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CAPACITARY STRONG TYPE ESTIMATES IN SEMILINEAR PROBLEMS

by D. R. ADAMS (*)(**) and M. PIERRE (**)

1. Introduction.

Let us consider the following semilinear boundary value problem

$$-\Delta u = u^\gamma + f \quad \text{on} \quad \Omega \tag{1.1}$$

$$u \geqslant 0 \quad \text{on} \quad \Omega \tag{1.2}$$

$$u = 0 \quad \text{on} \quad \partial\Omega \tag{1.3}$$

where Ω is a bounded open subset of $\mathbf{R}^N$ with a smooth boundary $\partial\Omega$, f a given nonnegative function on Ω and $\gamma \in (1, \infty)$. Two conditions on f are necessary for the existence of solutions to (1.1)-(1.3) :

(i) a regularity condition : f should be "regular" enough

(ii) a size condition : even if f is a C_0^∞-function, it should be small enough. For instance, if $f = \lambda g$ where $g \in C_0^\infty(\Omega)$, $g \geqslant 0$, $g \not\equiv 0$ and $\lambda > 0$, then there exists λ^* so that (1.1)-(1.3) does not have any solution for $\lambda > \lambda^*$.

A necessary and sufficient condition on f for the existence of solutions to (1.1)-(1.3) has been established by Baras-Pierre in [5]. It says that a certain "norm" of f, say $[f]_\gamma$, should be exactly less than or equal to

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$k(\gamma) = (\gamma - 1)/\gamma^{\gamma'}$ where $1/\gamma + 1/\gamma' = 1$. This quantity $[f]_\gamma$ is defined by duality through a functional which is naturally associated with the above problem, but whose expression looks rather awkward. Namely, the following result is proved in [5] : let f be a nonnegative measurable function on Ω. Then (1.1)-(1.3) has a solution (in a weak sense), if and only if

$$(1.4) \qquad \begin{cases} \forall \Psi \in C^\infty(\Omega) \text{ with } \Psi \geqslant 0 \text{ on } \Omega \text{ and } \Psi = 0 \text{ on } \partial\Omega \\ \displaystyle\int_\Omega \Psi f \leqslant k(\gamma) \int_\Omega |\Delta \Psi|^{\gamma'} \Psi^{1-\gamma'}. \end{cases}$$

Thus, one must deal with this functional if one wants to exactly describe the optimal size of the datum f. Actually, this result remains valid when f is replaced by a nonnegative Radon measure μ on Ω as also proved in [5].

Our purpose here is to better understand the regularity condition contained in (1.4). For instance, if $f = \lambda\mu$ where λ is a positive real number and μ is a nonnegative Radon measure, one might want to solve (1.1)-(1.3) at least for small λ. Then the measure μ should be "regular" enough, the exact regularity condition being expressed by (1.4). It turns out that this property has equivalent "simpler" forms – and apparently weaker – in terms of $W^{2,\gamma'}$-capacities (see [1], [2], [8], [10], [11]). It is our goal to state some of these forms and to prove their equivalence with (1.4).

2. The results.

We denote by $c_{2,p}$ the capacity associated with the norm $\|.\|_p$ of the space

$$W^{2,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j} \in L^p(\mathbb{R}^N), \forall i,j = 1, \ldots, N \right\}.$$

This means that, for any compact subset K of $\mathbb{R}^N$

$$c_{2,p}(K) = \inf\{\|u\|_p^p : u \in C_0^\infty(\mathbb{R}^N), u \geqslant 1 \text{ on } K, 0 \leqslant u \leqslant 1\}$$

where for instance

$$\|u\|_p^p := \|u\|_{L^p(\mathbb{R}^N)}^p + \sum_i \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\mathbb{R}^N)}^p + \sum_{i,j} \left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{L^p(\mathbb{R}^N)}^p.$$

When $2p < N$, this capacity is locally equivalent to the Riesz capacity defined by

$$R_{2,p}(K) = \inf\left\{ \|f\|_{L^p(\mathbb{R}^N)}^p ; f \geqslant 0, R_2 * f \geqslant 1 \text{ on } K \right\}$$

where

$$R_2 * f(x) = \int_{\mathbb{R}^N} |x-y|^{2-N} f(y)dy \tag{2.0}$$

is the Riesz potential of f (see for instance [12], [2]).

THEOREM 2.1. — *Assume $1 < p < N/2$ and let μ be a nonnegative Radon measure on $\mathbb{R}^N$. Then the following conditions are equivalent :*

$$\begin{cases} \textit{There exists } k_1 > 0 \textit{ such that for all } \Psi \in C_0^\infty(\mathbb{R}^N) \\ \textit{with } \Psi \geqslant 0 \\ \displaystyle\int_{\mathbb{R}^N} \Psi d\mu \leqslant k_1 \int_{\mathbb{R}^N} |\Delta\Psi|^p \Psi^{1-p} \end{cases} \tag{2.1}$$

$$\begin{cases} \textit{There exists } k_2 > 0 \textit{ such that for all } K \textit{ compact in } \mathbb{R}^N \\ \mu(K) \leqslant k_2 R_{2,p}(K) \end{cases} \tag{2.2}$$

$$\begin{cases} \text{There exists } k_3 > 0 \text{ such that for all } \varphi \in C_0^\infty(\mathbb{R}^N) \\ \displaystyle\int_{\mathbb{R}^N} |\varphi|^p d\mu \leqslant k_3 \int_{\mathbb{R}^N} |\Delta\varphi|^p \end{cases} \tag{2.3}$$

$$\begin{cases} \textit{For all } q \in [1,p], \textit{ there exists } k_4 > 0 \textit{ such that} \\ \text{for all } \Theta \in C_0^\infty(\mathbb{R}^N) \\ \displaystyle\int_{\mathbb{R}^N} |\Theta|^q d\mu \leqslant k_4 \int_{\mathbb{R}^N} |\Delta\Theta|^p |\Theta|^{q-p} \end{cases} \tag{2.4}$$

Remarks. — a) The equivalence (2.2) $\Leftrightarrow$ (2.3) is well known although not obvious (see [10], [1], [8]). Indeed (2.2) looks like a weak form of (2.3) and as a consequence, (2.3) is often referred to as "capacitary strong type estimate" although not stronger than (2.2).

b) The interest of the theorem lies in the equivalence between (2.2) (or (2.3)) and property (2.1) which appears in the characterization (1.4). Indeed, as a consequence, (2.2) provides a new and simpler characterization of the regularity of the measures μ for which problems of type (1.1)-(1.3) (with $f = \mu$) are solvable. Moreover, (2.2) is expressed in terms of a capacity which (at least locally) depends only on the $W^{2,p}$-norm rather than on the specific form of the operator Δ. This suggests that the solvability of

$$Lu = u^\gamma + \lambda\mu \tag{2.5}$$

for λ small, should be independent of the specific form of the uniformly elliptic operator L. This is precisely the purpose of the next theorem.

c) Another interest of the theorem is its proof which surprisingly relies (at least for the most difficult part (2.3) $\Rightarrow$ (2.1)) on deep results for singular integrals with A_p-weights.

d) Property (2.4) is just a natural generalization of (2.1) and (2.3) which correspond to the extreme cases $q = 1$ and $q = p$.

e) Theorem 2.1 is stated for $p < N/2$ since it is global in $\mathbf{R}^N$. This assumption will be dropped in the next theorem because of it local nature. In order to state it, some definitions are in order.

Let Ω be a bounded open subset of $\mathbf{R}^N$ and let L be a second order differential operator defined on open neighborhood ω of $\overline{\Omega}$ by

$$Lu = -\sum_{i,j}(a_{i,j}u_{x_i})_{x_j} + cu \tag{2.6}$$

where

$$a_{i,j} \in C^1(\omega), c \in L^\infty(\omega), \quad c \geqslant 0 \text{ on } \omega \tag{2.7}$$

$$\exists \alpha_0 > 0, \quad \sum_{i,j} a_{i,j}(x)\xi_i\xi_j \geqslant \alpha_0|\xi|^2 \qquad \forall \xi \in \mathbf{R}^N, \forall x \in \omega. \tag{2.8}$$

The adjoint L^* of L is defined

$$L^*\varphi = -\sum_{i,j}(a_{i,j}\varphi_{x_j})_{x_i} + c\varphi.$$

We introduce

$$Y(L) = \left\{\varphi \in W_0^{1,\infty}(\Omega); \varphi \geqslant 0, L^*\varphi \in L^\infty(\Omega) \text{ and has compact support}\right\}.$$

THEOREM 2.2. — *Assume $N > 2$ and $p > 1$. Let μ be a nonnegative measure with compact support in Ω. Then the following conditions are equivalent :*

$$\begin{cases} \text{There exists } k_1 > 0 \text{ such that for all } K \text{ compact in } \Omega \\ \mu(K) \leqslant k_1 c_{2,p}(K) \end{cases} \tag{2.9}$$

$$\begin{cases} \text{There exists } k_2 > 0 \text{ such that} \\ \forall \varphi \in Y(\Delta) \displaystyle\int_\Omega \varphi^p d\mu \leqslant k_2 \int_\Omega |\Delta\varphi|^p \end{cases} \tag{2.10}$$

$$\begin{cases} \text{There exists } k_3 > 0 \text{ such that} \\ \forall \varphi \in Y(\Delta) \displaystyle\int_\Omega \varphi d\mu \leqslant k_3 \int_\Omega |\Delta\varphi|^p \varphi^{1-p} \end{cases} \tag{2.11}$$

$$(2.12)\qquad \begin{cases} \textit{There exists } k_4 > 0 \textit{ such that} \\ \forall \varphi \in Y(L) \int_\Omega \varphi d\mu \leqslant k_4 \int_\Omega |L^*\varphi|^p \varphi^{1-p}. \end{cases}$$

Remark. — According to the results in [5], the property (2.12) characterizes the measures μ for which the problem (2.5) is solvable for λ small enough. Then, one can deduce the following applications from theorem 2.2.

COROLLARY 2.3. — *Let $N > 2$, $\gamma > 1$ and let μ be a nonnegative measure with compact support in Ω. Then the problem*

$$(2.13)\qquad \begin{cases} -\Delta u = u^\gamma + \lambda\mu \\ u \geqslant 0 \text{ on } \Omega, u = 0 \text{ on } \partial\Omega \end{cases}$$

has a solution for λ small if and only if the problem

$$(2.14)\qquad \begin{cases} Lu = u^\gamma + \lambda\mu \\ u \geqslant 0 \text{ on } \Omega, u = 0 \text{ on } \partial\Omega \end{cases}$$

has a solution for λ small.

Remark. — The solution in (2.13) or (2.14) is understood in a weak sense namely

$$u \in L^1_{\text{loc}}(\Omega), u(x) = \int_\Omega G_L(x,y)[u^\gamma(y)dy + \lambda d\mu(y)]$$

where G_L is the Green function of L.

COROLLARY 2.4 (Removable sets). — *Let $N > 2$, $\gamma > 1$ and let K be a compact subset of Ω. Then any solution of*

$$(2.15)\qquad \begin{cases} u \in L^\gamma_{\text{loc}}(\Omega \setminus K) \cap W^{1,1}_{\text{loc}}(\Omega \setminus K), u \geqslant 0 \\ Lu = u^\gamma \text{ in } \mathcal{D}'(\Omega \setminus K) \end{cases}$$

is a solution of

$$(2.16)\qquad \begin{cases} u \in L^\gamma_{\text{loc}}(\Omega) \cap W^{1,1}_{\text{loc}}(\Omega), u \geqslant 0 \\ Lu = u^\gamma \text{ in } \mathcal{D}'(\Omega) \end{cases}$$

if and only if

$$(2.17)\qquad c_{2,\gamma'}(K) = 0.$$

Remark. — The sufficiency of (2.17) was established in [4]. The necessity relies upon solving

$$Lu = u^\gamma + \lambda\mu_K \tag{2.18}$$

where $c_{2,\gamma'}(K) \neq 0$ and μ_K is the $c_{2,\gamma'}$-capacitary measure of K (see [12]). It is well known that μ_K satisfies (2.9) due to the uniform boundedness of nonlinear potentials (see [12], [2]). By theorem 2.2, (2.18) can be solved for λ small enough on some neighborhood of K. Therefore (2.18) provides a solution u of (2.15) which does not satisfy (2.16).

The result of corollary 2.4 had been obtained in the case $L = \Delta$ in [15] where it was directly established that capacitary measures satisfy condition (2.11).

3. The proof of theorem 2.1.

Preliminary remarks. — If (2.3) holds, it extends by density to all φ in $W^{2,p}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ or even to all φ in $W^{2,p}(\mathbb{R}^N)$ if one chooses the $W^{2,p}$-quasicontinuous representation of φ. It holds also for all Riesz-potentials $\varphi = R_2 * f$ where $f \in L^p(\mathbb{R}^N)$ (see (2.0)). Furthermore, the converse is true and (2.3) is even equivalent to

$$\forall f \in L^p(\mathbb{R}^N), f \geqslant 0 \qquad \int_{\mathbb{R}^N} (R_2 * f)^p d\mu \leqslant k_3' \int_{\mathbb{R}^N} f^p. \tag{2.3)$'$}$$

Indeed, if (2.3)$'$ holds and if $\varphi \in C_0^\infty(\mathbb{R}^N)$, we use that $\varphi = C(N) R_2 * (-\Delta\varphi)$ (where $C(N)^{-1} = (N-2)S_N$, S_N area of the unit sphere) and $|\varphi| \leqslant C(N) R_2 * |\Delta\varphi|$ so that

$$\int_{\mathbb{R}^N} |\varphi|^p d\mu \leqslant C(N) \int_{\mathbb{R}^N} (R_2 * |\Delta\varphi|)^p d\mu \leqslant C(N) k_3' \int_{\mathbb{R}^N} |\Delta\varphi|^p.$$

The fact that (2.1) and (2.4) are also equivalent to the corresponding Riesz-potential statement will be essential in the proof of the theorem :

$$\left\{ \begin{array}{l} \forall f \in L^p(\mathbb{R}^N), f \geqslant 0, f \text{ compactly supported} \\ \displaystyle\int_{\mathbb{R}^N} (R_2 * f)^q d\mu \leqslant k_4' \int_{\mathbb{R}^N} f^p (R_2 * f)^{q-p} \end{array} \right. \tag{2.4)$'$}$$

and similarly for (2.1)$'$ (take $q = 1$).

Indeed, (2.4)′ implies (2.4) since, for all $\Theta \in C_0^\infty(\mathbb{R}^N)$, $\Theta \geqslant 0$

$$|\Theta| \leqslant C_N R_2 * |\Delta\Theta|, |\Delta\Theta|^p (R_2 * |\Delta\Theta|)^{q-p} \leqslant |\Delta\Theta|^p |\Theta|^{q-p}.$$

The converse is obtained by regularization of f. Let $f_n \in C_0^\infty(\mathbb{R}^N)$, $f_n \geqslant 0$ converging to f in $L^p(\mathbb{R}^N)$ with support included in a fixed compact set K. One first shows that (2.4)′ holds for f_n by applying (2.4) to a sequence $\Theta_p \in C_0^\infty(\mathbb{R}^N)$ converging uniformly on compact sets to $R^2 * f_n$ and with $\Delta\Theta_p$ converging uniformly to f_n. Then, using that $R_2 * f_n$ remains positive on K (uniformly in n), we pass to the limit in n in (2.4)′.

Proof of theorem 2.1. — The equivalence (2.2) ⇔ (2.3) is proved in [1] and (2.1) (resp. (2.1)′) is a particular case of (2.4) (resp. (2.4)′). Therefore it is sufficient to prove "(2.1) ⇒ (2.3)′" and "(2.3) ⇒ (2.4)′".

"(2.1) ⇒ (2.3)′" : Let $f \in C_0^\infty(\mathbb{R}^N)$, $f \geqslant 0$ and $\varphi = R_2 * f$. Applying (2.1) to a sequence $\Psi_n \in C_0^\infty(\mathbb{R}^N)$ converging to φ^p in $C^2(\mathbb{R}^N)$ leads to (integrals are taken on $\mathbb{R}^N$)

$$\int \varphi^p d\mu \leqslant k_1 \int |\Delta(\varphi^p)|^p \varphi^{p(1-p)} \leqslant kC_p \int |\Delta\varphi|^p + |\nabla\varphi|^{2p}\varphi^{-p}. \tag{3.1}$$

LEMMA 3.1 (Hedberg [9]). — *There exists $C = C(N)$ such that*

$$\forall x \in \mathbb{R}^N \qquad |\nabla\varphi(x)|^2 \leqslant C_N \varphi(x) M(f)(x) \tag{3.2}$$

where

$$M(f)(x) = \sup_{r>0} \frac{1}{r^N} \int_{B(x,r)} |f(y)| dy. \tag{3.3}$$

From this we deduce

$$|\nabla\varphi|^{2p} \leqslant C_N^{2p} \varphi^p M(f)^p. \tag{3.4}$$

Using the maximal theorem, that is (see e.g. [16])

$$\int_{\mathbb{R}^N} M(f)^p \leqslant C(p,N) \int_{\mathbb{R}^N} f^p \quad \forall p \in (1,\infty), \tag{3.5}$$

with (3.1) and (3.4) we get

$$\int \varphi^p d\mu \leqslant kC_p \left[\int |\Delta\varphi|^p + C_N^{2p} \int M(f)^p \right] \leqslant kC(p,N) \int f^p. \tag{3.6}$$

This gives (2.3)′ for $f \in C_0^\infty(\mathbb{R}^N)$. The result follows by a density argument.

"(2.3) $\Rightarrow$ (2.4)$'$" : We may assume $1 \leqslant q < p$. Let $f \in C_0^\infty(\mathbf{R}^N)$, $f \geqslant 0$ and $\Psi = R_2 * f$. Applying (2.3) to a sequence $\varphi_n \in C_0^\infty(\mathbf{R}^N)$ converging to $\Psi^{q/p}$ in $C^2(\mathbf{R}^N)$ leads to

$$(3.7) \quad \int \Psi^q d\mu \leqslant k_3 \int |\Delta \Psi^{q/p}|^p \leqslant k_3 c(p,q) \int |\Delta\Psi|^p \Psi^{q-p} + |\nabla\Psi|^{2p}\Psi^{q-2p}.$$

Arguing as before, we use Hedberg's lemma to get

$$(3.8) \qquad |\nabla\Psi|^{2p}\Psi^{q-2p} \leqslant C_N^p \Psi^{q-p} M(f)^p.$$

Here estimate (3.5) is not sufficient for the result. We need its generalized version with A_p-weights.

LEMMA 3.2 (Muckenhoupt [13]). — *Let $\omega \in L^1_{\text{loc}}(\mathbf{R}^N)$, $\omega \geqslant 0$ such that*

$$(3.9) \qquad \sup_Q \left(\fint_Q \omega\right)\left(\fint_Q \omega^{-1/(p-1)}\right)^{p-1} \leqslant K \leqslant \infty, \quad 1 < p < \infty,$$

where the supremum is taken over all cubes Q and $\fint_Q$ denotes the average over Q. Then there exists $C = C(K,p,N)$ such that

$$(3.10) \qquad \int_{\mathbf{R}^N} M(f)^p(x)\omega(x)dx \leqslant C \int_{\mathbf{R}^N} |f|^p(x)\omega(x)dx$$

for all $f \in L^p(\omega(x)dx)$.

We will apply this lemma with $\omega = \Psi^{q-p}$ which turns out to satisfy (3.9) with a constant K independent of Ψ due to the fact that $\Psi = R_2 * f$, $f \geqslant 0$. Indeed, since $f \geqslant 0$, by Harnack's inequality, there exists $C = C(N)$ such that, for all cubes Q

$$(3.11) \qquad \inf_Q \Psi \geqslant C \fint_Q \Psi.$$

Therefore, since $\Psi \not\equiv 0$ and $q - p \leqslant 0$

$$\forall x \in Q \qquad \Psi^{q-p}(x) \leqslant \left[\inf_Q \Psi\right]^{q-p} \leqslant \left(C \fint_Q \Psi\right)^{q-p}$$

which after integration on Q implies

$$(3.12) \qquad \fint_Q \Psi^{q-p} \leqslant C^{q-p}\left[\fint_Q \Psi\right]^{q-p}$$

or

$$\left(\fint_Q \Psi^{q-p}\right)\left(\fint_Q \Psi\right)^{p-q} \leqslant C^{q-p}. \tag{3.13}$$

Now, since $1 \leqslant q \leqslant p$, Hölder's inequality implies

$$\fint_Q \Psi^{\frac{p-q}{p-1}} \leqslant \left(\fint_Q \Psi\right)^{\frac{p-q}{p-1}}.$$

Plugging this into (3.13), we have

$$\left(\fint_Q \Psi^{q-p}\right)\left(\fint_Q \Psi^{\frac{p-q}{p-1}}\right)^{p-1} \leqslant C^{q-p} \tag{3.14}$$

which is (3.9) with $\omega = \Psi^{q-p}$ and $K = C^{q-p}$.

We now apply (3.7), (3.8), (3.14) and lemma 3.2 to obtain

$$\int \Psi^q d\mu \leqslant k_3 C \int_{\mathbf{R}^N} f^p \Psi^{q-p}$$

which establishes (2.4)′ for all $f \in C_0^\infty(\mathbf{R}^N)$, $f \geqslant 0$. We finish with a density argument as in the preliminary remarks.

4. The proof of theorem 2.2.

We will prove (2.12) $\Rightarrow$ (2.9) $\Rightarrow$ (2.10) $\Rightarrow$ (2.12). The proof will then be complete since $-\Delta$ is a particular operator L. Let us start with the easy part.

Proof of (2.12) $\Rightarrow$ (2.9). — We denote by Ψ a fixed function in $C_0^\infty(\Omega)$ such that

$$0 \leqslant \Psi \leqslant 1, \Psi \equiv 1 \text{ on a neighborhood of the support of } \mu. \tag{4.1}$$

Let $K \subset \Omega$ be compact. By the definition of $c_{2,p}$, there exists a sequence $\varphi_n \in C_0^\infty(\mathbf{R}^N)$ such that

$$0 \leqslant \varphi_n \leqslant 1, \qquad \varphi_n \geqslant 1 \text{ on } K \tag{4.2}$$

$$\lim_{n\to\infty} \|\varphi_n\|_p^p = c_{2,p}(K). \tag{4.3}$$

We apply (2.12) with φ replaced by $\Psi^{2p}\varphi_n^{2p}$. Expanding $L^*\left(\Psi^{2p}\varphi_n^{2p}\right)$ gives various terms depending on $\varphi_{nx_ix_j}$, φ_{nx_i}, φ_n. Let us treat the main term as an example :

$$A=\sum_{i,j}\left[a_{ij}\left(\varphi_n^{2p}\right)_{x_j}\right]_{x_i}\Psi^{2p}=2p\sum_{i,j}\left[a_{ij}\varphi_n^{2p-1}\varphi_{nx_j}\right]_{x_i}\Psi^{2p}$$
$$=2p\Psi^{2p}\sum_{i,j}a_{ij}\varphi_n^{2p-1}\varphi_{nx_jx_i}+2p(2p-1)\Psi^{2p}\sum_{i,j}a_{ij}\varphi_n^{2p-2}\varphi_{nx_j}\varphi_{nx_i}$$
$$+2p\Psi^{2p}\sum_{i,j}(a_{ij})_{x_i}\varphi_n^{2p-1}\varphi_{nx_j}.$$

We deduce (using in particular $\Psi\leqslant 1$)

$$(4.4)\quad A^p[\varphi_n^{2p}\Psi^{2p}]^{1-p}\leqslant C(p,L^*)\left[\varphi_n^p\sum_{i,j}|\varphi_{nx_jx_i}|^p\right.$$
$$\left.+\varphi_n^p\sum_{i,j}|\varphi_{nx_j}|^p+|\nabla\varphi_n|^{2p}\right].$$

By Gagliardo-Nirenberg inequality [14],

$$(4.5)\qquad \int_\Omega|\nabla\varphi_n|^{2p}\leqslant C\|\varphi_n\|_{L^\infty}^p\sum_{i,j}|\varphi_{nx_ix_j}|^p.$$

Using (4.2) and (4.3) we then obtain

$$(4.6)\qquad \int_\Omega A^p\left[\varphi_n^{2p}\Psi^{2p}\right]^{1-p}\leqslant C\|\varphi_n\|_p^p\leqslant cc_{2,p}(K).$$

Others terms in $L^*(\Psi^{2p}\varphi_n^{2p})$ are treated in the same way and give an estimate similar to (4.6) with a constant C depending on the derivatives of Ψ. The inequality (2.9) follows.

Remark. — Note that the constant k_1 obtained depends on k_2, L, p as well as on the distance of the support of μ to the boundary of Ω.

Proof of (2.9) $\Rightarrow$ (2.10). — This is essentially the content of the classical capacitary strong type estimates ("weak $\Rightarrow$ strong" as proved in [8], [2]). We will not reproduce the proof here. Let us just indicate how the local version here can be deduced from the usual result for Bessel capacities [8] [2].

If μ satisfies (2.9), its extension by 0 outside to the whole space $\mathbb{R}^N$ satisfies the same inequality for all compact sets K in $\mathbb{R}^N$. By the results for Bessel capacities in [8], this implies the existence of a constant k such that

$$\forall \Theta \in W^{2,p}(\mathbb{R}^N) \cap C(\mathbb{R}^N) \quad \int_{\mathbb{R}^N} |\Theta|^p d\mu \leqslant k \int_{\mathbb{R}^N} |\Theta - \Delta\Theta|^p. \tag{4.7}$$

Let $\Psi \in C_0^\infty(\Omega)$ as in (4.1). For $\varphi \in Y(\Delta)$, we apply (4.7) to $\Theta = \varphi\Psi$ to obtain

$$\begin{aligned} \int_\Omega \varphi d\mu = \int_\Omega \varphi\Psi d\mu \leqslant k \int_\Omega |\varphi\Psi - \Delta(\varphi\Psi)|^p \\ \leqslant kC(\varphi) \int_\Omega |\Delta\varphi|^p + |\nabla\varphi|^p + \varphi^p. \end{aligned} \tag{4.8}$$

Since Ω is bounded, there exists C_1, C_2 depending on Ω and p such that (see e.g. [7])

$$\forall \varphi \in Y(\Delta) \int_\Omega \varphi^p \leqslant C_1 \int_\Omega |\nabla\varphi|^p \leqslant C_2 \int_\Omega |\Delta\varphi|^p. \tag{4.9}$$

Then (2.10) follows from (4.8) and (4.9).

Proof of (2.10) $\Rightarrow$ (2.12). — This is the main part of the proof. The main difficulty will be solved by using the singular integrals theory with A_p-weights as in theorem 2.1. But here there will be more technicalities due to the generality of L and to the necessity of working with A_p-weights in the whole space $\mathbb{R}^N$.

LEMMA 4.1. — *It is sufficient to prove (2.12) for operators L such that*

$$c = 0. \tag{4.10}$$

Proof of lemma 4.1. — Recall that

$$L^*\varphi = L_0^*\varphi + c\varphi$$

where

$$L_0^*\varphi = -\sum_{i,j} (a_{ij}\varphi_{x_j})x_i. \tag{4.11}$$

Then for $\varphi \in Y(L)$

$$\int_\Omega |L_0^*\varphi|^p \varphi^{1-p} \leqslant c_p \int_\Omega |L^*\varphi|^p \varphi^{1-p} + c_p \|c\|_\infty \int_\Omega \varphi. \tag{4.12}$$

Therefore, the proof of lemma 4.1 reduces to showing the existence of k such that

$$\forall \varphi \in Y(L) \int_\Omega \varphi \leqslant k \int_\Omega |L^*\varphi|^p \varphi^{1-p}. \tag{4.13}$$

(The difference between $Y(L_0)$ and $Y(L)$ is easily taken care of by a density argument.) In order to prove (4.13), let us set

$$w := |L^*\varphi|\varphi^{-1/p'}. \tag{4.14}$$

By assumptions (2.7), (2.8), L is an elliptic operator satisfying the maximum principle (see [7]). In particular, the mapping $f \to u$ where u is the solution of

$$\begin{cases} u \in W_0^{1,\infty}(\Omega) \cap W^{2,p}(\Omega) \\ Lu = f \text{ on } \Omega \end{cases} \tag{4.15}$$

is continuous from L^∞ into L^∞. By duality, $(L^*)^{-1}$ is continuous from L^1 into L^1, which implies the existence of $k = k(\Omega, L)$ such that (see 4.14)

$$\int_\Omega \varphi \leqslant k \int_\Omega |L^*\varphi| \leqslant k \int_\Omega w\varphi^{1/p'} \leqslant k \left[\int_\Omega w^p\right]^{1/p} \left[\int_\Omega \varphi\right]^{1/p'}. \tag{4.16}$$

This yields (4.13) and completes the proof of lemma 4.1.

We will now assume

$$c = 0, \qquad i.e.\ L^* = L_0^*. \tag{4.17}$$

We need to extend L^* to $\mathbb{R}^N$. For $\delta > 0$, we set

$$\Omega_\delta = \{x \in \mathbb{R}^N; d(x, \partial\Omega) < \delta\} \cup \Omega \quad (d = \text{euclidian distance}) \tag{4.18}$$

and we assume that (see (2.7), (2.8))

$$\Omega_{2\delta} \subset\subset \omega. \tag{4.19}$$

We extend L^* to $\mathbb{R}^N$ as follows. For $\varphi \in C_0^\infty(\mathbb{R}^N)$, we set

$$B\varphi = -\sum_{i,j} (b_{ij}\varphi_{x_i})_{x_j} \tag{4.20}$$

where

$$\begin{cases} b_{ij}(x) = a_{ij}(x) & \forall x \in \overline{\Omega} \\ b_{ij}(x) = \delta_{ij} & \forall x \notin \Omega_\delta\,. \\ b_{ij}(x) = \delta^{-1}[d(x,\partial\Omega)\delta_{ij} + (\delta - d(x,\partial\Omega))a_{ij}(x)] & \forall x \in \Omega_\delta \setminus \Omega \end{cases} \tag{4.21}$$

Here the b_{ij} are Lipschitz continuous on $\mathbf{R}^N$ and they satisfy the ellipticity property (2.8) with α_0 replaced by $\min(\alpha_0, 1)$. Note that

$$\begin{cases} B = L^* \text{ on } \overline{\Omega} \\ B = -\Delta \text{ outside } \Omega_\delta \end{cases}. \tag{4.22}$$

We will also use the following expanded form of B

$$B\varphi = -\sum_{i,j} b_{ij}\varphi_{x_i x_j} + \sum_i c_i \varphi_{x_i} \tag{4.23}$$

where

$$c_i = -\sum_j b_{ij x_j} \in L^\infty(\Omega). \tag{4.24}$$

LEMMA 4.2. — *For all $\varphi \in Y(L)$, there exists Ψ such that*

$$\Psi \in W^{1,2}_{\text{loc}}(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N) \tag{4.25}$$

$$B\Psi = \begin{cases} |L^*\varphi| \text{ on } \Omega \\ 0 \text{ outside } \Omega \end{cases} \tag{4.26}$$

$$\Psi \geqslant \varphi \text{ on } \Omega. \tag{4.27}$$

Moreover, for all $q > 1$, there exists $C = C(B, N, q)$ such that

$$\left[\int_{\mathbf{R}^N} \Psi^{qN/(N-2)}\right]^{(N-2)/N} \leqslant \int_\Omega |L^*\varphi|^q \tag{4.28}$$

and there exists $c = c(B, N)$ such that

$$\inf_Q \Psi \geqslant c \fint_Q \Psi \qquad \forall Q \text{ cube in } \mathbf{R}^N. \tag{4.29}$$

Proof of Lemma 4.2. — Let $\varphi \in Y(L)$ and $f = \begin{cases} |L^*\varphi| \text{ on } \Omega \\ 0 \text{ outside } \Omega \end{cases}$. We set $B_s = \{x \in \mathbf{R}^N : |x| < s\}$ and for s large enough so that $\overline{\Omega}_{2\delta} \subset B_s$, we solve

$$\begin{cases} \Psi_s \in W^{1,2}_0(B_s) \\ B\Psi_s = f \text{ on } B_s \end{cases}. \tag{4.30}$$

It is classical (see e.g. [7]) that Ψ_s exists and by maximum principle that

$$s \to \Psi_s \text{ is increasing and } \Psi_s \geqslant \varphi \text{ on } \Omega. \tag{4.31}$$

To bound Ψ_s from above, multiply (4.30) by Ψ_s^{q-1}, $q > 1$ to obtain

$$(q-1)\int_{\mathbb{R}^N} \sum_{i,j} b_{ij}\Psi_{sx_i}\Psi_{sx_j}\Psi_s^{q/2} = \int_\Omega \Psi_s^{q-1} f \tag{4.32}$$

$$(q-1)\min(1,\alpha_0)\int_{\mathbb{R}^N} |\nabla\Psi_s^{q/2}|^2 \leqslant \int_\Omega \Psi_s^{q-1} f. \tag{4.33}$$

By Sobolev's imbedding theorem and Hölder's inequality, there exists $C = C(q, B, N)$ such that

$$C\left[\int_{\mathbb{R}^N} \Psi_s^{qN/(N-2)}\right]^{\frac{N-2}{N}} \leqslant \left[\int_\Omega \Psi_s^{qN/(N-2)}\right]^{1/r}\left[\int_\Omega f^s\right]^{1/s} \tag{4.34}$$

$$\text{with } r = qN/(q-1)(N-2), \qquad s = qN/(N+2(q-1)). \tag{4.35}$$

This estimate together with (4.31) proves that Ψ_s converges to some Ψ satisfying the same inequality (4.34) and such that $\nabla\Psi^{q/2}$ is in $L^2(\mathbb{R}^N)$ by (4.33). Moreover Ψ is solution of (4.26) by passing to the limit in (4.30) and $\Psi \geqslant \varphi$ on Ω by (4.31). The fact that $\Psi \in L^\infty(\mathbb{R}^N)$ is a consequence of the maximum principle and $f \in L^\infty(\Omega)$.

Finally, since

$$B\Psi \geqslant 0 \text{ on } \mathbb{R}^N \tag{4.36}$$

and because of the structure of B, Ψ satisfies the one-sided Harnack inequality (4.29) (see [7] th. 8.18).

LEMMA 4.3. — *Let $\omega \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, $\omega \geqslant 0$, satisfying (3.9). Then there exists $C = C(p, B, K)$ such that for all $\Psi \in W^{2,\infty}(\mathbb{R}^N)$*

$$\int_{\mathbb{R}^N} |\Psi_{x_i x_j}|^p\omega \leqslant C\int_{\mathbb{R}^N} |B\Psi|^p\omega + \int_{\Omega_{2\delta}} (|\nabla\Psi|^p + \Psi^p)\omega. \tag{4.37}$$

Proof of lemma 4.3. — If B was equal to Δ on the whole space $\mathbb{R}^N$, this would be the classical weighted L^p-estimate for singular integrals except that the last integral on $\Omega_{2\delta}$ would not be needed (see Coifman-Fefferman [6]). Using the continuity of the b_{ij} and (4.23), we can extend it to B. Let us indicate how.

We set $$M = \max_{i,j} \|b_{ij}\|_{L^\infty(\mathbb{R}^N)}.$$

If b_{ij} are constant on $\mathbb{R}^N$, by a change of coordinates, from [6] we have the existence of $C = C(\alpha_0, M, p, K)$ such that

$$\int_{\mathbb{R}^N} |\Psi_{x_i x_j}|^p \omega \leqslant C \int_{\mathbb{R}^N} |B\Psi|^p \omega. \tag{4.38}$$

Now, for $\varepsilon > 0$ given, by continuity of the b_{ij}, there exist $\eta > 0$ and $\{x_k, \Omega_k\}_{k=1,\ldots,q}$ such that

$$x_k \in \overline{\Omega}_\delta, \Omega_k = \{x \in \mathbb{R}^N : |x - x_k| < \eta\}, \quad \eta \leqslant \delta \tag{4.39}$$

$$\overline{\Omega}_\delta \subset \bigcup_{k=1}^{q} \Omega_k \subset \overline{\Omega}_{2\delta} \tag{4.40}$$

$$\sup_{x \in \Omega_k} |b_{ij}(x) - b_{ij}(x_k)| < \varepsilon \text{ for all } k, i, j. \tag{4.41}$$

We denote $\Omega_0 = \mathbb{R}^N \setminus \overline{\Omega}_\delta$. We introduce a partition of unity ε_k, $k = 0, 1, \ldots, q$ subordinated to $\{\Omega_k\}_{k=0,1,\ldots,q}$ and with C^∞-functions. For $\Psi \in W^{2,\infty}(\mathbb{R}^N)$, we write

$$\Psi = \sum_{k=0}^{p} \varepsilon_k \Psi = \sum_{k=0}^{p} \Psi_k. \tag{4.42}$$

If $B_k = \sum_{i,j} b_{i,j}(x_k) \dfrac{\partial^2}{\partial_{x_i x_j}}$, by (4.38) we have

$$\int_{\mathbb{R}^N} |\Psi_{k x_i x_j}|^p \omega \leqslant C \int_{\mathbb{R}^N} |B_k \Psi_k|^p \omega \quad (C = C(\alpha_0, M, p, K)). \tag{4.43}$$

But

$$B_k \Psi_k = B\Psi_k + \sum_{i,j} (b_{ij}(x_k) - b_{ij}(x)) \Psi_{k x_i x_j} - \sum_i c_i \Psi_{k x_i}. \tag{4.44}$$

Since $\Psi_k = \varepsilon_k \Psi$ is supported in Ω_k, (4.41), (4.43), (4.44) imply

$$\max_{i,j} \int_{\mathbb{R}^N} |\Psi_{k x_i x_j}|^p \omega \leqslant C \left[\int_{\mathbb{R}^N} |B\Psi_k|^p \omega + \varepsilon^p \max_{i,j} \int_{\mathbb{R}^N} |\Psi_{k x_i x_j}|^p \omega \right.$$
$$\left. + \int_{\Omega_{2\delta}} |\nabla \Psi_k|^p \omega \right].$$

If ε has been chosen so that $C\varepsilon^p \leqslant \frac{1}{2}$, that is $\varepsilon = \varepsilon(\alpha_0, M, p, K)$, we have

$$\max_{i,j} \int_{\mathbb{R}^N} |\Psi_{k x_i x_j}|^p \omega \leqslant 2C \left[\int_{\mathbb{R}^N} |B\Psi_k|^p \omega + \int_{\Omega_{2\delta}} |\nabla \Psi_k|^p \omega \right]. \tag{4.45}$$

Now, we use

$$B\Psi_k = \varepsilon_k B\Psi + \Psi B\varepsilon_k + \sum b_{ij}(\varepsilon_{k_{x_j}}\Psi_{xi} + \varepsilon_{k_{x_i}}\Psi_{xj}).$$

Plugging this in (4.45) and using (4.39)-(4.42), we have (4.37). (Note that $\varepsilon_0 \equiv 1$ outside $\overline{\Omega}_{2\delta}$ by (4.40) (4.42)).

End of the proof of (2.10) ⇒ (2.12). — Let $\Theta \in C_0^\infty(\Omega)$ such that

(4.46) $\quad 0 \leqslant \Theta \leqslant 1, \Theta \equiv 1$ on a neighborhood of the support of μ.

Let now $\varphi \in Y(L)$ and let Ψ be the function associated with φ by lemma 4.2. Applying (2.10) with φ replaced by $\Theta\Psi^{1/p}$ gives (recall (4.27))

$$(4.47)\quad \begin{cases} \int_\Omega \varphi d\mu = \int_\Omega \varphi\Theta^p d\mu \leqslant \int_\Omega \Psi\Theta^p d\mu \leqslant k_2 \int_\Omega |\Delta(\Theta\Psi^{1/p})|^p \\ \leqslant C(\Theta,p)\int_\Omega \Psi + |\nabla\Psi|^p\Psi^{1-p} + |\nabla\Psi|^{2p}\Psi^{1-2p} + |\Delta\Psi|^p\Psi^{1-p} \end{cases}.$$

As in the proof of theorem 2.1, we use Harnack's inequality (4.29) to prove that $\omega = \Psi^{1-p}$ satisfies (3.9) with a constant K depending only on B and N. The, by lemma 4.3, we obtain that

$$(4.48)\quad \int_\Omega |\Psi_{x_ix_i}|^p\Psi^{1-p} \leqslant C\left[\int_\Omega |L^*\varphi|^p\varphi^{1-p} + \int_{\Omega_{2\delta}} |\nabla\Psi|^p\Psi^{1-p} + \Psi\right].$$

Now, using (4.47), (4.48) and lemma 4.3, the proof will be complete after proving the three following lemmas.

LEMMA 4.4. — *There exists $C = C(p)$ such that*

$$(4.49)\quad \int_{\mathbf{R}^N} |\nabla\Psi|^{2p}\Psi^{1-2p} \leqslant C\sum_i \int_{\mathbf{R}^N} |\Psi_{x_ix_i}|^p\Psi^{1-p}.$$

LEMMA 4.5. — *There exists $C = C(B, p, \Omega_\delta)$ such that*

$$(4.50)\quad \int_{\Omega_{2\delta}} \Psi \leqslant \int_\Omega |L^*\varphi|^p\varphi^{1-p}.$$

LEMMA 4.6. — *There exists $C = C(B, p, \Omega_\delta)$ such that*

$$(4.51)\quad \int_{\Omega_{2\delta}} |\nabla\Psi|^p\Psi^{1-p} \leqslant C\int_\Omega |L^*\varphi|^p\varphi^{1-p}.$$

Indeed, from (4.47), (4.48) we obtain

$$\int_\Omega \varphi d\mu \leqslant C\left[\int_\Omega |L^*\varphi|^p\varphi^{1-p} + \int_{\Omega_{2\delta}} |\nabla\Psi|^p\Psi^{1-p} + \Psi + \int_\Omega |\nabla\Psi|^{2p}\Psi^{1-2p}\right]; \tag{4.52}$$

We now use (4.49) from lemma 4.4 and (4.48) to deduce from (4.52)

$$\int_\Omega \varphi d\mu \leqslant C\left[\int_\Omega |L^*\varphi|\varphi^{1-p} + \int_{\Omega_{2\delta}} |\nabla\Psi|^p\Psi^{1-p} + \Psi\right]. \tag{4.53}$$

Finally we use lemmas 4.5 and 4.6 to obtain (2.12) from (4.53).

Proof of lemma 4.4. — We use an integration by parts as in [10], [1] to which we refer for the details

$$\begin{aligned}\int_{\mathbf{R}^N} |\Psi_{x_i}|^{2p}\Psi^{1-2p} &= \int_{\mathbf{R}^N} \Psi_{x_i}\Psi_{x_i}|\Psi_{x_i}|^{2p-2}\Psi^{1-2p}\\ &= -\int_{\mathbf{R}^N} \Psi\,\frac{\partial}{\partial x_i}\,[\Psi_{x_i}|\Psi_{x_i}|^{2p-2}\Psi^{1-2p}]\\ &= -(2p-1)\int_{\mathbf{R}^N} \Psi^{2-2p}|\Psi_{x_i}|^{2p-2}\Psi_{x_ix_i}\\ &\quad -(1-2p)\int_{\mathbf{R}^N} \Psi^{1-2p}|\Psi_{x_i}|^{2p}.\end{aligned}$$

$$2(p-1)\int_{\mathbf{R}^N} |\Psi_{x_i}|^{2p}\Psi^{1-2p} \leqslant (2p-1)\left[\int_{\mathbf{R}^N} |\Psi_{x_ix_i}|^p\Psi^{1-p}\right]^{1/p} \left[\int_{\mathbf{R}^N} |\Psi_{x_i}|^{2p}\Psi^{1-2p}\right]^{1/p'}.$$

Estimate (4.49) follows with $C = [(2p-1)/2(p-1)]^p$.

Proof of lemma 4.5. — Set $w = |L^*\varphi|\varphi^{-1/p'}$. By (4.28) and (4.27), for all $q \in (1, q)$, we have

$$\left[\int_{\Omega_{2\delta}} \Psi^{qN/(N-2)}\right]^{(N-2)/N} \leqslant C\int_\Omega w^q\varphi^{q/p'} \leqslant C\left[\int_\Omega w^p\right]^{q/p}\left[\int_\Omega \Psi^r\right]^{(p-q)/p}$$

for $r = q(p-1)/(p-q)$. We choose q so that $r = qN/(N-2)$, that is $q = 1 + 2(p-1)/N$ ($< p$). We then have

$$\int_{\Omega_{2\delta}} \Psi \leqslant C(\Omega)\left[\int_{\Omega_{2\delta}} \Psi^{qN/(N-2)}\right]^{(N-2)/Nq} \leqslant C\int_\Omega w^p \leqslant C\int_\Omega |L^*\varphi|^p\varphi^{1-p}.$$

Proof of lemma 4.6. — We apply Young's inequality to obtain

$$\int_{\Omega_{2\delta}} |\nabla\Psi|^p\Psi^{1-p} \leqslant \varepsilon \int_{\mathbf{R}^N} |\nabla\Psi|^{2p}\Psi^{1-2p} + C_\varepsilon \int_{\Omega_{2\delta}} \Psi.$$

The first integral on the right-hand side is estimated by (4.49) and (4.48), and the last one by (4.50). We deduce

$$\int_{\Omega_{2\delta}} |\nabla\Psi|^p\Psi^{1-p} \leqslant C_\varepsilon \left[\int_{\mathbf{R}^N} |L^*\varphi|^p\varphi^{1-p} + \int_{\Omega_{2\delta}} |\nabla\Psi|^p\Psi^{1-p}\right] + C_\varepsilon \int |L^*\varphi|^p\varphi^{1-p}.$$

Choosing ε small enough depending on C yields (4.51).

5. The proof of corollaries.

Proof of corollary 2.4. — By the results in [5], if (2.13) has a solution, then (2.11) holds with p replaced by γ' and $\lambda k_3 \leqslant (\gamma-1)/\gamma^{\gamma'}$. By theorem 2.2, (2.12) then holds with some constant k_4 and $p = \gamma'$. Again by the results in [5], (2.14) will have a solution if $\lambda k_4 \leqslant (\gamma-1)/\gamma^{\gamma'}$.

The converse is obtained in the same way.

Proof of corollary 2.5. — The sufficiency of (2.17) is proved in [4]. The necessity is obtained as indicated in the remark following corollary 2.5.

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