ANNALES DE L'INSTITUT FOURIER

MA LI

On equivariant harmonic maps defined on a Lorentz manifold

Annales de l'institut Fourier, tome 41, nº 2 (1991), p. 511-518

<http://www.numdam.org/item?id=AIF_1991__41_2_511_0>

Ann. Inst. Fourier, Grenoble
42, 2 (1991), 511-518.

ON EQUIVARIANT HARMONIC MAPS DEFINED ON A LORENTZ MANIFOLD

by MA LI

1. Introduction.

It is interesting to study harmonic maps from a Lorentz manifold into a Riemannian manifold. In this case, the harmonic map equation is a Hyperbolic system of second order. In this paper, we look for equivariant harmonic maps defined on a specific Lorentz manifold ; namely, the Lorentz manifold $M = M_0 \times R$ with the space-time metric

$$ds^2 = dt^2 - S^2(t)\, d\sigma^2$$

where $(M_0, d\sigma^2)$ be the symmetric space for a compact Lie group G with a bi-invariant Riemannian metric $d\sigma^2$ and $S(t)$ is a smooth positive function defined on R. The target manifold is a compact Riemannian manifold (N,h) admitting an isometric group action of G. This kind of problem is called a σ-model in Physics literature and one may see [G] and [EL] for further datum. Without loss of generality, we may assume that N is a submanifold of some Euclidean space R^k by Nash's isometrical imbedding theorem, so we may think of G as $\subset SO(k)$ with its Lie algebra $LG \subset so(k)$ the Lie algebra of $SO(k)$ whose elements are skew-matrices.

By definition, a smooth map u from M to N is called a harmonic map if it is a critical point of the following action integral

$$\begin{aligned} E_I(u) &= \int_{M_0 \times I} (Tr_{ds^2} u^* h)\, S^n(t)\, d\mu\, dt \\ &= \int_I \int_{M_0} (|\partial_t u|^2 - S^{-2}(t)|\nabla_0 u|^2)\, S^n(t)\, d\mu\, dt \end{aligned}$$

Key-words : Equivariant - Harmonic map - Minimax principle - Lorentz manifold.
A.M.S. Classification : 53C50 - 58E05 - 58E10 - 58F21.

for every interval $I = [a,b] \subset R$ among all maps of its class, here $n = \dim M_0$, $d\mu$ is the invariant measure of $(M_0, d\sigma)$, $|\cdot|$ is the usual norm induced by R^k and ∇_0 is the covariant derivative induced by $d\sigma^2$ on M_0.

We will prove the following

THEOREM. — *Let M and N be the manifold defined above. Suppose $S(t)$ is a smooth positive periodic function of period 2π, then, there exist infinitely many G-equivariant harmonic maps which are of period 2π in t from M to N.*

By equivariant, we mean that the map $u : M \to N$ satisfies

$$u(g \cdot m, t) = g \cdot u(m,t)$$

for every $g \in G$ and $(m,t) \in M_0 \times R$. We denote the set of equivariant maps $\mathcal{M}$ and it is non-empty by our assumptions on M and N. Select a basis $\{e_j\}_{j=1}^n$ (note $n = \dim M_0 = \dim_{\mathbb{R}} G$) of the Lie group G and let $\{A_j\}_{j=1}^n$ denote the corresponding basis of its Lie algebra. Fix $m \in M_0$ and write $x(t) = u(m,t)$. Because u is an equivariant map, $u(\exp(sA_j)m,t) = \exp(sA_j)\, u(m,t)$. Differentiating it *w.r.t.* s at $s = 0$ we get that $\nabla_0 u(m,t)(A_j) = A_j u(m,t)$ (matrix multiplication in R^k). From this and the invariance of the metric $d\sigma^2$, the action integral $E_I(\cdot)$ for the G-equivariant map u becomes

$$\begin{aligned} E_I(u) &= \int_{M_0} d\mu \int_I \Big(|u_t(m,t)|^2 - S^{-2}(t) \sum_{j=1}^{n} |A_j(u(m,t))|^2\Big) S^n(t)\, dt \\ &= \mathrm{Vol}(M_0)\, F_I(x), \end{aligned}$$

where the last integral factor $F_I(x)$ will be written as $F(x)$ when $I = S^1$.

It will be shown by the minimax principle that there exist infinitely many critical points of $F(\cdot)$ just like closed geodesics in N. But here we should mention that it is conceptually different from the closed geodesic case because the Euler-Lagrange equation for our $F(\cdot)$ is a non-autonomous one (see Lemma 2 below).

2. Some well-known facts.

Since the A_j is a skew-symmetric matrix, there exists a non-negative symmetric matrix A such that

$$\sum_{j=1}^{n} A_j^2 = -\sum_{j=1}^{n} A_j A_j^* = -A^2 .$$

So

$$(1) \qquad F(x) = \int_0^{2\pi} (|x'(t)|^2 - S^{-2}(t)|Ax(t)|^2)\, S^n(t)\, dt.$$

Think of $M_0 \times S^1$ as a Riemannian manifold with the metric $dt^2 + S^2(t)\, d\sigma^2$, we may define a Hilbert manifold $H = W^{l,2}(M_0 \times S^1, N)$ for l large enough. Now, H admits an isometric group action $(u,g) \to g^{-1} \cdot u \cdot g$ of G. Applying the theorem in page 23 of R. S. Palais [P2] to F on H and to the fixed point set of the map $u \to g^{-1} \cdot u \cdot g$, we find

LEMMA 1. — *If $u \in \mathcal{M}$, then u is harmonic if and only if $x(t) = u(m,t)$ is a critical point for $F_I(x)$ for all intervals $I \subset R$.*

Let $\mathcal{O}$ be an open uniform tubular neighborhood of N in R^k such that the $P : \mathcal{O} \to N$ given by $P(y) =$ the nearest point in N to y, is a smooth fibration.

LEMMA 2. — *The Euler-Lagrange equations for an equivariant harmonic map from M to N are*

$$(2) \qquad S^{-n}(t)(S^n(t)x')' - D^2P(x', x') + S^{-2}(t)A^2 x = 0,$$

which is a non-autonomous system except if $S(t) =$ const.

Proof. — Suppose x is the critical point of $F(\cdot)$ which corresponds to the equivariant harmonic map we consider. For $\eta \in W^{1,2}(S^1, R^k)$, if $\varepsilon > 0$ is small enough, we have that $P(x(\cdot) + s\eta(\cdot))$ is a smooth curve in $W^{1,2}(S^1, N) := \{y \in W^{1,2}(S^1, R^k);\, y(t) \in N\}$ passing through x for $s \in (-\varepsilon, \varepsilon)$. Hence

$$\begin{aligned}
0 &= 2^{-1}\, d/ds\,|_{s=0}\, F(P(x+s\eta)) \\
&= \frac{1}{2}\frac{d}{ds}\bigg|_{s=0} \int_{S^1} |DP_{x+s\eta} \cdot (x'(t) + s\eta'(t))|^2\, S^n(t)\, dt \\
&\quad - \int_{S^1} |AP(x(t) + s\eta(t))|^2\, S^{n-2}(t)\, dt \\
&= \int_{S^1} (\langle DP_x . x'(t), D^2P_x(x'(t), \eta(t)) + DP_x . \eta'(t) \rangle)\, S^n(t)\, dt \\
&\quad - \int_{S^1} \langle A^2 x(t), \eta(t) \rangle\, S^{n-2}(t)\, dt
\end{aligned}$$

$$= \int_{S^1} \langle x'(t), \eta'(t) \rangle \, S^n(t) \, dt$$

$$+ \int_{S^1} (\langle D^2 P_x(x'(t), x'(t)) - S^{-2}(t) A^2 x(t), \eta(t) \rangle) \, S^n(t) \, dt .$$

Since $P(x) = x$, we have that $DP_x(x') = x'$. So by integration by part we get (2). Since, for $S(\cdot) \neq$ const.,

$$(|x'(t)|^2 + S^{-2}(t) |Ax(t)|^2) \, S^n(t)$$

is not conserved, (2) is a non-autonomous system. □

Define

$$\Lambda^1 = \Lambda^1(N) = W^{1,2}(S^1, N) .$$

It is well-known that $\Lambda^1(N)$ is a Hilbert manifold [P1]. Since N is compact, there exist constants $c_i > 0 \, (i = 1, 2, 3)$ such that

$$c_1 D(y) - c_3 \leqslant F(y) \leqslant c_2 D(y) - c_3 , \tag{3}$$

here $D(y) := |y|_1^2 = \int_{S^1} |y'|^2$ for every $y \in \Lambda^1$. We will also need the following inequality

$$|y|_\infty \leqslant |y(0)| + c_4 |y|_1 , \tag{4}$$

for every $y \in \Lambda^1$ and the Sobolev imbedding $W^{1,2}(S^1, R^k) \to C^0(S^1, R^k)$ is compact.

LEMMA 3. — i) *$F(\cdot)$ satisfies Palais and Smale condition C;*

ii) *For every $c > 0$, there exists an integer $\bar{n} = \bar{n}(c)$ such that*

$$H^n(I_c) = 0$$

for $n > \bar{n}$, where $I_c = D^{-1}(-\infty, c]$.

Proof. — i) Suppose $\{x_m\} \subset \Lambda^1$ is a sequence such that

$$F(x_m) \to c \tag{5}$$

and

$$dF(x_m) \to 0 , \quad \text{in } H^{-1} . \tag{6}$$

Since N is compact, we may assume that $x_m(0) \to p$. By inequality (3), we get that, there exist a constant $C > 0$ such that

$$D(x_m) \leqslant C. \tag{7}$$

So we may assume that $x_m \to x$ in $C^0(S^1, N)$. Now,

$$\langle dF(x_m), \eta \rangle = 2\int_{S^1} (\langle x', \eta' \rangle + \langle D^2P(x', x'), \eta \rangle - S^{-2}(t)\langle A^2x, \eta \rangle)\, S^n(t).$$

Take $\eta = x_m - x_n$ and $x = x_m,\ x_n$ in (6), we get by (7) and (4) that

$$\begin{aligned} o(1) &= 2^{-1}\langle dF(x_m) - dF(x_n), x_m - x_n \rangle \\ &\geqslant c_1 D(x_m - x_n) - 2C|x_m - x_n|_\infty \\ &\quad - \int_{S^1} S^{n-2}(t)\langle A^2(x_m - x_n), x_m - x_n \rangle \\ &\geqslant c_1 D(x_m - x_n) - 2C_5 o(1). \end{aligned}$$

Here we implicity used boundness of the positive function $S(t)$. Hence, $D(x_m - x_n) = o(1)$.

ii) This is borrowed from Milnor's book (see theorem 16.2 in [M]). Since I_c is a strong deformation retract of a finite dimensional manifold, whose dimension n depends on c, then, we get the conclusion if we let $\bar{n}(c) = n$. □

Now, let us recall a result of M. Vigue-Poirrier and D. Sullivan [V-P S] about the topology of Λ^1.

PROPOSITION 4. — *If N is compact and simply connected, then there exists an infinite set of positive integers $\mathbb{M} \subset \mathbb{N}$ such that*

$$H^q(\Lambda^1) \neq 0$$

for $q \in \mathbb{M}$.

3. Final argument.

Consider a non-trivial $\alpha \in H^*(\Lambda^1)$ and set

$$\bar{\alpha} = \{B \subset \Lambda^1; i_B^*(\alpha) \neq 0\}, \tag{8}$$

where

$$i_B^* : H^*(\Lambda^1) \to H^*(B)$$

is the homomorphism induced by the inclusion

$$i_B : B \to \Lambda^1 .$$

Remark 5. — $\bar{\alpha}$ defined in (8) is non-empty and contains the compact support of a k-chain $a \in \alpha$, $k = \deg \alpha$, which is not homologous to constant by the nontrivial property of α.

LEMMA 6. — *Let* $\alpha \in H^*(\Lambda^1)$, $\alpha \neq 0$ *and define*

$$c_\alpha = \inf_{B \in \bar{\alpha}} \sup F(B) . \tag{9}$$

Then, c_α *is a critical value of* F *on* Λ^1; *moreover, if we assume that* $H^q(\Lambda^1) \neq 0$ *for infinitely many* q, *there exists a sequence* $\{c_\alpha\}$ *of critical values of* F *defined as in* (9) *which satisfies that*

$$c_\alpha \to +\infty , \qquad \textit{as} \deg \alpha \to +\infty . \tag{9'}$$

Proof. — By our Remark 5 we have

$$c_\alpha < +\infty .$$

Suppose some c_α is not a critical value of F, then by lemma 3 i) and a well-known deformation lemma in page 125 of R. S. Palais [P1], we know that there exists a positive number ε and a homeomorphism η on Λ^1 such that

$$\eta(F^{-1}_{c_\alpha+\varepsilon}) \subset F^{-1}_{c_\alpha-\varepsilon} . \tag{10}$$

Since

$$\eta^* : H^q(\eta(\Lambda^1)) \to H^q(\Lambda^1)$$

is an isomorphism, we have that

$$i^*_{\eta(B)}(\alpha) = (\eta^*)^{-1} \cdot i_B(\alpha) \neq 0$$

for all $B \in \bar{\alpha}$. Hence η leaves $\bar{\alpha}$ invariant. But, by the definition of c_α, there exists $B \in \bar{\alpha}$ such that

$$\sup \mathrm{F}(B) < c_\alpha + \varepsilon .$$

So by (10) and $\eta(B) \in \bar{\alpha}$ we have

$$\sup F(\eta(B)) < c_\alpha - \varepsilon .$$

It is absurd.

To get (9′), we take $k \in \mathbb{N}$. By lemma 3 ii), there exists $\bar{n} = \bar{n}(k) \in \mathbb{N}$ such that $H^q(I_k) = 0$ for $q > \bar{n}$. By our assumption on $H^*(\Lambda^1)$ we may take $q_k > \bar{n}$ with $H^{q_k}(\Lambda^1) \neq 0$ and consider $\alpha \in H^{q_k}(\Lambda^1)$, $\alpha \neq 0$. Denote

$$I^k = \{x \in \Lambda^1 ; D(x) > k\},$$

we claim that

$$\forall B \in \bar{\alpha},\, B \cap I^k \neq 0. \tag{11}$$

Suppose it is not true, then, there exists $B \in \bar{\alpha}$ such that

$$B \subset \Lambda^1 \backslash I^k := I_k,$$

then

$$H^{q_k}(\Lambda') \xrightarrow{i_2^*} H^{q_k}(I_k) \xrightarrow{i_1^*} H^{q_k}(B), \tag{12}$$

where i_2^*, i_1^* are the homomorphisms induced by the inclusion maps

$$i_2 : I_k \to \Lambda^1, \qquad i_1 : B \to I_k .$$

Then, by $B \in \bar{\alpha}$ we have that

$$i_1^* \cdot i_2^*(\alpha) = i_B^*(\alpha) \neq 0 . \tag{13}$$

From (12) and (13) we obtain that $H^{q_k}(I_k) \neq 0$, a contradiction to our assumption on q_k. So (11) is true.

By (11) and our choices of c_α we have that

$$c_\alpha \geqslant c_1 k - C$$

which implies our conclusion. □

Proof of Theorem. — 1) If N is simply-connected, then the result follows from Proposition 4 and Lemma 6.

2) If $\pi_1(N) \neq 0$ and finite. Then the universal covering $(\tilde{N}, \Pi)$ is compact. By 1) we have infinitely many critical points $\tilde{x}_n : S^1 \to \tilde{N}$ of F such that

$$F(\tilde{x}_n) \to +\infty, \qquad \text{as } n + \infty .$$

Therefore, set $x_n = \Pi(\tilde{x}_n)$, we obtain the existence of infinitely many critical points of F, and infinitely distinct harmonic maps of periodic 2π in t from M to N by Lemma 1.

3) If $\pi_1(N) = \infty$. We may get a minimizer of F in each homotopy class by the Palais-Smale condition in lemma 3 i). □

Remark 7. – (1) Suppose $S(t)$ is not periodic in t. Take $I = [0,1]$, $x(0)$ and $x(1)$ two point in N, we can prove as in our theorem that there are infinitely many geometrical distinct critical points of F. It is interesting to consider the behavior of the orbit of some critical point of F just like that of the geodesic in N.

(2) It is an open question to obtain our theorem when $S(t) = 1 - \cos(t)$. In this case, the Lorentz manifold M is called Friedman-Robertson-Walker space-time in general relativity.

Acknowledgement. – The paper was prepared when the author was a visiting member of ICTP at Trieste, the author would like to thank Professors J. Eells Jr and A. Verjovsky for their hospitality. The author would like to thank the referee for his comments on the first version of this paper.

BIBLIOGRAPHY

[E] J. Eells Jr, Proc 1981 Shanghai-Hefei Symps. Diff. Geom. Diff. Eq., Sci. Press, Beijing, (1984), 55-73.

[EL] J. Eells Jr and L. Lemaire, Another Report on Harmonic Maps, Bull. London Math. Soc., 20 (1988), 385-524.

[G] Gu Chao-hao, On the Two-dimensional Minkowski space, Comm. Pure and Appl. Math., 33 (1980), 727-738.

[M] J. Milnor, Morse Theory, Princeton, 1963.

[P1] R. S. Palais, Lusternik-Schnirelmann theory on Banach Manifold, Topology, 5 (1966), 115-132.

[P2] R. S. Palais, The Principle of Symmetric Criticality, Comm. Math. Phys., 69 (1979), 19-30.

[V-PS] M. Vigue-Poirrier, D. Sullivan, The Homology Theory of the Closed Geodesic Problem, J. Diff. Geom., 11 (1976), 633-644.

Manuscrit reçu le 16 mai 1990,
révisé le 5 avril 1991.

Ma Li,
Peking university,
Dept of Mathematics
Institute of Mathematics
Beijing 100871 (Rép. Pop. Chine).