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FIBRATION OF THE PHASE SPACE FOR THE KORTEWEG-DE VRIES EQUATION

by Thomas KAPPELER(*)

1. Introduction and summary of the results.

It is well known that the Korteweg-de Vries equation (KdV) $u_t + u_{xxx} + uu_x = 0$, considered on the circle, is a completely integrable, infinite dimensional Hamiltonian system. The periodic eigenvalues of the Schrödinger operator $-y'' + u(\bullet, t)y = \lambda y$ are invariant under the flow by KdV and give a complete set of conserved quantities. Thus the level sets of KdV are the isospectral sets $\operatorname{Iso} q$ of potentials, where $\operatorname{Iso} q$ consists of all potentials p such that $-d_x^2 + p$ and $-d_x^2 + q$ have the same periodic spectrum. These isospectral sets are compact and connected and are generically an infinite product of circles.

For finite dimensional completely integrable Hamiltonian systems with regular compact, connected level sets, Liouville's theorem implies that the phase space is fibrated by the level sets. I would like to examine in which sense this result can be generalized to KdV and what are the global properties of this fibration. Taking various properties of isospectral sets into account, I introduce for this purpose a model space, $\mathcal{M}$, consisting of sequences $R = (R_k)_{k\geq 1}$ of 2×2, symmetric, trace free matrices with $\sum_{k\geq 1} \|R_k\|^2 < \infty$. For $R = (R_k)_{k\geq 1}$ in $\mathcal{M}$, denote by $\operatorname{Iso} R := \{(R'_k)_{k\geq 1}$:

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$\operatorname{spec} R'_k = \operatorname{spec} R_k, \forall k \geq 1\}$. It is immediate that $\operatorname{Iso} R$ is compact, connected and, generically, an infinite product of circles. In this paper I prove that the space of L^2-potentials with average 0 can be mapped into the model space by a real analytic isomorphism Φ with $\Phi(\operatorname{Iso} q) = \operatorname{Iso}(\Phi(q))$. This shows that the infinite dimensional fibration by isospectral sets of potentials is trivial. Recall that the phase-space of KdV can be chosen to be $C_0^\infty(S^1) := \{q \in C^\infty(S^1) : \int_0^1 q(x)dx = 0\}$ with symplectic structure given by $\frac{\partial}{\partial x}$. Thus a C^∞-version of the above result would be needed in order to apply it to KdV. To avoid technicalities I restrict myself to N-gap potentials. As it will turn out, the above result directly applies in this case.

In order to define the map Φ from the space of potentials into $\mathcal{M}$, I use the following properties of the 1-dimensional Schrödinger operator

$$-y''(x) + q(x)y(x) = \lambda y(x)\,; \quad y(x+1) = y(x) \tag{1}$$

where q is in $L^2 := L^2[0,1]$, periodically extended to all of $\mathbf{R}$:

(i) $\int_0^1 q dx$ is a spectral invariant. Thus I may choose $L_0^2 := \{q \in L_0^2 : \int_0^1 q dx = 0\}$ as space of potentials.

(ii) The spectrum $\operatorname{spec} q$ of (1) (with multiplicities) determines the antiperiodic spectrum, *i.e.* the spectrum of the operator $-y'' + qy = \lambda y\,;\ y(x+1) = -y(x)$.

(iii) Denote by $(\lambda_n)_{n\geq 0}$ the union of the periodic and antiperiodic eigenvalues arranged in increasing order. Then $\lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 \ldots$ *i.e.* $(\lambda_n)_{n\geq 1}$ comes in isolated pairs.

(iv) For q in L_0^2, $\sum_{n\geq 1} (\lambda_{2n} - \lambda_{2n-1})^2 < \infty$.

Then the map $\Phi : L_0^2 \to \mathcal{M}$ is defined by $\Phi = (\Phi_n)_{n\geq 1}$ where $\Phi_n(q)$ is a trace free version of the restriction $-d_x^2 + q$ on the 2-dimensional subspace $E_n(q)$ generated by an orthonormal pair of eigenfunctions f_{2n-1} and f_{2n} corresponding to the eigenvalues λ_{2n-1} and λ_{2n}. More precisely, I provide an orthonormal basis $\{G_{2n-1}, G_{2n}\}$ of $E_n(q)$, depending analytically on q and defined on all of L_0^2. $\Phi_n(q)$ is then given by expressing the restriction of $-d_x^2 + q - (\lambda_{2n-1} + \lambda_{2n})/2$ to $E_n(q)$ with respect to this basis. Clearly, one cannot choose for $\{G_{2n-1}, G_{2n}\}$ the two eigenfunctions f_{2n-1} and f_{2n} as they do not depend smoothly on q, due to the possibility of double eigenvalues $\lambda_{2n-1} = \lambda_{2n}$. I would like to point out that the Dirichlet eigenvalues $(\mu_n)_{n\geq 1}$ which are often used, together with additional variables, as coordinates of isospectral sets, are not part of the restriction to isospectral sets of the above global coordinates, provided by Φ. For a potential p in an isospectral set $\operatorname{Iso} q$, the μ_n's have to stay within the interval $[\lambda_{2n-1}(q), \lambda_{2n}(q)]$, whose end points do not depend smoothly on q.

The main results of this paper are the following :

THEOREM. — (1) Φ *is a real analytic isomorphism*

(2) Φ *preserves isospectrality,* $\operatorname{Iso} \Phi(q) = \Phi(\operatorname{Iso} q)$

(3) $\Phi = (\Phi_n)_{n\geq 1}$ *is closely related to the Fourier transform :*

$$\Phi_n(q) = \begin{pmatrix} \hat{q}_{2n} & \hat{q}_{2n-1} \\ \hat{q}_{2n-1} & -\hat{q}_{2n} \end{pmatrix} + O\left(\frac{\log n}{n}\right)$$

uniformly on bounded sets of potentials in L_0^2 *where*

$$q = \sum_{n\geq 1} \hat{q}_{2n} \cos 2\pi n x + \hat{q}_{2n-1} \sin 2\pi n x.$$

The main work of the proof consists in showing that the derivative $d_q\Phi : L_0^2 \to \mathcal{M}$ is a linear isomorphism. To show this one has to prove that certain expressions involving products of eigenfunctions form a basis of L_0^2. I provide a new general method to do that (*cf.* section 6).

This theorem can be applied to the so-called finite gap potentials.

Define $\mathrm{Gap}_N := \{q \in L_0^2 : \lambda_{2n}(q) = \lambda_{2n-1}(q),\ \forall n \geq N+1\}$ and $\mathrm{Gap}_{N,r} := \{q \in \mathrm{Gap}_N : \lambda_{2n-1}(q) < \lambda_{2n}(q),\ 1 \leq n \leq N\}$. Elements in $\mathrm{Gap}_{N,r}$ are called regular N-gap potentials. It is well known that potentials in Gap_N are, in fact, real analytic. Observe that $\mathrm{Gap}_N = \Phi^{-1}\{R = (R_k)_{k\geq 1} \in \mathcal{M} : R_k = 0\ \forall k \geq N+1\}$ and thus Gap_N is a $2N$ dimensional manifold. Clearly $\mathrm{Gap}_{N,r}$ is an open set of Gap_N and $\Phi(\mathrm{Gap}_{N,r}) = (\mathbf{R}^+)^N \times T^N$ (diffeomorphically) where $\mathbf{R}^+ = \{x : x > 0\}$ and T^N denotes the N-torus $(S^1)^N$. Obviously $\mathrm{Gap}_{N,r}$ is invariant by KdV. Therefore, with the symplectic structure coming from KdV, it follows from the above theorem that $(\mathbf{R}^+)^N \times T^N$ is a symplectic manifold of dimension $2N$ with a trivial fibration by Lagrangian tori T^N. Adopting a definition of global action-angle variables due to Duistermaat [Du] one obtains the following

COROLLARY. — *When restricted to regular N-gap potentials* KdV *admits global action-angle variables.*

The paper is organized as follows :

2. Model space
3. Auxiliary results
4. Global coordinates : Definition and first properties
5. The derivative of Φ
6. Local properties of Φ
7. Global properties of Φ.

In a subsequent paper this technique is applied to obtain various results concerning the spectrum of Schrödinger operators on 2 dimensional flat tori.

Notation. — $L^2 := L^2[0,1]$ denotes the space of square integrable real valued functions on the unit interval with inner product $\langle f, g\rangle = \int_0^1 fg dx$. Denote $L_0^2 := \{f \in L^2 : \int_0^1 f dx = 0\}$. For q in L^2, denote by $(\lambda_n)_{n\geq 0}$ the union of periodic and antiperiodic eigenvalues of (1) with multiplicities arranged in increasing order. Further introduce $\tau_n = (\lambda_{2n}+\lambda_{2n-1})/2$. Let $(f_n)_{n\geq 0}$ be a L^2-orthonormal system of eigenfunctions corresponding to the eigenvalues $(\lambda_n)_{n\geq 0}$ with the properties : (i) $f_n(0) > 0$ or $f_n(0) = 0$ and $f_n'(0) > 0$ and (ii) if $\lambda_{2n-1} = \lambda_{2n}$, then $f_{2n-1}(0) = 0$. They satisfy $f_j(x+1) = (-1)^n f_j(x)$ for $j \in \{2n-1, 2n\}$. $E_n = E_n(q)$ denotes the 2-dimensional subspace of L^2 generated by f_{2n-1} and f_{2n} and $P_n = P_n(q)$ the orthogonal projection $L^2 \to E_n$. As usual, $y_1(x) = y_1(x, \lambda, q)$ and $y_2(x) = y_2(x, \lambda, q)$ denote the solutions of $-y'' + qy = \lambda y$ (x in $\mathbf{R}$) with $(y_1(0), y_1'(0)) = (1,0)$ and $(y_2(0), y_2'(0)) = (0,1)$. $\Delta(\lambda) = \Delta(\lambda, q)$ denotes the discriminant, $\Delta(\lambda) = y_1(1,\lambda) + y_2'(1,\lambda)$. Further denote by $(\mu_n)_{n\geq 1}$ the Dirichlet eigenvalues of q, *i.e.* the eigenvalues of the operator $-y''+qy = \lambda y$ with $y(0) = y(1) = 0$. Then $\mu_n = \mu_n(q)$ depends analytically on q and satisfies $\lambda_{2n-1}(q) \leq \mu_n(q) \leq \lambda_{2n}(q)$. Denote by $(g_n)_{n\geq 1}$ the orthonormal system of eigenfunctions corresponding to the eigenvalues $(\mu_n)_{n\geq 1}$ with the property that $g_n'(0) > 0$. Finally denote by $(\nu_n)_{n\geq 0}$ the Neumann eigenvalues of q, *i.e.* the eigenvalues of the operator $-y'' + qy = \lambda y$ with $y'(0) = y'(1) = 0$. Then $\nu_n = \nu_n(q)$ depends analytically on q and satisfies $\lambda_{2n-1}(q) \leq \nu_n(q) \leq \lambda_{2n}(q)$ $(n \geq 1)$. Denote by $(h_n)_{n\geq 0}$ the orthonormal system of eigenfunctions corresponding to the eigenvalues $(\nu_n)_{n\geq 0}$ with the property that $h_n(0) > 0$ (all $n \geq 0$). More details about these eigenvalues and eigenfunctions can be found in [CL], [MW], [Ma], [PT]. I denote by H^k_{per} the space of functions f in $H^k_{\text{loc}}(\mathbf{R})$ which are periodic of period 1. By $H^k[0,1]$, I denote the space of functions in $H^k_{\text{loc}}(\mathbf{R})$ restricted to the interval $[0,1]$. By $\ell^2_k(\mathbf{N})$ I denote the space of sequences $(x_n)_{n\geq 1}$ such that $\|x\|_k = \big(\sum_{n\geq 1} n^{2k} x_n^2\big)^{1/2} < \infty$. For two Banach spaces X_1 and X_2, I denote by $\mathcal{L}(X_1, X_2)$ the space of linear operators from X_1 to X_2 with the uniform norm. For functions $f = f(x,\lambda)$ depending on a real variable x and a possibly complex spectral parameter λ, the partial derivative $\partial f/\partial x$ with respect to x is denoted by f' and the partial derivative $\partial f/\partial \lambda$ with respect to λ is denoted by $\dot{f}$.

Let X be Banach space. A sequence $(x_n)_{n\geq 1}$ in X is said to converge weakly to $x \in X$ if $\lim_{n\to\infty} L(x_n) = L(x)$ for all L in the dual of X.

2. Model space.

In this section I describe the model space, define what it means for two elements in the model space to be isospectral and describe the isospectral set. Denote by M_0 the 2-dimensional $\mathbb{R}$-vector space of all symmetric trace free 2×2 matrices, *i.e.* matrices of the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$, with norm $\left\| \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \right\| = \sqrt{a^2 + b^2}$. Denote by $\mathcal{M}$ the Hilbert sum of M_0, *i.e.* the space of all sequences $R = (R_k)_{k\geq 1}$ in M_0 such that $\|R\| := \left(\sum_{k\geq 1} \|R_k\|^2 \right)^{1/2} < \infty$. Clearly $\mathcal{M}$ can be identified isometrically with $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ and thus, via Fourier transform, to L_0^2. Two elements

$$R = (R_k)_{k\geq 1}, \quad R_k = \begin{pmatrix} a_k & b_k \\ b_k & -a_k \end{pmatrix},$$

and

$$S = (S_k)_{k\geq 1}, \quad S_k = \begin{pmatrix} \alpha_k & \beta_k \\ \beta_k & -\alpha_k \end{pmatrix},$$

are said to be isospectral if $\operatorname{spec} R_k = \operatorname{spec} S_k$ for $k \geq 1$ where $\operatorname{spec} R_k$ denotes the spectrum of the 2×2 matrix R_k. Clearly R and S are isospectral if and only if $a_k^2 + b_k^2 = \alpha_k^2 + \beta_k^2$ for all $k \geq 1$. Define for R in $\mathcal{M}$

$$\operatorname{Iso} R := \{S \in \mathcal{M} : S \text{ and } R \text{ are isospectral}\}.$$

Then $\operatorname{Iso} R$ is an infinite product of circles, the radii of which are given by $\sqrt{a_k^2 + b_k^2}$. This implies that $\operatorname{Iso} R$ is compact and, generically, not a manifold. However in a straightforward way one can define tangent and normal spaces at each point of $\operatorname{Iso} R$. I summarize these results in the following

PROPOSITION 1. — *Let $R = (R_k)_{k\geq 1}$ be in $\mathcal{M}$. Then*

(1) $\|R_k\|$ *is a spectral invariant (all $k \geq 1$).*

(2) $\operatorname{Iso} R$ *is a compact connected torus in $\mathcal{M}$, consisting generically of an infinite product of circles.*

3. Auxiliary results.

For a potential q in L_0^2 periodically extended to all of $\mathbf{R}$, consider Hill's operator $-d_x^2 + q$ on $[0,1]$. Let $(f_n)_{n\geq 0}$ be the orthonormal system of eigenfunctions corresponding to the eigenvalues $(\lambda_n)_{n\geq 0}$ as described in the introduction. It is well known that both f_{2n-1} and f_{2n} have precisely n zeroes in $[0,1)$ all of which are simple. Fix n and denote the zeroes of f_{2n-1} and f_{2n} by $0 \leq y_1 < y_2 \ldots < y_n < 1$ and $0 \leq z_1 < z_2 < \ldots < z_n < 1$. By a standard deformation argument, considering the 1-parameter family of potentials $\tau q(x)$ $(0 \leq \tau \leq 1)$, one can prove that $(y_j)_{1\leq j\leq n}$ and $(z_j)_{1\leq j\leq n}$ interlace.

Next we will need the following

LEMMA 1. — *For q in L_0^2 and $n \geq 1$, $E_n(q) \to \mathbf{R}^2$, $f \mapsto (f(0), f'(0))$ is a linear isomorphism.*

Proof. — Fix q and n. It suffices to show that $f_{2n}(0)f'_{2n-1}(0) - f_{2n-1}(0)f'_{2n}(0) \neq 0$. To prove it, introduce the Wronskian

$$W(x) = f_{2n}(x)f'_{2n-1}(x) - f'_{2n}(x)f_{2n-1}(x).$$

Observe that

$$W' = (\lambda_{2n} - \lambda_{2n-1})f_{2n}f_{2n-1}.$$

Denote the zeroes of f_{2n-1} and f_{2n} by $0 \leq y_1 < \ldots < y_n < 1$ and $0 \leq z_1 < \ldots < z_n < 1$. According to Lemma 1 we may assume, to make notation easier, that $y_1 < z_1 < y_2 < \ldots < y_n < z_n$. Then

$$W(x) = W(y_1) + (\lambda_{2n} - \lambda_{2n-1}) \int_{y_1}^{x} f_{2n}f_{2n-1}.$$

Further observe that $f_{2n}f_{2n-1}$ is a periodic function of period 1. It thus suffices so show that $W(x)$ never vanishes for $y_1 \leq x \leq 1+y_1$. Without loss of generality we might assume that $f'_{2n-1}(y_1) > 0$ and $f_{2n}(y_1) > 0$. This can always be achieved by a suitable renormalization of f_{2n-1} and f_{2n}.

Then $f_{2n-1}(y_2) = 0$, $f'_{2n-1}(y_2) < 0$ and $f_{2n}(y_2) < 0$, *etc.* . One concludes that $W(y_j) = f_{2n}(y_j)f'_{2n-1}(y_j) > 0$. In the case $\lambda_{2n} = \lambda_{2n-1}$, one obtains $W(x) = W(y_1) > 0$. In the case where $\lambda_{2n} > \lambda_{2n-1}$ one observes that $W(x)$ is increasing in the intervals $[y_j, z_j]$ and decreasing in the intervals $[z_j, y_{j+1}]$. This shows that $\min_x W(x) = \min_{1 \le j \le n} W(y_j) > 0$. □

For q in L^2_0, denote by $P_n(q)$ the orthogonal projection of L^2 into $E_n(q)$. $P_n(q)$ has a representation of the form $P_n(q) = -\frac{1}{2\pi i}\int_{\Gamma_n} R(\lambda, q)d\lambda$ where $R(\lambda, q)$ denotes the resolvent $(-d_x^2 + q - \lambda)^{-1}$ and where Γ_n is a circle in the complex plane such that λ_{2n} and λ_{2n-1} are inside and all other eigenvalues outside Γ_n. For the next results, consider [Ka] as a general reference.

LEMMA 2. — *(1) For $n \ge 1$, $P_n : L^2_0 \to \mathcal{L}(L^2, H^2_{\mathrm{per}})$ is real analytic.*

(2) P_n is compact, i.e. if $(q_j)_{j\ge1}$ is a sequence in L^2_0, converging weakly to q, then $\lim_{j\to\infty} \sup_{f\in L^2} \|P_n(q_j)f - P_n(q)f\|_{L^2}/\|f\|_{L^2} = 0$.

(3) The derivative of $d_{q_0}P_n[p]$ of P_n at q_0 in direction p is given by

$$d_{q_0}P_n[p] = \frac{1}{2\pi i}\int_{\Gamma_n} R(\lambda, q_0)pR(\lambda, q_0)d\lambda.$$

(4) $P_n(q_0)d_{q_0}P_n[p]P_n(q_0) = 0$.

Now I introduce the so-called transformation operators. Fix q_0 in L^2_0. Following Kato [Ka] I denote by $U_n(q)$ the transformation operator $U_n(q) : E_n(q_0) \to H^2_{\mathrm{per}}$ given by $(\mathrm{Id} - (P_n(q) - P_n(q_0))^2)^{1/2}P_n(q)$. $U_n(q)$ is defined for q in a sufficiently small neighborhood V of q, which might depend on n. It turns out that the image of $U_n(q)$ is $E_n(q)$.

LEMMA 3. — *Let $n \ge 1$ be given.*

(1) U_n is real analytic as function from V into $\mathcal{L}(E_n(q_0), H^2_{\mathrm{per}})$.

(2) U_n is compact i.e. if $(q_j)_{j\ge1}$ is a sequence in V converging weakly to a limit q in V then $\lim_{j\to\infty} U_n(q_j) = U_n(q)$ in the operator norm of $\mathcal{L}(E_n(q_0), L^2)$.

(3) $d_{q_0}U_n[p] = d_{q_0}P_n[p]$.

Next, for the convenience of the reader, I collect a few well known results concerning Hill's equation. For reference *cf.* [Ma], [MW] and [PT].

LEMMA 4. — *(1) For $k \geq 0$, $\lambda_k : L_0^2 \to \mathbf{R}$ is weakly continuous.*

(2) The λ_k's have the following asymptotics :

$$\lambda_{2k} = k^2\pi^2 + \Big| \int_0^1 e^{-2\pi ikx} q(x)dx \Big| + O\big(\frac{1}{k}\big)$$

$$\lambda_{2k-1} = k^2\pi^2 - \Big| \int_0^1 e^{-2\pi ikx} q(x)dx \Big| + O\big(\frac{1}{k}\big)$$

where both error terms are uniform on bounded sets of potentials in L_0^2.

LEMMA 5. — *(1) For $k \geq 1$, $\mu_k : L_0^2 \to \mathbf{R}$ is weakly continuous.*

(2) The μ_k's have the following asymptotics

$$\mu_k = k^2\pi^2 - \int_0^1 \cos 2\pi kx q(x)dx + O\big(\frac{1}{k}\big).$$

(3) For $k \geq 0$, $\nu_k : L_0^2 \to \mathbf{R}$ is weakly continuous.

(4) The ν_k's have the following asymptotics

$$\nu_k = k^2\pi^2 + \int_0^1 \cos 2\pi kx q(x)dx + O\big(\frac{1}{k}\big).$$

LEMMA 6. — *(1) $y_2(1,\lambda) = \prod_{k\geq 1} \frac{\mu_k - \lambda}{k^2\pi^2}$*

(2) $\dfrac{y_2(1,\lambda_{2n})}{\lambda_{2n} - \mu_n} = \dfrac{1}{2}\dfrac{(-1)^n}{n^2\pi^2}\big(1 + O\big(\dfrac{\log n}{n}\big)\big)$

(3) $\dfrac{y_2(1,\lambda_{2n-1})}{\mu_n - \lambda_{2n-1}} = \dfrac{1}{2}\dfrac{(-1)^{n+1}}{n^2\pi^2}\big(1 + O\big(\dfrac{\log n}{n}\big)\big)$ where the error terms are uniform on bounded sets of potentials in L_0^2 and where for $\mu_n = \lambda_{2n-1}$ or $\mu_n = \lambda_{2n}$ the formulae $\dfrac{y_2(1,\lambda_{2n-1})}{\mu_n - \lambda_{2n-1}}$ resp $\dfrac{y_2(1,\lambda_{2n})}{\lambda_{2n} - \mu_n}$ are replaced by the corresponding derivative $\dot{y}_2(1,\mu_n)$.

LEMMA 7. — *(1) $y_1'(1,\lambda) = (\nu_0 - \lambda) \prod_{k\geq 1} \frac{\nu_k - \lambda}{k^2\pi^2}$*

(2) $\dfrac{y_1'(1,\lambda_{2n})}{\lambda_{2n} - \nu_n} = \dfrac{1}{2}(-1)^{n+1}\big(1 + O\big(\dfrac{\log n}{n}\big)\big)$

(3) $\dfrac{y_1'(1,\lambda_{2n-1})}{\nu_n - \lambda_{2n-1}} = \dfrac{1}{2}(-1)^{n}\big(1 + O\big(\dfrac{\log n}{n}\big)\big)$ with corresponding explanations as in Lemma 6.

Next recall that the discriminant is defined by $\Delta(\lambda) = y_1(1,\lambda) + y_2'(1,\lambda)$. Then

LEMMA 8.

(1) $\Delta(\lambda)^2 - 4 = 4(\lambda_0 - \lambda) \prod_{n\geq 1} \frac{(\lambda_{2n} - \lambda)(\lambda_{2n-1} - \lambda)}{n^4\pi^4}$

(2) $\dot{\Delta}(\lambda_{2n}) = (-1)^{n+1} \frac{1}{4} \frac{\lambda_{2n} - \lambda_{2n-1}}{n^2\pi^2} (1 + O(\frac{\log n}{n}))$

(3) $\dot{\Delta}(\lambda_{2n-1}) = (-1)^{n} \frac{1}{4} \frac{\lambda_{2n} - \lambda_{2n-1}}{n^2\pi^2} (1 + O(\frac{\log n}{n}))$

(4) $\Delta(\mu)^2 - 4 = \frac{1}{n^2\pi^2}(\lambda_{2n} - \mu)(\mu - \lambda_{2n-1})(1 + O(\frac{\log n}{n}))$ *for* $\lambda_{2n-1} < \mu < \lambda_{2n}$.

As a consequence of Lemmas 5, 6, 7 and 8, one gets

COROLLARY 9. — *If* $\lambda_{2n-1} < \lambda_{2n}$, *then for* $j \in \{2n-1, 2n\}$

(1) $-\frac{y_2(1,\lambda_j)}{\dot{\Delta}(\lambda_j)} = \frac{|\lambda_j - \mu_n|}{(\lambda_{2n} - \lambda_{2n-1})/2} (1 + O(\frac{\log n}{n}))$

(2) $\frac{y_1'(1,\lambda_j)}{\dot{\Delta}(\lambda_j)} = n^2\pi^2 \frac{|\lambda_j - \nu_n|}{(\lambda_{2n} - \lambda_{2n-1})/2} (1 + O(\frac{\log n}{n}))$.

Finally I will need the following well known representation of the eigenfunction f_n corresponding to a simple eigenvalue.

LEMMA 10. — *For* q *in* L_0^2, *and* λ_n *a simple eigenvalue*

$$f_n(x,q) = (-y_2(1,\lambda_n)/\dot{\Delta}(\lambda_n))^{1/2} y_1(x,\lambda_n,q) + \sigma_n (y_1'(1,\lambda_n)/\dot{\Delta}(\lambda_n))^{1/2} y_2(x,\lambda_n,q)$$

where the sign σ_n *of the second radical is given by the sign of*

$$(-1)^{[\frac{n}{2}]+1}(y_1(1,\lambda_n) - y_2'(1,\lambda_n)).$$

4. Global coordinates : Definition and first properties.

In this section I introduce a map Φ from the space of potentials L_0^2 into the model space $\mathcal{M}$ which preserves isospectrality. I show among other things, that Φ is real analytic and prove that asymptotically Φ

is closely related to the Fourier transform. Fix q in L^2_0. According to Lemma 3.1 there exists a unique element $G_{2n-1}(x) = G_{2n-1}(x,q)$ in $E_n(q)$ such that $G_{2n-1}(0) = 0$, $G'_{2n-1}(0) > 0$ and $\|G_{2n-1}\|_{L^2} = 1$. Define $G_{2n}(x) = G_{2n}(x,q)$ in $E_n(q)$ by requiring that $\langle G_{2n}, G_{2n-1}\rangle = 0$, $\|G_{2n}\|_{L^2} = 1$. The sign of $G_{2n}(0)$ is determined by requiring that the oriented angle between G_{2n-1} and G_{2n} is $-\pi/2$ where the orientation is provided through the map $E_n(q) \to \mathbb{R}^2$, $f \mapsto (f(0), f'(0))$. Clearly G_{2n} and G_{2n-1} are linear combinations of f_{2n} and f_{2n-1} where $(f_k)_{k\geq 0}$ denotes the orthonormal system of eigenfunctions of $-d_x^2 + q$ as specified in the introduction.

DEFINITION. — *$\Phi(q) := (\Phi_n(q))_{n\geq 1}$ where $\Phi_n(q)$ is given by*

$$\begin{pmatrix} \langle G_{2n}, (-d_x^2 + q - \tau_n)G_{2n}\rangle & \langle G_{2n}, (-d_x^2 + q - \tau_n)G_{2n-1}\rangle \\ \langle G_{2n-1}, (-d_x^2 + q - \tau_n)G_{2n}\rangle & \langle G_{2n-1}, (-d_x^2 + q - \tau_n)G_{2n-1}\rangle \end{pmatrix}.$$

First let us show that $\Phi(q)$ is an element in $\mathcal{M}$. For this purpose, express G_{2n} and G_{2n-1} in terms of f_{2n} and f_{2n-1}. Define ε_n to be the signature of the Wronskian $W[f_{2n}, f_{2n-1}](0)$. Then .

$$\begin{pmatrix} G_{2n} \\ G_{2n-1} \end{pmatrix} = \begin{pmatrix} \cos\vartheta_n & -\sin\vartheta_n \\ \sin\vartheta_n & \cos\vartheta_n \end{pmatrix} \begin{pmatrix} f_{2n} \\ \varepsilon_n f_{2n-1} \end{pmatrix}$$

where ϑ_n, up to 2π, is uniquely determined by the chosen normalizations of the f's and G's. A simple computation shows that

$$\Phi_n(q) = \left(\frac{\lambda_{2n} - \lambda_{2n-1}}{2}\right) \begin{pmatrix} \cos 2\vartheta_n & \sin 2\vartheta_n \\ \sin 2\vartheta_n & -\cos 2\vartheta_n \end{pmatrix}.$$

Thus $\Phi_n(q)$ is symmetric and trace free and its eigenvalues are $\pm(\lambda_{2n} - \lambda_{2n-1})/2$. Moreover it is well known that for $q \in L^2_0$, $\sum_{n\geq 1} (\lambda_{2n} - \lambda_{2n-1})^2 < \infty$ uniformly on bounded sets of potentials. Thus I have proved

LEMMA 1. — *Φ maps L^2_0 into $\mathcal{M}$ and is bounded.*

Next I want to show that Φ preserves isospectrality.

PROPOSITION 2. — *Let p and q be in L^2_0. Then* $\operatorname{spec}(-d_x^2 + p) = \operatorname{spec}(-d_x^2 + q)$ *if and only if $\Phi(p)$ and $\Phi(q)$ are isospectral.*

Proof. — Assume $\operatorname{spec}(-d_x^2 + p) = \operatorname{spec}(-d_x^2 + q)$. Then $\lambda_n(p) = \lambda_n(q)$ $(n \geq 1)$. In particular $\lambda_{2n}(p) - \lambda_{2n-1}(p) = \lambda_{2n}(q) - \lambda_{2n-1}(q)$. Thus, by the representation of $\Phi_n(q)$ above, we see that $\Phi(p)$ and $\Phi(q)$

are isospectral. Conversely, if $\Phi(p)$ and $\Phi(q)$ are isospectral, then $\lambda_{2n}(p) - \lambda_{2n-1}(p) = \lambda_{2n}(q) - \lambda_{2n-1}(q)$ for all $n \geq 1$. But this implies $\lambda_n(p) = \lambda_n(q)$ $(n \geq 0)$ (*cf.* e.g. [Kp] for an elementary proof).

The next results concern the identification of the range of Φ when restricted to even potentials in L_0^2, *i.e.* potential satisfying $q(x) = q(1-x)$.

PROPOSITION 3. — *Let q be in L_0^2. Then*

(1) *$\Phi_n(q)$ is diagonal if and only if $\mu_n \in \{\lambda_{2n-1}, \lambda_{2n}\}$ where μ_n, as usual, denotes the n'th Dirichlet eigenvalue of $-d_x^2 + q$.*

(2) *q is even if and only if $\Phi_n(q)$ is diagonal for all $n \geq 1$.*

Proof. — (1) If μ_n is a periodic or antiperiodic eigenvalue, then $G_{2n-1} \in \{f_{2n-1}, f_{2n}\}$ and thus $\{G_{2n-1}, \pm G_{2n}\} = \{f_{2n-1}, f_{2n}\}$. This implies that $\Phi_n(q)$ is diagonal. Conversely if

$$\Phi_n(q) = \left(\frac{\lambda_{2n} - \lambda_{2n-1}}{2}\right) \begin{pmatrix} \cos 2\vartheta_n & \sin 2\vartheta_n \\ \sin 2\vartheta_n & -\cos 2\vartheta_n \end{pmatrix}$$

is diagonal then either $\lambda_{2n} - \lambda_{2n-1} = 0$ and thus $f_{2n-1} = G_{2n-1}$, $f_{2n} = G_{2n}$ or $\vartheta_n \in \{k\pi/2 : k \in \mathbb{Z}\}$. But then $\{\pm G_{2n}, G_{2n-1}\} = \{f_{2n}, f_{2n-1}\}$ and thus $G_{2n-1}(1) = G_{2n-1}(0) = 0$ and thus μ_n is a periodic eigenvalue.

(2) It is a well known fact that q being even implies $\{\mu_n, \nu_n\} = \{\lambda_{2n}, \lambda_{2n-1}\}$ for all $n \geq 1$, where ν_n denotes the n'th Neumann eigenvalue of $-d_x^2 + q$. Thus by (1), $\Phi_n(q)$ is diagonal. The converse follows from [PT], Lemma 3.4.

Next let us investigate the analytic properties of $\Phi(q)$. First we need to study certain properties of G_n $(n \geq 1)$. Observe that f_{2n} and f_{2n-1} are eigenfunctions of $-d_x^2 + q$ but do not depend smoothly on q. In contrast to that G_{2n} and G_{2n-1} are not necessarily eigenfunctions, but they depend analytically on q as the following result shows :

LEMMA 4. — *For all $n \geq 1$, $G_n(\bullet, q)$ is real analytic when considered as a map form L_0^2 into $H^2[0,1]$.*

Proof. — Fix a potential q_0 in L_0^2. According to Lemma 3.4, there exists an open neighborhood V of q_0 in L_0^2 where one can define a canonical transformation $U_n(q) : E_n(q_0) \to H^2[0,1]$, $U_n(q)$ being a real analytic function on V with range $E_n(q)$ such that $U_n(q) : E_n(q_0) \to E_n(q)$ is $1-1$ and onto. Clearly it suffices to prove that $G_{2n-1}(\bullet, q)$ is real analytic. For

q in V, $G_{2n-1}(\bullet, q)$ can be expressed as a linear combination

$$G_{2n-1}(\bullet, q) = \alpha_n(q)U_n(q)f_{2n}(\bullet, q_0) + \beta_n(q)U_n(q)f_{2n-1}(\bullet, q_0)$$

where $(\alpha_n(q), \beta_n(q)) = \left(\tilde{\alpha}_n(q)^2 + \tilde{\beta}_n(q)^2\right)^{1/2}\left(\tilde{\alpha}_n(q), \tilde{\beta}_n(q)\right)$ and $(\tilde{\alpha}_n\tilde{\beta}_n)$ is given by $T_n(q)(0,1)$ where $T_n(q)$ is the inverse of the 2×2 matrix

$$\begin{pmatrix} (U_n(q)f_{2n}(\bullet, q_0))(0) & (U_n(q)f_{2n-1}(\bullet, q_0))(0) \\ (U_n(q)f_{2n}(\bullet, q_0))'(0) & (U_n(q)f_{2n-1}(\bullet, q_0))'(0) \end{pmatrix}.$$

Thus $\alpha_n(q)$ and $\beta(q)$ are real analytic functions on V. This proves Lemma 4.

Next we would like to prove the following

LEMMA 5. — *$G_k(\bullet, q)$ is a weakly continuous function when considered as a map from L_0^2 into $H^2[0,1]$ for all $k \geq 1$, i.e. if $(p_n)_{n\geq 1}$ is a sequence in L_0^2, with $p_n \rightharpoonup p$ weakly then $G_k(\bullet, p_n) \rightharpoonup G_k(\bullet, p)$ weakly in $H^2[0,1]$.*

Proof. — Fix $k \geq 1$. It suffices to prove that $\lim_{n\to 0} G_{2k-1}(\bullet, p_n) = G_{2k-1}(\bullet, p)$ weakly in $H^2[0,1]$. First, by a well known result, $\lambda_k(q)$ is a compact function of q. Thus $\lim_{n\to 0} \lambda_k(p_n) = \lambda_k(p)$. This is used to prove that $(G_{2k-1}(\bullet, p_n))_{n\geq 1}$ is a bounded sequence in $H^2[0,1]$. Thus there exists a subsequence, again denoted by $(G_{2k-1}(\bullet, p_n))_{n\geq 1}$ which converges weakly in $H^2[0,1]$. Therefore, $\lim_{n\to 0} G_{2k-1}(\bullet, p_n) = f$ in $C^1[0,1]$. In particular $\|f\|_{L^2} = 1$, $f(0) = 0$ and $f'(0) \geq 0$. From section 3 we learn that $\lim_{n\to 0} P_k(p_n) = P_k(p)$ in the operator norm and thus $f = P_k(p)f$, *i.e.* $f \in E_k(p)$. This proves that $f = G_{2k-1}(\bullet, p)$. But for every subsequence of $(G_{2k-1}(\bullet, p_n))_{n\geq 1}$ we can argue as above and extract another subsequence which converges to G_{2k-1}. Thus $\lim_{n\to 0} G_{2k-1}(\bullet, p_n) = G_{2k-1}(\bullet, p)$ weakly in $H^2[0,1]$.

THEOREM 6. — (1) *$\Phi : L_0^2 \to \mathcal{M}, q \mapsto (\Phi_n(q))_{n\geq 1}$ is real analytic.*

(2) *For each n, $\Phi_n : L_0^2 \to M_0$ is compact, i.e. if $p_k \rightharpoonup p$ weakly in L_0^2, then $\Phi_n(p) = \lim_{k\to\infty} \Phi_n(p_k)$ strongly in M_0.*

Proof. — (1) Φ is locally bounded and thus it suffices to prove that for any n, each coefficient of Φ_n is real analytic. Observe that $\lambda_{2n} + \lambda_{2n-1}$ is real analytic, being a symmetric expression in λ_{2n} and λ_{2n-1}. The analyticity of each of the coefficients of Φ_n then follows from Lemma 4.

(2) It is to show that each coefficient of Φ_k is a compact function on L^2_0. Let $(p_n)_{n\geq 1}$ be a sequence in L^2_0, weakly convergent to $p \in L^2_0$. Then e.g.,

$$\lim_{n\to\infty} \langle G_{2k-1}(\bullet, p_n), (-d_x^2 + p_n - \tau_n)G_{2k-1}(\bullet, p_n)\rangle$$
$$= \langle G_{2k-1}(\bullet, p), (-d_x^2 + p - \tau_n)G_{2k-1}(\bullet, p)\rangle$$

where we used Lemma 5 and the fact that λ_k is compact.

The last results of this section concern asymptotic properties of the Φ_n's and G_n's.

THEOREM 7. — $\Phi_n(q) = \begin{pmatrix} \widehat{q}_{2n} & \widehat{q}_{2n-1} \\ \widehat{q}_{2n-1} & -\widehat{q}_{2n} \end{pmatrix} + O(\frac{\log n}{n})$ *where* $\widehat{q}_{2n}$ *and* $\widehat{q}_{2n-1}$ *denote the Fourier coefficients of* q,

$$\widehat{q}_{2n} = \int_0^1 q(x)\cos 2\pi nx dx$$

and

$$\widehat{q}_{2n-1} = \int_0^1 q(x)\sin 2\pi nx dx.$$

The error estimates are uniform on bounded sets of potentials.

Proof. — Recall that Φ_n can be written as

$$\Phi_n = \frac{\lambda_{2n} - \lambda_{2n-1}}{2}\begin{pmatrix} \cos 2\vartheta_n & \sin 2\vartheta_n \\ \sin 2\vartheta_n & -\cos 2\vartheta_n \end{pmatrix}$$

where I set $\vartheta_n = 0$ in the case the eigenvalues λ_{2n} is double. For n with $\lambda_{2n-1} < \lambda_{2n}$, ϑ_n was defined such that $0 = G_{2n-1}(0) = \sin\vartheta_n f_{2n}(0) + \varepsilon_n \cos\vartheta_n f_{2n-1}(0)$. From Lemma 3.10

$$f_j(0) = (-y_2(1, \lambda_j)/\dot{\Delta}(\lambda_j))^{1/2}$$

for $j = 2n-1$ or $2n$. Thus $\sin^2\vartheta_n = \dfrac{f_{2n-1}(0)^2}{f_{2n}(0)^2 + f_{2n-1}(0)^2}$ and $\cos^2\vartheta_n = \dfrac{f_{2n}(0)^2}{f_{2n}(0)^2 + f_{2n-1}(0)^2}$. By Corollary 3.9

$$\cos^2\vartheta_n - \sin^2\vartheta_n = \frac{\lambda_{2n} + \lambda_{2n-1} - 2\mu_n}{(\lambda_{2n} - \lambda_{2n-1})}\big(1 + O(\frac{\log n}{n})\big).$$

Using Corollary 3.4 and 3.5 one obtains

$$\frac{\lambda_{2n} - \lambda_{2n-1}}{2}\cos 2\vartheta_n = -(\mu_n - n^2\pi^2)\big(1 + O(\frac{\log n}{n})\big) = \widehat{q}_{2n} + O(\frac{\log n}{n}).$$

Next

$$|2\sin\vartheta_n\cos\vartheta_n| = \frac{2\sqrt{(\mu_n-\lambda_{2n-1})(\lambda_{2n}-\mu_n)}}{\lambda_{2n}-\lambda_{2n-1}} + O\left(\frac{\log n}{n}\right)$$

$$= \frac{\sqrt{(-\widehat{q}_{2n}+\sqrt{\widehat{q}_{2n}^2+\widehat{q}_{2n-1}^2})(\sqrt{\widehat{q}_{2n}^2+\widehat{q}_{2n-1}^2}+\widehat{q}_{2n})}}{\sqrt{\widehat{q}_{2n}^2+\widehat{q}_{2n-1}^2}}$$

$$+ O\left(\frac{\log n}{n}\right)$$

$$= \frac{|\widehat{q}_{2n-1}|}{\sqrt{\widehat{q}_{2n}^2+\widehat{q}_{2n-1}^2}} + O\left(\frac{\log n}{n}\right).$$

Thus it remains to determine the sign of $\sin\vartheta_n\cos\vartheta_n$. Recall that

$$f_n'(0) = \sigma_n(y_1'(1,\lambda_n)/\dot{\Delta}(\lambda_n))^{1/2}$$

with $\sigma_n = \operatorname{sgn}(-1)^{[n/2]+1}(y_1(1,\lambda_n) - y_2'(1,\lambda_n))$ and further that

$$y_1(1,\lambda_{2n}) = \cos\sqrt{\lambda_{2n}} + \int_0^1 \frac{\sin\sqrt{\lambda_{2n}}(1-t)}{\sqrt{\lambda_{2n}}}\cos\sqrt{\lambda_{2n}}t q(t)dt + O\left(\frac{1}{n^2}\right)$$

$$y_2'(1,\lambda_{2n}) = \cos\sqrt{\lambda_{2n}} + \int_0^1 \cos\sqrt{\lambda_{2n}}(1-t)\cdot\frac{\sin\sqrt{\lambda_{2n}}t}{\sqrt{\lambda_{2n}}}q(t)dt + O\left(\frac{1}{n^2}\right).$$

Thus

$$y_1(1,\lambda_{2n}) - y_2'(1,\lambda_{2n})$$

$$= \frac{1}{\sqrt{\lambda_{2n}}}(-1)^{n+1}\int_0^1 \sin 2n\pi t q(t)dt + O\left(\frac{1}{n^2}\right).$$

Similarly

$$y_1(1,\lambda_{2n-1}) - y_2'(1,\lambda_{2n-1}) = \frac{1}{\sqrt{\lambda_{2n-1}}}(-1)^{n+1}\int_0^1 \sin 2n\pi t q(t)dt + O\left(\frac{1}{n^2}\right).$$

Thus $\sigma_{2n} = (-1)^{2n}\operatorname{sgn} b_n$, $\sigma_{2n-1} = (-1)^{2n-1}\operatorname{sgn} b_n$ for n sufficiently large. Next observe that

$$0 < \varepsilon_n W[f_{2n}, f_{2n-1}](0) = W[f_{2n}, \varepsilon_n f_{2n-1}](0)$$
$$= W[G_{2n}, G_{2n-1}](0) = G_{2n}(0)G_{2n-1}'(0).$$

As $G_{2n-1}'(0) > 0$, this implies that $G_{2n}(0) > 0$. Thus together with

$$0 < G_{2n-1}'(0) = \sin\vartheta_n f_{2n}'(0) + \varepsilon_n\cos\vartheta_n f_{2n-1}'(0)$$

it follows that $\cos\vartheta_n \geq 0$; from $G_{2n-1}(0) > 0$ we then obtain that $\sin\vartheta_n\cos\vartheta_n$ and b_n have the same sign and Theorem 7 is proved.

The last results of this section concern asymptotic properties of the functions G_n.

PROPOSITION 8. — *(1)* $G_{2n-1}(x) = \sqrt{2}\sin\pi nx + O(\frac{1}{n})$ *and* $G'_{2n-1}(x) = \sqrt{2}\pi n\cos\pi nx + O(1)$.

(2) $G_{2n}(x,q) = \sqrt{2}\cos\pi nx + O(\frac{1}{n})$ *and* $G'_{2n}(x,q) = -\sqrt{2}n\pi\sin\pi nx + O(1)$.

Proof. — It is well known that $E_n(q)$ has an orthonormal basis H_{2n-1} and H_{2n} of the form $H_{2n-1}(x) = \sqrt{2}\sin n\pi x + O(\frac{1}{n})$, $H'_{2n-1}(x) = \sqrt{2}n\pi\cos n\pi x + O(1)$ and $H_{2n}(x) = \sqrt{2}\cos n\pi x + O(\frac{1}{n})$, $H'_{2n}(x) = -\sqrt{2}n\pi\sin n\pi x + O(1)$. Thus, due to the normalization of H_{2n-1} and H_{2n}, we have $G_{2n-1} = H_{2n-1} + O(\frac{1}{n})$ and $G_{2n} = H_{2n} + O(\frac{1}{n})$ and the result follows.

5. Derivative of Φ.

In this section I compute the derivative of Φ and study its asymptotic behavior. It turns out that it is convenient to write Φ in a slightly different form. For q in L_0^2 denote its Fourier series by $\sum_{n\geq 1}\widehat{q}_{2n}\cos 2\pi nx + \widehat{q}_{2n-1}\sin 2\pi nx$. Then $\widehat{q} = (\widehat{q}_n)_{n\geq 1} \in \ell^2(\mathbb{N})$ and I write Φ as a map $\Psi : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ with $\Psi(\widehat{q}) = (\Psi_n(\widehat{q}))_{n\geq 1}$ where

$$\Psi_{2n-1}(\widehat{q}) := \langle G_{2n}, (-d_x^2 + q - \tau_n)G_{2n-1}\rangle$$

and

$$\Psi_{2n}(\widehat{q}) := -\langle G_{2n-1}, (-d_x^2 + q - \tau_n)G_{2n-1}\rangle.$$

To make notation easier, we write simply $\Psi(q)$ and $\Psi_n(q)$. For illustration let us start by computing the derivative of Ψ at $q = 0$. For $q = 0$, $G_{2n-1}(x) = \sqrt{2}\sin\pi nx$ and $G_{2n}(x) = \sqrt{2}\cos\pi nx$ and $(-d_x^2 + q - \tau_n)G_{2n-1} = 0$.

$$\begin{aligned} d_{q=0}\Psi_{2n}[p] &= -\langle G_{2n-1}, pG_{2n-1}\rangle + \langle G_{2n-1}, G_{2n-1}\rangle d_{q=0}\tau_n[p] \\ &= \int_0^1 p(x)\cos 2\pi nx dx = \widehat{p}_{2n}. \end{aligned}$$

Similarly $d_{q=0}\Psi_{2n-1}[p] = \widehat{p}_{2n-1}$. Thus $d_{q=0}\Psi = \mathrm{Id}$. In particular $d_{q=0}\Psi$ is $1-1$ and onto.

To compute the derivative for general q let me recall that

$$\begin{pmatrix} G_{2n} \\ G_{2n-1} \end{pmatrix} = \begin{pmatrix} \cos\vartheta_n & -\sin\vartheta_n \\ \sin\vartheta_n & \cos\vartheta_n \end{pmatrix} \begin{pmatrix} f_{2n} \\ \varepsilon_n f_{2n-1} \end{pmatrix}.$$

Thus $\varepsilon_n f_{2n-1} = -\sin\vartheta_n G_{2n} + \cos\vartheta_n G_{2n-1}$ and $f_{2n}^2 + f_{2n-1}^2 = G_{2n}^2 + G_{2n-1}^2$.

PROPOSITION 1.

$$(1) \qquad d_q\Psi_{2n}[p] = \int_0^1 \frac{G_{2n}^2 - G_{2n-1}^2}{2} p\,dx - 2\Psi_{2n-1}(q)\int_0^1 d_qG_{2n-1}[p]G_{2n}dx$$

and

$$(2) \qquad d_q\Psi_{2n-1}[p] = \int_0^1 G_{2n}G_{2n-1}p\,dx + 2\Psi_{2n}(q)\int_0^1 d_qG_{2n-1}[p]G_{2n}dx\,.$$

Proof. — Write

$$\begin{aligned}
&(-d_x^2 + q - \tau_n)G_{2n-1} \\
&= \sin\vartheta_n \frac{\lambda_{2n} - \lambda_{2n-1}}{2} f_{2n} - \varepsilon_n \cos\vartheta_n \frac{\lambda_{2n} - \lambda_{2n-1}}{2} f_{2n-1} \\
&= \frac{\lambda_{2n} - \lambda_{2n-1}}{2}(-\cos 2\vartheta_n)G_{2n-1} + \sin 2\vartheta_n \frac{\lambda_{2n} - \lambda_{2n-1}}{2} G_{2n}.
\end{aligned}$$

To compute $d_q\Psi_n[p]$ note that $2\langle d_qG_{2n}, G_{2n}\rangle = d_q\langle G_{2n}, G_{2n}\rangle = 0$ and $\langle d_qG_{2n}, G_{2n-1}\rangle = -\langle d_qG_{2n-1}, G_{2n}\rangle$. Thus

$$\begin{aligned}
-d_q\Psi_{2n}[p] &= 2\langle d_qG_{2n-1}[p], (-d_x^2 + q - \tau_n)G_{2n-1}\rangle \\
&\quad + \langle G_{2n-1}, pG_{2n-1}\rangle - \frac{1}{2}\langle f_{2n}^2 + f_{2n-1}^2, p\rangle \\
&= \sin 2\vartheta_n(\lambda_{2n} - \lambda_{2n-1})\langle d_qG_{2n-1}[p], G_{2n}\rangle \\
&\quad - \langle p, \frac{G_{2n}^2 - G_{2n-1}^2}{2}\rangle
\end{aligned}$$

and (1) follows. Similarly one proves (2) :

$$\begin{aligned}
d_q\Phi_{2n-1}[p] &= \langle p, G_{2n}G_{2n-1}\rangle \\
&\quad + (\lambda_{2n} - \lambda_{2n-1})\cos 2\vartheta_n\langle d_qG_{2n-1}[p], G_{2n}\rangle.
\end{aligned}$$

This proves Proposition 1.

The derivates $d_q\Phi_{2n}$ and $d_q\Psi_{2n-1}$ can be expressed in terms of f_{2n} and f_{2n-1} instead of G_{2n} and G_{2n-1}. Observe that

$$G_{2n}^2 - G_{2n-1}^2 = \cos 2\vartheta_n(f_{2n}^2 - f_{2n-1}^2) - \varepsilon_n \sin 2\vartheta_n 2f_{2n}f_{2n-1}$$

and

$$G_{2n}G_{2n-1} = \sin 2\vartheta_n \frac{f_{2n}^2 - f_{2n-1}^2}{2} + \cos 2\vartheta_n \varepsilon_n f_{2n} f_{2n-1}.$$

Thus we obtain

COROLLARY 2.

$$\begin{pmatrix} d_q\Psi_{2n}[p] \\ d_q\Psi_{2n-1}[p] \end{pmatrix} = \int_0^1 \frac{f_{2n}^2 - f_{2n-1}^2}{2} p dx \begin{pmatrix} \cos 2\vartheta n \\ \sin 2\vartheta_n \end{pmatrix}$$
$$+ \left(\varepsilon_n \int_0^1 f_{2n} f_{2n-1} p dx + (\lambda_{2n} - \lambda_{2n-1}) \int_0^1 d_q G_{2n-1}[p] G_{2n} dx \right) \begin{pmatrix} -\sin 2\vartheta_n \\ \cos 2\vartheta_n \end{pmatrix}.$$

To study the asymptotics of $d_q\Psi_n$ it will be useful to bring

$$\int_0^1 d_q G_{2n-1}[p] G_{2n} dx$$

into another form. In section 3, I introduced unitary transformations $U_n(p) : E_n(q) \to H^2[0,1]$ with range in $E_n(p)$ for p in a neighborhood V of q such that $U_n(p)$ is real analytic in p and satisfies $U_n(q) = P_n(q)$, $P_n(q) d_q U_n = 0$ as well as $d_q U_n[p] = (d_q P_n[p]) U_n(q)$. Define $\alpha(p)$ and $\beta(p)$ by

$$G_{2n-1}(\bullet, p) = \alpha(p) U_n(p) G_{2n}(\bullet, q) + \beta(p) U_n(p) G_{2n-1}(\bullet, q).$$

LEMMA 3.

$$\int_0^1 d_q G_{2n-1}[p] G_{2n} dx = \sum_{\substack{j \geq 0 \\ j \neq 2n, 2n-1}} \varepsilon_n \cos \vartheta_n f_j(0) \frac{\int_0^1 f_j f_{2n-1} p dx}{\lambda_{2n-1} - \lambda_j}$$
$$+ \sin \vartheta_n \sum_{\substack{j \geq 0 \\ j \neq 2n, 2n-1}} f_j(0) \frac{\int_0^1 f_j f_{2n} p dx}{\lambda_{2n} - \lambda_j}.$$

Proof. — Clearly, $\alpha(p)^2 + \beta(p)^2 = 1$ and $(\alpha(q), \beta(q)) = (0, 1)$. $\alpha(p)$ and $\beta(p)$ are real analytic functions of p, thus

$$P_n(q) d_q G_{2n-1}[p] = d_q \alpha[p] G_{2n}(\bullet, q) + d_q \beta[p] G_{2n-1}(\bullet, q).$$

It follows that $\int_0^1 d_q G_{2n-1}[p] G_{2n} dx = d_q \alpha[p]$. Now

$$(\alpha, \beta) = (\widehat{\alpha}^2 + \widehat{\beta}^2)^{-1/2} (\widehat{\alpha}, \widehat{\beta})$$

where $(\widehat{\alpha}, \widehat{\beta})$ is determined by

$$0 = \widehat{\alpha}(p)(U(p)G_{2n}(\bullet, q))(0) + \widehat{\beta}(p)(U(p)G_{2n-1}(\bullet, q))(0)$$
$$1 = \widehat{\alpha}(p)(U(p)G_{2n}(\bullet, q))'(0) + \widehat{\beta}(p)(U(p)G_{2n-1}(\bullet, q))'(0).$$

Observe that $\widehat{\alpha}(q) = 0$ and $\widehat{\beta}(q) = 1/G'_{2n-1}(0, q)$ and thus the derivative of the first equation above yields

$$G'_{2n-1}(0, q)d_q\widehat{\alpha}[p] = -(d_qP_n[p]G_{2n-1}(\bullet, q))(0).$$

Together with $d_q\widehat{\alpha}[p] = (1/G'_{2n-1}(0, q))d_q\alpha[p]$ one obtains

$$d_q\alpha[p] = -(d_qP_n[p]G_{2n-1}(\bullet, q))(0).$$

By Cauchy's formula $d_qP_n[p] = -\dfrac{1}{2\pi i}\displaystyle\int_{\Gamma_n} d_qR[p](z)dz$ where Γ_n is a circle in $\mathbb{C}$ including λ_{2n-1} and λ_{2n} and $R(z)$ is the resolvent,

$$R(z) = \left(- d_x^2 + q - z\right)^{-1} = \sum_{j\geq 0} \frac{1}{\lambda_j - z}\langle f_j, \bullet\rangle f_j.$$

As $d_qR[p]$ is given by $-R(z)pP(z)$, this leads to

$$\begin{aligned} d_qP_n[p]G_{2n-1} &= \frac{1}{2\pi i}\sum_{j\geq 0}\int_{\Gamma_n} \frac{1}{\lambda_j - z}\langle f_j, pR(z)G_{2n-1}\rangle dz\, f_j \\ &= \sum_{j\geq 0} f_j\langle f_j, pf_{2n-1}\rangle\varepsilon_n \cos\vartheta_n \frac{1}{2\pi i}\int_{\Gamma_n} \frac{1}{\lambda_j - z}\frac{1}{\lambda_{2n-1} - z}dz \\ &\quad + \sum_{j\geq 0} f_j\langle f_j, pf_{2n}\rangle \sin\vartheta_n \frac{1}{2\pi i}\int_{\Gamma_n} \frac{1}{\lambda_j - z}\frac{1}{\lambda_{2n} - z}dz. \end{aligned}$$

By Cauchy's theorem, Lemma 3 then follows.

I will now study the asymptotics of $d_q\Psi_n$ as $n \to \infty$.

PROPOSITION 4.

$$\begin{aligned} &(d_q\Psi_{2n}[p], d_q\Psi_{2n-1}[p]) \\ &\qquad = \Big(\int_0^1 p(x)\cos 2\pi nxdx, \int_0^1 p(x)\sin 2\pi nxdx\Big) + O\big(\frac{1}{n}\big) \end{aligned}$$

where the error terms are bounded uniformly for bounded sets of potentials q and p.

Proof. (1) In section 4, I derived the following estimates :

$$\int_0^1 \frac{G_{2n}^2 - G_{2n-1}^2}{2}pdx = \int_0^1 p\cos 2\pi nxdx + O\big(\frac{1}{n}\big)$$

$$\int_0^1 G_{2n} G_{2n-1} p dx = \int_0^1 p \sin 2\pi n x dx + O\left(\frac{1}{n}\right)$$

and from section 3, I recall

$$\frac{\lambda_{2n} - \lambda_{2n-1}}{2} = \left| \int_0^1 q(x) e^{2\pi i n x} dx \right| + O\left(\frac{1}{n}\right)$$

where the error terms are bounded uniformly for bounded sets of potentials of q's and of p's. According to Proposition 1, it thus suffices to bound $(\lambda_{2n} - \lambda_{2n-1}) \int_0^1 d_q G_{2n-1}[p] G_{2n} dx$ appropiately. In view of Lemma 3 we need to estimate $\sum_{\substack{j \geq 0 \\ j \neq 2n, 2n-1}} f_j(0) \frac{\int_0^1 f_j f_k p dx}{\lambda_k - \lambda_j}$ for $k \in \{2n-1, 2n\}$.

$$\begin{aligned} &\Big| \sum_{\substack{j \geq 0 \\ j \neq 2n, 2n-1}} f_j(0) \frac{\int_0^1 f_j f_k p dx}{\lambda_k - \lambda_j} \Big| \\ &\leq \Big(\sum_{\substack{j \geq 0 \\ j \neq 2n, 2n-1}} f_j(0)^2 |\langle f_j, f_{2n-1} p \rangle|^2 \Big)^{1/2} \Big(\sum_{\substack{j \geq 0 \\ j \neq 2n, 2n-1}} \frac{1}{(\lambda_k - \lambda_j)^2} \Big)^{1/2} \\ &\leq \sup_j \|f_j\|_{L^\infty} \|f_{2n-1}\|_{L^\infty} \|p\|_{L^2} \Big(\sum_{\substack{j \geq 0 \\ j \neq 2n, 2n-1}} \frac{1}{(\lambda_k - \lambda_j)^2} \Big)^{1/2}. \end{aligned}$$

It remains to prove that for $k \in \{2n-1, 2n\}$ $\sup_{n \geq 1} n^2 \sum_{\substack{j \geq 0 \\ j \neq 2n, 2n-1}} \frac{1}{(\lambda_k - \lambda_j)^2}$ is bounded uniformly for bounded sets of potentials. Thus it suffices to show that for a given bounded set B of potentials there exist $N \geq 1$ and $K > 0$ such that

$$\sup_{n \geq N} n^2 \sum_{\substack{j \geq N \\ j \neq n}} \frac{1}{(\lambda_k - \lambda_{2j})^2} \leq K$$

and

$$\sup_{n \geq N} n^2 \sum_{\substack{j \geq N \\ j \neq n}} \frac{1}{(\lambda_{2j-1} - \lambda_k)^2} \leq K$$

for $k \in \{2n-1, 2n\}$ and q in B. E.g. let us consider

$$\sup_{n \geq N} n^2 \sum_{\substack{j \geq N \\ j \neq n}} \frac{1}{(\lambda_{2j} - \lambda_{2n})^2}.$$

Choose $N \geq 1$ such that for $q \in B$

(a) $\lambda_{2n}, \lambda_{2n-1} \geq 1$ $(n \geq N)$ and

(b) $\left|\sqrt{\lambda_{2n}} - n\right| \le \frac{1}{4}$, $\left|\sqrt{\lambda_{2n-1}} - n\right| \le \frac{1}{4}$ $(n \ge N)$.

Then

$$n^2 \sum_{j \ge n+1} \frac{1}{(\lambda_{2j} - \lambda_{2n})^2} \le \sum_{j \ge n+1} \frac{1}{\left(\sqrt{\lambda_{2j}} - \sqrt{\lambda_{2n}}\right)^2} \frac{n^2}{\left(\sqrt{\lambda_{2j}} + \sqrt{\lambda_{2n}}\right)^2}.$$

Further

$$n^2/\left(\sqrt{\lambda_{2j}} + \sqrt{\lambda_{2n}}\right)^2 \le 2$$

and

$$1/\left(\sqrt{\lambda_{2j}} - \sqrt{\lambda_{2n}}\right)^2 \le 1/(j - n - 1/2)^2 \quad (j, n \ge N).$$

Thus

$$\sum_{j \ge n+1} n^2/(\lambda_{2j} - \lambda_{2n})^2 \le 2 \sum_{\ell \ge 1} 1/(\ell - 1/2)^2$$

for all $n \ge N$, q in B. Similarly,

$$n^2 \sum_{j=N}^{n-1} \frac{1}{(\lambda_{2j} - \lambda_{2n})^2} \le \sum_{j=N}^{n-1} \frac{n^2}{(j - n + 1/2)^2} \frac{n^2}{\left(\sqrt{\lambda_{2j}} + \sqrt{\lambda_{2n}}\right)^2}$$
$$\le 2 \sum_{\ell \ge 1} \frac{1}{(\ell - 1/2)^2}.$$

These estimates prove Proposition 4.

For the last result of this section, I first need to introduce some more notation. Let q be in L_0 and define $J := \{n \ge 1 : \lambda_{2n-1} < \lambda_{2n}\}$. Observe that for $n \in J$, $f_{2n} = \cos\vartheta_n G_{2n} + \sin\vartheta_n G_{2n-1}$ and $\varepsilon_n f_{2n-1} = -\sin\vartheta_n G_{2n} + \cos\vartheta_n G_{2n-1}$. These relations remain true for $n \notin J$ if we set $\vartheta_n = 0$ and $\varepsilon_n = +1$ for $n \notin J$. Then for all $n \ge 1$

$$\left(f_{2n}^2 - f_{2n-1}^2\right)/2 = \cos 2\vartheta_n\left(G_{2n}^2 - G_{2n-1}^2\right)/2 + \sin 2\vartheta_n G_{2n} G_{2n-1}$$

and

$$\varepsilon_n f_{2n} f_{2n-1} = -\sin 2\vartheta_n\left(G_{2n}^2 - G_{2n-1}^2\right)/2 + \cos 2\vartheta_n G_{2n} G_{2n-1}.$$

Now introduce

$$F_{2n} := \frac{f_{2n}^2 - f_{2n-1}^2}{\sqrt{2}} \ (n \ge 1) \text{ and } F_{2n-1} := \sqrt{2}\varepsilon_n f_{2n} f_{2n-1} \text{ for } n \notin J$$

as well as

$$F_{2n-1} := \sqrt{2}(\lambda_{2n} - \lambda_{2n-1}) d_q\vartheta_n \text{ for } n \in J,$$

where, by slight abuse of notation (*cf.* Remark after Lemma 6.8), we define

$$d_q\vartheta_n = \frac{1}{\lambda_{2n} - \lambda_{2n-1}} \varepsilon_n f_{2n} f_{2n-1} + \int_0^1 d_q G_{2n-1}[\cdot] G_{2n} dx.$$

Further introduce the orthonormal trigonometric basis $(T_n)_{n\geq 1}$ of L_0^2,

$$T_{2n}(x) := \cos 2\vartheta_n \sqrt{2} \cos 2\pi nx + \sin 2\vartheta_n \sqrt{2} \sin 2\pi nx$$

and

$$T_{2n-1}(x) := -\sin 2\vartheta_n \sqrt{2} \cos 2\pi nx + \cos 2\vartheta_n \sqrt{2} \sin 2\pi nx.$$

From the asymptotics for

$$G_{2n}(x) = \sqrt{2} \cos 2n\pi x + O\left(\frac{1}{n}\right) \quad \text{and} \quad G_{2n-1}(x) = \sqrt{2} \sin 2n\pi x + O\left(\frac{1}{n}\right)$$

derived in section 4, the following result is then immediate :

PROPOSITION 5. — *$(F_n)_{n\geq 1}$ and $(T_n)_{n\geq 1}$ are quadratically close, i.e.* $\sum\limits_{n\geq 1} \|F_n - T_n\|_{L^2}^2 < \infty$.

6. Local properties of Φ.

In this section I prove that $d_q\Phi$ is a linear isomorphism for any q in L_0^2. I include a proof for finite band potentials, *i.e.* potentials q in L_0^2 with $J := \{n \geq 1 : \lambda_{2n-1} < \lambda_{2n}\}$ finite, as the proof simplifies in that case.

THEOREM 1. — *$d_q\Phi$ is $1-1$ and onto.*

First I need to derive a few auxiliary results. Recall that the set $\mathrm{Iso}\, q$ of isospectral potentials is a countable intersection of manifolds. So one can define the normal space N_q and tangent space T_q of $\mathrm{Iso}\, q$ at q.

LEMMA 2. — *Let p_n denote the potential $\dfrac{d}{dx}\dfrac{\partial\Delta(\mu)}{\partial q(x)}$ with $\mu = \mu_n(q)$. Then*

(1) $\int_0^1 (f_{2k}^2 - f_{2k-1}^2) p_n dx = 0 \ \ \forall n \in J, \ \forall k \geq 1$

(2) $\int_0^1 f_{2k} f_{2k-1} p_n dx = 0 \ \ \forall n \in J, \ \forall k \notin J.$

Proof. — (1) and (2) are proved in a similar way. I show that for $j \geq 0$ $n \in J$, $2\int_0^1 f_j^2 p_n dx = 0$. One might assume that q satisfies $\lambda_{2n-1}(q) < \mu_n(q) < \lambda_{2n}(q)$ as the general case follows by continuity (in q). Then with $\mu = \mu_n(q)$,

$$\frac{\partial\Delta(\mu)}{\partial q(x)} = y_2(1,\mu) f_+(x,\mu) f_-(x,\mu)$$

where $f_\pm(x,\mu) := y_1(x,\mu) + \left[\frac{m_\pm - y_1(1,\mu)}{y_2(1,\mu)}\right] y_2(x,\mu)$ with $m_\pm(\mu) := \frac{\Delta(\mu)}{2} \pm \frac{1}{2}\sqrt{\Delta(\mu)^2 - 4}$ (cf. [FIT]). Then $f_\pm(x+1,\mu) = m_\pm f_\pm(x,\mu)$ and $m_+ m_- = 1$. Thus

$$\begin{aligned} 2\int_0^1 f_j^2 p_n dx &= y_2(1,\mu)\int_0^1 \left(f_j^2 \frac{d}{dx} f_+ f_- - \left(\frac{d}{dx} f_j^2\right) f_+ f_-\right) dx \\ &= \frac{y_2(1,\mu)}{\lambda_j - \mu}\int_0^1 \frac{d}{dx}\left(W[f_j, f_+] W[f_j, f_-]\right) dx = 0 \end{aligned}$$

where we used that $\lambda_j - \mu \neq 0$. The last equality follows from

$$\begin{aligned} &W\left[f_j(1), f_+(1)\right] W\left[f_j(1), f_-(1)\right] \\ &\qquad = m_+ m_- W\left[f_j(0), f_+(0)\right] W\left[f_j(0), f_-(0)\right]. \end{aligned}$$

LEMMA 3. — *$\int_0^1 f_k^2 \frac{d}{dx} f_n^2 dx = 0$ for $k = n$ and all k with $\lambda_k \neq \lambda_n$.*

Proof. — $2\int_0^1 f_k^2 \frac{d}{dx} f_n^2 dx = 2\int_0^1 f_k f_n W[f_k, f_n] dx$ where $W[f,g]$ denotes the Wronskian. As $\frac{d}{dx} W[f_k, f_n] = (\lambda_k - \lambda_n) f_k f_n$ we obtain $2(\lambda_k - \lambda_n)\int_0^1 f_k^2 \frac{d}{dx} f_n^2 dx = 0$ by the periodicity of f_k and f_n. The case $n = k$ is trivial.

COROLLARY 4. — *$\int_0^1 (f_{2k}^2 - f_{2k-1}^2)\frac{d}{dx}(f_{2\ell}^2 - f_{2\ell-1}^2) = 0$ (all k, ℓ).*

Denote by J the set $\{n \geq 1 : \lambda_{2n-1} < \lambda_{2n}\}$. Then

LEMMA 5.

(1) *For $k \notin J$, $n \geq 1$ with $n \neq k$, $\int_0^1 f_{2k} f_{2k-1} \frac{d}{dx}(f_{2n}^2 - f_{2n-1}^2) dx = 0$.*

(2) *For $k \notin J$,*

$$\begin{aligned} &\int_0^1 f_{2k} f_{2k-1} \frac{d}{dx}(f_{2k}^2 - f_{2k-1}^2) dx \\ &\qquad = -\left(\int_0^1 y_1^2(x, \lambda_{2k})\right)^{-1/2} \left(\int_0^1 y_2^2(x, \lambda_{2k})\right)^{-1/2} \end{aligned}$$

where y_1 and y_2 denote, as usual, the fundamental solutions.

Proof. — (1) and (2) are proved in a very similar way, thus we concentrate on (2) only, which follows from the following statement

$$\int_0^1 y_1(x,\lambda_{2k})y_2(x,\lambda_{2k})\frac{d}{dx}y_j^2(x,\lambda_{2k})dx=(-1)^j\int_0^1 y_j^2(x,\lambda_{2k})dx$$
$$(1\le j\le 2).$$

To make notation easier, choose $j=1$. Then

$$\int_0^1 y_1(x,\lambda_{2k})y_2(x,\lambda_{2k})\frac{d}{dx}y_1^2(x,\lambda_{2k})dx$$
$$=\lim_{\mu\to\lambda_{2k}}\int_0^1 y_1(x,\mu)y_2(x,\mu)\frac{d}{dx}y_1^2(x,\lambda_{2k})dx.$$

For $\mu\neq\lambda:=\lambda_{2k}$ we have, by a similar argument as in the proof of Lemma 2,

$$2(\mu-\lambda)\int_0^1 y_1(x,\mu)y_2(x,\mu)\frac{d}{dx}y_1^2(x,\lambda)dx=y_1'(1,\mu)y_2'(1,\mu).$$

Clearly,

$$\lim_{\mu\to\lambda}y_2'(1,\mu)=1\quad\text{and}\quad\lim_{\mu\to\lambda}\frac{y_1'(1,\mu)}{\mu-\lambda}=\frac{d}{d\mu}|_{\mu=\lambda}y_1'(1,\mu).$$

Thus

$$\lim_{\mu\to\lambda}\int_0^1 y_1(x,\mu)y_2(x,\mu)\frac{d}{dx}y_1^2(x,\lambda)dx=\frac{1}{2}\frac{d}{d\mu}|_{\mu=\lambda}y_1'(1,\mu)$$
$$=-\frac{1}{2}\int_0^1 y_1^2(x,\lambda).$$

Thus Lemma 4 follows.

Denote by T_t translation by t, i.e. $T_tf(x):=f(x+t)$. Then

LEMMA 6. — *(1) T_t leaves Iso_q invariant.*

(2) Given $q\in L_0^2$ there exists a countable set $A\subseteq[0,1]$ such that for $n\in J$ and all t in $[0,1]\setminus A$, $\lambda_{2n-1}(q)<\mu_n(T_tq)<\lambda_{2n}(q)$.

Proof. — (1) is immediate by applying T_t to the equation $-y''+qy=\lambda y$.

(2) It is well known that $f_{2n}(x,q)$ and $f_{2n-1}(x,q)$ have precisely n zeroes in $0\le x<1$. Observe that $f_n(x,q)=\pm f_n(0,T_xq)$ for all $n\ge 0$ with $\lambda_{2n-1}\neq\lambda_{2n}$. Thus the claim follows by observing that for $n\in J$ and $j\in\{2n-1,2n\}$, $f_j(0,T_xq)=0$ if and only if $\lambda_j(q)=\mu_n(T_xq)$.

LEMMA 7. — (1) *There exists* $0 \le t < 1$ *such that for any finite subset* $J' \subseteq J$, *the matrix*

$$\Big(\int_0^1 \frac{d}{dx}(f_{2n}^2 - f_{2n-1}^2)(x) g_k^2(x - t, T_t q) dx \Big)_{n,k \in J'}$$

is non-singular where $(g_k)_{k \ge 1}$ *denotes the* L^2 *normalized system of Dirichlet eigenfunctions, defined in the introduction.*

(2) *Any finite collection of* $(f_{2n}^2 - f_{2n-1}^2)_{n \ge 1}$, $(f_{2n} f_{2n-1})_{n \notin J}$, $\frac{d}{dx}(f_{2n}^2 - f_{2n-1}^2)_{n \in J}$ *is linearly independent.*

Proof. — In view of Lemma 5 and Corollary 4, to show (2), it suffices to prove that any finite collection of $\big(\frac{d}{dx}(f_{2n}^2 - f_{2n-1}^2)\big)_{n \in J}$ is linearly independent. By Lemma 6, I may assume that q has the property that $\lambda_{2n-1}(q) < \mu_n(q) < \lambda_{2n}(q)$ for all n in J. (1) and (2) then follow, once I have shown that for any finite subset $J' \subseteq J$, $\det \big(\int_0^1 g_k^2 \frac{d}{dx}(f_{2n}^2 - f_{2n-1}^2) dx \big)_{k,n \in J'} \neq 0$. To prove it observe that

$$\begin{aligned} 2(\mu_k - \lambda_n) \int_0^1 g_k^2 \frac{d}{dx} f_n^2 &= \int_0^1 2(\mu_k - \lambda_n) g_k f_n W[g_k, f_n] dx \\ &= W[g_k, f_n]^2 \Big|_0^1 = f_n(0)^2 \big(g_k'(1)^2 - g_k'(0)^2\big). \end{aligned}$$

Thus

$$\begin{aligned} 2 \int_0^1 & g_k^2 \frac{d}{dx}(f_{2n}^2 - f_{2n-1}^2) dx \\ &= \big(g_k'(1)^2 - g_k'(0)^2\big) \Big(\frac{f_{2n}(0)^2}{\mu_k - \lambda_{2n}} - \frac{f_{2n-1}(0)^2}{\mu_k - \lambda_{2n-1}} \Big). \end{aligned}$$

Observe that as $\lambda_{2k-1} < \mu_k < \lambda_{2k}$, $g_k'(1) \neq \pm g_k'(0)$ and thus $g_k'(1)^2 - g_k'(0)^2 \neq 0$. Moreover, again as $\lambda_{2n-1} < \mu_n < \lambda_{2n}$ it follows that $f_{2n}(0)^2 \neq 0$ and $f_{2n-1}(0)^2 \neq 0$. It remains to show that

$$A := \det \Big(- \frac{f_{2n}^2(0)}{\mu_k - \lambda_{2n}} + \frac{f_{2n-1}^2(0)}{\mu_k - \lambda_{2n-1}} \Big)_{k,n \in J'} \neq 0.$$

First, as the determinant is multilinear,

$$A = \sum_x (-1)^{|\varepsilon|} \prod_{-x_n = \lambda_{2n}} f_{2n}^2(0) \prod_{-x_n = \lambda_{2n-1}} f_{2n-1}^2(0) \det \Big(\frac{1}{\mu_k + x_n} \Big)_{k,n \in J'}$$

where $x = (x_k)_{k \in J'}$ with $-x_k \in \{\lambda_{2k-1}, \lambda_{2k}\}$ and where $\varepsilon = (\varepsilon_k)_{k \in J'}$ with $\varepsilon_i = 0$ if $x_k = \lambda_{2k-1}$ and $\varepsilon_i = 1$ if $x_k = \lambda_{2k}$. Moreover $|\varepsilon| = \sum_{k \in J'} \varepsilon_k$. By an introduction argument (*cf.* [PS], p. 98)

$$\det\Big(\frac{1}{\mu_n + x_k}\Big)_{n,k\in J'} = \frac{\prod\limits_{j>k}(\mu_j-\mu_k)(x_j-x_k)}{\prod\limits_{j,k\in J'}(\mu_j+x_k)}.$$

Now observe that $\prod\limits_{j>k}(\mu_j - \mu_k) > 0$ and that $\operatorname{sgn}\prod\limits_{j>k}(x_j - x_k)$ is independent of x. Moreover

$$\operatorname{sgn}\prod_{n,k\in J'}(\mu_n + x_k) = (-1)^{|\varepsilon|}\operatorname{sgn}\prod_{n,k\in J'}(\mu_n - \lambda_{2k-1}).$$

This last equality is verified by observing that given $x = (x_k)$ and $y = (y_k)$ with $x_k = y_k$ for $k \neq n$ and $\{x_n, y_n\} = \{\lambda_{2n-1}, \lambda_{2n}\}$, $\operatorname{sgn}(\mu_j + y_k) = \operatorname{sgn}(\mu_j + x_k)$ except when $j = k = n$; in that case $\mu_n + y_n = -(\mu_n + x_n)$. This proves Lemma 7.

Recall that for $n \in J$ the angular coordinate ϑ_n was introduced in section 4 by $G_{2n-1}(x) = \sin\vartheta_n f_{2n}(x) + \varepsilon_n \cos\vartheta_n f_{2n-1}(x)$ and $G_{2n}(x) = \cos\vartheta_n f_{2n}(x) - \varepsilon_n \sin\vartheta_n f_{2n-1}(x)$. First I derive an expression for the directional derivative $d_q\vartheta_n[p]$ when p is in the tangent space T_q.

LEMMA 8. — *For $p \in T_q$ and $n \in J$ with $\lambda_{2n-1}(q) < \mu_n(q) < \lambda_{2n}(q)$*

$$d_q\vartheta_n[p] = \frac{1}{2}G_{2n}(0)^{-1}\varepsilon_n \cos\vartheta_n f_{2n-1}(0)$$
$$\sum_{j\geq 1}\int_0^1 g_j^2 p\,dx\Big(\frac{1}{\mu_j - \lambda_{2n-1}} - \frac{1}{\mu_j - \lambda_{2n}}\Big).$$

Proof. — By definition $G_{2n-1}(0) = 0$ and thus

$$G_{2n}(0)d_q\vartheta_n[p] = \sin\vartheta_n d_q f_{2n}(0)[p] + \cos\vartheta_n\varepsilon_n d_q f_{2n-1}(0)[p].$$

Further, as $\lambda_{2n-1} < \lambda_{2n}$, one has for $k \in \{2n-1, 2n\}$

$$f_k(x) = +\sqrt{-\frac{y_2(1,\lambda_k)}{\dot\Delta(\lambda_k)}}\,y_1(x,\lambda_k) + \sigma\sqrt{\frac{y_1'(1,\lambda_k)}{\dot\Delta(\lambda_k)}}\,y_2(x,\lambda_k)$$

where the sign σ of the last radical is given by

$$\operatorname{sgn}(-1)^{[\frac{k}{2}]+1}(y_1(1,\lambda_k) - y_2(1,\lambda_k)).$$

Observe that $(-1)^{k+1}\dot\Delta(\lambda_{2k}) > 0$, $(-1)^k\dot\Delta(\lambda_{2k-1}) > 0$ and $-\dfrac{y_2(1,\lambda_k)}{\dot\Delta(\lambda_k)} \geq 0$ as well as $\dfrac{y_1'(1,\lambda_k)}{\dot\Delta(\lambda_k)} \geq 0$. Next it is well known (*cf.* [PT]) that $y_2(1,\lambda)$ is given by $y_2(1,\lambda) = \prod_{j\geq 1} \frac{\mu_j - \lambda}{j^2\pi^2}$. Thus $d_q y_2(1,\lambda) = \sum_{j\geq 1} \frac{y_2(1,\lambda)}{\mu_j - \lambda} g_j^2$. Due to the assumption that $p \in T_q$ one has $d_q\lambda_k[p] = d_q\dot{\Delta}(\lambda_k)[p] = 0$ and thus

$$d_q f_k(0) = \frac{1}{2} f_k(0) \sum_{j\geq 1} \frac{1}{\mu_j - \lambda_k} \int_0^1 g_j^2 p dx.$$

The Lemma now follows by observing that

$$0 = \sin\vartheta_n f_{2n}(0) + \varepsilon_n \cos\vartheta_n f_{2n-1}(0).$$

Remark. — For potentials $q \in L_0^2$ with $\lambda_{2n-1} < \mu_n < \lambda_{2n}$ one can choose $\vartheta_n(q)$ to depend analytically on the potential in a sufficiently small neighborhood of q. Then

$$\begin{aligned}(\lambda_{2n} - \lambda_{2n-1}) & d_q\vartheta_n[p] \\ &= \varepsilon_n \int_0^1 f_{2n} f_{2n-1} p dx + (\lambda_{2n} - \lambda_{2n-1}) \int_0^1 d_q G_{2n-1}[p] G_{2n} dx.\end{aligned}$$

By a slight abuse of notation, we denote the right hand side of the equality above by $(\lambda_{2n} - \lambda_{2n-1}) d_q\vartheta_n[p]$ even if $\mu_n(q) \in \{\lambda_{2n-1}, \lambda_{2n}\}$.

Recall that I have introduced the potentials

$$p_n(x) = \frac{d}{dx} \frac{\partial\Delta(\lambda)}{\partial q(x)}\Big|_{\lambda=\mu_n(q)}.$$

LEMMA 9. — *Let $J' \subseteq J$ be finite with the property that $\lambda_{2n-1}(q) < \mu_n(q) < \lambda_{2n}(q)$ $\forall n \in J'$. Then*

$$\det(d_q\vartheta_k[p_n])_{k,n\in J'} \neq 0.$$

Proof. — Using Wronskians one verifies that

$$\int_0^1 g_j^2 p_n dx = \delta_{jn} \frac{1}{2} (y_1(1,\mu_n) - y_2'(1,\mu_n))$$

(*cf.* [MT], p. 164) where δ_n denotes the Kronecker delta function. Thus, by Lemma 8,

$$\begin{aligned}G_{2k}(0) d_q\vartheta_k[p] = \frac{1}{2}\varepsilon_k \cos\vartheta_k f_{2k-1}(0) \frac{1}{2} (y_1(1,\mu_n) - y_2'(1,\mu_n)) \\ \Big(\frac{1}{\mu_n - \lambda_{2k-1}} - \frac{1}{\mu_n - \lambda_{2k}}\Big).\end{aligned}$$

Clearly, as for all $k \in J'$,

$$\lambda_{2k-1}(q) < \mu_k(q) < \lambda_{2k}(q),$$

$$(y_1(1,\mu_n) - y_2'(1,\mu_n))^2 = \Delta^2(\mu_k) - 4 \neq 0$$

and

$$\cos\vartheta_k f_{2k-1}(0) \neq 0.$$

Thus it suffices to prove that $\det\left(\frac{1}{\mu_n - \lambda_{2k-1}} - \frac{1}{\mu_n - \lambda_{2k}}\right)_{n,k\in J'} \neq 0$. This is shown by the same method as in the proof of Lemma 7.

LEMMA 10. — *Let $n \in J$ with $\mu_n(q) \in \{\lambda_{2n-1}(q), \lambda_{2n}(q)\}$. Then $d_q\vartheta_k[p_n] = \delta_{kn}c_n$ with $c_n \neq 0$.*

Proof. — Clearly $\Delta(\mu_n)^2 - 4 = 0$. Thus

$$\Big|\int_0^1 g_j^2 p_n dx\Big| = \delta_{jn}\frac{1}{2}\sqrt{\Delta^2(\mu_n) - 4} = 0,$$

and by Lemma 8 $d_q\vartheta_k[p_n] = 0$ for $k \neq n$. It remains to show that $c_n \neq 0$. To make notation easier, let me assume that $\mu_n = \lambda_{2n-1}$. Define a sequence $q_j \in \operatorname{Iso} q$ such that $q = \lim_{j\to\infty} q_j$ in L_0^2 and for all j, $\lambda_{2n-1} < \mu_n(q_j) < \lambda_{2n}$. (E.g. $q_j(x) := q(x + t_j)$ will do for an appropriate sequence t_j with $t_j \downarrow 0$). Define $p_{jn}(x) := \frac{d}{dx}\frac{\partial\Delta(\lambda)}{\partial q(x)}\big|_{\lambda=\mu_n(q_j)}$. From Lemma 8 we then obtain

$$d_{q_j}\vartheta_n[p] = \frac{1}{2}G_{2n}(0)^{-1}\varepsilon_n\cos\vartheta_n f_{2n-1}(0)\frac{1}{2}(y_1(1,\mu_n) - y_2'(1,\mu_n)) \left(\frac{1}{\mu_n - \lambda_{2n-1}} - \frac{1}{\mu_n - \lambda_{2n}}\right)$$

where the right hand side is evaluated at q_j. Clearly

$$\lim_{j\to\infty}|y_1(1,\mu_n) - y_2'(1,\mu_n)| = \lim_{j\to\infty}\sqrt{\Delta(\mu_n)^2 - 4} = 0$$

and $\lim_{j\to\infty} G_{2n}(0,q_j) = G_{2n}(0,q) \neq 0$ as well as $\lim_{j\to\infty}\cos\vartheta_n(q_j) = 1$. It remains to compute

$$\lim_{j\to\infty} f_{2n-1}(0)\frac{\sqrt{\Delta^2(\mu_n) - 4}}{\mu_n - \lambda_{2n-1}} = \lim_{j\to\infty}\sqrt{-\frac{1}{\dot\Delta(\lambda_{2n-1})}\frac{y_2(1,\lambda_{2n-1})}{\mu_n - \lambda_{2n-1}}} \sqrt{\frac{\Delta^2(\mu_n) - 4}{\mu_n - \lambda_{2n-1}}}.$$

Clearly

$$\frac{y_2(1,\lambda_{2n-1})}{\mu_n - \lambda_{2n-1}} = \frac{1}{n^2\pi^2}\prod_{k\neq n}\frac{\mu_k(q_j) - \lambda_{2n-1}}{k^2\pi^2}$$

converges to

$$\frac{1}{n^2\pi^2}\prod_{k\neq n}\frac{\mu_k(q)-\lambda_{2n-1}}{k^2\pi^2} = -\frac{\partial}{\partial\lambda}\Big|_{\lambda=\mu_n(q)} y_2(1,\lambda) \neq 0.$$

On the other hand

$$\lim_{j\to\infty}\frac{\Delta^2(\mu_n)-4}{\mu_n-\lambda_{2n-1}} = \dot{\Delta}(\mu_n)(-1)^n 4 \neq 0$$

and it follows that $c_n \neq 0$. This proves Lemma 10.

Proof of Theorem 1 for a finite band potential. — As $d_q\Phi$ is a Fredholm operator of index 0 (*cf.* Proposition 5.5), it suffices to prove that $d_q\Phi$ is $1-1$. Assume that for some $p \in L_0^2$, $d_q\Phi[p] = 0$. It is to show that $p = 0$. From Proposition 5.2 one learns that $\int_0^1 F_n p dx = 0$ for all $n \geq 1$ where $F_{2n} := (f_{2n}^2 - f_{2n-1}^2)/\sqrt{2}$, $F_{2n-1} := \sqrt{2}\varepsilon_n f_{2n} f_{2n-1}$ $(n \notin J)$ and $F_{2n-1} := \sqrt{2}(\lambda_{2n}-\lambda_{2n-1})d_q\vartheta_n$ $(n \in J)$. Clearly it suffices to show that $(F_n)_{n\geq 1}$ is a basis of L_0^2. From Corollary 5.5 we learn that $(F_n)_{n\geq 1}$ is quadratically close to an orthonormal basis. Thus according to a result of Bari (*cf.* [GK], p. 317) it suffices to show that $(F_n)_{n\geq 1}$ is ω-linearly independent, *i.e.* if $(\alpha_n)_{n\geq 1}$ is a sequence in $\mathbf{R}$ such that

(i) $\sum_{n\geq 1}\alpha_n^2\|F_n\|^2 < \infty$ and (ii) $\sum_{n\geq 1}\alpha_n F_n = 0$, then $\alpha_n = 0$ for all $n \geq 1$. First observe that $(F_{2n})_{n\geq 1}$ and $(F_{2n-1})_{n\notin J}$ are all in the normal space N_q. As $p_n \in T_q$, this implies

$$0 = \sum_{k\geq 1}\alpha_k\langle F_k, p_n\rangle = \sum_{k\in J}\alpha_{2k-1}\langle F_{2k-1}, p_n\rangle$$

where $\langle f, g\rangle$ denotes the inner product $\int_0^1 fg dx$ in L^2. Now

$$\langle F_{2k-1}, p_n\rangle_{k,n\in J}$$

is a finite matrix as q is a finite band potential and is regular according to Lemmas 9 and 10. Thus $\alpha_{2k-1} = 0$ for all $k \in J$. For $n \notin J$, define

$$r_{2n} := \frac{d}{dx}(f_{2n}f_{2n-1}) \quad\text{and}\quad r_{2n-1} := \frac{d}{dx}(f_{2n}^2 - f_{2n-1}^2).$$

According to Lemma 5, $\langle F_{2k}, r_{2n}\rangle = c_{2n}\delta_{nk}$ with $c_{2n} \neq 0$ and $k \geq 1$ and $\langle F_{2k-1}, r_{2n}\rangle = 0$ for all $k \notin J$. Thus

$$0 = \sum_{k\geq 1}\alpha_{2k}\langle F_{2k}, r_{2n}\rangle + \sum_{k\notin J}\alpha_{2k-1}\langle F_{2k-1}, r_{2n}\rangle = \alpha_{2n}c_{2n}$$

and thus $\alpha_{2n} = 0$ for $n \notin J$. Similarly, for all $k \geq 1$, $\langle F_{2k}, r_{2n-1}\rangle = 0$ and $\langle F_{2k-1}, r_{2n-1}\rangle = c_{2n-1}\delta_{nk}$ with $c_{2n-1} \neq 0$, again according to Lemma 5.

This implies that $\alpha_{2n-1} = 0$ for $n \notin J$, and thus $\sum_{n\in J} \alpha_{2n}F_{2n} = 0$. By Lemma 7, $\alpha_{2n} = 0\ \forall n \in J$.

Proof of Theorem 1 for a general potential. — As in the proof for the case where q is a finite band potential it suffices to show that $(F_n)_{n\geq 1}$ is ω-linearly independent. Let $(\alpha_n)_{n\geq 1}$ be a sequence of real numbers such that $\sum_{n\geq 1} \alpha_n^2 \|F_n\|^2 < \infty$ and $\sum_{n\geq 1} \alpha_n F_n = 0$. It is to prove that $\alpha_n = 0$ $\forall n \geq 1$. Introduce $J_1 := \{n \geq 1 : \lambda_{2n-1} < \mu_n < \lambda_{2n}\}$. Again, $(F_{2n})_{n\geq 1}$ and $(F_{2n-1})_{n\notin J}$ are elements of the normal space N_q and the potentials $p_n := \frac{d}{dx}\frac{\partial\Delta(\lambda)}{\partial q(x)}|_{\lambda=\mu_n}$ are in T_q.

(a) It follows from Lemma 10, that $\alpha_{2n-1} = 0$ for $n \in J \setminus J_1$.

(b) Next I want to prove that $\alpha_{2n-1} = 0$ for $n \in J_1$. For that purpose define $A_{nk} := d_q\vartheta_k[4n\pi p_n]$ for $k, n \in J_1$. According to Lemma 8, A_{nk} is given by

$$A_{nk} = G_{2k}(0)^{-1}\varepsilon_k \cos\vartheta_k f_{2k-1}(0) n\pi(y_1(1,\mu_n) - y_2'(1,\mu_n)) \frac{\lambda_{2k} - \lambda_{2k-1}}{(\mu_n - \lambda_{2k-1})(\lambda_{2k} - \mu_n)}.$$

Define $B_{nk} := A_{nk} - A_{nn}\delta_{nk}$ and $C_{nk} := A_{nk}\delta_{nk}$.

LEMMA 11.

(1) $B : \ell^2(J_1) \to \ell^2(J_1),\quad (x_k)_{k\in J_1} \mapsto \big(\sum_{k\in J_1} B_{nk}x_k\big)_{n\in J_1}$ *is a linear operator of trace class.*

(2) $C : \ell^2(J_1) \to \ell^2(J_1)$, $(x_k)_{k\in J_1} \mapsto (A_{nn}x_n)_{n\in J_1}$ *is a bounded invertible linear operator with a bounded inverse.*

Proof. — First let us consider the asymptotics : $G_{2k}(0)^{-1} = \frac{1}{\sqrt{2}} + O(\frac{1}{k})$. Further $|y_1(1,\mu_n) - y_2'(1,\mu_n)| = \sqrt{\Delta(\mu_n)^2 - 4}$ and thus by Lemma 3.8

$$n\pi|y_1(1,\mu_n) - y_2'(1,\mu_n)| = \sqrt{(\lambda_{2n} - \mu_n)}\sqrt{\mu_n - \lambda_{2n-1}}\big(1 + O(\frac{\log n}{n})\big).$$

Further as $\cos^2\vartheta_k = f_{2k}(0)^2/(f_{2k-1}(0)^2 + f_{2k}(0)^2)$ and $f_j(0)^2 = -\frac{y_2(1,\lambda_j)}{\dot{\Delta}(\lambda_j)}$. Corollary 3.9 implies that

$$\cos\vartheta_k f_{2k-1}(0)(\lambda_{2k} - \lambda_{2k-1}) = \sqrt{2}\sqrt{\lambda_{2k} - \mu_k}\sqrt{\mu_k - \lambda_{2k-1}}\big(1 + O(\frac{\log k}{k})\big).$$

To prove (1) it suffices to show that

$$\sum_{\substack{k\neq n\\ k,n\in J_1}} \frac{1}{|\mu_n - \lambda_{2k-1}||\mu_n - \lambda_{2k-1}|} < \infty.$$

This follows immediately from the asymptotics $|\mu_n - \lambda_j| = \pi^2|n^2-k^2|+O(1)$ for $j \in \{2k-1, 2k\}$. (2) follows immediately from the fact that $A_{nn} \neq 0$ for all $n \in J_1$ and from the asymptotics derived above : $|A_{nn}| = 1 + O(\frac{\log n}{n})$.

Back to the proof of (b), it follows from Lemma 11 that $C^{-1}A = \mathrm{Id} + C^{-1}B$ is a bounded operator of determinant class, *i.e.* has a Fredholm determinant $\det(C^{-1}A)$. To prove (b) it suffices to show that $C^{-1}A$ is $1-1$ or, equivalently, that $\det C^{-1}A \neq 0$. The Fredholm determinant $\det C^{-1}A$ is a limit of determinants of finite matrices, *i.e.* $\det C^{-1}A = \lim (C^{-1}A)_{J'}$ where $(C^{-1}A)_{J'}$ denotes the $J' \times J'$ matrix $(C^{-1}A)_{k,n\in J'}$ with $J' \subseteq J_1$ finite. As C^{-1} is diagonal, $\det(C^{-1}A)_{J'} = \det A_{J'} / \det C_{J'}$. Thus

$$\begin{aligned}\det A_{J'} / \det C_{J'} &= \det\Big(\frac{1}{\mu_n - \lambda_{2k-1}} - \frac{1}{\mu_n - \lambda_{2k}}\Big)_{n,k\in J'}\\ &\qquad / \prod_{n\in J'}\Big(\frac{1}{\mu_n - \lambda_{2n-1}} + \frac{1}{\lambda_{2n} - \mu_n}\Big).\end{aligned}$$

As in the proof of Lemma 7, one writes

$$\begin{aligned}&\det\Big(\frac{1}{\mu_n - \lambda_{2k-1}} - \frac{1}{\mu_n - \lambda_{2k}}\Big)_{n,k\in J'}\\ &= \sum_x (-1)^{|\varepsilon|} \det\Big(\frac{1}{\mu_n + x_k}\Big)_{n,k\in J'}\\ &= \sum_x (-1)^{|\varepsilon|} \frac{\prod_{n>k}(\mu_n - \mu_k)\prod_{n>k}(x_n - x_k)}{\prod_{n,k}(\mu_n + x_k)}\end{aligned}$$

where $x = (x_k)_{k\in J'}$ with $-x_k \in \{\lambda_{2k-1}, \lambda_{2k}\}$ and $\varepsilon = (\varepsilon_k)_{k\in J'}$ with $\varepsilon_k = 0$ if $-x_k = \lambda_{2k-1}$ and $\varepsilon_k = 1$ if $-x_k = \lambda_{2k}$. Finally $|\varepsilon| = \sum_{k\in J'} \varepsilon_k$. Then

$$\begin{aligned}&\det\Big(\frac{1}{\mu_n - \lambda_{2k-1}} - \frac{1}{\mu_n - \lambda_{2k}}\Big)_{n,k\in J'}\\ &\quad = \sum_x \Big(\prod_{n\in J'} \frac{1}{|\mu_n + x_n|}\Big) \prod_{n\in J'}\prod_{k>n}\Big(1 - \frac{x_k + \mu_k}{\mu_n + x_k}\Big)\Big(1 - \frac{x_k + \mu_k}{x_n + \mu_k}\Big).\end{aligned}$$

Now observe that for $n + 1 \leq k$,

$$\Big(1 - \frac{\mu_k + x_k}{\mu_n + x_k}\Big)\Big(1 - \frac{\mu_k + x_k}{x_n + \mu_k}\Big) = 1 - \frac{(x_k + \mu_k)(\mu_n + x_n)}{(\mu_n + x_k)(x_n + \mu_k)}$$

is always strictly positive and that there exists $K' > 0$ such that

$$\frac{|x_k + \mu_k||\mu_n + x_n|}{|\mu_n + x_k||x_n + \mu_k|} \le \frac{K'}{(k^2 - n^2)^2}.$$

Then

$$\sum_{n\ge 1}\sum_{k\ge n+1} \frac{|x_k + \mu_k||\mu_n + x_n|}{|\mu_n + x_k||x_n + \mu_k|} \le K' \sum_{n\ge 1} \frac{1}{(2n)^2} \sum_{\ell\ge 1} \frac{1}{\ell^2} < \infty.$$

Choose N such that $\frac{K'}{(k^2 - n^2)^2} \le \frac{1}{2}$ for all $k \ge n + 1 \ge N + 1$. Using that $\log(1 + t) \ge \log(1 - |t|) \ge -1$ for $|t| \le 1/2$ one deduces that there exists a constant $K > 0$, independent of $J' \subseteq J_1$ and $x = (x_k)_{k\in J_1}$ with $-x_k \in \{\lambda_{2k-1}, \lambda_{2k}\}$ such that

$$0 < K \le \prod_{\substack{n\ge N+1\\ n\in J'}} \prod_{\substack{k\ge n+1\\ k\in J'}} \left(1 - \frac{|x_k + \mu_k||x_n + \mu_n|}{|\mu_n + x_k||x_n + \mu_k|}\right),$$

$$0 < K \le \prod_{\substack{1\le n\le N\\ n\in J'}} \prod_{\substack{k\ge N+1\\ k\in J'}} \left(1 - \frac{|x_k + \mu_k||x_n + \mu_n|}{|\mu_n + x_k||x_n + \mu_k|}\right)$$

and

$$0 < K \le \prod_{\substack{1\le n\le N\\ n\in J'}} \prod_{\substack{n+1\le k\le N\\ k\in J'}} \left(1 - \frac{(x_k + \mu_k)(x_n + \mu_n)}{(\mu_n + x_k)(x_n + \mu_k)}\right).$$

Thus, for all $J' \subseteq J_1$ finite

$$\det\left(\frac{1}{\mu_n - \lambda_{2k-1}} - \frac{1}{\mu_n - \lambda_{2k}}\right)_{n,k\in J'} \ge K^3 \sum_x \prod_{n\in J'} \frac{1}{|\mu_n + x_n|}.$$

But

$$\det C_{J'} = \sum_x \prod_{n\in J'} \frac{1}{|\mu_n + x_n|}.$$

Thus I have shown that $\det(C^{-1}A)_{J'} \ge K^3$ independent of $J' \subseteq J_1$. This implies that $\det C^{-1}A \ge K^3 > 0$ and (b) follows.

(c) As in the proof of Theorem 1 for finite band potentials one shows that $\alpha_{2n} = \alpha_{2n-1} = 0$ for all $n \notin J$.

(d) It remains to show that $\alpha_{2n} = 0$ for $n \in J$. By Lemma 6 I may assume that $\lambda_{2n-1} < \mu_n < \lambda_{2n}$ for all $n \in J$, as the property of $(f_{2n}^2 - f_{2n-1}^2)_{n\in J}$ being ω-linearly independent is invariant under translation of the potential. The argument is similar to the one of (b). I introduce for $n, k \in J = J_1$

$$A_{nk} := -\dot{y}_2(1,\mu_n)(\lambda_{2n} - \lambda_{2n-1})/\sqrt{(\lambda_{2n} - \mu_n)(\mu_n - \lambda_{2n-1})} \int_0^1 \frac{d}{dx}(f_{2k}^2 - f_{2k-1}^2) g_n^2 dx.$$

As in the proof of Lemma 7, A_{nk} can be computed, using Wronskian identities, to give

$$((\lambda_{2n} - \lambda_{2n-1})/2) \cdot \dot{y}_2(1,\mu) \frac{g_n'(1)^2 - g_n'(0)^2}{\sqrt{(\lambda_{2n} - \mu_n)(\mu_n - \lambda_{2n-1})}} \Big(- \frac{f_{2k}(0)^2}{\mu_n - \lambda_{2k}} + \frac{f_{2k-1}(0)^2}{\mu_n - \lambda_{2k-1}} \Big).$$

I define $B_{nk} := A_{nk} - A_{nk}\delta_{nk}$ and $C_{nk} := A_{nk}\delta_{nk}$.

LEMMA 12. — (1) $B : \ell^2(J) \to \ell^2(J)$, $(x_k)_{k \in J} \mapsto \big(\sum_{k \in J} B_{nk} x_k \big)_{n \in J}$ *is a linear operator of trace class.*

(2) $C : \ell^2(J) \to \ell^2(J)$, $(x_k)_{k\in J} \mapsto (A_{nn}x_n)_{n\in J}$ *is a bounded invertible linear operator with a bounded inverse.*

Proof. — First let us consider the asymptotics. Recall that $f_j(0)^2 = -y_2(1,\lambda_j)\dot{\Delta}(\lambda_j)$; thus Corollary 3.9 implies that

$$f_j(0)^2 = \frac{|\lambda_j - \mu_n|}{(\lambda_{2n} - \lambda_{2n-1})/2}(1 + O(\frac{\log n}{n}))$$

for $j \in \{2n-1, 2n\}$. Moreover,

$$g_n'(1)^2 - g_n'(0)^2 = \frac{y_2'(1,\mu_n)^2 - 1}{\|y_2(\cdot,\mu_n)\|^2}.$$

But $\|y_2(\cdot,\mu_n)\|^2 = y_2(1,\mu_n)y_2'(1,\mu_n)$ (*cf.* e.g. [PT], p.30) and, by the Wronskian identity $1 = y_1(1,\mu_n)y_2'(1,\mu_n)$, one deduces that

$$|\dot{y}_2(1,\mu_n)(g_n'(1)^2 - g_n'(0)^2)| = |y_1(1,\mu_n) - y_2'(1,\mu_n)| = \sqrt{\Delta^2(\mu_n) - 4}.$$

By Lemma 3.8,

$$\sqrt{\Delta^2(\mu_n) - 4} = \sqrt{(\lambda_{2n} - \mu_n)(\mu_n - \lambda_{2n-1})}(1 + O(\frac{\log n}{n})).$$

Thus

$$\frac{|\dot{y}_2(1,\mu_n)(g_n'(1)^2 - g_n'(0)^2)|}{\sqrt{(\lambda_{2n} - \mu_n)(\mu_n - \lambda_{2n-1})}} = 1 + O(\frac{\log n}{n}).$$

Next

$$- \frac{f_{2k}(0)^2}{\mu_n - \lambda_{2k}} + \frac{f_{2k-1}(0)^2}{\mu_n - \lambda_{2k-1}}$$
$$= \Big(\frac{\lambda_{2k} - \mu_k}{(\lambda_{2k} - \lambda_{2k-1})/2} \frac{1}{\lambda_{2k} - \mu_n} + \frac{\mu_k - \lambda_{2k-1}}{(\lambda_{2k} - \lambda_{2k-1})/2} \frac{1}{\mu_n - \lambda_{2k-1}} \Big)$$
$$(1 + O(\frac{\log n}{n})).$$

To prove (1) it thus suffices to show that

$$\sum_{\substack{k\neq n\\ k,n\in J}} \Big(\frac{\lambda_{2n}-\lambda_{2n-1}}{|\lambda_{2k}-\mu_n|}+\frac{\lambda_{2n}-\lambda_{2n-1}}{|\lambda_{2k-1}-\mu_n|}\Big)<\infty.$$

Observe that

$$|\lambda_{2k-1}-\mu_n|, |\lambda_{2k}-\mu_n| = \pi^2|k^2-n^2|+O(1).$$

Thus

$$\sum_{n\geq 1}\sum_{k\geq n+1}\frac{\lambda_{2n}-\lambda_{2n-1}}{(k-n)(k+n)} = \sum_{n\geq 1}(\lambda_{2n}-\lambda_{2n-1})\Big(\frac{1}{n}\Big)^{7/12}$$

$$\sum_{k\geq n+1}\frac{1}{(k-n)}\Big(\frac{1}{k+n}\Big)^{5/12}$$

By Hölder's inequality for $p=3/2$ and $p'=3$ one obtains

$$\sum_{k\geq n+1}\frac{1}{k-n}\Big(\frac{1}{k+n}\Big)^{3/12} \geq \Big(\sum_{k\geq n+1}\Big(\frac{1}{k-n}\Big)^p\Big)^{1/p}$$

$$\Big(\sum_{k\geq n+1}\Big(\frac{1}{k+n}\Big)^{p'5/12}\Big)^{1/p'}<\infty.$$

Similarly one shows that $\sum_{n\geq 1}\sum_{k\leq n-1}\frac{\lambda_{2n}-\lambda_{2n-1}}{(n-k)(k+n)}<\infty$. This proves (1).

(2) follows from the fact that $A_{nn}\neq 0$ for all $n\in J$ and the asymptotics $|A_{nn}|=1+O\big(\frac{\log n}{n}\big)$.

Back to the proof of (d), it follows from Lemma 12 that $C^{-1}A = \mathrm{Id}+C^{-1}B$ is a bounded operator of determinant class. By the same argument as in (b) it suffices to prove that $\det C^{-1}A\neq 0$. But $\det C^{-1}A = \lim \det A_{J'}/\det C_{J'}$ where $A_{J'}$ denotes the $J'\times J'$ matrix $(A_{nk})_{n,k\in J'}$ with $J'\subseteq J$ finite and where $C_{J'}$ is defined similarly.

Thus $\det A_{J'}/\det C_{J'}$ is given by

$$\det\Big(-\frac{f_{2k}(0)^2}{\mu_n-\lambda_{2k}}+\frac{f_{2k-1}(0)^2}{\mu_n-\lambda_{2k-1}}\Big)_{n,k\in J'}$$

$$\Big/\prod_{n\in J'}\Big(-\frac{f_{2n}(0)^2}{\mu_n-\lambda_{2n}}+\frac{f_{2n-1}(0)^2}{\mu_n-\lambda_{2n-1}}\Big).$$

As in the proof of Lemma 7, one writes

$$\det\Big(-\frac{f_{2k}(0)^2}{\mu_n-\lambda_{2k}}+\frac{f_{2k-1}(0)^2}{\mu_n-\lambda_{2k-1}}\Big)_{n,k\in J'}$$

$$=\sum_x(-1)^{|\varepsilon|}\prod_{-x_k=\lambda_{2k}}f_{2k}(0)\prod_{-x_k=\lambda_{2k-1}}f_{2k-1}(0)\det\Big(\frac{1}{\mu_n+x_k}\Big)_{n,k\in J'}$$

where $x = (x_k)_{k\in J'}$ and $\varepsilon = (\varepsilon_k)_{k\in J'}$ are defined as in Lemma 7. Similarly

$$\prod_{n\in J'} \Big(- \frac{f_{2n}(0)^2}{\mu_n - \lambda_{2n}} + \frac{f_{2n-1}(0)^2}{\mu_n - \lambda_{2n-1}} \Big)$$
$$= \sum_x \prod_{-x_k=\lambda_{2k}} f_{2k}(0) \prod_{-x_k=\lambda_{2k-1}} f_{2k-1}(0) \prod_n \frac{1}{|\mu_n + x_n|}.$$

Arguing as in the proof of (b) we see that

$$\det A_{J'} / \det C_{J'} = \big(\sum_x R_x \cdot S_x \big) / \sum_x R_x$$

where

$$R_x = \prod_{-x_k=\lambda_{2k}} f_{2k}(0) \prod_{-x_k=\lambda_{2k}} f_{2k-1}(0) \prod_n \frac{1}{|\mu_n + x_n|}$$

and where

$$S_x = \prod_{n\in J'} \prod_{k>n} \Big(1 - \frac{x_k + \mu_k}{\mu_n + x_k} \Big) \Big(1 - \frac{x_k + \mu_k}{x_n + \mu_k} \Big).$$

Observe that for all x, $R_x > 0$ and that from the proof of (b), $S_x > K^3$ where K does not depend on $(x_k)_{k\in J'}$ and $J' \subseteq J$. Thus $\det A_{J'} / \det C_{J'} \geq K^3 > 0$ for all $J' \subseteq J$ or $\det C^{-1} A \geq K^3 > 0$. Now (d) follows and the proof of Theorem 1 is finished.

7. Global properties of Φ.

In this section I show

THEOREM 1. — *Φ is $1-1$ and onto.*

As an immediate consequence we obtain, by applying Theorem 6.1

THEOREM 2. — *Φ and its inverse are real analytic isomorphisms.*

Proof (of Theorem 1). — Denote by E the subspace of all potentials q in L_0^2 with $q(x) = q(1-x)$. From [GT1] together with Proposition 4.3 follows that $\Phi|_E$ is $1-1$ and $\Phi(E) = \{R = (R_n)_{n\geq 1} \in \mathcal{M} : R_n$ diagonal $\forall n \geq 1\}$. Further it is well known that for all $q \in L_0^2$, $\mathrm{Iso}\, q \cap E \neq \emptyset$ (*cf.* e.g. [GT2]). In view of Proposition 4.2 it then suffices to prove that $\Phi|_{\mathrm{Iso}\, q}$ is $1-1$ and that $\Phi(\mathrm{Iso}\, q) = \mathrm{Iso}\, \Phi(q)$. Using that $\mathrm{Iso}\, p$ is compact, I show that $\Phi|_{\mathrm{Iso}\, q}$ is $1-1$ as follows (*cf.* [GT2] for a similar argument) :

Let K be the set of points in $\operatorname{Iso} q$ such that $\Phi(q)$ has more than one preimage. As Φ is a local homeomorphism, K is open in $\operatorname{Iso} q$. K contains no even potentials and $K \neq \operatorname{Iso} q$, as $E \cap \operatorname{Iso} q \neq \emptyset$. Further K is closed. Indeed assume there exists a sequence $(q_j)_{j\geq 1}$ in K. Then there exists a convergent subsequence, again denoted by $(q_j)_{j\geq 1}$, and a convergent sequence $(p_j)_{j\geq 1}$ in $\operatorname{Iso} q$ such that $\Phi(q_j) = \Phi(p_j)$, but $q_j \neq p_j$ $\forall j \geq 1$. Then $\lim\limits_{j\to\infty} q_j \neq \lim\limits_{j\to\infty} p_j$ as Φ is a local homeomorphism. Thus K is open and closed and properly contained in $\operatorname{Iso} q$, hence empty. To prove that $\Phi(\operatorname{Iso} q) = \operatorname{Iso} \Phi(q)$ observe that both $\operatorname{Iso} q$ and $\operatorname{Iso} \Phi(q)$ are connected tori of the same, generically infinite, genus. If q has the property that $J := \{n \geq 1 : \lambda_{2n-1}(q) < \lambda_{2n}(q)\}$ is finite, then both $\operatorname{Iso} q$ and $\operatorname{Iso} \Phi(q)$ are of finite genus and thus $\Phi(\operatorname{Iso} q) = \operatorname{Iso} \Phi(q)$. To prove $\Phi(\operatorname{Iso} q) = \operatorname{Iso} \Phi(q)$ for arbitrary q, let $R = (R_k)_{k\geq 1} \in \operatorname{Iso} \Phi(q)$ and assume without loss of generality that $\mu_n(q) = \lambda_{2n-1}(q)$ $\forall n \geq 1$. I have to show that $R \in \Phi(\operatorname{Iso} q)$. Define a sequence $(R^{(j)})_{j\geq 1}$ in $\mathcal{M}$ as follows : $R_k^{(j)} = R_k$ for $1 \leq k \leq j$ and $R_k^{(j)} = 0$ for $k \geq j+1$. Then $\lim\limits_{j\to\infty} R^{(j)} = R$ in $\mathcal{M}$. Define q_j to be the unique even potential with $\lambda_{2k-1}(q) = \mu_k(q_j) = \lambda_{2k-1}(q_j)$ and $\lambda_{2k}(q) = \lambda_{2k}(q_j)$ for $1 \leq k \leq j$ and $\lambda_{2k-1}(q_j) = \lambda_{2k}(q_j)$ for $k \geq j+1$. Then $\Phi(q_j) \in \operatorname{Iso}(R^{(j)})$ and $\lim\limits_{j\to\infty} \Phi(q_j) = \Phi(q)$ in $\mathcal{M}$. As $\Phi|_E : E \to \{S \in \mathcal{M} : S_k \text{ diagonal } \forall k\}$ is a homeomorphism, one concludes that $\lim\limits_{j\to\infty} q_j = q$ in L_0^2. Define $p_j \in \operatorname{Iso} q_j$ to be the unique potential with $\Phi(p_j) = R^{(j)}$. Then $\|q_j\|_{L^2} = \|p_j\|_{L^2}$ and thus there exists a subsequence, again denoted by p_j which is weakly convergent to $p \in L_0^2$. Clearly $\Phi_n(p) = \lim\limits_{j\to\infty} \Phi_n(p_j) = \lim\limits_{j\to\infty} R_n^{(j)} = R_n$ for all $n \geq 1$, as Φ_n is compact. This proves that $\Phi(p) = R$.

Denote by H_0^n the Sobolev space $\{f \in H_{\text{per}}^n : \int_0^1 f(x)dx = 0\}$. Clearly $H_0^{n+1} \subseteq H_0^n \subseteq H_0^0 \equiv L_0^2$. It is a well known result that $q \in H_0^n$ if and only if $q \in L_0^2$ and $(\lambda_{2k} - \lambda_{2k-1})_{k\geq 1} \in \ell_n^2$. From the representation

$$\Phi_k(q) = \frac{\lambda_{2k} - \lambda_{2k-1}}{2} \begin{pmatrix} \cos 2\vartheta_k & \sin 2\vartheta_k \\ \sin 2\vartheta_k & -\cos 2\vartheta_k \end{pmatrix}$$

one deduces from Theorem 2 the following

COROLLARY 3. — *$\Phi : H_0^n \to \mathcal{M}^n$ is a real analytic, $1-1$ and onto where*

$$\mathcal{M}^n := \{(R_k)_{k\geq 1} \in \mathcal{M} : (R_k)_{k\geq 1} \in \ell_n^2\}.$$

Remark. — It is very likely that $\Phi : H_0^n \to \mathcal{M}^n$ is bianalytic for all n. However I have not verified this statement.

As the last result in this section I want to discuss the S^1 action on $\mathrm{Iso}\, q$ generated by translations.

THEOREM 4. — *Let $q \in H_0^1$. Then for all $n \geq 1$ with $\lambda_{2n-1} < \lambda_{2n}$, there exists a continuouly differentiable function $\varphi_n : \mathbf{R} \to \mathbf{R}$ such that*

$$\Phi_n(T_t q) = \frac{\lambda_{2n} - \lambda_{2n-1}}{2} \begin{pmatrix} \cos 2\varphi_n(t) & \sin 2\varphi_n(t) \\ \sin 2\varphi_n(t) & -\cos 2\varphi_n(t) \end{pmatrix}.$$

Moreover the winding number $(2\varphi_n(1) - 2\varphi_n(0))/2\pi$ is equal to n.

Proof. — Observe that for $n \geq 1$ with $\lambda_{2n-1} < \lambda_{2n}$, $f_k(x, T_t q) = \pm f_k(x+t, q)$ for $k \in \{2n-1, 2n\}$. Instead of expressing $G_{2n-1}(x, T_t q)$ and $G_{2n}(x, T_t q)$ in terms of $f_{2n-1}(x, T_t q)$ and $f_{2n}(x, T_t q)$ I use $f_{2n-1}(x+t, q)$ and $f_{2n}(x+t, q)$. It was proved in section 3, that $W[f_{2n-1}(x,q), f_{2n}(x,q)] \neq 0$ for all x. Denote the zeroes of $f_{2n-1}(x,q)$ and $f_{2n}(x,q)$ by $0 \leq y_1 < \cdots < y_n < 1$ and $0 \leq z_1 < \cdots < z_n < 1$ respectively. These zeroes interlace. To make notation easier I assume that $0 = y_1 < z_1 < \cdots < y_n < z_n < 1$. Recall that by the definition of f_k's, $f'_{2n-1}(0) > 0$ and $f_{2n}(0) > 0$. It follows that there exists a continuously differentiable function $\varphi_n(t)$ such that

$$\begin{aligned} G_{2n-1}(x, T_t q) &= \cos \varphi_n(t) f_{2n-1}(x+t, q) - \sin \varphi_n(t) f_{2n}(x+t, q) \\ G_{2n}(x, T_t q) &= \sin \varphi_n(t) f_{2n-1}(x+t, q) + \cos \varphi_n(t) f_{2n}(x+t, q). \end{aligned}$$

Taking the derivative of the first equation with respect to t at $x = 0$ leads to, using that q is in H_0^1,

$$0 = -\frac{d}{dt}\varphi_n(t) G_{2n}(0, T_t q) + \frac{d}{dx}|_{x=0} G_{2n-1}(x, T_t q).$$

By definition $G'_{2n-1}(0, T_t q) > 0$ for all t. Further, by a simple verification, $G_{2n}(0, T_t q) > 0$ for all t. This implies $\frac{d}{dt}\varphi_n(t) > 0$ $\forall t$. Moreover $\varphi_n(1) - \varphi_n(0) = \pi k$ for some $k \geq 1$. As $f_{2n-1}(x,q)$ has precisely n zeroes in $[0,1)$, it follows that $\varphi_n(1) - \varphi_n(0) = \pi n$.

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