

# ANNALES DE L'INSTITUT FOURIER

JEAN-MICHEL BISMUT

E. VASSEROT

## **The asymptotics of the Ray-Singer analytic torsion of the symmetric powers of a positive vector bundle**

*Annales de l'institut Fourier*, tome 40, n° 4 (1990), p. 835-848

[http://www.numdam.org/item?id=AIF\\_1990\\_\\_40\\_4\\_835\\_0](http://www.numdam.org/item?id=AIF_1990__40_4_835_0)

© Annales de l'institut Fourier, 1990, tous droits réservés.

L'accès aux archives de la revue « Annales de l'institut Fourier » (<http://annalif.ujf-grenoble.fr/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

## THE ASYMPTOTICS OF THE RAY-SINGER ANALYTIC TORSION OF THE SYMMETRIC POWERS OF A POSITIVE VECTOR BUNDLE

by J.-M. BISMUT and E. VASSEROT

---

Let  $X$  be a compact complex manifold, equipped with a smooth Hermitian metric. Let  $(E, \|\cdot\|_E)$ ,  $(\xi, \|\cdot\|_\xi)$  be holomorphic Hermitian vector bundles on  $X$ . Assume that  $(E, \|\cdot\|_E)$  is positive, i.e. if  $L^E$  is the curvature of the holomorphic Hermitian connection on  $(E, \|\cdot\|_E)$ , for any  $U \in TX \setminus \{0\}$ ,  $e \in E \setminus \{0\}$ , then  $\langle L^E(U, \bar{U})e, \bar{e} \rangle > 0$ . By [K] Theorem III 6.19,  $E$  is an ample vector bundle on  $X$ .

For  $p \in \mathbb{N}$ , let  $S^p(E)$  be the  $p^{\text{th}}$  symmetric tensor power of  $E$ . Then by a result of Le Potier [LP], [K] Theorem III.6.25, for  $p$  large enough, and  $q > 0$ ,  $H^q(S^p(E) \otimes \xi) = 0$ . Let  $\tau_p$  be the Ray-Singer analytic torsion of the Dolbeault complex  $\Omega^{(0,\cdot)}(S^p(E) \otimes \xi)$  [RS]. The purpose of this paper is to establish an asymptotic formula for  $\text{Log}(\tau_p)$  as  $p \rightarrow +\infty$ . This extends an earlier result by ourselves [BV] Theorem 8, in the case where  $E$  is a positive line bundle.

The general strategy is the same as in [BV]. Namely if  $\bar{\square}_p^{X,q}$  denotes the Hodge Laplacian acting on  $\Omega^{(0,q)}(S^p(E) \otimes \xi)$ , we first establish in Theorem 1 an asymptotic formula for  $\text{Tr} \left[ \exp \left( -\frac{t}{p} \bar{\square}_p^{X,q} \right) \right]$  as  $p \rightarrow +\infty$ . In Theorem 8 we prove that if  $\lambda_p^q$  is the lowest eigenvalue of  $\bar{\square}_p^{X,q}$ , if  $q > 0$ , as  $p \rightarrow +\infty$ ,  $\lambda_p^q$  grows at least like  $p$ . The combination of these two results leads us in Theorem 11 to an asymptotic formula for  $\text{Log}(\tau_p)$  very much like in [BV].

To establish these intermediary results, we use a trick due to Getzler [Ge] in a different context. In [Ge], Getzler extended a result of Bismut [B2] on the asymptotics of certain heat equation operators, which

---

*Key-words* : Sheaves and cohomology of sections of holomorphic vector bundles.

*A.M.S. Classification* : 32L10.

is valid for line bundles, to vector bundles, associated with representations of the structure group with weight  $p\lambda$  as  $p \rightarrow +\infty$ . Here if  $\mu$  is the dual of the universal line bundle on  $\mathbb{P}(E^*)$ , we consider  $S^p(E)$  as the direct image of  $\mu^{\otimes p}$  by the map  $\pi: \mathbb{P}(E^*) \rightarrow X$ . We then use Getzler's trick to lift our initial problem to a corresponding problem on the line bundle  $\mu^{\otimes p}$  on  $\mathbb{P}(E^*)$ , to which the techniques of [BV] can be applied.

Our paper is organized as follows. In §1, we introduce our main assumptions and notation. In §2, we calculate the asymptotics of  $\text{Tr} \left[ \exp \left( -\frac{t}{p} \bar{\square}_p^{X,q} \right) \right]$  as  $p \rightarrow \infty$ . Finally in §3, we establish our main result on the asymptotics of  $\text{Log}(\tau_p)$  as  $p \rightarrow +\infty$ .

As was pointed out by the referee, the results contained in this paper can be extended to other irreducible representations of  $E$ , which are associated with the weights  $pa$  (where  $a$  is a given weight) when  $p$  tends to  $+\infty$ . The corresponding vector bundles can be expressed as direct images of the  $p^{\text{th}}$  power of a certain line bundle over the corresponding flag manifold. Arguments of Demailly [De] Lemma 3.7, can then be used to prove the positivity of this Hermitian line bundle when  $(E, \|\cdot\|_E)$  is positive, and the trick of Getzler [Ge] together with the techniques used in our paper still apply. This extension of our main result is left to the reader.

### 1. Assumptions and notation.

Let  $X$  be a compact complex manifold of complex dimension  $\ell$ . Let  $TX$  be the complex holomorphic tangent space.

Let  $E$  be a holomorphic vector bundle on  $X$ , of complex dimension  $k$ . Let  $E^*$  be the dual of  $E$ . For  $p \in \mathbb{N}$ ,  $S^p(E)$  denotes the  $p^{\text{th}}$  symmetric tensor power of  $E$ .

Let  $\mathbb{P}(E^*)$  denote the projectivization of  $E^*$ , and let  $\pi$  be the projection  $\mathbb{P}(E^*) \rightarrow X$ . Let  $\mu$  be the dual of the universal line bundle on  $\mathbb{P}(E^*)$ .

Let  $\|\cdot\|_E$  be a smooth Hermitian metric on  $E$ . Let  $\|\cdot\|_{S^p(E)}$ ,  $\|\cdot\|_{E^*}$ ,  $\|\cdot\|_\mu$  be the Hermitian metrics on  $S^p(E)$ ,  $E^*$ ,  $\mu$  induced by the metric  $\|\cdot\|_E$ .

Let  $\nabla^E, \nabla^{E^*}$  be the holomorphic Hermitian connections on  $(E, \|\cdot\|_E), (E^*, \|\cdot\|_{E^*})$ , and let  $L^E, L^{E^*}$  be the corresponding curvatures. Let  $\nabla^\mu$  be the holomorphic Hermitian connection on  $(\mu, \|\cdot\|_\mu)$  and let  $r$  be its curvature.

We first calculate  $r$ . Let  $T^V\mathbb{P}(E^*)$  be the relative tangent bundle to the fibres of  $\pi : \mathbb{P}(E^*) \rightarrow X$ . The connection  $\nabla^{E^*}$  induces a horizontal subbundle  $T^H\mathbb{P}(E^*)$  of  $T\mathbb{P}(E^*)$ .

Let  $r^V$  be the restriction of  $r$  to  $T^V\mathbb{P}(E^*)$ .  $r^V$  is explicitly known by a formula given in [GrH] p. 30, and defines the Fubini-Study metric along the fibres of  $\mathbb{P}(E^*)$ .  $r^V$  extends into a (1,1) form on  $\mathbb{P}(E^*)$  such that if  $U \in T^H\mathbb{P}(E^*), i_U r^V = 0$ .

Set

$$(1) \quad r^H = -\pi^* \frac{\langle L^{E^*} y, \bar{y} \rangle}{|y|^2}; \quad y \in E^* \setminus \{0\}.$$

Then  $r^H$  is a (1,1) form on  $\mathbb{P}(E^*)$ . Also by [K] p. 90,

$$(2) \quad r = r^V + r^H$$

$r^H$  is then the restriction of  $r$  to  $T^H\mathbb{P}(E^*) \times T^H\mathbb{P}(E^*)$ .

For  $p \geq 1$ , the connection  $\nabla^E$  induces on  $S^p(E)$  the holomorphic Hermitian connection  $\nabla^{S^p(E)}$  on  $(S^p(E), \|\cdot\|_{S^p(E)})$ .

Let  $(\xi, \|\cdot\|_\xi)$  be a holomorphic Hermitian vector bundle on  $X$ . Let  $\nabla^\xi$  be the corresponding holomorphic Hermitian connection.

Let  $\|\cdot\|_{TX}$  be a Hermitian metric on  $X$ .

We then equip  $\Lambda(T^{*(0,1)}X) \otimes S^p(E) \otimes \xi$  with the tensor product of the metrics induced by  $\|\cdot\|_{TX}$  on  $\Lambda(T^{*(0,1)}X)$ , of the metric  $\|\cdot\|_{S^p(E)}$  and of the metric  $\|\cdot\|_\xi$ .

For  $0 \leq q \leq \ell$ , let  $\Omega^{(0,q)}(S^p(E) \otimes \xi)$  be the set of  $C^\infty$  sections of  $\Lambda^q(T^{*(0,1)}X) \otimes S^p(E) \otimes \xi$  over  $X$ . Set  $\Omega^{(0,\cdot)}(S^p(E) \otimes \xi) = \bigoplus_{q=0}^{\ell} \Omega^{(0,q)}(S^p(E) \otimes \xi)$ .

Let  $dx$  be the volume form on  $X$  associated with the metric  $\|\cdot\|_{TX}$ . We equip  $\Omega^{(0,\cdot)}(S^p(E) \otimes \xi)$  with the  $L_2$  Hermitian product

$$(3) \quad \alpha, \alpha' \in \Omega^{(0,\cdot)}(S^p(E) \otimes \xi) \rightarrow \langle \alpha, \alpha' \rangle = \int_X \langle \alpha, \alpha' \rangle(x) \frac{dx}{(2\pi)^{\dim X}}$$

Let  $\bar{\partial}_p^X$  be the Dolbeault operator acting on  $\Omega^{(0,\cdot)}(S^p(E)\otimes\xi)$ , and let  $\bar{\partial}_p^{X*}$  be the formal adjoint of  $\bar{\partial}_p^X$  with respect to the Hermitian product (3). Set

$$(4) \quad \bar{\square}_q^X = (\bar{\partial}_p^X + \bar{\partial}_p^{X*})^2.$$

For  $0 \leq q \leq \ell$ , let  $\bar{\square}_p^{X,q}$  be the restriction of  $\bar{\square}_p^X$  to  $\Omega^{(0,q)}(S^p(E)\otimes\xi)$ .

**2. The asymptotics of the trace of certain heat kernels as  $p \rightarrow +\infty$ .**

If  $U \in T_R X$ , let  $U^H$  be its horizontal lift in  $T_R^H \mathbb{P}(E)$ , so that  $U^H \in T_R^H \mathbb{P}(E)$ ,  $\pi_* U^H = U$ .

Let  $w_1, \dots, w_\ell$  be an orthonormal base of  $TX$ , let  $w^1, \dots, w^\ell$  be the corresponding dual base of  $T^*X$ . If  $z \in \mathbb{P}(E^*)$ ,  $r_z$  acts as a derivation  $r_{d,z}^H$  of  $\Lambda_{\pi z}^{(0,1)}(T^{*(0,1)}X)$  by the formula

$$(5) \quad r_{d,z}^H = - \sum r_z(w_i^H, \bar{w}_j^H) \bar{w}^j \wedge i_{\bar{w}_i}.$$

We identify  $r_z^H$  with the self-adjoint matrix  $r_z^H \in \text{End}_{\pi x}(\overline{TX})$  such that if  $U, V \in T_{\pi z} X$ ,

$$r_z^H(U^H, \bar{V}^H) = \langle U^H, r_z^H \bar{V}^H \rangle.$$

For  $0 \leq q \leq \ell$ , let  $r_d^{H,q}$  be the restriction of  $r_d^H$  to  $\Lambda^q(T^{*(0,1)}X)$ . As  $t \rightarrow 0$ , we have the asymptotic expansion

$$(6) \quad \int_{\mathbb{P}(E^*)} \frac{\text{Tr}[e^{tr_d^{H,q}}]}{\det(1 - e^{-tr^H})} \exp\left(\frac{-r}{2i\pi}\right) = \sum_{j=-\ell}^m a_j^q t^j + o(t^m).$$

Also for  $p \in \mathbb{N}$ ,  $0 \leq q \leq \ell$ ,  $t > 0$ , let  $\text{Tr}[\exp(-t \bar{\square}_p^{X,q})]$  be the trace of the operator  $\exp(-t \bar{\square}_p^{X,q})$ . For any  $m \in \mathbb{N}$ , as  $t \rightarrow 0$

$$(7) \quad p^{-(\dim X + \dim E - 1)} \text{Tr}\left[\exp\left(\frac{-t}{p} \bar{\square}_p^{X,q}\right)\right] = \sum_{j=-\ell}^m a_{p,j}^q t^j + o(t^m).$$

**THEOREM 1.** — *For any  $t > 0$ ,  $0 \leq q \leq \ell$ , the following identity holds*

$$(8) \quad \lim_{p \rightarrow +\infty} p^{-(\dim X + \dim E - 1)} \text{Tr}\left[\exp\left(\frac{-t}{p} \bar{\square}_p^{X,q}\right)\right] = rk(\xi) \int_{\mathbb{P}(E^*)} \frac{\text{Tr}[e^{r_d^{H,q}}]}{\det(1 - e^{-tr^H})} \exp\left(\frac{-r}{2i\pi}\right)$$

and the convergence in (8) is uniform as  $t$  varies in compact sets of  $R_+^*$ . For any  $j \geq -\dim X$ ,  $0 \leq q \leq \dim X$ , as  $p \rightarrow +\infty$

$$(9) \quad a_{p,j}^q = \text{rk}(\xi) a_j^q + O\left(\frac{1}{\sqrt{p}}\right).$$

In (7), for any  $m \in \mathbb{N}$ ,  $o(t^m)$  is uniform with respect to  $p \in \mathbb{N}$ .

*Proof.* — By [GrH] p. 165 and [K] Theorem III.4.10, for any  $x \in X$ ,  $p \in \mathbb{N}$

$$(10) \quad \begin{aligned} H^q(\mathbb{P}_x(E^*), \mu_{\mathbb{P}_x(E^*)}^{\otimes p}) &= S^p(E)_x & \text{if } q = 0 \\ H^q(\mathbb{P}_x(E^*), \mu_{\mathbb{P}_x(E^*)}^{\otimes p}) &= 0 & \text{if } q > 0. \end{aligned}$$

To prove (8), we will use (10) together with a procedure used by Getzler [Ge] in a similar situation, to transform the initial problem into a corresponding problem on  $\mathbb{P}(E^*)$  associated with  $\mu^{\otimes p}$ , to which we can apply results of Bismut [B2] and Bismut-Vasserot [BV].

Let  $U(E^*)$  be the bundle of orthonormal frames in  $E^*$ . We identify  $U(E^*)$  with the set of linear isometries from  $\mathbb{C}^k$  into  $E^*$ . Clearly

$$\mathbb{P}(E^*) = U(E^*) \times_{U(k)} \mathbb{P}(\mathbb{C}^k).$$

The connection  $\nabla^{E^*}$  on  $U(E^*)$  induces a connection on the fibration  $\pi: \mathbb{P}(E^*) \rightarrow X$ . The associated horizontal subbundle of  $T\mathbb{P}(E^*)$  is exactly the vector bundle  $T^H\mathbb{P}(E^*)$  considered in § 1.

We then have the identification of  $C^\infty$  vector bundles

$$(11) \quad \begin{aligned} T\mathbb{P}(E^*) &\cong T^H\mathbb{P}(E^*) \oplus T^V\mathbb{P}(E^*) \\ T^H\mathbb{P}(E^*) &\cong \pi^*TX. \end{aligned}$$

From (11), we deduce the identification of  $C^\infty$  vector bundles

$$(12) \quad \Lambda(T^{*(0,1)}\mathbb{P}(E^*)) \cong \pi^*(\Lambda(T^{*(0,1)}X)) \widehat{\otimes} \Lambda(T^{V*(0,1)}\mathbb{P}(E^*)).$$

We equip  $T^V\mathbb{P}(E^*)$  with the Fubini-Study metric  $\|\cdot\|_{T^V\mathbb{P}(E^*)}$ . Let  $\|\cdot\|_{T\mathbb{P}(E^*)}$  be the metric on  $T\mathbb{P}(E^*) \cong T^H\mathbb{P}(E^*) \oplus T^V\mathbb{P}(E^*)$  which is the orthogonal sum of  $\pi^*\|\cdot\|_{TX}$  and  $\|\cdot\|_{T^V\mathbb{P}(E^*)}$ . For  $p \in \mathbb{N}$ , set

$$(13) \quad \xi_p = \mu^{\otimes p} \otimes \pi^*\xi.$$

Let  $\bar{\partial}_p^V$  be the  $\bar{\partial}$  operator along the fibres of  $\mathbb{P}(E^*)$  acting on smooth sections of  $\pi^*(\Lambda(T^{*(0,1)}X)) \widehat{\otimes} \Lambda(T^{V*(0,1)}\mathbb{P}(E^*)) \otimes \xi_p$ , and let  $\bar{\partial}_p^{V*}$  be its formal adjoint with respect to the considered metrics.

Let  $\nabla^{T^V\mathbb{P}(E^*)}$ ,  $\nabla^{\xi_p}$  be the holomorphic Hermitian connections on  $T^V\mathbb{P}(E^*)$ ,  $\xi_p$  respectively. These connections induce a natural connection on  $\Lambda(T^{V*(0,1)}\mathbb{P}(E^*)) \otimes \xi_p$ , which we note  $\nabla^{\Lambda(T^{V*(0,1)}\mathbb{P}(E^*)) \otimes \xi_p}$ .

DEFINITION 2. — *If  $\alpha$  is a smooth section of  $\Lambda(T^{V*(0,1)}\mathbb{P}(E^*)) \otimes \xi_p$  over  $\mathbb{P}(E^*)$ , if  $U \in T_R X$ , set*

$$(14) \quad \tilde{\nabla}_{p,U} \alpha = \nabla_{U^H}^{\Lambda(T^{V*(0,1)}\mathbb{P}(E^*)) \otimes \xi_p} \alpha.$$

We extend  $\tilde{\nabla}_p$  to a differential operator acting on smooth sections of  $\pi^*(\Lambda(T_R^*X)) \widehat{\otimes} \Lambda(T^{V*(0,1)}\mathbb{P}(E^*)) \otimes \xi_p$ , with the convention that if  $\omega$  is a smooth section of  $\Lambda(T_R^*X)$ , and if  $\alpha$  is a smooth section of  $\Lambda(T^{V*(0,1)}\mathbb{P}(E^*)) \otimes \xi_p$ , then

$$(15) \quad \tilde{\nabla}_p(\omega\alpha) = \pi^*(d\omega)\alpha + (-1)^{\text{deg } \omega} \omega \wedge \tilde{\nabla}_p \alpha.$$

Let  $\tilde{\nabla}'_p, \tilde{\nabla}''_p$  be the holomorphic and antiholomorphic parts of  $\tilde{\nabla}_p$ , so that

$$(16) \quad \tilde{\nabla}_p = \tilde{\nabla}'_p + \tilde{\nabla}''_p.$$

For  $0 \leq q' \leq k + \ell - 1$ , let  $\Omega^{(0,q)}(\xi_p)$  be the set of smooth sections of  $\Lambda(T^{*(0,1)}\mathbb{P}(E^*)) \otimes \xi_p$  over  $\mathbb{P}(E^*)$ . Set  $\Omega^{(0,\bullet)}(\xi_p) = \bigoplus_0^{k+\ell-1} \Omega^{(0,q)}(\xi_p)$ . Let  $\bar{\partial}_p^{\mathbb{P}(E^*)}$  be the classical Dolbeault operator acting on  $\Omega^{(0,\bullet)}(\xi_p)$ .

Using the identification (12), it is clear that  $\tilde{\nabla}''_p, \bar{\partial}_p^V, \bar{\partial}_p^{V*}$  act on  $\Omega^{(0,\bullet)}(\xi_p)$ .

If  $A, B$  are operators acting on the  $\mathbb{Z}$ -graded vector space  $\Omega^{(0,\bullet)}(\xi_p)$ ,  $[A,B]$  denotes the supercommutator of  $A$  and  $B$  in the sense of [Q].

PROPOSITION 3. — *The following identities of operators acting on  $\Omega^{(0,\bullet)}(\xi_p)$  hold*

$$(17) \quad \begin{aligned} (\tilde{\nabla}''_p)^2 &= 0 \\ [\tilde{\nabla}'_p, \bar{\partial}_p^V] &= [\tilde{\nabla}''_p, \bar{\partial}_p^{V*}] = 0 \\ \bar{\partial}_p^{\mathbb{P}(E^*)} &= \tilde{\nabla}''_p + \bar{\partial}_p^V. \end{aligned}$$

*Proof.* — Assume first that  $\xi = \mathbb{C}$ . Let  $F$  be the vector space of smooth sections of  $\Lambda(T^{V^*(0,1)}\mathbb{P}(E^*)) \otimes \mu^{\otimes p}$  over  $\mathbb{P}(E^*)$ . As in [B1], Section 1f), we view  $F$  as an infinite dimensional vector bundle over  $X$ . If  $x \in X$ , the fibre  $F_x$  is simply the vector space of smooth sections  $\Lambda(T^{V^*(0,1)}\mathbb{P}(E^*)) \otimes \mu^{\otimes p}$  on the fibre  $\mathbb{P}(E^*)_x$ .

Clearly  $F$  is a  $U(k)$ -equivariant vector bundle on  $X$ , in the sense it comes from a representation space for  $U(k)$ .  $\tilde{\nabla}_p$  is then a connection on the vector bundle  $F$ , which is inherited from the original connection  $\nabla^E$  on  $U(E)$ . Since  $(\nabla^{E''})^2 = 0$ , then  $(\tilde{\nabla}_p'')^2 = 0$ . It is now trivial to prove in full generality the equation  $(\tilde{\nabla}_p'')^2 = 0$ .

Since  $U(k)$  acts on  $\mathbb{P}(\mathbb{C}^{k^*})$  by holomorphic isometries which lift unitarily to the dual of the universal bundle on  $\mathbb{P}(\mathbb{C}^{k^*})$ , we get

$$(18) \quad [\tilde{\nabla}_p, \bar{\partial}_p^V] = 0; \quad [\tilde{\nabla}_p, \bar{\partial}_p^{V^*}] = 0.$$

In particular, the second equation in (17) holds.

If  $B \in \text{End}(E^*)$  is skew-adjoint, let  $By$  be the holomorphic Killing vector field on  $\mathbb{P}(E^*)$  induced by the corresponding vector field on  $E^*$ . The vector field  $By$  lies in  $T^V\mathbb{P}(E^*)$ . Then  $L^{E^*}y$  is a (1,1) form on  $X$  taking values in vector fields in  $T^V\mathbb{P}(E^*)$ .  $L^{E^*}y$  lifts to a (1,1) form on  $\mathbb{P}(E^*)$ .

Assume first that  $p = 0$ , and  $\xi = \mathbb{C}$ . Let  $d^V$  be the de Rham operator along the fibres of  $\mathbb{P}(E^*)$ , and let  $d$  be the de Rham operator on  $\mathbb{P}(E^*)$ . Similarly  $\tilde{\nabla}_0$  can be made to act on the de Rham complex of  $\mathbb{P}(E^*)$ . Then by [B1] eq. (1.30) and [BGS1] eq. (1.26), we find that

$$(19) \quad d = \tilde{\nabla}_0 + d^V + i_{L^{E^*}y}.$$

Since  $L^{E^*}$  is of type (1,1), we deduce from (19) that we have the identity of operators acting on  $\Omega^{(0,\cdot)}(\xi_0)$

$$(20) \quad \bar{\partial}_0^{P(E^*)} = \tilde{\nabla}_0'' + \bar{\partial}^V.$$

Extending (20) to  $\Omega^{(0,\cdot)}(\xi_p)$  is easy and is left to the reader. □

*Remark 4.* — The fibration  $\pi : \mathbb{P}(E^*) \rightarrow X$  is locally Kähler in the sense of Bismut-Gillet-Soulé [BGS1], [BGS2]. Part of the identities in (17) follows from [BGS1], Theorem 2.6.



Let  $\tilde{\nabla}_p^{''*}, \bar{\partial}_p^{V*}, \bar{\partial}_p^{P(E^*)}$ , be the formal adjoints of  $\tilde{\nabla}_p'', \bar{\partial}_p^V, \bar{\partial}_p^{P(E^*)}$ , with respect to the obvious Hermitian product on  $\Omega^{(0,\cdot)}(\xi_p)$  associated to the various metrics.

Observe that  $\bar{\partial}_p^{V*}$  restricts on each fibre of  $\pi : \mathbb{P}(E^*) \rightarrow E^*$  to the fibrewise adjoint of  $\bar{\partial}_p^V$ .

**THEOREM 5.** — *The following identities of operators acting on  $\Omega^{(0,\cdot)}(\xi_p)$  hold*

$$(21) \quad (\bar{\partial}_p^{P(E^*)} + \bar{\partial}_p^{P(E^*)*})^2 = (\tilde{\nabla}_p'' + \tilde{\nabla}_p^{''*})^2 + (\bar{\partial}_p^V + \bar{\partial}_p^{V*})^2$$

$$[(\tilde{\nabla}_p'' + \tilde{\nabla}_p^{''*})^2, (\bar{\partial}_p^V + \bar{\partial}_p^{V*})^2] = 0.$$

*Proof.* — (21) follows from Proposition 3. □

For  $0 \leq q \leq \dim X, 0 \leq q' \leq \dim E - 1$ , let  $\Omega^{(0,q,q')}(\xi_p)$  be the set of smooth section of  $\pi^*(\Lambda^q(T^{*(0,1)}X)) \widehat{\otimes} \Lambda^{q'}(T^{V*(0,1)}\mathbb{P}(E^*)) \otimes \xi_p$ . By (12), we know that  $\Omega^{(0,q,q')}(\xi_p)$  is a vector subspace of  $\Omega^{(0,q+q')}(\xi_p)$ . More precisely, for any  $q, 0 \leq q \leq \dim X + \dim E - 1$

$$(22) \quad \Omega^{(0,q)}(\xi_p) = \bigoplus_{q'+q''=q} \Omega^{(0,q',q'')}(\xi_p).$$

Set

$$(23) \quad \square_p^{P(E^*)} = (\bar{\partial}_p^{P(E^*)} + \bar{\partial}_p^{P(E^*)*})^2.$$

By Theorem 5, we find that

$$(24) \quad \square_p^{P(E^*)} = \tilde{\nabla}_p'' \tilde{\nabla}_p^{''*} + \tilde{\nabla}_p^{''*} \tilde{\nabla}_p'' + \bar{\partial}_p^V \bar{\partial}_p^{V*} + \bar{\partial}_p^{V*} \bar{\partial}_p^V.$$

From (24), it is clear that the operator  $\square_p^{P(E^*)}$  acts on each  $\Omega^{(0,q,q')}(\xi_p)$ . Let  $\square_p^{P(E^*),q,q'}$  be the restriction of  $\square_p^{P(E^*)}$  to  $\Omega^{(0,q,q')}(\xi_p)$ .

We now have the following result directly inspired by Getzler [Ge].

**THEOREM 6.** — *For any  $p \in N, 0 \leq q \leq \dim X, t > 0$ , the following identity holds*

$$(25) \quad \text{Tr} [\exp(-t \square_p^{X,q})] = \sum_{q'=0}^{\dim E-1} (-1)^{q'} \text{Tr} [\exp(-t \square_p^{P(E^*),q,q'})].$$

*Proof.* — Let  $F_p$  be the vector space of smooth sections of  $\Lambda(T^{V*(0,1)}\mathbb{P}(E^*)) \otimes \xi_p$  over  $\mathbb{P}(E^*)$ . As in the proof of Proposition 3, we regard  $F_p$  as an infinite dimensional vector bundle over  $X$ . If  $x \in X$ ,

the fibre  $F_{p,x}$  is the set of smooth sections of  $\Lambda(T^{V^*(0,1)}\mathbb{P}(E^*)) \otimes \xi_p$  over  $\mathbb{P}(E^*)_x$ .  $F_p$  is a  $U(k)$ -equivariant Hermitian vector bundle on  $X$ , and  $\tilde{\nabla}_p$  is the corresponding holomorphic Hermitian connection. Also  $\Omega^{(0,\cdot)}(F_p)$  is canonically isomorphic to  $\Omega^{(0,\cdot)}(\xi_p)$ .

The operator  $(\bar{\partial}_p^V + \bar{\partial}_p^{V^*})^2$  is  $U(k)$ -equivariant. Therefore the spectrum of  $(\bar{\partial}_p^V + \bar{\partial}_p^{V^*})^2$  acting on a fibre  $F_{p,x}$  does not depend on  $x \in X$ . In the sequel  $\lambda \geq 0$  varies in the spectrum of  $(\bar{\partial}_p^V + \bar{\partial}_p^{V^*})^2$ .

The vector bundle  $F_p$  over  $X$  then splits into a direct orthonormal sum of finite dimensional vector spaces  $F_p^\lambda$  which are eigenspaces of  $(\bar{\partial}_p^V + \bar{\partial}_p^{V^*})^2$  associated with the eigenvalues  $\lambda$ , i.e.

$$(26) \quad F_p = \bigoplus_{\lambda \geq 0} F_p^\lambda.$$

For  $0 \leq q' \leq \dim E - 1$ , let  $F_p^{q'}$  be the set of smooth sections of  $\Lambda^{q'}(T^{V^*(0,1)}\mathbb{P}(E^*)) \otimes \xi_p$  over  $\mathbb{P}(E^*)$ . Clearly  $F_p = \bigoplus_{q'=0}^{\dim E - 1} F_p^{q'}$ . Also the operator  $(\bar{\partial}_p^V + \bar{\partial}_p^{V^*})^2$  preserves each  $F_p^{q'}$ . To the splitting (26) of  $F_p$  corresponds the splitting

$$(27) \quad F_p^{q'} = \bigoplus_{\lambda \geq 0} F_p^{q',\lambda}$$

of each  $F_p^{q'}$ . Using (10) and Hodge theory, we know that

$$(28) \quad \begin{aligned} F_p^{q',(0)} &= S^p(E) \otimes \xi & \text{if } q' = 0 \\ &= 0 & \text{if } q' > 0. \end{aligned}$$

Moreover since  $U(k)$  acts irreducibly on  $S^p(E)$ , the metric on  $S^p(E)$  induced from the  $L_2$  metric on the fibers of  $\mathbb{P}(E^*)$  coincides (up to an irrelevant constant) with the metric  $\| \cdot \|_{S^p(E)}$ .

Let  $\tilde{\square}_p^{q,q',\lambda}$  be the restriction of the operator  $(\tilde{\nabla}_p'' + \tilde{\nabla}_p''^*)^2$  to the set of smooth sections of  $\Lambda^q(T^*(0,1)X) \widehat{\otimes} F_p^{q',\lambda}$  over  $X$ . From Theorem 5, we get

$$(29) \quad \begin{aligned} \sum_{q'=0}^{\dim E - 1} (-1)^{q'} \text{Tr} [\exp(-t \tilde{\square}_p^{(E^*),q,q'})] \\ = \sum_{\lambda \geq 0} \exp(-t\lambda) \sum_{q'=0}^{\dim E - 1} (-1)^{q'} \text{Tr} [\exp(-t \tilde{\square}_p^{q,q',\lambda})]. \end{aligned}$$

From (28) and from the considerations which follow, we find that

$$(30) \quad \sum_{q'=0}^{\dim E-1} (-1)^{q'} \operatorname{Tr} [\exp(-t \tilde{\square}_p^{q, q', 0})] = \operatorname{Tr} [\exp(-t \bar{\square}_p^{X, q})].$$

On the other hand, for  $\lambda > 0$ , we have a  $U(k)$ -equivariant exact sequence of vector bundles on  $X$

$$(31) \quad 0 \rightarrow F_p^{0, \lambda} \xrightarrow{\partial_p^V} F_p^{1, \lambda} \rightarrow \dots \xrightarrow{\partial_p^V} F_p^{\dim E-1, \lambda} \rightarrow 0.$$

From (31), we easily deduce that for  $\lambda > 0$

$$(32) \quad \sum_{q'=0}^{\dim E-1} (-1)^{q'} \operatorname{Tr} [\exp(-t \tilde{\square}_p^{q, q', \lambda})] = 0.$$

Using (30), (32), we get (25). □

*Remark 7.* — As  $t \rightarrow 0$ , the left-hand side of (25) has a singularity  $t^{-\dim X}$ . A priori, the right-hand side has a singularity  $t^{-(\dim X + \dim E - 1)}$ . Therefore a cancellation process occurs in the right-hand side of (25) as  $t \rightarrow 0$ .

*Proof of Theorem 1.* — Let  $r_d$  be the analogue of  $r_d^H$  on  $\mathbb{P}(E^*)$ . Namely if  $w'_1, \dots, w'_{\ell+k-1}$  is an orthonormal base of  $T\mathbb{P}(E^*)$ , if  $w'^1, \dots, w'^{\ell+k-1}$  is the corresponding base of  $T^*\mathbb{P}(E^*)$ , set

$$(33) \quad r_d = - \sum r(w'_i, \bar{w}'_j) \bar{w}'^{ij} \wedge i_{\bar{w}'_i}.$$

Then  $r_d$  acts as a derivation of

$$\Lambda(T^{*(0,1)\mathbb{P}}(E^*)) = \pi^*(\Lambda(T^{*(0,1)X})) \widehat{\otimes} \Lambda(T^{V^*(0,1)\mathbb{P}}(E^*)).$$

We identify  $r$  with the self-adjoint matrix  $\mathring{r} \in \operatorname{End}(\overline{T\mathbb{P}(E)})$  such that  $U, V \in T\mathbb{P}(E)$

$$(34) \quad r(U, V) = \langle U, \mathring{r} \bar{V} \rangle.$$

By (1), (2), it is clear that  $r_d$  preserves  $\pi^*(\Lambda^q(T^{*(0,1)X})) \otimes \Lambda^{q'}(T^{V^*(0,1)\mathbb{P}}(E^*))$ . Let  $r_d^{q, q'}$  be the corresponding restriction of  $r_d$ .

Let  $dz$  be the volume form on  $\mathbb{P}(E^*)$  with respect to the metric  $\|\cdot\|_{T\mathbb{P}(E^*)}$ .

Clearly as  $t \rightarrow 0$ , we have the asymptotic expansion

$$(35) \quad (2\pi)^{-(\dim X + \dim E - 1)} \int_{\mathbb{P}(E^*)} \det(\tilde{r}) \frac{\text{Tr}[e^{q,q'}_d]}{\det(1 - e^{-tr})} dz = \sum_{j=-\ell-k+1}^m b_j^{q,q'} t^j + o(t^m).$$

For any  $p \in \mathbb{N}$ ,  $0 \leq q \leq \dim X$ ,  $0 \leq q' \leq \dim E - 1$ , as  $t \rightarrow 0$ , we have the asymptotic expansion

$$(36) \quad p^{-(\dim X + \dim E - 1)} \text{Tr} \left[ \exp \left( -\frac{t}{p} \square_p^{\mathbb{P}(E^*), q, q'} \right) \right] = \sum_{j=-\ell-k+1}^m b_j^{q,q'} t^j + o(t^m).$$

By a straightforward adaptation of [B2]. Theorem 1.5, and [BV] Theorem 2, we know that for any  $t > 0$

$$(37) \quad \lim_{p \rightarrow +\infty} p^{-(\dim X + \dim E - 1)} \text{Tr} \left[ \exp \left( -\frac{t}{p} \square_p^{\mathbb{P}(E^*), q, q'} \right) \right] = (2\pi)^{-(\dim X + \dim E - 1)} rk(\xi) \int_{\mathbb{P}(E^*)} \frac{\det(\tilde{r}) \text{Tr}[e^{q,q'}_d]}{\det(1 - e^{-tr})} dz$$

and the convergence is uniform as  $t$  varies in compact subsets of  $R_+^*$ . Also as  $p \rightarrow +\infty$

$$(38) \quad b_{p,j}^{q,q'} = rk(\xi) b_j^{q,q'} + O\left(\frac{1}{\sqrt{p}}\right).$$

Moreover in (36),  $o(t^m)$  is uniform with respect to  $p \in \mathbb{N}$ .

By (2)  $\tilde{r}$  map  $T^V \mathbb{P}(E^*)$  into itself. Let  $\tilde{r}^V$  be the restriction of  $\tilde{r}$  to  $T^V \mathbb{P}(E^*)$ . We then find that

$$(39) \quad \sum_0^{\dim E - 1} (-1)^{q'} \text{Tr}[e^{tr,q,q'}_d] = \text{Tr}[e^{tr,H,q}_d] \det(1 - e^{-tr^V})$$

$$\det(1 - e^{-tr}) = \det(1 - e^{-tr^H}) \det(1 - e^{-tr^V}).$$

By (25), (37), (39), we get

$$(40) \quad \lim_{p \rightarrow +\infty} p^{-(\dim X + \dim E - 1)} \text{Tr} \left[ \exp \left( -\frac{t}{p} \square_p^{X, q, q'} \right) \right] = rk(\xi) \int_{\mathbb{P}(E^*)} \frac{\text{Tr}[e^{tr,H,q}_d]}{\det(1 - e^{-tr^H})} \det\left(\frac{\tilde{r}}{2\pi}\right) dz.$$

Clearly

$$(41) \quad \det \left( \frac{\dot{r}}{2\pi} \right) dz = \left[ \exp \left( \frac{-r}{2i\pi} \right) \right]^{\max}.$$

Using (40), (41), we get (8). From the previous considerations, we also obtain the full proof of Theorem 1. □

### 3. The asymptotics of the Ray-Singer analytic torsion as $p \rightarrow \infty$ .

From now on, we assume that the holomorphic Hermitian vector bundle  $(E, \| \cdot \|_E)$  is positive, i.e. that if  $U \in TX \setminus \{0\}$ ,  $e \in E \setminus \{0\}$

$$(42) \quad \langle L^E(U, \bar{U})e, \bar{e} \rangle > 0.$$

From (2), we find that if  $y \in E^* \setminus \{0\}$  represent  $z \in \mathbb{P}(E^*)$ , then

$$(43) \quad r_z = r^V + \pi^* \frac{\langle L^E \bar{y}, y \rangle}{|y|^2}.$$

Classically [GrH] p. 30, the restriction of the line  $(\mu, \| \cdot \|_\mu)$  to the fibres  $\mathbb{P}(E^*)$  is positive, i.e. if  $U \in T^V \mathbb{P}(E^*) \setminus \{0\}$ ,  $r^V(U, \bar{U}) > 0$ . From (43), we deduce that the Hermitian line bundle  $(\mu, \| \cdot \|_\mu)$  is positive on  $\mathbb{P}(E^*)$ . This is of course a well-known result [K] Theorem III 6.19.

**THEOREM 8.** — *There exists  $C > 0$ ,  $c > 0$ ,  $c' > 0$  such that for any  $p \in \mathbb{N}$ ,  $1 \leq q \leq \ell$ ,  $t \geq 1$ , then*

$$(44) \quad p^{-(\dim X + \dim E - 1)} \text{Tr} \left[ \exp \left( - \frac{t}{p} \square_p^{X, q, q} \right) \right] \leq C \exp \left( - \left( c - \frac{c'}{p} \right) t \right).$$

*Proof.* — By [BV] Theorems 1 and 2, there exist  $C > 0$ ,  $c > 0$ ,  $c' > 0$  such that for  $p \in \mathbb{N}$ ,  $0 \leq q \leq \ell$ ,  $0 \leq q' < k - 1$ ,  $q + q' \geq 1$ ,  $t \geq 1$

$$(45) \quad \text{Tr} \left[ \exp \left( - \frac{t}{p} \square_p^{P(E^*), q, q'} \right) \right] \leq C \exp \left( - \left( c - \frac{c'}{p} \right) t \right).$$

Using (25) and (45), (44) follows. □

*Remark 9.* — Let  $\lambda_p^q$  be the lowest eigenvalue of  $\bar{\square}_p^{X,q}$ . From (44), we deduce that if  $q \geq 1$

$$(46) \quad \lambda_p^q \geq cp - c'.$$

(46) is also an easy consequence of Theorem 5, of the considerations in the proof of Theorem 6 and of [BV], Theorem 1. [BV] Theorem 1 is itself a consequence of the Bochner-Kodaira-Nakano formula of Demailly [De] for the operator  $\bar{\square}_p^{p(E^*)}$ . Strangely enough, (46) does not seem to be a straightforward consequence of a similar formula for  $\bar{\square}_p^X$ .

By Theorem 8 or by (46), there exists  $p_0 \in \mathbb{N}$  such that if  $p \geq p_0$ ,  $1 \leq q \leq \ell$ , the operator  $\bar{\square}_p^{p(E^*),q}$  is invertible.

DEFINITION 10. — For  $p \geq p_0$ ,  $s \in \mathbb{C}$ ,  $\text{Re}(s) \geq \ell$ , set

$$(47) \quad \zeta_p(s) = \frac{-1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \left( \sum_1^\ell (-1)^q q \text{Tr} [\exp(-t \bar{\square}_p^{X,q})] \right) dt.$$

By a well-known result of Seeley [Se],  $\zeta_p(s)$  extends into a meromorphic function of  $s \in \mathbb{C}$  which is holomorphic at  $s = 0$ . By definition  $\exp(-\zeta_p'(0))$  is the Ray-Singer analytic torsion [RS] of the Hermitian vector bundle  $S^p(E) \otimes \xi$ .

We now state the main result of this paper.

THEOREM 11. — As  $p \rightarrow +\infty$

$$(48) \quad \zeta_p'(0) = rk(\xi) \frac{1}{2} \int_{\mathbb{P}(E^*)} \text{Log} \left[ \det \left( \frac{p^{\circ H}}{2\pi} \right) \right] \exp \left( \frac{-pr}{2i\pi} \right) + o(p^{(\dim X + \dim E - 1)}).$$

In particular as  $p \rightarrow +\infty$

$$(49) \quad \zeta_p'(0) = O(p^{\dim X + \dim E - 1} \text{Log } p).$$

*Proof.* — In view of Theorems 1 and 8, which are the obvious extensions of [BV] Theorem 2, the proof of Theorem 11 proceeds formally as the proof of [BV] Theorems 4 and 8. Details are left to the reader. □

## BIBLIOGRAPHY

- [B1] J. M. BISMUT, The index Theorem for families of Dirac operators : two heat equation proofs, *Invent. Math.*, 83 (1986), 91-151.
- [B2] J. M. BISMUT, Demailly's asymptotic Morse inequalities : a heat equation proof, *J. Funct. Anal.*, 72 (1987), 263-278.
- [BGS1] J. M. BISMUT, H. GILLET, C. SOULÉ, Analytic torsion and holomorphic determinant bundles. II, *Comm. Math. Phys.*, 115 (1988), 79-126.
- [BGS2] J. M. BISMUT, H. GILLET, C. SOULÉ, Analytic torsion and holomorphic determinant bundles. III, *Comm. Math. Phys.*, 115 (1988), 301-351.
- [BV] J. M. BISMUT, E. VASSEROT. The asymptotics of the Ray-Singer analytic torsion associated with high powers of a positive line bundle, *Comm. Math. Phys.*, 125 (1989), 355-367.
- [De] J. P. DEMAILLY, Vanishing theorems for tensor powers of a positive vector bundle. In *Geometry and Analysis*, T. Sunada, ed., pp. 86-106, *Lecture Notes in Math.* Berlin-Heidelberg-New York, Springer-Verlag, 1988.
- [Ge] E. GETZLER, Inégalités asymptotiques de Demailly pour les fibrés vectoriels, *C.R. Acad. Sci., Série I. Math.*, 304 (1987), 475-478.
- [GrH] P. GRIFFITHS, J. HARRIS, *Principles of algebraic geometry*, New York, Wiley, 1978.
- [K] S. KOBAYASHI, *Differential geometry of complex vector bundles*, Iwanami Shoten and Princeton University Press, 1987.
- [LP] J. LE POTIER, Théorèmes d'annulation en cohomologie, *C.R. Acad. Sci. Paris, Série A*, 276 (1976), 535-537.
- [Q] D. QUILLEN, Superconnections and the Chern character, *Topology*, 24 (1985), 89-95.
- [RS] D. B. RAY, I. M. SINGER, Analytic torsion for complex manifolds, *Ann. of Math.*, 98 (1973), 154-177.
- [Se] R. T. SEELEY, Complex powers of an elliptic operator, *Proc. Symp. Pure and Appl. Math.*, Vol. 10, 288-307, Providence, Am. Math. Soc., (1967).

Manuscrit reçu le 10 octobre 1990.

J. M. BISMUT,  
Université de Paris-Sud  
Mathématiques  
Bâtiment 425  
91405 Orsay Cedex.

E. VASSEROT,  
École Normale Supérieure  
DMI  
45, rue d'Ulm  
75005 Paris.