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ON THE GENERIC SPECTRUM OF A RIEMANNIAN COVER

by Steven ZELDITCH

This paper has two related purposes. The first is to analyze the spectrum of the generic Laplacian on certain finite C^∞ riemannian covers. The second is to give applications to isospectral theory: in particular, to construct isospectral pairs (M_1, g_1), (M_2, g_2) with simple eigenvalue spectra.

The model for problems on generic properties of spectra is the work of J. Albert, K. Uhlenbeck and others on the eigenvalues and eigenfunctions of generic Laplacians on a riemannian manifold ([Al], [U], [Be1], [BaUr]). A typical result of their work is that the generic spectrum is simple (all eigenvalues have multiplicity one). In this paper, we will consider what happens in the case of a finite riemannian cover $p : (M, g) \to (M_0, g_0)$ $(p^*(g_0) = g)$. When p is a normal cover, with covering group G, it is evident that the real eigenspaces $E_\lambda^{\mathbb{R}}$ of the Laplacian Δ_g on M are orthogonal representations of G. A natural (and apparently common [Wig]) conjecture is that, for the generic g_0, the $E_\lambda^{\mathbb{R}}$ should be irreducible. Our first main result (Theorem A) is a proof of this conjecture under a « high dimension-low degree » assumption: namely, that $\dim M > \deg(\sigma)$ for all orthogonal irreducibles σ of G (*). Our method of proof breaks down at every step without this hypothesis, and it is far from clear at present if the conjecture is true without it.

Theorem A has a fairly straightforward generalization to covers which are not necessarily normal. Such covers arise often in isospectral theory (cf. [Su]). To state the result, let $p_1 : M_1 \to M_0$ be a given finite

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(*) We thank G. Besson for pointing out that our result holds as well if dim M $\geqslant \deg \sigma$ for all σ.

cover, and let G be its monodromy group. A straightforward construction produces a tower $M \to M_1 \to M_0$ of finite covers, with $M \to M_i$ normal $(i=0,1)$ and with G equal to the covering group of $p: M \to M_0$. We will refer to p as the « normal closure » of p_1. Then the properties of the generic spectrum of $(M_1, p_1^*(g_0))$ are determined by the structure of the G-space $L^2(G/H, \mathbb{R})$, where H is the group of $M \to M_1$. Of particular interest in this paper are cases where the eigenvalues of the generic Laplacian of $(M_1, p_1^*(g_0))$ are simple. It turns out that two assumptions are necessary and sufficient for this property : first, that $L^2(G/H, \mathbb{R})$ be multiplicity free (each orthogonal irreducible σ of G occurs in it at most once) ; and second that it be « completely of real type » (each occurring σ must be of real type). We then have Theorem B : Let $p_1 : M_1 \to M_0$ be a given finite cover, and let $p: M \to M_0$ be its normal closure. If

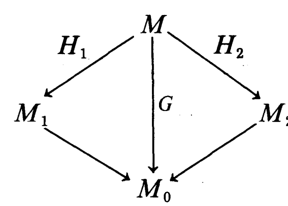
(i) p satisfies the « high dimension-low degree » hypothesis.

(ii) $L^2(G/H, \mathbb{R})$ is multiplicity free, and completely of real type ; then :

$$\operatorname{spec}(M_1, p_1^*(g_0)) \text{ is generically simple.}$$

Theorem B can be combined with Sunada's method for constructing isospectral pairs of riemannian manifolds to produce a « simple isospectral pair » : a pair $(M_1, g_1), (M_2, g_2)$ with $\operatorname{spec}(M_1, g_1) = \operatorname{spec}(M_2, g_2)$, and with $\operatorname{spec}(M_i, g_i)$ simple. Sunada's method is to search for isospectral pairs among commensurate pairs : that is, among pairs M_1, M_2 which fit into a diagram of finite covers :

(1.0)



Here, $M \to M_i (i=0,1,2)$ are normal covers, with covering groups as shown. Sunada observed that if $L^2(G/H_1) \cong L^2(G/H_2)$ as real G-modules, then for any metric g_0 on M_0, $\operatorname{spec}(M_1, p_1^*(g_0)) = \operatorname{spec}(M_2, p_2^*(g_0))$; typically, these pairs are not isometric. To construct a simple isospectral pair, we need a diagram satisfying this condition and also the conditions of Theorem B. Fortunately, an example due to Brooks [Bro] has all these properties. This gives our Theorem C : there exist (non-isometric) simple-isospectral pairs.

The interest of Theorem C (at least to us) is that, at present, Sunada's method is the only systematic method for constructing isospectral pairs. Since his pairs always have a common quotient, it was far from clear that they could have simple spectra : hence, that any isospectral pairs could be simple.

Remarks and Acknowledgements.

(1) Dual to the problem of constructing simple isospectral pairs is that of constructing isospectral pairs with simple length spectra (the length spectrum being the set of lengths of closed geodesics). Under certain conditions, simple length spectrum implies that a riemannian manifold is not a nontrivial cover. Hence, Sunada's method can never produce such examples. It seems quite doubtful that any such examples can exist in the Fourier Integral category [Z].

(2) This paper leaves open many interesting cases of the generic irreducibility question for Laplace eigenspaces on a normal, riemannian cover. In particular, it does not touch the case of graphs, and barely touches a few examples of surfaces. Moreover, the methods fail completely to extend to principal bundles with non-discrete structure groups : in particular, on a manifold with a positive degree of symmetry, does the generic invariant Laplacian have irreducible eigenspaces ?

(3) Finally, we would like to thank the many mathematicians with whom we discussed this work as it evolved. Special thanks are due to B. Brooks, H. Duistermaat and G. Mess for suggestions that significantly improved this paper. It is also a pleasure to thank MSRI and in particular the organizers of the symplectic geometry program for financial and other support and a good atmosphere while this work was done.

1. PRELIMINARIES

Our purpose in this section is to prepare the way for § 2, in which we will prove the generic irreducibility of Laplace eigenspaces on a normal riemannian cover. Here we will set up notation and background for :

(a) Perturbation theory of general Laplacians.

(b) Harmonic analysis on a normal, riemannian cover.

(c) Perturbation theory on a normal riemannian cover.

1a. Perturbation theory of general Laplacians.

The material we summarize here is all well-known. A very detailed and readable account of it can be found in [BaUr], and we will closely follow the notation and terminology of that paper. Other references include [U], [Al], [BlWil], [Be1].

Let X denote a compact, C^∞ manifold of dimension n. We let $S(X)$ denote the Fréchet space of C^∞ symmetric covariant 2-tensors on $X: h \in S(X)$ if, locally,

$$(1.1) \qquad h = \sum_{i,j=1}^{n} h_{ij}\, dx_i \otimes dx_j, \qquad h_{ij} = h_{ji}, \qquad h_{ij} \in C^\infty.$$

$S(X)$ carries a Fréchet norm $|\,.\,|$, obtained by suitably summing local C^k norms [BaUr]. Let ρ' denote the resulting complete metric on $S(X)$.

Next, we let $\mathcal{M} = \mathcal{M}(X)$ denote the Fréchet space of C^∞ riemannian metrics on X. $\mathcal{M}(X)$ also carries a complete metric ρ, for which the metric and C^∞ topologies coincide ([BaUr]). Indeed, let $P_x(X)$ be the cone of positive definite elements of $S_x(X)$. A complete metric ρ''_x may be defined on P_x by setting:

$$(1.2) \qquad \rho''_x(\phi, \psi) = \inf\{\delta > 0 : e^{-\delta}\varphi < \psi < e^{\delta}\varphi\}.$$

One then sets: $\rho''(h_1, h_2) = \sup_x \rho''_x(h_1(x), h_2(x))$, and $\rho = \rho' + \rho''$. For further details, see [BaUr].

Recall that a subset of $\mathcal{M}$ is called *residual* if it contains a countable intersection of open dense sets. Residual subsets of complete metric spaces are dense. A property of metrics in $\mathcal{M}$ will be called *generic* if it holds on a residual subset.

Now let $g_0 \in \mathcal{M}$. *A real analytic deformation* of g_0 is a real analytic curve $g(t) : (-\varepsilon, \varepsilon) \to \mathcal{M}$ with $g(0) = g_0$. Only linear deformations $g(t) = g_0 + th$, $h \in S(X)$, need to be considered in this paper.

The Laplacian $\Delta(g)$ of $g \in \mathcal{M}$ is the essentially self-adjoint operator on $C^\infty(X)$ given in local coordinates by:

$$(1.3) \qquad \Delta(g) = -\frac{1}{\sqrt{|g|}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\sqrt{|g|}\, g^{ij} \frac{\partial}{\partial x_j}\right),$$

where $g = (g_{ij})$, $(g_{ij})^{-1} = (g^{ij})$ and $|g| = \det(g_{ij})$. The sign convention in (1.3) implies that $\Delta(g) \geqslant 0$. The spectrum spec (X, g) of the riemannian manifold (X, g) is the spectrum of $\Delta(g)$: i.e. the eigenvalues $0 = \lambda_0(g) < \lambda_1(g) \leqslant \lambda_2(g)$. The corresponding normalized *real-valued* eigenfunctions will be denoted by $\varphi_k(g)$:

$$\text{(1.4)} \qquad \begin{cases} \Delta(g)\varphi_k(g) = \lambda_k(g)\varphi_k(g) \\ \varphi_k(g) \in C^\infty(X, \mathbb{R}), \qquad (\varphi_k, \varphi_j)_g = \delta_{kj}. \end{cases}$$

Here the inner product $(\varphi, \psi)_g$ is of course that of $L^2\ (X, \mathrm{dvol}_g)$.

We will let $E^{\mathbb{R}}_{\lambda_k}(g)$ denote the real eigenspace of real-valued eigenfunctions of eigenvalue λ_k. The complex eigenspace will be written $E^{\mathbb{C}}_{\lambda_k}(g)$. Of course, $E^{\mathbb{C}}_{\lambda_k} = E^{\mathbb{R}}_{\lambda_k} \otimes \mathbb{C}$. The multiplicity $m(\lambda_k, g)$ of the k^{th} eigenvalue $\lambda_k(g)$ is the dimension (over $\mathbb{R}$) of $E^{\mathbb{R}}_{\lambda_k}(g)$.

The reader should reflect at this point on the fact that complex eigenspaces of normal riemannian covers cannot generally be split into unitary irreducibles by metric perturbations. This will be discussed further in § *c*. For the time being, it should explain the close attention we pay to real versus complex eigenspaces.

One always has the orthogonal decompositions:

$$\text{(1.5i)} \qquad L^2(X, \mathbb{R}; \mathrm{dvol}_g) = \bigoplus_k E^{\mathbb{R}}_{\lambda_k}(g)$$

$$\text{(1.5ii)} \qquad L^2(X, \mathbb{C}; \mathrm{dvol}_g) = \bigoplus_k E^{\mathbb{C}}_{\lambda_k}(g).$$

Now let $g(t)$ be a real analytic deformation of g. The Laplacian of $g(t)$ will be denoted $\Delta(t)$. If λ_k is a simple eigenvalue of $\Delta(0)$ (i.e. mult $(\lambda_k, g_0) = 1$), then there are real analytic functions $\lambda_k(t)$, $\varphi_k(t)$ of $t \in (-\varepsilon, \varepsilon)$ for some ε so that $\Delta(t)\varphi_k(t) = \lambda_k(t)\varphi_k(t)$. The eigenfunction $\varphi_k(t)$ is real-valued and $(\varphi_k(t), \varphi_k(t))_t \equiv 1$. If λ_k is a multiple eigenvalue, say mult $(\lambda_k, g_0) = \ell$, then there always exists an orthonormal basis of $E^{\mathbb{R}}_{\lambda_k}(g_0)$ which extends analytically along $g(t)$. More precisely, there exist ε, and real analytic $\lambda^i_k(t)$ and $\varphi^i_k(t)$, so that

(1.6i) $\Delta(t)\varphi^i_k(t) = \lambda^i_k(t)\varphi^i_k(t),\ |t| < \varepsilon,\ i = 1, \ldots, \ell$;

(1.6ii) $\lambda^i_k(0) = \lambda_k(g)$, $\{\varphi^i_k(0)\}$ is an orthonormal basis of $E^{\mathbb{R}}_{\lambda_k}$;

(1.6iii) $(\varphi^i_k(t), \varphi^j_k(t))_t \equiv \delta_{ij}$.

The initial basis $\{\varphi^i_k(0)\}$ is sometimes called a *Kato basis* for $E^{\mathbb{R}}_{\lambda_k}(g_0)$ and for the deformation $g(t)$ [PSa]). To emphasize that the $\varphi^i_k(t)$ are real-valued we will call it a *real Kato basis*.

Taking the t-derivative at $t = 0$ of (1.6i) we get the variational formula :

$$(1.7) \qquad (\dot{\Delta}-\dot{\lambda}_k^i)\varphi_k^i + (\Delta-\lambda_k)\dot{\varphi}_k^i = 0 .$$

For simplicity, we do not put either $g(0)$ or the infinitesimal deformation $\dot{g}(0)$ explicitly into the notation.

The infinitesimal deformations $\dot{\Delta}$ of the Laplacian are given by the following well-known formulae ([U], p. 1075-6, [BaUr], Lemma 4.4)

(1.8a) When $g(t) = (1+tr)g$ ($r \in C^\infty(X)$) is a conformal deformation, then $\dot{\Delta} = -\frac{n}{2} r\Delta + \left(\frac{n}{2}-1\right) \operatorname{div}(r\nabla)$.

(1.8b) When $g(t)$ is volume preserving, then

$$\dot{\Delta} = -\frac{1}{\sqrt{|g|}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\sqrt{|g|}\, \dot{g}^{ij} \frac{\partial}{\partial x_j}\right).$$

Here $(\dot{g}^{ij})$ is a traceless, contravariant symmetric 2-tensor: $\sum_{i,j} \dot{g}_{ij} g^{ij} = 0$.

1b. Harmonic analysis on a normal, riemannian cover.

Let $p: M \to M_0$ be a finite normal cover, with covering group G. In other words: a principal G-bundle over M_0, where G is a finite group.

Let $\mathcal{M}_0 = \mathcal{M}(M_0)$ be the C^∞ metrics on M_0, let $p^*\mathcal{M}_0 \subset \mathcal{M}(M)$ be their pullbacks to M and let $\mathcal{M}_G$ be the G-invariant metric on M. Normality of p implies :

$$(1.9) \qquad \mathcal{M}_G = p^*\mathcal{M}_0 .$$

When M and M_0 are equipped with metrics g and g_0, p is said to be *riemannian* precisely when $p^*(g_0) = g$. This terminology of course makes sense for any cover, normal or not. Only such metrics will be considered on covers M in this paper.

Given $f \in L^1(M, \mathrm{dvol}_g)$, we let f^0 denote its projection to $L^1(M_0, \mathrm{dvol}_{g_0})$

$$(1.10) \qquad f^0(x) \stackrel{\text{def}}{=} \sum_{g \in G} f(gx) .$$

Here we expect no confusion to arise between the notation g for group elements and for metrics. We also note that in (1.10) we have tacitly identified elements of $L^1(M_0)$ with G-invariant elements of $L^1(M)$. We also will use the notation:

$$(1.11) \qquad f^{\text{ave}}(x) \overset{\text{def}}{=} \frac{1}{|G|} f^0(x), \qquad |G| = \operatorname{card} G.$$

Now let $g = p^*(g_0)$, and let $\langle , \rangle$ be the inner product on $L^2(M, \mathbb{C}; \mathrm{dvol}_g)$. We then have:

$$(1.12) \qquad \langle f_1, f_2 \rangle = \int_{M_0} (f_1 \bar{f}_2)^0 \, \mathrm{dvol}_{g_0}(x).$$

Of course $(f_1\bar{f}_2)^0(x)$ is the inner product on $L^2(G)$ of the functions $f_i^x(g) \overset{\text{def}}{=} f_i(gx)$. Formula (1.12) accounts for all the special features distinguishing analysis on normal covers from that on non-normal ones.

When $g = p^*(g_0)$, it is clear that the real eigenspaces $E_\lambda^{\mathbb{R}}(g)$, resp. the complex eigenspaces $E_\lambda^{\mathbb{C}}(g)$ are orthogonal, resp. unitary representations of G. In general they are reducible. Thus, we have G-isomorphisms:

$$(1.13\text{a}) \qquad E_\lambda^{\mathbb{R}}(g) \simeq \bigoplus_\sigma m(\lambda, \sigma; g) W_\sigma$$

$$(1.13\text{b}) \qquad E_\lambda^{\mathbb{C}}(g) \simeq \bigoplus_\rho m(\lambda, \rho; g) V_\rho$$

where: (i) (σ, W_σ) runs over the set $\hat{G}_0$ of (equivalence classes of) real, orthogonal irreducibles of G; (ii) (ρ, V_ρ) runs over the set $\hat{G}_u$ of unitary irreducibles; (iii) $m(\lambda, \sigma; g)$ is the multiplicity of (σ, W_σ) in $E_\lambda^{\mathbb{R}}$ (likewise for $\hat{G}_u$).

Under a fixed isomorphism in (1.13a), $E_\lambda^{\mathbb{R}}(g)$ is decomposed into an orthogonal sum of isotypic summands:

$$(1.14) \qquad E_\lambda^{\mathbb{R}} = \bigoplus_\sigma E_\lambda^\sigma \quad (\sigma \in \hat{G}_0).$$

E_λ^σ is the subspace of eigenfunctions « transforming according to σ ». Note that we have dropped explicit mention of $\mathbb{R}$ and g in the notation for E_λ^σ, and will continue to do so when no confusion seems likely. A similar decomposition to (1.14) of course holds for $E_\lambda^{\mathbb{C}}$.

Let us also set:

$$(1.15\text{i}) \qquad L_\sigma^2(M, \mathbb{R}; \mathrm{dvol}_g) = \bigoplus_\lambda E_\lambda^\sigma,$$

so that

$$(1.15\text{ii}) \qquad L^2(M) = \bigoplus_{\sigma} L^2_\sigma .$$

Our main problem in § 2 will be to show that under our « high dimension-low degree » hypothesis, generically $E^{\mathbb{R}}_\lambda(g)$ is irreducible for all $\lambda \in \operatorname{spec}(M,g)$. It will clarify matters considerably if we now translate this problem in terms of equivariant eigenvectors for the Laplacian on M. To facilitate their definition, we will give some more terminology regarding the decomposition in (1.13a) and also to distinguish certain special kinds of bases for E^σ_λ.

First, we obviously have:

$$(1.16) \qquad E^\sigma_\lambda \simeq m(\lambda,\sigma) W_\sigma .$$

Let us call a fixed isomorphism in (1.16) a *splitting* of E^σ_λ (into irreducibles). We then let $W^\sigma_{\lambda,j}$ denote the irreducible subspaces of E^σ_λ coming from the splitting, so that:

$$(1.17) \qquad E^\sigma_\lambda = \bigoplus_{j=1}^{m(\lambda,\sigma)} W^\sigma_{\lambda,j} .$$

Second, we distinguish certain orthonormal bases for the $W^\sigma_{\lambda,j}$. To do this, we begin by selecting from each equivalence class in $\hat{G}_0$ a specific representative, or *model*, (σ, W_σ). This is of course a specific matrix representation $\sigma : G \to 0(\mathbb{R}^p)$, $p = \deg \sigma$. Let $(\sigma_{ij}(g))$ be the matrix of σ relative to the standard basis e_j of $\mathbb{R}^p$.

We then say:

(1.18) Definition. — *$\{\varphi^\sigma_{\lambda,j;1}, \cdots, \varphi^\sigma_{\lambda,j;p}\}$ is a normalized σ-basis of $W^\sigma_{\lambda,j}$ if:*

(i) $\varphi^\sigma_{\lambda,j;k}(gx) = \sum_{l=1}^{p} \sigma_{kl}(g)\varphi^\sigma_{\lambda,j;l}(x)$,

(ii) $(\varphi^\sigma_{\lambda,j;k}, \varphi^\sigma_{\lambda,j;l}) = \delta_{kl}$.

Equivalently, the isomorphism: $W^\sigma_{\lambda,j} \mapsto \mathbb{R}^p$ taking $\varphi^\sigma_{\lambda,j;k}$ to ε_k is a G-isomorphism to the model. Note also that (ii) is redundant by the Schur orthogonality relations (see the Appendix to this section).

We now introduce *equivariant vectors :*

(1.19) DEFINITION. – *If* (σ, W_σ) *is the model above, let* $\varepsilon^\sigma = \varepsilon^\sigma(M, \mathbb{R})$ *be the space of :*

$$\begin{cases} \Phi : M \to \mathbb{R}^p, & \|\Phi\|_2^2 < \infty \\ \Phi(\sigma g) = \sigma(g)\Phi(g) \end{cases}$$

where : $p = \deg \sigma$, *and* $\|\Phi\|_2^2 = \int_M \langle \Phi(x), \Phi(x) \rangle \, \mathrm{dvol}_g$, $\langle . \rangle$ *denoting here the inner product on* $\mathbb{R}^p$.

If we write $\Phi = (\varphi_1, \ldots, \varphi_p)'$, with $\varphi_j : M \to \mathbb{R}$, then the equivariance conditions means exactly that the φ_j transform according to (1.18i).

It is obvious that the Laplacian is essentially self-adjoint on $C^\infty \cap \varepsilon^\sigma$, and hence we get an orthogonal decomposition

(1.20) $$\varepsilon^\sigma = \bigoplus_\lambda \varepsilon^\sigma_\lambda$$

into spaces of equivariant eigenvectors :

(1.21) $$\varepsilon^\sigma_\lambda = \{\Phi \in \varepsilon^\sigma : \Delta\Phi = \lambda\Phi\}.$$

The choices above in (1.17) and (1.18) of splittings and σ-bases correspond to the choices of certain orthonormal bases for $\varepsilon^\sigma_\lambda$. To describe them, we need to recall some elementary facts about real, orthogonal representations [BröT-D], [Ad], [K]).

First, for an orthogonal irreducible representation $\sigma : G \to O(\mathbb{R}^p)$, let

(1.22) $$K(\sigma) = \{A \in \mathrm{End}\,(\mathbb{R}^p) : A\sigma(g) = \sigma(g)A, \forall g \in G\}.$$

$K(\sigma)$ is the intertwinning algebra, or centralizer, of the σ-action. For such $\sigma \in \hat{G}_0$, $K(\sigma)$ is a division algebra, i.e. $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$ (the quaternions). σ is said to be of real, complex of quaternionic type accordingly (the letter « K » is supposed to suggest « field »).

An equivalent description of types is in terms of the complexification $\sigma_{\mathbb{C}}$ of σ. Letting ρ denote a *unitary* irreducible of degree p, one has :

(1.23i) $\sigma_{\mathbb{C}} = \rho$, σ of $\mathbb{R}$-type

(1.23ii) $\sigma_{\mathbb{C}} = \rho \oplus \bar{\rho}$, σ of $\mathbb{C}$-type

(1.23iii) $\sigma_{\mathbb{C}} = \rho \oplus \rho$, σ of $\mathbb{H}$-type.

Let us then define:

(1.24) DEFINITION. — *Let* $K = K(\sigma)$. *A set* $\{\Phi^\sigma_{\lambda,1}, \ldots, \Phi^\sigma_{\lambda,m}\}$ *of equivariant eigenvectors is called a normalized K-basis of* $\varepsilon^\sigma_\lambda$ *if:*

(i) $\langle \Phi^\sigma_{\lambda,i}, \Phi^\sigma_{\lambda,j} \rangle = p\delta^i_j$.

(ii) $\varepsilon^\sigma_\lambda = \bigoplus_{i=1}^{m} K\Phi^\sigma_{\lambda,i}$.

Our main result in this section is the following.

(1.25) PROPOSITION. — *There is a* $1-1$ *correspondence between*

(a) *a choice of splitting* $E^\sigma_\lambda = \bigoplus_{j=1}^{m(\lambda,\sigma)} W^\sigma_{\lambda,j}$ *together with a choice of normalized* σ*-basis for each* $W^\sigma_{\lambda,j}$, *and*

(b) *a choice of orthonormal* $K(\sigma)$*-basis for* $\varepsilon^\sigma_\lambda$.

Proof. — Of course, the proof is just to confirm that the correspondence $\Phi = (\varphi_1, \ldots, \varphi_p) \mapsto \{\varphi_1, \ldots, \varphi_p\}$ between equivariant vectors and σ-bases carries the data of (a) to (b) and vice-versa.

First, by the Schur orthogonality relations for real, orthogonal representations (cf. the Appendix to this section), the σ-basis $\{\varphi_1, \ldots, \varphi_p\}$ is automatically orthonormal if $\|\Phi\|_2^2 = p$.

Now, suppose $\{\Phi^\sigma_{\lambda,1}, \ldots, \Phi^\sigma_{\lambda,m}\}$ is an orthonormal K-basis of $\varepsilon^\sigma_\lambda$. To each $\Phi^\sigma_{\lambda,j}$ we may associate the irreducible subspace $E(\Phi^\sigma_{\lambda,j})$ of E^σ_λ spanned by its components:

$$(1.24) \qquad E(\Phi^\sigma_{\lambda,j}) \overset{\text{def}}{=} \bigoplus_{k=1}^{p} \mathbb{R}\varphi^\sigma_{\lambda,j;k}.$$

Obviously $E(\Phi^\sigma_{\lambda,j}) \perp E(\Phi^\sigma_{\lambda,k})$ if $j \neq k$. Further, $\{\varphi^\sigma_{\lambda,j;k} : k=1, \ldots, p\}$ is a normalized σ-basis for $E(\Phi^\sigma_{\lambda,j})$ by the remark above. To show that $\{\Phi^\sigma_{\lambda,1}, \ldots, \Phi^\sigma{}_{\lambda,m}\}$ determines a complete splitting of E^σ_λ, and a normalized σ-basis for each irreducible, we only have to show that $m = m(\lambda,\sigma)$. But obviously $m \leqslant m(\lambda,\sigma)$. Suppose then that $m < m(\lambda,\sigma)$, and let $W^\sigma_{\lambda,0}$ be a irreducible in E^σ_λ which is orthogonal to $\bigoplus_{j=1}^{m} E(\Phi^\sigma_{\lambda,j})$. Let $\{\psi_1, \ldots, \psi_p\}$ be a normalized σ-basis for $W^\sigma_{\lambda,0}$, and let Ψ be the corresponding equivariant eigenvector. By assumption, $\Psi = \sum_{j=1}^{m} A_j\Phi^\sigma_{\lambda,j}$, with $A_j \in K$. This clearly implies that $W^\sigma_{\lambda,0} \subset \bigoplus_{j=1}^{m} E(\Phi^\sigma_{\lambda,j})$, a contradiction.

So $m = m(\lambda,\sigma)$, $\bigoplus_{j=1}^{m} E(\Phi^\sigma_{\lambda,j})$ is a complete splitting of E^σ_λ, and the components of the Φ's give the σ-bases for their summand.

Conversely, given the data of (a), we use the σ-bases of $W^\sigma_{\lambda,j}$ to form equivariant eigenvectors $\{\Phi^\sigma_{\lambda,1}, \ldots, \Phi^\sigma_{\lambda,m}\}$, where now it is clear that $m = m(\lambda,\sigma)$. We need to show that $\{\Phi^\sigma_{\lambda,j}\}$ is a K-basis of $\varepsilon^\sigma_\lambda$. By orthogonality of the $W^\sigma_{\lambda,j}$, it follows that for any $A \in \mathrm{End}(\mathbb{R}^p)$, $\langle A\Phi^\sigma_{\lambda,i}, \Phi^\sigma_{\lambda,j}\rangle = 0\,(i\neq j)$. This applies in particular if $A \in K$. So $\bigoplus_{j=1}^{m} K\Phi^\sigma_{\lambda,j}$ is an orthogonal sum in $\varepsilon^\sigma_\lambda$, and $\{\Phi^\sigma_{\lambda,j}\}$ is a normalized basis for it. If it doesn't equal $\varepsilon^\sigma_\lambda$, there is a $\Psi \in \varepsilon^\sigma_\lambda$ with $\Psi \perp \bigoplus_{j=1}^{m} K\Phi^\sigma_{\lambda,j}$. Then the components $\{\psi_1, \ldots, \psi_p\}$ of Ψ form a σ-basis for an irreducible subspace $E(\Psi)$ of E^σ_λ. By assumption, $E^\sigma_\lambda = \bigoplus_{j=1}^{m} E(\Phi^\sigma_{\lambda,j})$, so there are constants $(A^j)^l_k$ $(j=1,\ldots,m\,;\,k,l=1,\ldots,p)$ with $\psi_k = \sum_{j,k} (A^j)^l_k \varphi^\sigma_{\lambda,j;l}$. Equivalently: $\Psi = \sum_{j=1}^{m} A^j\Phi^\sigma_{\lambda,j}$. We reach a contradiction if $A^j \in K$. But $\sigma(g)^{-1}\Psi(\sigma g) = \Psi$ implies that $\sum_j [\sigma(g)^{-1}A^j\sigma(g) - A^j]\Phi^\sigma_{\lambda,j} = 0$. By an obvious orthogonality (used above), we must have that $\sigma(g)^{-1}A^j\sigma(g) - A^j = 0$ for all j. So $A^j \in K$, concluding the proof. □

(1.25) Corollary. – $\dim \varepsilon^\sigma_\lambda = (\dim_{\mathbb{R}} K(\sigma))m(\lambda,\sigma)$.

The proposition allows us to restate our criterion on irreducibility of eigenspaces:

(1.26) The eigenspace $E^{\mathbb{R}}_\lambda(g)$ is irreducible if and only if:

(i) There is at most one $\sigma \in \hat{G}_0$ so that $\varepsilon^\sigma_\lambda \neq \{0\}$.

(ii) If $\varepsilon^\sigma_\lambda \neq \{0\}$, then $\varepsilon^\sigma_\lambda = K(\sigma)\Phi^\sigma_\lambda$ for some Φ^σ_λ.

Finally, we end this section by explaining our earlier remark on the non-splittability of complex eigenspace. To do so, we reconsider (1.23). Suppose that $\sigma \in \hat{G}_0$ is of $\mathbb{C}$- or $\mathbb{H}$-type. As is well-known (cf. [Do]), for any metric g, there is non-zero proportion of eigenvalues λ for which $E^\sigma_\lambda \neq \phi$. With no real loss of generality, we may therefore assume $E^{\mathbb{R}}_\lambda(g) = E^\sigma_\lambda(g)$. Then we recall that $E^{\mathbb{C}}_\lambda(g) = E^{\mathbb{R}}_\lambda(g) \otimes \mathbb{C}$, i.e. $E^{\mathbb{R}}_\lambda(g)$ consists of the real and imaginary parts of elements of $E^{\mathbb{C}}_\lambda$. It follows that no metric perturbation can split up the unitary irreducibles of $E^{\mathbb{C}}_\lambda$ corresponding to (1.23ii-iii).

1c. Appendix.

In this appendix we state and sketch the proof of the Schur orthogonality relations for real, orthogonal irreducibles $\sigma \in \hat{G}_0$. We have already needed these in (1.18) and Proposition (1.25). Unfortunately, we were unable to find a standard reference for them.

Since the orthogonality relations depend on the type of representation, we will organize this appendix accordingly.

(1) σ of real type $(\deg \sigma = p)$. Then:

$$\text{(1A.1)} \qquad \frac{1}{|G|} \sum_g \sigma_{i_2 j_2}(g) \sigma_{i_1 j_1}(g) = \delta^{i_2}_{i_1} \delta^{j_2}_{j_1} \frac{1}{\deg \sigma}.$$

(The proof is exactly as for unitary irreducibles.)

(2) σ of $\mathbb{C}$-type $(\deg \sigma = p)$; then $K(\sigma) = \{a + b\mathscr{I}\}$, where: $\mathscr{I} \in O(\mathbb{R}^p)$; $\mathscr{I}^2 = -\mathrm{id}$; $a, b \in \mathbb{R}$.

Since $\mathbb{C}$ acts non-trivially on $\mathbb{R}^p$, we must have $p = 2m$. Let $\{e_1, \ldots, e_m, f_1, \ldots, f_m\}$ be a symplectic basis for the form $g_0(\mathscr{I}\cdot, \cdot)$, g_0 being the Euclidean metric on $\mathbb{R}^p$. Then $\mathscr{I}$ has matrix

$$\left[\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right],$$

where I is the $m \times m$ identity matrix. Since $\sigma(g)$ commutes with $\mathscr{I}$, it has matrix

$$\left[\begin{array}{c|c} A & B \\ \hline -B & A \end{array}\right]$$

with A, $B \in \mathrm{End}(\mathbb{R}^m)$.

We claim:

$$\text{(1A.2)} \quad \text{(i)} \ \frac{1}{|G|} \sum_g a_{i_1 j_1}(g) a_{i_2 j_2}(g) = \frac{1}{|G|} \sum_g b_{i_1 j_1}(g) b_{i_2 j_2}(g) = \delta^{i_1}_{i_2} \delta^{j_1}_{j_2}$$

$$\text{(ii)} \ \sum_g a_{i_1 j_1}(g) b_{i_2 j_2}(g) = 0 \quad (\forall i_1, i_2, j_1, j_2).$$

Here a_{ij} (resp. b_{ij}) are the matrix elements of A (resp. B).

Indeed, for any $M \in \mathrm{End}(\mathbb{R}^p)$, we set:

$$M^0 \overset{\text{def}}{=} \frac{1}{|G|}\sum_g \sigma(g)^{-1} M \sigma(g), \tag{1A.3}$$

Then $M^0 \in K(\sigma)$, so $M^0 = a + b\mathcal{I}$. Clearly $a = \mathrm{Tr}\, M$, $b = -\mathrm{Tr}\, \mathcal{I} M$. Now let

$$M = \left[\begin{array}{c|c} X & 0 \\ \hline 0 & 0 \end{array}\right],$$

with $X \in \mathrm{End}(\mathbb{R}^n)$.

As $\sigma(g)^{-1} = \sigma(g)^t$, we get:

$$\sigma(g)^t M \sigma(g) = \begin{bmatrix} A^t XA & A^t XB \\ -B^t XB & B^t XB \end{bmatrix}. \tag{1A.4}$$

Then $M^0 = (\mathrm{tr}\, X)I$, hence

(1A.5) (i) $\frac{1}{|G|}\sum_g A(g)^t XA(g) = \frac{1}{|G|}\sum_g B(g)^t XB(g) = \mathrm{tr}\, XI$

(ii) $\sum_g A(g)^t XB(g) = \sum_g B(g)^t XA(g) = 0$.

Plugging in $X = E^{i_1 j_1}$ (1 in the (i_1, j_1)-position and zero elsewhere), we get (A1.2) above.

(3) σ of $\mathbb{H}$-type.

By an argument similar to (2), $p = 4m$ and

$$K(\sigma) = \{A + b\mathcal{I} + c\mathcal{J} + d\mathcal{K}\},$$

where $\mathcal{I}, \mathcal{J}, \mathcal{K}$ generate the quaternions. One can choose a basis of $\mathbb{R}^p$ so that:

$$\mathcal{I} = \left[\begin{array}{cc|cc} 0 & 1 & & \\ -1 & 0 & & \\ \hline & & 0 & 1 \\ & & -1 & 0 \end{array}\right] \tag{1A.6 (i)}$$

$$\text{(ii)}\quad \mathscr{J} = \left[\begin{array}{cc|cc} & & 1 & 0 \\ & & 0 & 1 \\ \hline -1 & 0 & & \\ 0 & -1 & & \end{array}\right]$$

$$\text{(iii)}\quad \mathscr{K} = \left[\begin{array}{cc|cc} & & 0 & -1 \\ & & 1 & 0 \\ \hline 0 & -1 & & \\ 1 & 0 & & \end{array}\right].$$

From the fact that σ commutes with $K(\sigma)$, one gets that the matrix of σ for this basis is:

$$\text{(1A.7)} \qquad \sigma(g) = \left[\begin{array}{cc|cc} A & B & C & D \\ -B & A & -D & C \\ \hline -C & D & A & B \\ -D & -C & -B & A \end{array}\right].$$

Letting $M \in \mathrm{End}(\mathbb{R}^p)$, we find that:

(1A.8)

$$M^0 = (\mathrm{Tr}\, M)I + (-\mathrm{Tr}\, \mathscr{I}M)\mathscr{I} + (-\mathrm{Tr}\, \mathscr{J}M)\mathscr{J} + (-\mathrm{Tr}\, \mathscr{K}M)\mathscr{K}.$$

If we let

$$M = \left[\begin{array}{cc|c} X & 0 & \\ 0 & 0 & 0 \\ \hline & 0 & 0 \end{array}\right],$$

with $X = E^{i_1 j_1}$ as before, we get

$$\text{(1A.8)} \qquad \frac{1}{|G|}\sum_g r_{i_1 j_1}(g) r_{i_2 j_2}(g) = \frac{1}{p}\,\delta_{i_1}^{i_2}\delta_{j_1}^{j_2}$$

where r_{ij} are the matrix elements of any of A, B, C, D. Further, the elements of any distinct pair from A, B, C, D are orthogonal.

This completes the discussion of the Schur relations. We now show that the elements of a σ-basis are always orthonormal. The proof is well-illustrated with the care of σ of $\mathbb{C}$-type; the general case is left to the reader.

Suppose then that σ has $\mathbb{C}$-type. We write $\mathbb{R}^p = \mathbb{R}^m \otimes \mathbb{R}^2$ and let $\varepsilon_0 = (1,0)$, $\varepsilon_1 = (0,1)$ be the standard basis of $\mathbb{R}^2$. Then any $v \in \mathbb{R}^p$ may be written : $v = v_1 \otimes \varepsilon_1 + v_2 \otimes \varepsilon_2$, with $v_i \in \mathbb{R}^m$. So an equivariant eigenvector $\Phi : \mathrm{M} \to \mathbb{R}^p$ is $\Phi = \Phi_0 \otimes \varepsilon_0 + \Phi_1 \otimes \varepsilon_1$. The equivariance condition is :

$$\begin{aligned} (1A.9) \qquad \Phi_0(gx) &= A(g)\Phi_0(x) + B(g)\Phi_1(x) \\ \Phi_1(gx) &= -B(g)\Phi_0(x) + A(g)\Phi_1(x). \end{aligned}$$

Let $\Phi_0 = (\varphi_{01}, \ldots, \varphi_{0p})$, $\Phi_1 = (\varphi_{11}, \ldots, \varphi_{1p})$. Then our claim is : $\langle \varphi_{0i}, \varphi_{0j} \rangle = \delta_{ij}\|\varphi_{0i}\|_2^2$, $\langle \varphi_{0i}, \varphi_{1j} \rangle = 0$, $\langle \varphi_{1i}, \varphi_{1j} \rangle = \delta_{ij}\|\varphi_{1i}\|_2^2$. We do only the first. But by (1.12) :

$$(1A.10) \qquad \langle \varphi_{0i}, \varphi_{0j} \rangle = \int_{M_0} \left(\sum_g \varphi_{0i}(gx)\varphi_{0j}(gx)\right) dx ,$$

Then (1A.9) combines with (1A.2) to give the stated result.

1d. Perturbation theory on a normal riemannian cover.

We now adapt the material of § 1a, and some other material from [BaUr], to the situation of a normal, riemannian cover.

Suppose, then, that $g = p^*(g_0)$. Let $g(t) = g + tp^*(h)$, where $h \in S(M_0)$. According to (1.6), there is a real Kato basis for $E_\lambda^{\mathbb{R}}(g)$ along $g(t)$. In fact, the proof in (say) [BaUr] shows more. Namely, if $E_\lambda^{\mathbb{R}}(g) = \bigoplus_\sigma E_\lambda^\sigma(g)$, then each $E_\lambda^\sigma(g)$ has its own real Kato basis along $g(t)$. Indeed, one just applies the same proof to the deformation $\Delta(t)$ on L_σ^2. So let $\{\varphi_k^{\sigma,i}\}$ be the real Kato basis of $E_\lambda^\sigma(g)$. It is then clear that $E(\varphi_k^{\sigma,i}(t))$ is a real analytic family of irreducible eigenspaces of $\Delta(t)$. Let us say :

(1.27) Definition. — *A splitting* $E_\lambda^\sigma = \bigoplus_{j=1}^{m(\lambda,\sigma)} W_{\lambda,j}^\sigma$ *is a Kato-splitting (along* $g(t)$*), if the irreducibles* $W_{\lambda,j}^\sigma$ *extend to a real analytic family* $W_{\lambda,j}^\sigma(t)$ *of eigenspaces for* $\Delta(t)$.

Thus, $\bigotimes_{j=1}^{m(\lambda,\sigma)} E(\varphi_k^{\sigma,i}(0))$ is a Kato splitting of $E_{\lambda_k}^\sigma(g)$.

We now remark that *any* orthonormal basis for the summands $W^{\sigma}_{\lambda,j}$ of a Kato splitting will form a Kato basis along $g(t)$. Indeed, this follows from the irreducibility of the $W^{\sigma}_{\lambda,j}$. We may therefore fix a real σ-basis for each $W^{\sigma}_{\lambda,j}$ which extends analytically along $g(t)$. We will call such a basis a *real Kato σ-basis* (for $g(t)$).

Such a real Kato σ-basis is equivalent to a *$K(\sigma)$-Kato basis* for $\varepsilon^{\sigma}_{\lambda}$; i.e. a normalized $K(\sigma)$-basis $\{\Phi^{\sigma,i}_k\}$ of $\varepsilon^{\sigma}_{\lambda}$ which extends analytically to $\{\Phi^{\sigma,i}_k(t)\}$ along $g(t)$. The proof is immediate from Proposition (1.25). Summing up, we have

(1.28) Proposition. — *Let $g(t) = g + tp^*(h)$. Then for all $\sigma \in \hat{G}_0$, and $k \in \mathbb{Z}^+$, there exists ε and real analytic $\lambda^i_k(t)$, $\Phi^{\sigma,i}_k(t) \in \varepsilon^{\sigma}_{\lambda_k(t)}$ so that :*

(i) $\Delta(t)\Phi^{\sigma,i}_k(t) = \lambda^i_k(t)\Phi^{\sigma,i}_k(t)$;

(ii) $\lambda^i_k(0) = \lambda_k(g)$;

(iii) $\{\Phi^{\sigma,i}_k(0)\}$ *is a normalized K-basis of* $\varepsilon^{\sigma}_{\lambda_k}(g)$;

(iv) $\langle \Phi^{\sigma,i}_k(t), \Phi^{\sigma,j}_k(t)\rangle = (\deg \sigma)\delta^i_j$.

Our second order of business in this part is to recall (and adapt to our setting) some material on the continuity of eigenvalues and upper semi-continuity of multiplicities. Our reference for this is (again) [BaUr].

To begin with, let us define

(1.29) Definition. — (i) $U_{\varepsilon}(g_0) = \{g'_0 \in \mathcal{M}_0 : \rho(g_0, g'_0) < \varepsilon\}$

(ii) $V_{\varepsilon}(g_0) = \{g'_0 \in \mathcal{M}_0 : \rho''(g_0, g'_0) < \varepsilon\}$ (cf. (1.2), (1.9)).

We then have (following [BaUr], theorem 2.2) :

(1.30) Proposition. — *$g'_0 \in V_{\varepsilon}(g_0)$ implies, for all $k \in \mathbb{Z}^+$,*

$$e^{-(n+1)\varepsilon} \leqslant \lambda_k(p^*(g_0))/\lambda_k(p^*(g'_0)) \leqslant e^{(n+1)\varepsilon}.$$

Proof. — Precisely as in [BaUr]. For completeness, we give some details. The main point is to use the mini-max characterization of eigenvalues :

$$\lambda_k(g) = \inf_{L_{k+1}} \Lambda_g(L_{k+1}) \tag{1.31}$$

where L_{k+1} is a $(k+1)$-dimensional subspace of $C^{\infty}(M)$, and where (in an obvious notation)

$$\Lambda_g(L_{k+1}) = \sup\{\|df\|^2_g/\|f\|^2_g \,;\, 0 \neq f \in L_{k+1}\}. \tag{1.32}$$

To compare $\lambda_k(p^*(g_0))$ and $\lambda_k(p^*(g'_0))$, one thus only needs to compare the ratios $\|df\|_g^2/\|f\|_g^2$ for $g = p^*(g_0)$ and $g' = p^*(g'_0)$. By the assumption that $g'_0 \in V_\varepsilon(g_0)$, one has :

$$(1.33) \qquad e^{-\varepsilon}(g'_{0ij}) \leqslant (g_{0ij}) \leqslant e^{\varepsilon}(g'_{0ij})$$

(in some coordinate chart X). Obviously (1.33) lifts to $p^*(g_0)_{ij}$ and $p^*(g'_0)_{ij}$. It follows that for any f supported in the $p^{-1}(X)$:

$$(1.34) \quad e^{-(n+1)\varepsilon}\frac{\|df\|^2_{p^*(g'_0)}}{\|f\|^2_{p^*(g'_0)}} \leqslant \frac{\|df\|^2_{p^*(g_0)}}{\|f\|^2_{p^*(g_0)}} \leqslant e^{(n+1)\varepsilon}\frac{\|df\|^2_{p^*(g'_0)}}{\|f\|^2_{p^*(g'_0)}}.$$

(Compare [BaUr], pp. 161-162).

The proposition now follows from (1.32). □

A small modification of the proposition allows us to relativize it to a representation σ. In other words, let $\lambda_k(\sigma, g_0)$ be the k^{th} eigenvalue of $\Delta(p^*(g_0))$ in $L^2_\sigma(M,\mathbb{R})$, and let spec $(M, p^*(g_0), \sigma)$ be the set of these. We obviously have :

$$(1.35) \qquad \lambda_k(\sigma, g_0) = \inf_{L_{k+1}} \Lambda_{p^*(g_0)}(L^\sigma_{k+1})$$

where L^σ_{k+1} runs over $(k+1)$-dimensional subspaces of $C^\infty \cap L^2_\sigma$. Following thru (1.30), we get :

(1.36) Corollary. — *$g'_0 \in V_\varepsilon(g_0)$ implies, for all $k \in \mathbb{Z}^+$:*

$$e^{-(n+1)\varepsilon} \leqslant \frac{\lambda_k(\sigma, g_0)}{\lambda_k(\sigma, g'_0)} \leqslant e^{(n+1)\varepsilon}.$$

Proposition (1.30) and Corollary (1.36) immediately imply some useful results on upper semi-continuity of eigenvalues.

(1.37) Proposition (Compare [BaUr], Corollary 2.3).

(a) $(\forall g_0 \in \mathcal{M}_0)(\forall k)(\exists \varepsilon)$: $g'_0 \in V_\varepsilon(g_0)$ *implies :*

$$m(\lambda_k, p^*(g'_0)) \leqslant m(\lambda_k, p^*(g_0))$$

(b) $(\forall g_0 \in \mathcal{M}_0)(\forall \sigma)(\forall k)(\exists \varepsilon)$: $g'_0 \in V_\varepsilon(g_0)$ *implies :*

$$m(\lambda_k, \sigma, p^*(g'_0)) \leqslant m(\lambda_k, \sigma, p^*(g_0)).$$

Proof. — (a) Write λ_k for $\lambda_k(p^*(g_0))$ and λ'_k for $\lambda_k(p^*(g'_0))$. Then, if $m(\lambda'_k) > m(\lambda_k)$, there must exist k_0 so that $\lambda_{k_0} = \lambda_k$, $\lambda_{k_0+1} > \lambda_k$, $\lambda'_{k_0} = \lambda'_k$, $\lambda'_{k_0+1} = \lambda'_k$. Hence : $1 < \frac{\lambda_{k_0+1}}{\lambda_{k_0}} = \frac{\lambda_{k_0+1}}{\lambda'_{k_0+1}}\frac{\lambda'_{k_0}}{\lambda_{k_0}} \leqslant e^{(2n+1)\varepsilon}$. We get a contradiction if we can let $\varepsilon \to 0$.

(b) Similar. □

2. GENERIC SPECTRUM OF A NORMAL COVER

Our object in this section is the proof of :

(2.1) THEOREM A. — *Let $p : M \to M_0$ be a normal cover, with covering group G. Assume that p satisfies the following « high dimension — low degree » hypothesis :*

(2.2) (HDLD) [1] $\dim M > \max\{\deg \sigma, \sigma \in \hat{G}_0\}$.

Then for the generic G-invariant Laplacian on M, all eigenspaces are irreducible.

Proof. — We begin by stating the conclusion more precisely. Let us make the

(2.3) DEFINITION. — $S = \{g_0 \in \mathcal{M}_0 : (\forall k=0,1,2,\dots)\ E^{\mathbb{R}}_{\lambda_k}(p^*(g_0))$ *is an irreducible, real G-module*$\}$.

Equivalently (by Proposition (1.25) and (1.28)) :

(2.4) $S = \{g_0 \in \mathcal{M}_0 : (\forall k=0,1,2,\dots) : \varepsilon^{\sigma}_{\lambda_k} \neq \{0\}$ for at most one most one $\sigma \in \hat{G}_0$, and then $\varepsilon^{\sigma}_{\lambda_k} = K(\sigma)\Phi^{\sigma}_{\lambda_k}\}$.

Our claim is that S is residual in $\mathcal{M}_0$.

Now, it is clear that

(2.5) $$S = \bigcap_k S_k,$$

[1] G. Besson has pointed out that all the results of this section are valid if only dim M $\geqslant$ max {deg σ, $\sigma \in \hat{G}_0$}.

where

(2.6) $S_k = \{g_0 \in \mathcal{M}_0 : E^{\mathbb{R}}_{\lambda_j}(p^*(g_0))$ is irreducible for $j=0,1,\dots,k\}$.

Hence we need to show that each S_k is open and dense.

Open-ness of S_k

This is quite straightforward, and can be adapted from the similar assertion in [BaUr] (Theorem 3.2). First, let us set :

$$\text{(2.7i)} \quad S(\sigma) = \{g_0 \in \mathcal{M}_0 : E^{\sigma}_{\lambda_k}(g) \text{ is irreducible } (\forall k)\} = \{g_0 \in \mathcal{M}_0 : E^{\sigma}_{\lambda_k}(g) = K(\sigma)\Phi^{\sigma}_{\lambda_k}(\forall k)\}$$

$$\text{(2.7ii)} \quad S_k(\sigma) = \{g_0 \in \mathcal{M}_0 : E^{\sigma}_{\lambda_k}(g) \text{ is irreducible for } j=0,1,\dots,k\}.$$

Precisely as in [BaUr], p. 164], it follows directly from Proposition (1.37b) that :

$$\text{(2.8)} \qquad (\forall g_0 \in S_k(\sigma))(\exists \varepsilon) : \ V_\varepsilon(g_0) \subset S_k(\sigma).$$

Evidently $S_k(\sigma)$ is open $(\forall k \in \mathbb{N})$. We can now prove :

(2.9) PROPOSITION. – S_k is open $(\forall k \in \mathbb{N})$.

Proof. – Obviously, $S_k \subset \bigcap_\sigma S_k(\sigma)$ $(\sigma \in \hat{G}_0)$. Suppose then that $g_0 \in S_k$. One has :

$$\text{(2.10i)} \qquad (\forall j \leqslant k)(\forall \sigma) : \ m(\lambda_j, \sigma, p^*(g_0)) \leqslant 1$$

$$\text{(2.10ii)} \qquad (\forall j \leqslant k)(\exists ! \sigma) : \ m(\lambda_j, \sigma, p^*(g_0)) \neq 0.$$

By upper semi-continuity of eigenvalues (Proposition (1.37b)), one has :

(2.11) $(\forall j \leqslant k)(\forall \sigma) \exists \in (j,\sigma) : g'_0 \in V_{\in(j,\sigma)}(g_0)$ implies (2.10i-ii) hold for g'_0. Then $\bigcap_{(j \leqslant k,\sigma)} V_{\in(j,\sigma)}(g_0) \subset S_k$.

It follows that S_k is open. □

Denseness of S_k

It will suffice to show that S_k is dense in S_{k+1}. So suppose $g_0 \in S_k$. Then, for all $j \leqslant k$, $E^{\mathbb{R}}_{\lambda_j}(p^*(g_0))$ is irreducible. Suppose that $E^{\mathbb{R}}_{\lambda_{k+1}}(p^*(g_0)) = \bigoplus_\sigma E^{\sigma}_{\lambda_{k+1}}$ and that $E^{\sigma}_{\lambda_{k+1}} = \bigoplus_{j=1}^{m(\lambda_{k+1},\sigma)} W^{\sigma}_{\lambda_{k+1},j}$ the $W^{\sigma}_{k+1,j}$ being irreducible). We must show that there is a $g_0' \in S_{k+1}$ arbitrarily close to g_0. By an obvious inductive argument, it is enough to show:

(2.12) $(\forall \in > 0)\,(\exists g_0' \in S_k)$:

(*a*) $(\forall \sigma \in \hat{G}_0)\, m(\lambda_{k+1}, \sigma, g_0') \leqslant m(\lambda_{k+1}, \sigma, g_0)$, and

(*b*) If $g_0 \notin S_{k+1}$, then $\exists \sigma \in \hat{G}_0 : m(\lambda_{k+1}, \sigma, g_0') < m(\lambda_{k+1}, \sigma, g_0)$.

In other words, if $E^{\mathbb{R}}_{\lambda_{k+1}}(p^*(g_0))$ is reducible, then there are arbitrarily small deformations in S_k which split off at least one irreducible from the eigenspace. The equivalent version in terms of the equivariant eigenspace $\varepsilon_{\lambda_{k+1}}(p^*(g_0)) = \bigoplus_{\sigma} \varepsilon^{\sigma}_{\lambda_{k+1}}$ is: there are arbitrarily small deformations in S_k which split off a summand $K(\sigma)\Phi^{\sigma}_{\lambda_{k+1}}$.

Assume not. Then let $g(t) = g + th$ be an analytic deformation of $g = p^*(g_0)$ in S_k. Choose a $K(\sigma)$-Kato-basis $\{\Phi^{\sigma,i}_{\lambda_{k+1}}\}$ for each $\varepsilon^{\sigma}_{\lambda_{k+1}}$ along $g(t)$. Then of course:

$$\text{(2.13i)} \qquad \Delta(t)\Phi^{\sigma,i}_{\lambda_{k+1}}(t) = \lambda^{\sigma,i}_{k+1}(t)\Phi^{\sigma,i}_{\lambda_{k+1}}(t) \quad (\forall \sigma, i)$$

$$\text{(2.13ii)} \qquad (\Delta - \lambda^{\sigma,i}_{\lambda_{k+1}})\dot{\Phi}^{\sigma,i}_{\lambda_{k+1}} + (\dot{\Delta} - \dot{\lambda}_{k+1})\Phi^{\sigma,i}_{\lambda_{k+1}} = 0 \quad (\forall \sigma, i).$$

If the deformation fails to split of a $K(\sigma)\Phi^{\sigma}_{\lambda_{k+1}}$, then:

$$\text{(2.14)} \qquad (\forall \sigma, \sigma', i, j) : \dot{\lambda}^{\sigma,i}_{k+1} = \dot{\lambda}^{\sigma',j}_{k+1}.$$

Taking the inner product in (2.13ii) with $\Phi^{\sigma,i}_{\lambda_{k+1}}$ we get (by normality of the basis):

$$\text{(2.15)} \qquad \dot{\lambda}^{\sigma,i}_{k+1} = (\dot{\Delta}\Phi^{\sigma,i}_{\lambda_{k+1}}, \Phi^{\sigma,i}_{\lambda_{k+1}}) \cdot \frac{1}{\deg \sigma}.$$

Combining (2.14) and (2.15) we conclude:

$$\text{(2.16)} \qquad (\forall \sigma, \sigma', i, j) : \dot{\Delta}\Phi^{\sigma,i}_{\lambda_{k+1}}, \Phi^{\sigma,i}_{\lambda_{k+1}}) \frac{1}{\deg \sigma} = (\dot{\Delta}\Phi^{\sigma',j}_{\lambda_{k+1}}, \Phi^{\sigma',j}_{\lambda_{k+1}}) \frac{1}{\deg \sigma'}.$$

Further, taking the inner product in (2.13ii) with $\Phi^{\sigma,i}_{\lambda_{k+1}}$, $j \neq i$, we get

$$\text{(2.16a)} \qquad (\forall \sigma, \forall i \neq j) \quad (\dot{\Delta}\Phi^{\sigma,i}_{\lambda_{k+1}}, \Phi^{\sigma,j}_{\lambda_{k+1}}) = 0.$$

In fact, under our assumption, (2.16) holds for every small enough deformation in $p^*\mathcal{M}_0$.

From now on, we fix one pair of equivariant eigenvectors $\Phi^{\sigma,i}_{\lambda_{k+1}}$, resp. $\Phi^{\sigma',j}_{\lambda_{k+1}}$ and write them for notational simplicity as Φ^{σ}_{λ}, resp. Ψ^{τ}_{λ}. We then use (1.8) to convert (2.16) into a pointwise statement.

(2.17) (2) PROPOSITION. — *Assume (HDLD) and assume that* (2.13*ii*), (2.14), (2.15) *hold for all G-invariant analytic deformations. Then*

(a) $\dfrac{1}{\deg \sigma} |\Phi_\lambda^\sigma(x)|^2 = \dfrac{1}{\deg \tau} |\Psi_\lambda^\tau(x)|^2$

(b) $\dfrac{1}{\deg \sigma} \sum_{j=1}^{\deg \sigma} d\varphi_j \circledS d\varphi_j = \dfrac{1}{\deg \tau} \sum_{j=1}^{\deg \tau} d\psi_j \circledS d\psi_j .$

Further, if $\tau = \sigma$, then :

(c) $\langle \Phi_\lambda^\sigma(x), \Psi_\lambda^\sigma(x) \rangle = 0$

(d) $\dfrac{1}{\deg \sigma} \sum_{j=1}^{\deg \sigma} d\varphi_j \circledS d\psi_j = 0 .$

Here $\Phi_\lambda^\sigma = (\varphi_1, \ldots, \varphi_p)$, $p = \deg \sigma$, similarly for Ψ_λ^σ, and $df \circledS dg = \frac{1}{2}(df \otimes dg + dg \otimes df)$.

Proof. — First we consider conformal deformations. If $\deg \sigma = \deg \tau$, we have (by 1.8a)) :

$$(2.18) \qquad (\dot{\Delta}\Phi, \Psi) = -\lambda \frac{n}{2} (r\Phi, \Psi) + \left(\frac{n}{2} - 1\right) (r\nabla\Phi, \nabla\Psi) .$$

Second, consider volume preserving deformations. From (1.8b) we get :

$$(2.19\text{i}) \qquad (\dot{\Delta}\Phi, \Psi) = \int_M h(\nabla\Phi, \nabla\Psi) \,\text{dvol}.$$

In terms of the inner product $\langle , \rangle$ on symmetric covariant 2-tensors associated to g, we may rewrite (2.8ii) by :

$$(2.19\text{ii}) \qquad (\dot{\Delta}\Phi, \Psi) = \int_M \langle h, \operatorname{tr} d\Phi \circledS d\Psi \rangle \,\text{dvol}$$

with $\operatorname{tr} d\Phi \circledS d\Psi = \sum_i d\varphi_i \circledS d\psi_i$. From (2.16) we get, setting $\Phi = \Psi$ in (2.18) :

$$(2.20\text{a}) \qquad \frac{1}{\deg \sigma} \{ -\lambda \frac{n}{2} (r\Phi, \Phi) + \left(\frac{n}{2} - 1\right) (r\nabla\Phi, \nabla\Phi) \}$$
$$= \frac{1}{\deg \tau} \{ -\lambda \frac{n}{2} (r\Psi, \Psi) + \left(\frac{n}{2} - 1\right) (r\nabla\Psi, \nabla\Psi) \}$$

(2) G. Besson has shown that the hypothesis (HDLD) may be eliminated from the proposition.

for all G-invariant $r \in C^\infty(M)$. Hence:

$$\text{(2.20b)}\quad \frac{1}{\deg \sigma}\left[-\lambda\frac{n}{2}|\Phi(x)|^2+\left(\frac{n}{2}-1\right)|\nabla\Phi(x)|^2\right] = \frac{1}{\deg \tau}\left[-\lambda\frac{n}{2}|\Psi(x)|^2+\left(\frac{n}{2}-1\right)|\nabla\Psi(x)|^2\right].$$

Moreover, setting $\Phi = \Psi$ in (2.19ii), we also get:

$$\text{(2.21a)}\quad \frac{1}{\deg \sigma}\int_M \langle h, \operatorname{tr} d\Phi \circledS d\Phi\rangle \, d\mathrm{vol} = \frac{1}{\deg \tau}\int_M \langle h, \operatorname{tr} d\Psi \circledS d\Psi\rangle \, d\mathrm{vol},$$

for all G-invariant $h \in p^*S(M_0)$.

It follows from (2.21a) that:

$$\text{(2.21b)}\quad \operatorname{tr}\left(\frac{1}{\deg \sigma} d\Phi \circledS d\Phi - \frac{1}{\deg \tau} d\Psi \circledS d\Psi\right) = \mu(x) g$$

where $\mu(x) = \dfrac{1}{\deg \sigma}|\nabla\Phi(x)|^2 - \dfrac{1}{\deg \tau}|\nabla\Psi(x)|^2$ (compare [U], p. 1075).

We now claim that $\mu = 0$. This will use (HDLD) for the first time. To see this, set

$$\text{(2.22)}\qquad Q_\Phi = \frac{1}{\deg \sigma}\operatorname{tr} d\Phi \circledS d\Phi.$$

$Q_\Phi \in S_G(M)$ (G-invariant symmetric covariant 2-tensors), and obviously $Q_\Phi \geqslant 0$. We also let V_Φ denote the nullspace of Q_Φ:

$$\text{(2.23)}\qquad (V_\Phi)_x = \{V \in T_xM : Q_\Phi(V, \cdot) \equiv 0 \text{ on } T_xM\}.$$

Obviously $\dim (V_\Phi)_x \geqslant n - \deg \sigma$, with strict inequality on the *degeneracy set* D_Φ where the differentials $\{d\phi_1, \ldots, d\phi_p, p = \deg \sigma\}$ fail to be independent. Let Q_Ψ, V_Ψ and D_Ψ denote the corresponding objects for Ψ.

From (2.21b) we have:

$$\text{(2.24)}\qquad (V_\Phi)_x \cap (V_\Psi)_x \neq \{0\} \Leftrightarrow \mu(x) = 0.$$

Indeed, if $0 \neq v \in (V_\Phi)_x \cap (V_\Psi)_x$, then (2.11b) implies $\mu(x)g(v,v) = 0$. Likewise the converse. So we need

$$(V_\Phi)_x \cap (V_\Psi)_x \neq \{0\}. \tag{2.25}$$

Suppose then that $(V_\Phi)_x \cap (V_\Psi)_x = \{0\}$. Since $(V_\Phi)_x \neq \{0\}$ by (HDLD), there is a v so that $Q_\Phi(v,v) = 0$ but $Q_\Psi(v,v) > 0$. By (2.21b), $\mu(x) < 0$. Similarly there is a w so $Q_\Psi(w,w) = 0$ but $Q_\Phi(w,w) > 0$. Hence $\mu(x) > 0$. The contradiction proves (2.25), and hence by (2.24) that $\mu(x) = 0$. We thus have:

$$Q_\Phi = Q_\Psi, \tag{2.26}$$

and, using (2.20b),

$$\frac{1}{\deg \sigma} |\Phi(x)|^2 = \frac{1}{\deg \tau} |\Psi(x)|^2. \tag{2.27}$$

This proves (a) and (b) above.

For (c) and (d) we note that (2.16a), (2.18) and (2.19ii) imply:

$$-\lambda \frac{n}{2} \langle \Phi(x), \Psi(x) \rangle + \left(\frac{n}{2} - 1\right) \langle \nabla\Phi(x), \nabla\Psi(x) \rangle = 0 \tag{2.28i}$$

$$\frac{1}{p} \operatorname{Tr} d\Phi \; \textcircled{s} \; d\Psi = \mu(x) g \tag{2.28ii}$$

where $p = \deg \sigma = \deg \tau$ and $\mu(x) = \frac{1}{p} \langle \nabla\Phi(x), \nabla\Psi(x) \rangle$. But the rank of $\operatorname{Tr} d\Phi \; \textcircled{s} \; d\Psi$ is at most p so by (HDLD), $\mu(x) = 0$. (c) and (d) follow. □

The first corollary of Proposition (2.17) is simply:

$$V_\Phi = V_\Psi. \tag{2.29}$$

To analyze (2.29) further, we will need the main technical lemma of this section:

(2.30) LEMMA. — *Assume (HDLD)* (3), *and fix* $k \in \mathbb{Z}^+$. *Then there is an open dense set* $\mathscr{B}_k$ *in* $\mathscr{M}_0$ *with the property: if* $g_0 \in \mathscr{B}_k$, *and* $0 \not\equiv \Phi^\sigma_{\lambda_k} \in \varepsilon^\sigma_{\lambda_k}(g_0)$ *(for some* σ*), then there exists an open dense set* $V \subset M$ *so that* $\Phi^\sigma_{\lambda_k}|_V$ *is a submersion into its image.*

(3) It suffices to assume here that $\dim \mathscr{M} \geqslant \max \{\deg \sigma,\ \sigma \in \hat{G}_0\}$.

Since the proof of Lemma (2.30) is somewhat lengthy, we will postpone it until the end of this section.

Assuming (2.30) for the moment, there exist open dense sets U_Φ (resp. U_Ψ) on which Q_Φ (resp. Q_Ψ) has rank exactly equal to $n-\deg\sigma$ (resp. $n-\deg\tau$). (2.26) obviously implies then that $\deg\sigma = \deg\tau = p$ (say). Moreover the submersion property on U_Φ implies:

$$(2.31) \qquad (V_\Phi)_x = T_x(\Phi^{-1}(y)), \qquad x \in U_\Phi, \qquad y = \Phi(x).$$

Thus V_Φ is the vertical bundle of the submersion $\Phi|_{U_\Phi}$. A connection for this bundle is given by defining the horizontal spaces as the orthogonal complements of the vertical:

(2.32) DEFINITION. – $(H_\Phi)_x = (V_\Phi)_x^\perp$.

Similarly, we set $(H_\Psi)_x = (V_\Psi)_x^\perp$. Since U_Φ and U_Ψ are both open dense, they have an open dense intersection $U = U_\Phi \cap U_\Psi$. By shrinking U, if necessary, we may assume that both submersions $\Phi : U \to \Phi(U) \subset \mathbb{R}^p$ and $\Psi : U \to \Psi(U) \subset \mathbb{R}^p$ are conjugate to coordinate projections. We then have a double fibration:

$$(2.33) \qquad \Phi(U) \xleftarrow{\;\Phi\;} U \xrightarrow{\;\Psi\;} \Psi(U)$$

such that the fibers are connected pieces of level sets $\Phi^{-1}(c_1)$, respectively $\Psi^{-1}(c_2)$. Since $V_\Phi = V_\Psi$, it must be the case that there exists a smooth map $f : \Phi(U) \to \Psi(U)$ so that $\Phi^{-1}(c_1) = \Psi^{-1}(f(c_1))$. Furthermore, it must be the case that f is a local isometry. Indeed, fix $x \in U$ and consider the diagram:

$$(2.34) \qquad \begin{array}{ccc} (H_\Phi)_x & \overset{\text{id}}{=} & (H_\Psi)_x \\ \downarrow{\scriptstyle d\Phi_x} & & \downarrow{\scriptstyle d\Psi_x} \\ T_{\Phi(x)}\mathbb{R}^p & \xrightarrow{df_c} & T_{\Psi(x)}\mathbb{R}^p \end{array} \qquad (c = \Phi(x)).$$

If ds_0^2 is the Euclidean metric on $\mathbb{R}^p$, then by definition of Q_Φ we have:

$$(2.35\text{i}) \qquad \Phi^*(ds_0^2) = Q_\Phi.$$

Similarly

$$(2.35\text{ii}) \qquad \Psi^*(ds_0^2) = Q_\Psi.$$

It follows that all linear isomorphisms in (2.34) are isometries.

Thus, f is a local isometry on $\mathbb{R}^p$, and so $f(c) = Ac + B$ with $A \in 0(\mathbb{R}^p)$ and $B \in \mathbb{R}^p$. Consequently, $\Psi(x) = A\Phi(x) + B$, $x \in U$. Since Ψ and Φ are equivariant eigenfunctions, we must have that $B = 0$ and that A intertwines σ and τ. In the first place we get $\sigma = \tau$, and that $A \in K(\sigma)$. Further, by the unique continuation theorem for solutions of second order elliptic equations ([H]), we get that $\Psi(x) = A\Phi(x)$ for all $x \in M$. But by assumption Ψ was supposed to come from an orthogonal summand in $\varepsilon_{\lambda_k}(g)$ to $K(\sigma)\Phi$. Modulo (2.30), this contradiction completes the proof of Theorem A (4). □

Remark. — The proof above was simplified a good deal from the original, following some remarks of H. Duistermaat.

Finally we give the *Proof of Lemma* (2.30). — Let $\Phi = \Phi_\lambda^\sigma \in \varepsilon_\lambda^\sigma(g_0)$, and write $\Phi = (\varphi_1, \ldots, \varphi_p)$, $p = \deg \sigma$, as usual. Then $d\Phi_x$ is a submersion if and only if $d\phi_1 \wedge \ldots \wedge d\phi_p \neq 0$ at x. Thus, we want to prove: for all $\sigma \in \hat{G}_0$

(2.35) $\mathscr{B}_k^{(\sigma)} \overset{\text{def}}{=} \{g_0 \in \mathscr{M}_0 : \forall \Phi_{\lambda_k}^\sigma \in \varepsilon_{\lambda_k}^\sigma(g_0),\ \Phi_{\lambda_k}^\sigma \not\equiv 0$ implies $d\phi_1 \wedge \ldots \wedge d\phi_p \neq 0$ on an open dense set$\}$ is open dense in $\mathscr{M}_0$.

First we note a simplification: namely, it suffices to show $d\phi_1 \wedge \ldots \wedge d\phi_p \neq 0$ on *some* open set. Indeed, let $\Delta = d\delta + \delta d$ on p-forms. Evidently $\Delta(d\phi_1 \wedge \ldots \wedge d\phi_p) = d\delta(d\phi_1 \wedge \ldots \wedge d\phi_p)$. Now δ does not act as a derivation on $\wedge^p$, but a straightforward computation shows:

$$(2.36)\quad \delta(d\phi_1 \wedge \ldots \wedge d\phi_p) = \sum_{i=1}^{p} d\phi_1 \wedge \ldots \wedge \delta(d\phi_i) \wedge \ldots \wedge d\phi_p + B(d\phi_1 \wedge \ldots \wedge d\phi_p),$$

where B is a linear algebraic (i.e. 0^{th} order differential) operator from p-forms to $(p\text{-}1)$-forms. Hence:

$$(2.37)\quad \Delta(d\phi_1 \wedge \cdots \wedge d\phi_p) = \lambda(d\phi_1 \wedge \cdots \wedge d\phi_p) + dB(d\phi_1 \wedge \cdots \wedge d\phi_p).$$

Evidently, $d\phi_1 \wedge \cdots \wedge d\phi_p$ satisfies a second order elliptic equation. By the unique continuation theorem ([H]), if $d\phi_1 \wedge \cdots \wedge d\phi_p \not\equiv 0$ then it is non-vanishing on an open dense set.

Thus we need: for all σ,

(2.38) $\mathscr{B}'_k(\sigma) \overset{\text{def}}{=} \{g_0 \in \mathscr{M}_0 : \forall \Phi_{\lambda_k}^\sigma \in \varepsilon_{\lambda_k}^\sigma(g_0), \Phi_{\lambda_k}^\sigma \not\equiv 0 \Rightarrow d\phi_1 \wedge \cdots \wedge d\phi_p \neq 0\}$ is open dense in $\mathscr{M}_0$.

(4) Again, it suffices that $\dim \mathscr{M} \geqslant \max\{\deg \sigma,\ \sigma \in \hat{G}_0\}$.

It is clear that $\mathscr{B}'_k(\sigma)$ is open. To prove denseness, we introduce some notions of *stability* of eigenspaces.

(2.39) DEFINITION. — (i) *An eigenspace* $\varepsilon^\sigma_{\lambda_k}(g_0)$ *is stable if there exists a* $\delta = \delta(g_0)$ *so that for all* $h \in S(M_0)$ *with* $|h| = 1$ (*see* (1.1)), *and for all branches* $\lambda^i_k(t)$ *of eigenvalues along* $g_0 + tp^*(h)$ (*see* (1.6)), *with* $|t| < \delta$, *we have,* $\lambda^i_k(t) = \lambda^j_k(t)$ *for all* i, j.

In other words, a stable eigenspace $\varepsilon^\sigma_{\lambda_k}(g_0)$ fails to split along any ray in the ball of some radius $\delta(g_0)$ around g_0.

(ii) $\varepsilon^\sigma_{\lambda_k}(g_0)$ *is* (n^{th} *order*) *infinitesimally stable if in* (i) *we replace* $\lambda^i_k(t) = \lambda^j_k(t)$ *by equality of the first n derivatives at* $t = 0$.

If $\varepsilon^\sigma_{\lambda_k}(g_0)$ is not of the form $K(\sigma)\Phi^\sigma_{\lambda_k}$, we will speak of a *stably degenerate* eigenspace (which are not supposed to exist).

Evidently, stable $\Rightarrow$ infinitesimally stable.

Let $S\mathscr{T}_k(\sigma) = \{g_0 : \varepsilon^\sigma_{\lambda_k}(g_0) \text{ is stable}\}$, we claim:

(2.40) PROPOSITION. — $S\mathscr{T}_k(\sigma)$ *is open-dense in* $\mathscr{M}_0$.

Proof. — Open: This is obvious, as δ may depend on g_0. Dense: Suppose to the contrary that there is an open set U in $\mathscr{M}_0$ containing no $g \in S\mathscr{T}_k(\sigma)$. Let $g_0 \in U$. Since $\varepsilon^\sigma_{\lambda_k}(g_0)$ is unstable, it must be the case that for all $\varepsilon > 0$, there is a $g_1 \in B_\varepsilon(g_0)$ (ball of radius ε) so that $\dim \varepsilon^\sigma_{\lambda_k}(g_1) < \dim \varepsilon^\sigma_{\lambda_k}(g_0)$. Choose ε_1 so that $B_{\varepsilon_1}(g_0) \subset U$. Then $g_1 \in U$, so $g_1 \notin S\mathscr{T}_k(\sigma)$. We then repeat the process, producing $g_2 \in B_{\varepsilon_1}(g_1)$ with mult $(\lambda_k, \sigma; g_2) <$ mult $(\lambda_k, \sigma; g_1)$. After a finite number of repetitions, we must end up with a g_m for which $m(\lambda_k, \sigma; g_m)$ is a local minimum. But $\varepsilon^\sigma_{\lambda_k}(g_m)$ must then be stable. □

By virtue of proposition (2.40), (2.38) follows as long as

(2.41) $\mathscr{B}'_k(\sigma) \cap S\mathscr{T}_k(\sigma)$ is dense in $S\mathscr{T}_k(\sigma)$.

Proof. — We argue by contradiction. So suppose there is an open subset $\mathscr{V}_k(\sigma) \subset S\mathscr{T}_k(\sigma)$ so that $\mathscr{B}'_k(\sigma) \cap \mathscr{V}_k(\sigma) = \varnothing$. Then, for all $g \in \mathscr{V}_k(\sigma)$, there is a $\Phi^\sigma_{\lambda_k} \in \varepsilon^\sigma_{\lambda_k}(g)$ so that $d\phi_1 \wedge \cdots \wedge d\phi_p \equiv 0$. Existence of one such $\Phi^\sigma_{\lambda_k}$ actually implies that $d\psi_1 \wedge \cdots \wedge d\psi_p \equiv 0$ for all $\Psi \in \varepsilon^\sigma_{\lambda_k}(g)$. This follows from Proposition (2.17b), which in fact shows that

$$(2.42) \quad (\forall g \in \mathscr{V}_k(\sigma))(\exists \Phi^\sigma_{\lambda_k} \in \varepsilon^\sigma_{\lambda_k}(g))(\forall \Psi \in \varepsilon^\sigma_{\lambda_k}(g)) : \ d\psi_i = \sum_{j=1}^{p} a_{ij}\, d\varphi_j$$

(where $\Phi^\sigma_{\lambda_k} = (\varphi_1, \dots, \varphi_p)$, $\Psi = (\psi_1, \dots, \psi_p)$, $a_{ij} \in C^\infty(M)$). Thus our assumption implies $d\psi_1 \wedge \cdots \wedge d\psi_p \equiv 0$ for all $\Phi \in \varepsilon^\sigma_{\lambda_k}(g)$, hence

(2.43) $$(\forall g \in \mathscr{V}_k(\sigma))(\forall \Phi \in e^\sigma_{\lambda_k}(g))(\exists U \subset M)$$

s.t. U is open and

$$d\phi_p = \sum_{j=1}^{p-1} a_j \, d\phi_j .$$

Here we picked the p^{th} component only for convenience. However, in view of (2.42), we may fix one choice of nonzero $\Phi^\sigma_{\lambda_k} \in \varepsilon^\sigma_{\lambda_k}(g)$, and then the differentials of its components $\{d\phi_i : i = 1, \dots, p-1\}$ span, at each x, the subspace of $T^*_x M$ given by span $\{d\psi_j(x) : \psi_j \in E^\sigma_{\lambda_k}(g)\}$.

Our plan is to show now that

(2.44) $$d\phi_1 \wedge \cdots \wedge d\phi_{p-1} \equiv 0 .$$

If not, then on some open set U, the $\{\phi_1, \dots, \phi_{p-1}\}$ are the initial elements of a set of coordinates $\{x_1, \dots, x_n : x_i = \psi_i, i = 1, \dots, p-1\}$. Let now $g \in \mathscr{V}_k(\sigma)$ and let $g(t) = g + th$ be a deformation so that $g(t) \in \mathscr{V}_k(\sigma)$ for t in some interval $(-\varepsilon, \varepsilon)$. Fix a $K(\sigma)$-Kato basis $\{\Phi^\sigma_{\lambda_k, i}\}$ of $\varepsilon^\sigma_{\lambda_k}(g)$, and let $\Phi^\sigma_{\lambda_k}$ be the specific Φ above. Letting $\Phi(t)$ denote its real analytic extension along $g(t)$ we have by assumption:

(2.45) $$d\phi_1(t) \wedge \cdots \wedge d\phi_p(t) \equiv 0 .$$

Letting . denote $\left.\dfrac{d}{dt}\right|_{t=0}$, we get:

(2.46) $$\sum_i d\phi_1 \wedge \cdots \wedge d\dot{\phi}_i \wedge \cdots \wedge d\phi_p \equiv 0 \quad \text{(any deformation)}.$$

Now $\dot{\phi}_i$ satisfies:

(2.47) $$(\Delta - \lambda)\dot{\phi}_i = -(\dot{\Delta} - \dot{\lambda})\phi_i \quad (\forall i)$$

where we drop the subscript k in λ_k. Hence

(2.48) $$\dot{\phi}_i = -\pi^\perp_\lambda (\Delta - \lambda)^{-1} \pi^\perp_\lambda (\dot{\Delta}\phi_i) + \pi_\lambda(\dot{\phi}_i)$$

where π_λ is orthogonal projection into $E_\lambda(g)$, and $\pi^\perp_\lambda = I - \pi_\lambda$.

Let us set:

(2.49) $$G^\perp_\lambda = \pi^\perp_\lambda (\Delta - \lambda)^{-1} \pi^\perp_\lambda .$$

Putting (2.49) into (2.46), we get :

$$(2.50) \qquad \sum_i d\phi_1 \wedge \cdots \wedge d(G_\lambda^\perp \dot{\Delta}\phi_i) \wedge \cdots d\phi_p \equiv 0 .$$

Here we omitted the $d\pi_\lambda(\dot{\phi}_i)$ term since it contributes zero anyway (see below (2.43)), and also used the orthogonality of the spaces $L^2_{\sigma_1}$, $L^2_{\sigma_2}$ if $\sigma_1 \neq \sigma_2$.

(2.50) means that :

$$(2.51) \qquad \int_M d_x G_\lambda^\perp(x,y) \wedge \Big(\sum_i (-1)^i \dot{\Delta}\varphi_i(y)[d\phi_1 \wedge \cdots \wedge d\hat{\phi}_i \wedge \cdots d\phi_p]\Big) \equiv 0$$

for all variations $\dot{\Delta}$ of the Laplacian. The integrand in (2.51) is a (distributional) function of y with values in the p-forms in x, and the integration is with respect to the y-variable.

We now use (1.8a-b) to convert (2.51) into pointwise statements.

First, let $\dot{\Delta}$ come from a conformal variation. Following the argument of Proposition 2.17, we get :

$$(2.52) \qquad \int_M r(y) \Big\{ -\lambda \frac{n}{2} d_x G_\lambda^\perp(x,y) \wedge \omega_\Phi + \left(\frac{n}{2} - 1\right) d_x \otimes d_y G_\lambda^\perp \mathbin{\dot\wedge} d_y \omega_\Phi \Big\} = 0 \quad (\forall r \in p^* C^\infty(M_0^\bullet)),$$

where

$$(2.53) \qquad \omega_\Phi = \sum_{i=1}^{p} (-1)^i \phi_i(y)\, d\phi_1 \wedge \ldots \wedge d\hat{\phi}_i \wedge \ldots \wedge d\phi_p ,$$

where $\hat{}$ always refers to the x-variable and where $d_x \otimes d_y G_\lambda^\perp \mathbin{\dot\wedge} d_y \omega_\Phi$ means to contract the 2-tensor in the y-variable and $\wedge$ the resulting 1, resp. $(p\text{-}1)$, forms in the x-variable.

It follows from (2.52) that :

$$(2.54) \qquad \left[-\lambda \frac{n}{2} d_x G_\lambda^\perp (x,y) \wedge \omega_\Phi + \left(\frac{n}{2} - 1\right) d_x \otimes d_y G_\lambda^\perp \mathbin{\dot\wedge} d_y \omega_\Phi \right]^{\text{ave}} = 0$$

(all (x,y)), where for any covariant tensor, η,

$$(2.55) \qquad \eta^{\text{ave}} = \frac{1}{|G|} \sum_{g \in G} g^*(\eta) .$$

(Above g^* applies to the y-variable.)

Secondly, let us consider the volume preserving deformations. In this case we get:

$$(2.56) \qquad 0 = \int_M \langle h, d_x \otimes d_y G_\lambda^\perp(x,y) \underset{\textcircled{s}}{\wedge} d_y \omega_\Phi \rangle \, dy = 0$$

for all traceless $h \in p^*S(M_0)$. Here $d_x \otimes d_y G$ is a double (1,1)-form and $\underset{\textcircled{s}}{\wedge}$ means to take $\wedge$ in the x-variables and $\textcircled{s}$ in the y-variables. So $d_x \otimes d_y G \underset{\textcircled{s}}{\wedge} d_y \omega_\Phi$ is a symmetric covariant 2-tensor in y with values in the p-forms in x.

Since the integral (2.56) vanishes for all traceless h, is must be the case (apply Proposition (2.17) component by component) that:

$$(2.57) \qquad \{d_x \otimes d_y G_\lambda^\perp(x,y) \underset{\textcircled{s}}{\wedge} (d_y \omega_\Phi - d_x \otimes d_y G_\lambda^\perp \wedge d_y \omega_\Phi)\} \equiv 0 .$$

In fact, as in Proposition 2.17, both terms (2.57) must independently vanish. Indeed, we may write

$$(2.58) \qquad \omega_\Phi(x,y) = f_\Phi(x,y) \, dx_1 \wedge \ldots \wedge dx_{p-1}$$

where $x_i = \phi_i$ and where

$$(2.59) \qquad f_\Phi(x,y) = (-1)^{p-1} \sum_{j=1}^{p-1} a_j(x) \phi_j(y)$$

(cf. (2.43)-(2.44)).

Obviously, only p-vectors of the form

$$X_l \overset{\text{def}}{=} \frac{\partial}{\partial x_l} \wedge \frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_{p-1}}$$

do not automatically annihilate $d_x \otimes d_y G_\lambda^\perp(x,y) \underset{\textcircled{s}}{\wedge} d_y \omega_\Phi$. Plugging X_l into (2.57), we get (simplifying the notation):

$$(2.60) \qquad \left\{ d_y \frac{\partial}{\partial x_l} G(x,y) \textcircled{s} \, d_y f_\Phi \right\}^{\text{ave}} - \{d_y \frac{\partial}{\partial x_l} G . d_y f_\Phi\}^{\text{ave}} g_y \equiv 0 .$$

As the first term here has rank at most 2 (and if 2, a negative determinant), while g has rank n and is positive definite, we get:

$$(2.61) \qquad \{d_y \frac{\partial}{\partial x_l} G \textcircled{s} \, d_y f_\Phi\}^{\text{ave}} \equiv 0 .$$

We now show that (2.61) implies (2.44).

Indeed, we write (2.61) as:

$$\sum_{g\in G} I_{\Phi,\lambda}(x,gy) = 0 \tag{2.62}$$

where

$$\begin{aligned} I_{\Phi,\lambda}(x,y) &= d_x G_\lambda^\perp(x,y) \wedge \omega_\Phi \\ &= f_\Phi(x,y) d_x G_\lambda^\perp(x,y) \wedge dx_1 \wedge \ldots \wedge dx_{p-1} \end{aligned} \tag{2.63}$$

(cf. (2.44), (2.58)).

We now claim:

$$I_{\Phi,\lambda}(x,y) \equiv 0 \pmod{C^\infty}. \tag{2.64}$$

Proof. – Since $\operatorname{singsupp} G_\lambda^\perp(.,.) = \operatorname{diag}_{M\times M}$, it follows that $\operatorname{singsupp} G_\lambda^\perp(\cdot,g\cdot) = \operatorname{graph}(g^{-1}) = \{(gy,y) : y\in M\}$. Since $p: M \to M_0$ is normal, G acts freely on M, and hence $\operatorname{singsupp} G_\lambda^\perp(\cdot,g_1\cdot) \cap \operatorname{singsupp} G_\lambda^\perp(\cdot,g_2\cdot) = \emptyset$ if $g_1 \neq g_2$. Since $\operatorname{singsupp} I_{\Phi,\lambda}(x,g\cdot) \subset \operatorname{singsupp} G_\lambda^\perp(\cdot,g\cdot)$, we see that (2.62) must imply (2.64). □

Still under the assumption that $\phi_1,\ldots,\phi_{p-1}$ have independent differentials on some open set U, (2.64) implies:

$$f_\Phi(x,y)\frac{\partial}{\partial x_i} G_\lambda^\perp(x,y) \equiv 0 \pmod{C^\infty} \quad (\forall i = p,\ldots,n). \tag{2.65}$$

Now the singularities of $\frac{\partial}{\partial x_i} G_\lambda^\perp(x,y)$ are very explicitly described by the Hadamard parametrix method ([H], 17.4). One has (if $n>2$):

(2.66) (5) $\quad G_\lambda^\perp(x,y) = a_0(x,y,\lambda)s^{2-n} + a_1(x,y,\lambda)s^{3-n} \pmod{C^\infty}$

on a sufficiently small neighborhood W of the diagonal in $M\times M$, where $s = S(x,y)$ is the riemannian distance on W, and where $a_i(x,y,\lambda) \in C^\infty(W)$. When $n=2$, s^{2-n}(resp. s^{3-n}) should be replaced by $\log s$ (resp. $s\log s$). For the sake of completeness, we sketch a proof of (2.66):

Proof. – First, we fix:

(2.67i) W: a small enough neighborhood of the diagonal so that, for all $x\in M$, $\exp_x^{-1}: W_x \stackrel{\text{def}}{=} \{y : (x,y)\in W\} \to T_xM$ is a diffeomorphism onto its image;

(5) A more precise account of the singularities of the green's kernel near the diagonal can be found in [Be 2], sections B and App. D.

(2.67ii) χ: a smooth cutoff in $C_0^\infty(W)$, equal to 1 on a smaller neighborhood of the diagonal;

(2.68iii) ρ_λ: a smooth function on $\mathbb{R}^+$, which vanishes in a neighborhood of $\{r^2 \leqslant \lambda\}$ and equals one on $\{r^2 \geqslant \lambda + 1\}$.

(2.68iv) $F_\nu(x,\lambda) = (\nu!)(2\pi)^{-n} \int_{\mathbb{R}^n} (|\xi|^2 - \lambda)^{-\nu-1} \rho_\lambda(\xi) e^{i\langle x,\xi\rangle} d\xi$,

$\nu = 0, 1, 2, \ldots$. Note that $F_\nu(x,\lambda)$ is actually a (distributional) function, still denoted F_ν, of $|x|$.

One now seeks a solution (mod C^k) of the form

$$G_{\lambda,N}(x,y) = \chi(x,y) \sum_{\nu=0}^{N} U_\nu(x,y) F_\nu(s(x,y),\lambda) \tag{2.69}$$

for the equation

$$(\Delta_x - \lambda)\, G^\perp_{\lambda,N}(x,y) = \delta_y(x) - \pi_\lambda(x,y). \tag{2.70}$$

In (2.69) the coefficients U_ν are assumed to belong to $C^\infty(W)$. They may be constructed ([H], p. 34) so that

$$(\Delta_x - \lambda)\, G_{\lambda,N} = \delta_y(x) - R_N(x,y), \tag{2.71}$$

where $R_N \in C^{2N+1-n}$. It of course follows that

$$G^\perp_{\lambda,N} - G_\lambda \in C^{2N+3-n}, \tag{2.72}$$

and therefore, up to order $2N + 3 - n$, the singularities along the diagonal of $G^\perp_\lambda$ and $G_{\lambda,N}$ coincide. The singularities of the latter may be deduced from (2.69), after replacing $F_\nu(s,\lambda)$ by its asymptotic expansion as $s \to 0$. This expansion could be derived (for example) from the formulae ([H], loc. cit.):

(2.73) (i) $(-\Delta - \lambda) F_\nu \equiv \nu F_{\nu-1} \pmod{C^\infty}, \quad \nu > 0$

(ii) $(-\Delta - \lambda) F_0 = \delta_0 \pmod{C^\infty}$

(iii) $-2 \dfrac{\partial F_\nu}{\partial x} = x F_{\nu-1} \pmod{C^\infty}, \quad \nu > 0.$

If we group together the terms of the resulting expansion with powers of s of like parity, we get (2.66). □

(2.74) *Remark.* — For the generic metric on M, the coefficients a_0 and a_1 are non-vanishing in a neighborhood of the diagonal. This may be confirmed quite easily from the explicit formulae ([H], p. 33) for the coefficients U_ν in (2.69).

We now return to (2.65). Combining it with (2.66), we get

(2.75) f_Φ vanishes to infinite order on $\mathrm{diag}_{M\times M}$. Indeed, if f_0 only vanishes to order k on $\mathrm{diag}_{M\times M}$, then the term of (2.66) with odd power of s would only in C^{1-n+k} or C^{2-n+k}, depending on parity of n.

It follows that

(2.76) ω_Φ vanishes to infinite order on $\mathrm{diag}_{M\times M}$.

But ω_Φ is the solution of the second order elliptic equation

$$(2.77) \qquad [\Delta_y + \Delta_x^{p-1}]\omega_\Phi + dB_x\omega_\Phi = \lambda\omega_\Phi,$$

where B is from (2.36) and Δ^{p-1} is the Laplacian on $(p-1)$-forms. Hence $\omega_\Phi \equiv 0$.

By an obvious orthogonality, this implies

$$(2.78) \qquad d\phi_1 \wedge \ldots \wedge d\hat{\phi}_i \wedge \ldots \wedge d\phi_p \equiv 0 \quad (\forall i).$$

This gives (2.44).

Furthermore, we can repeat every step of the above argument with $d\phi_1 \wedge \ldots \wedge d\phi_{p-2}$ replacing $d\phi_1 \wedge \ldots \wedge d\phi_{p-1}$ of (2.44). Doing this $(p-2)$ times, we see that our above argument actually implies:

$$(2.79) \qquad d\phi_i = 0 \quad \text{for all } i.$$

This absurdity finishes the proof of Lemma (2.30). □

3. GENERIC SPECTRUM OF A COVER

In this section, we drop the assumption that the cover $p_1 : M_1 \to M_0$ is normal. In fact our main interest is with the opposite extreme, where the covering group is trivial. In that case, there are no manifest symmetries forcing metrics in $p^*\mathcal{M}_0$ to have multiple eigenvalues. However there may be « hidden » symmetries which do this. Our main result is *Theorem B*, which explains precisely when no such hidden symmetries occur.

The trick here (and for any riemannian cover) is to pass to the least normal $p : M \to M_0$ extending p_1. To define this cover (which we will call the normal closure of p_1), we need to recall some elementary facts about covers and their monodromy groups [SeT].

Let us fix $x_0 \in M_0$. Then path lifting gives a homomorphism τ : $\pi_1(M_0, x_0) \to \mathrm{Perm}(p_1^{-1}(x_0))$, where $\mathrm{Perm}(p_1^{-1}(x_0))$ is of course the permutation group of the fiber over x_0. The kernel of τ is a normal subgroup, equal to

$$\Gamma_0 \stackrel{\text{def}}{=} \bigcap_i (p_1)_* \pi_1(M_1, x_i) \tag{3.1}$$

where $p_1^{-1}(x_0) = \{x_i\}$.

Γ_0 consists of the loops at x_0 which lift to loops at all $x_i \in p_1^{-1}(x_0)$.

The monodromy group G is the image of τ, hence

$$G = \pi_1(M_0, x_0) | \Gamma_0 . \tag{3.2}$$

If $\tilde{M}_0$ is the universal cover of M_0, we set

$$M = \tilde{M}_0 | \Gamma_0 . \tag{3.3}$$

Evidently we get a tower of covers,

$$M \to M_1 \to M_0 \quad \text{(normal closure)} \tag{3.4}$$

with $M \to M_1$ and $M \to M_0$ normal. The covering group of $M \to M_0$ is G by construction; let us denote the group of $M \to M_1$ by H.

Our main result is

(3.5) THEOREM B. – *Let $p_1 : M_1 \to M_0$ be a finite cover, and let $p : M \to M_0$ be its normal closure. Assume p satisfies the HDLD assumption* (2.2). *Then the following are equivalent :*

(a) $\{g_0 \in \mathcal{M}_0 :$ spec $(M_1, p_1^* g_0)$ *is simple*$\}$ *is residual in* $\mathcal{M}_0$

(b) $L^2(G|H, \mathbb{R})$ *is multiplicity free and all orthogonal irreducibles occurring in it are of real type.*

Remark. – $L^2(G|H, \mathbb{R})$ is *multiplicity-free* if the multiplicity with which each irreducible $\sigma \in \hat{G}_0$ occurs in it is at most one.

Proof. – By Theorem A (2.1), we have that for a residual set of metrics $g_0 \in \mathcal{M}_0$, each real eigenspace $E_\lambda^{\mathbb{R}}(p^*(g_0))$ in $L^2(M)$ is irreducible. On the other hand, it is clear that the multiplicity of an eigenvalue $\lambda \in \operatorname{spec}(M_1, p_1^*(g_0))$ equals $\dim(\mathrm{E}_\lambda^{\mathbb{R}}(p^*(g_0))^H$ (H-invariant vectors). Thus, spectral simplicity holds for a residual set in $\mathcal{M}_0$ as long as mult $(\operatorname{res}_H^G \sigma : 1_H) \leqslant 1)$ for all $\sigma \in \hat{G}_0$. The Frobenius reciprocity takes the following form for real, orthogonal irreducible σ :

(3.6) $\quad \operatorname{mult}(\operatorname{res}_H^G \sigma, 1_H) = \dim_{\mathbb{R}} K(\sigma) . \operatorname{mult}(\sigma, \operatorname{ind}_H^G 1)$.

Multiplicity free means mult $(\sigma, \operatorname{ind}_H^G 1) \leqslant 1$, while real type means $\dim_{\mathbb{R}} \mathrm{K}(\sigma) = 1$, so we see that $(b) \Rightarrow (a)$.

Conversely, assume (a). We then have

(3.7) $\{g_0 : \dim(E_\lambda^{\mathbb{R}}(p^*(g_0))^H = 1\}$ is residual in $\mathcal{M}_0$. However, any $\sigma \in \hat{G}_0$ occurs (with positive density [D]) in some $E_\lambda^{\mathbb{R}}$. If obviously follows that $\dim V_\sigma^H \leqslant 1$ for all $\sigma \in \hat{G}_0$. By Frobenius, then, $(a) \Rightarrow (b)$. □

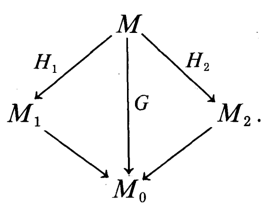
4. SIMPLE ISOSPECTRAL MANIFOLDS

Our object in this section is :

(4.1) THEOREM C. – *There exist isospectral (and non-isometric) pairs* (M_1, g_1), (M_2, g_2) *so that* spec (M_i, g_i) *is simple.*

Proof. – We follow the method of Sunada [Su], and seek such simple isospectral pairs among commensurate manifolds :

(4.2)



Here, $p_i : M \to M_i$ $(i=0,1,2)$ are assumed normal, and with the exhibited covering groups. By [Su], spec $(M_1, p_1^* g_0) = \operatorname{spec}(M_2, p_1^* g_0)$, for any $g_0 \in \mathcal{M}_0$, as long as $L^2(G/H_1, \mathbb{R}) \simeq L^2(G|H_2, \mathbb{R})$ (equivalence of real G-spaces). By Theorem B, we see that $(M_1, p_1^*(g_0))$ and $(M_2, p_2^*(g_0))$ are simple isospectral for $g_0 \in \mathcal{M}_0$ if M satisfies (HDLD) and additionally the $L^2(G|H_i, \mathbb{R})$ are multiplicity free and completely of real type.

An example of such a triple (G, H_1, H_2) is given in [Bro]:

$$G = SL_n(\mathbb{F}_p) \quad (n \geqslant 3) \tag{4.3}$$

$$H_1 = \left[\begin{array}{c|c} * & * \cdots * \\ \hline 0 & X \end{array}\right]$$

$$H_2 = H_1^t.$$

As explained in [Bro], $L^2(G/H_1, \mathbb{R}) \simeq L^2(G/H_2, \mathbb{R})$.

To see that $L^2(G/H_1, \mathbb{R})$ (say) is multiplicity free, we note that $G/H_1 = \mathbb{P}^n(\mathbb{F}_q)$. There are just two H_1-orbits on G/H_1: $\{e\}$ and $\mathbb{P}^n(\mathbb{F}_q) - \{e\}$. Hence there are just two double cosets in $H_1 \backslash G/H_1$. It follows that $L^2(G/H_1, \mathbb{C})$ splits into exactly two unitary irreducibles ([K]):

$$L^2(G/H_1, \mathbb{C}) = \mathbb{C} \oplus \mathbb{C}^\perp \tag{4.4}$$

(constants and orthogonals to the constants). Since $L^2(G/H_1, \mathbb{C}) = L^2(G/H_1, \mathbb{R}) \otimes \mathbb{C}$ we must have

$$L^2(G/H_1, \mathbb{R}) = \mathbb{R} \oplus \mathbb{R}^\perp. \tag{4.5}$$

Each of these irreducibles must be of real type by (1.12). Hence $L^2(G/H_1, \mathbb{R})$ is multiplicity free and completely of real type.

To construct the diagram (4.2) we now need M_0 satisfying (HDLD) and such that exists a surjective homomorphism from $\pi_1(M_0, x_0) \to G$. An example of such an M_0 is: $M_0 = N_0 \times S^n$, where N_0 is a Riemann surface of a genus g greater than the number of generators of G, and where S^n is the n-sphere. Following a standard argument [Su], a surjective homomorphism can first be defined from $\pi_1(N_0, n_0) \to G$. It immediately induces one from $\pi_1(M_0, x_0)$. Then, choosing n large enough, we may assume M_0 satisfies (HDLD). The proof is complete. □

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