections pull back to holomorphic $\varphi : \Delta \to \hat{\mathbb{C}}\setminus\bar{\Delta}$ satisfying (3.2); since the matrix of $m(g)$ may be written

$$\begin{pmatrix} a & b \\ \bar{a} & \bar{b} \end{pmatrix},$$

note that φ satisfies (3.2) if and only if its reflection $h \overset{\text{def}}{=} 1/\bar{\varphi}$ through the unit circle satisfies (3.2), so that exterior sections also correspond to antiholomorphic $h : \Delta \to \Delta$ satisfying (3.2).

We will summarize this last point by saying that exterior sections correspond under Schwarz reflection to antiholomorphic interior sections, but note that the antiholomorphic character of the graph of h is lost upon reverting to the original coordinates in $\Omega \times \mathbb{C}$.

4. Realizibility.

It is time now to note that *any* homomorphism $m : G \to \operatorname{Aut} \Delta$ gives rise to a so-called flat disc bundle over Ω by obtained by dividing the polydisc $\Delta \times \Delta$ by the G-action (3.1); this bundle is contained in an obvious way in a flat $\hat{\mathbb{C}}$-bundle over Ω. The preceding paragraph enables us to extend the discussion of interior, boundary, and exterior sections to this more abstract setting. It is natural to ask which homomorphisms m correspond to actual Levi-flat hypersurfaces with circular fibers in $\Omega \times \mathbb{C}$. The question is answered by the following result.

THEOREM 5. — *A homomorphism $m : G \to \operatorname{Aut} \Delta$ is the monodromy of a Levi-flat hypersurface with circular fibers in $\Omega \times \mathbb{C}$ if and only if the corresponding flat disc bundle over Ω admits an exterior section.*

Proof. — ($\Rightarrow$) Any D in $\Omega \times \mathbb{C}$ satisfying the conditions of Theorem 4 admits an exterior section $w \equiv \infty$, so the implication follows from the invariant nature of our terminology.

($\Leftarrow$) Let B_i denote the flat disc bundle over Ω, let B denote the corresponding $\hat{\mathbb{C}}$-bundle and let $e : \Omega \to B$ denote the given exterior section. It will suffice to find a holomorphic trivialization $\tau : B \cong \Omega \times \hat{\mathbb{C}}$ over Ω so that τ maps the graph of e to the ∞-section $\{w \equiv \infty\}$ of $\Omega \times \hat{\mathbb{C}}$, for then we can simply take our hypersurface to be the boundary of $D = \tau(B_i) \subset \Omega \times \mathbb{C}$.

It is easy to do this locally. Indeed, let $z_0 \in \Omega$, let U be a simply-connected neighborhood of z_0, and let $\nu : B|_U \cong U \times \hat{\mathbb{C}}$ be a trivialization of B over U. Let g be the meromorphic function g on U given by graph $g = \nu(\text{graph } e)$. If $g(z_0) \neq \infty$ then by shrinking U we may assume that g is holomorphic; then the composition of ν with the map $U \times \hat{\mathbb{C}} \to U \times \hat{\mathbb{C}}$, $(z,w) \mapsto (z, 1/(w-g(z))$ transforms the graph of g to the ∞-section as required. If $g(z_0) = \infty$ then we use $(z,w) \mapsto (z, 1/(w^{-1}-(g(z))^{-1})$ instead.

Once this has been done locally then any two local trivializations transforming e to the ∞-section are related by a coordinate transformation of the form $(z',w') = (z, a(z)w+b(z))$. The functions $a(z)$ obtained in this manner give rise to a multiplicative 1-cocyle with values in $\mathcal{O}^*$.

But the exact sequence

$$\begin{array}{ccccccccc} \cdots & \to & H^1(\mathcal{O}) & \to & H^1(\mathcal{O}^*) & \to & H^2(\mathbb{Z}) & \to & \cdots \\ & & \wr\| & & & & \wr\| & & \\ & & 0 & & & & 0 & & \end{array}$$

shows that $H^1(\mathcal{O}^*)$ vanishes, allowing us to redefine our local trivializations so that each $a(z) \equiv 1$. The functions $b(z)$ now define an additive 1-cocycle with values in $\mathcal{O}$ so that the vanishing of $H^1(\mathcal{O})$ allows us to arrange that each $b(z) \equiv 0$. The redefined local trivializations now paste together to define a global trivialization with the desired properties. □

Thus to prove Theorem 2 it will suffice to find a homomorphism $m: G \to \operatorname{Aut} \hat{\mathbb{C}}$ for which the associated flat disc bundle has no interior or boundary sections but does admit an exterior section. To ease the notation in Section 5 we observe that the homomorphism $m^*: G \to \operatorname{Aut} \hat{\mathbb{C}}$ given by $m^*(g)(z) = 1/m(g(1/z))$ has the property that interior sections of the bundle associated to m^* correspond naturally to exterior sections of the bundle associated to m and vice versa. Thus it will suffice to find m for which the associated flat disc bundle has no exterior or boundary sections but does admit an interior section.

5. Conformal mapping.

We now focus our attention on the special case of homomorphisms $m: G \to \operatorname{Aut} \hat{\mathbb{C}}$ having the property that the image of m is contained in a fixed-point free Fuchsian group $G' \subset \operatorname{Aut} \Delta$, so that Δ/G' is a Riemann surface Ω' with $\pi_1(\Omega') \cong G'$. Then from basic properties of covering spaces we may deduce that each $\varphi: \Delta \to \Delta$ satisfying (3.2) induces a holomorphic map $\psi: \Omega \to \Omega'$ such that the induced map ψ_* makes the following diagram commute:

$$\begin{array}{ccc} \pi_1(\Omega) & \xrightarrow{\psi_*} & \pi_1(\Omega') \\ \wr\| & & \wr\| \\ G & \xrightarrow[m]{} & G' \end{array} \tag{5.1}$$

(Here $\pi_1(\Omega)$ and $\pi_1(\Omega')$ are defined with respect to the basepoints $T(0)$ and $T'(\varphi(0))$.) Thus interior sections of the flat disc bundle defined by m correspond to holomorphic maps from Ω to Ω' satisfying the topological condition (5.1). Similarly, from our work in Section 3 we see that exterior sections correspond to anti-holomorphic mappings $\psi : \Omega \to \Omega'$ satisfying the same condition (5.1).

Thus it it now suffices to find G' and m so that

(i) the class of all continuous mappings : $\psi : \Omega \to \Omega'$ satisfying (5.1) contains a holomorphic mapping but does not contain an anti-holomorphic mapping,

and

(ii) the image of m has no common fixed points on the unit circle.

The simplest choice to make at this point is to take $G' = G$ and $m = \mathrm{Id}$ so that $\Omega = \Omega'$. Thus we get an interior section (generally unique) by taking ψ to be the identity. It is well-known that if G is non-abelian then G has no common fixed points on the unit circle so that our bundle admits no boundary sections. Thus by the considerations at the end of Section 4 it suffices now to rule out the existence of an exterior section. But this follows from the following mapping result.

THEOREM 6. – *If Ω is a Riemann surface with non-abelian fundamental group then there does not exist an anti-holomorphic self-map of Ω inducing the identity map on $\pi_1(\Omega)$.*

Proof. – For notational convenience we now regard Ω as the quotient of the upper half-plane H by a fixed-point free Fuchsian group $G \subset \mathrm{Aut}\, H \cong PSL(2,\mathbb{R}) = \{A \in GL(2,\mathbb{R}) : \det A = 1\}/ \pm \mathbb{1}$. Recall that the automorphism corresponding to A is hyperbolic, parabolic, or elliptic according to whether $|\mathrm{trace}\, A|$ is greater than, equal, or less than 2. G contains no elliptic elements since it is fixed-point free.

We claim that G must contain a hyperbolic element. Suppose to the contrary that all non-trivial elements of G are parabolic. Let $A \in SL(2,\mathbb{R})$ represent an element of G, $A \neq \mathbb{1}$. After conjugation we may assume that

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

so that A represents the translation $z \mapsto z + 1$. Let

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

represent an element of G which fails to commute with A. Then trace A^nB = trace $B + cn$, so we must have $c = 0$ if we are to avoid hyperbolic elements. Since B is also parabolic we must have $a = d = \pm 1$ so that B does commute with A, contrary to hypothesis.

So G does indeed contain a hyperbolic element. After conjugation we may assume that this element is a dilation $g_\lambda : z \mapsto \lambda z$ for some $\lambda > 1$.

Suppose now that there does exist an anti-holomorphic map $\psi : \Omega \to \Omega$ which induces the identity map on $\pi_1(\Omega)$. Then ψ lifts to an antiholomorphic map $\Psi : H \to H$ satisfying

$$\Psi(g \cdot z) = g \cdot \Psi(z) \text{ for } z \in \Delta \text{ and } g \in G. \tag{5.2}$$

Let G_λ be the subgroup of G generated by g_λ. Then since (5.2) holds in particular for g restricted to G_λ it follows that Ψ induces an anti-holomorphic self-map of the annulus H/G_λ which induces the identity map on $\pi_1(H/G_\lambda)$. Since it is known that any such map must be an anti-automorphism [Hu] (see also [K, p. 14]) it follows that Ψ must itself be an anti-automorphism of H. By (5.2) Ψ must commute with the dilation g_λ, so Ψ must be of the form $\Psi(z) = -\gamma\bar{z}$, $\gamma \in \mathbb{R}_+$. But by (5.2) again G consists of hyperbolic automorphisms which commute with Ψ, so all elements of G are dilations g_μ. But this contradicts the non-abelian nature of $\pi_1(\Omega)$. □

This completes the proof of Theorem 2. □

6. Remarks.

1) We note that the annulus $\Omega = \{z \in \mathbb{C} : r_1 < |z| < r_2\}$ does admit a topologically trivial anti-automorphism, namely the map $z \mapsto r_1 r_2 z^{-1}$. This map can be used to show that flat disc bundles over annuli admit interior sections if and only if they admit exterior sections. The same remark applies to the punctured plane $\mathbb{C}^*$.

2) The punctured disc does not admit a topologically trivial anti-automorphism, so the construction in Section 5 leads to a flat disc bundle over $\Delta \backslash \{0\}$ admitting an interior section but no exterior section. This disc bundle corresponds to the Levi-flat hypersurface $\{(z,w) \in (\Delta\backslash\{0\}) \times \mathbb{C} : |w - (\log|z|)^{-1}| < (\log|z|)^{-1}\}$, which does admit a boundary section $w \equiv 0$.

3) The function f constructed in the proof of Theorem 2 is not in general bounded, but a normal families argument shows that there is a subdomain of Ω on which f is bounded (as a multiple-valued function) and still fails to have an average $\tilde{f}$. The author has not yet solved the problem of characterizing those Riemann surfaces Ω for which the averaging principle holds for *bounded* f.

Added in proof: This implication (i) $\Rightarrow$ (iii) in Theorem 4 may also be proved by directly computing a holomorphically varying family of fractional linear transformations mapping any three leaves of S aver U to any three leaves of the product hypersurface $U \times \partial\Delta$. This argument appears in « Analytic multivalued functions and polynomically convex hulls », by Donna Kumagai.

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