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HOLOMORPHIC FOLIATIONS IN CP(2) HAVING AN INVARIANT ALGEBRAIC CURVE

by D. CERVEAU and A. LINS NETO

0. Introduction.

A holomorphic foliation $\mathcal{F}$ in $\mathbf{CP}(2)$ can be given in at least three equivalent ways :

(*a*) In affine coordinates $(x,y) \in \mathbf{C}^2$, by a polynomial vector field $P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$, g.c.d. $(P,Q) = 1$. The singular set in the coordinate system is by definition the variety $\{P=Q=0\}$. The holomorphic solutions of $\frac{dx}{dt} = P(x,y)\ \frac{dy}{dt} = Q(x,y)$ define the leaves of $\mathcal{F}/\mathbf{C}^2$ outside $\{P=Q=0\}$.

(*b*) In affine coordinates $(x,y) \in \mathbf{C}^2$ by a polynomial 1-form $\tilde{\omega} = P(x,y)\,dy - Q(x,y)\,dx$. In this case $\{P=Q=0\}$, and the leaves of $\mathcal{F}/\mathbf{C}^2$ outside $\{P=Q=0\}$ are the solutions of the differential equation $\tilde{\omega} = 0$.

(*c*) Let us suppose that $\tilde{\omega} = P\,dy - Q\,dx$ is as in (*b*) and $\max(dg(P), dg(Q)) = n \geqslant 0$. Let $\pi : \mathbf{C}^2 - \{(x,y,z); z=0\} \to \mathbf{C}^2$ given by $r(x,y,z) = (x/z, y/z)$. The $\pi^*\tilde{\omega} = \tilde{\omega}^*$ has poles at $z = 0$ and so we can write $\tilde{\omega}^* = z^{-k}\omega$, where ω is holomorphic and we choose k so that ω is not divisible by z. Remark that ω satisfies the following properties :

(*i*) $\omega \wedge d\omega = 0$ (integrability condition)

(*ii*) if $\omega = A\,dx + B\,dy + C\,dz$, then A, B, C are homogeneous of the same degree and $xA + yB + zC \equiv 0$. We will write this

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condition as $i_E\omega = 0$, where $E = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y} + z\dfrac{\partial}{\partial z}$ is the radial vector field and i_E is the interior productor by E. Clearly (*ii*) implies (*i*).

The singular set of $\mathcal{F}$ in homogenous coordinates is given by:

$$\operatorname{sing} \mathcal{F} = \{A=B=C=0\}.$$

Observe that if $\tilde{\omega} = P\,dy - Q\,dx$ as in (*b*) then $\{P=Q=0\} = \{A(x,y,1)=B(x,y,1)=C(x,y,1)=0\}$.

The *degree of the foliation* $\mathcal{F}$ is defined as follows : take an affine coordinate system and write $\mathcal{F}$ as the solutions of $P\,dy - Q\,dx = 0$. Let $\{L=0\}$ be a straight line in $\mathbf{C}^2$. If L is not invariant by $\mathcal{F}$, then the number of points $p \in L$ such that either $p \in \{P=Q=0\}$, or the leaf of $\mathcal{F}$ through p is tangent to L, is finite. The maximal number of such points is bounded by $\max(dg(P), dg(Q)) = l$. In fact for a generic line L, this number is constant ([4]). We call this number the degree of $\mathcal{F}$. If the degree of $\mathcal{F}$ is n and $\tilde{\omega}$ is as above then $P = p + xg$ and $Q = q + yg$, where $\max(dg(p), dg(q)) \leqslant n$, g is a homogeneous polynomial of degree n, and $\max(dg(p), dg(q)) = n$ if $g \equiv 0$ (cf. [4]). If ω is obtained from $\tilde{\omega}$ as in (*c*), then the degree of all coefficients of ω is exactly $n + 1$.

We will say that $\mathcal{F}$ is represented by $P\dfrac{\partial}{\partial x} + Q\dfrac{\partial}{\partial y}$, $P\,dy - Q\,dx$ or $\omega = A\,dx + B\,dy + C\,dz$ and that each one of these objects represent $\mathcal{F}$.

We are mainly interested here in the case in which there is an algebraic curve $S \subset \mathbf{CP}(2)$ invariant by $\mathcal{F}$. This condition can be expressed as follows : let $f = 0$ be a reduced equation of S in homogeneous coordinates and ω be a 1-form in $\mathbf{C}^3$ which represents $\mathcal{F}$. Then S is invariant by $\mathcal{F}$ if, and only if, $df \wedge \omega = f\theta$, where θ is a 2-form with homogeneous polynomial coefficients. We will say also that S (or f) is a *separatrix* of $\mathcal{F}$.

One of the problems we will consider here is the following :

Problem (Poincaré [5]). — Is it possible to bound the degree of S in terms of the degree of $\mathcal{F}$?

Of course, in general it is not possible as shows the example $px\dfrac{\partial}{\partial x} + qy\dfrac{\partial}{\partial y}$ ($p \neq q$ positive integers) which has $y^p - x^q = 0$ as a separatrix, but has degree one.

However, if we assume some hypothesis on S or $\mathcal{F}$ it is possible as we will see. The result we will prove next, will provide such a bound when the singularities of S are of nodal type, that is with singularities of normal crossing type.

Let S be an irreducible curve on $\mathbf{CP}(2)$ of degree m and $\mathcal{F}$ be a foliation of degree n having S as a separatrix. For each singularity p of $\mathcal{F}$ such that $p \in S$, and each local branch B of S passing through p we associate the *multiplicity of $\mathcal{F}$ at B*, which is defined as follows : take a vector field $X = P\dfrac{\partial}{\partial x} + Q\dfrac{\partial}{\partial y}$ which represents $\mathcal{F}$ in a neighbourhood of p and a minimal Puiseux's parametrization of B, say $\varphi : D \to \mathbf{C}^2$, $\varphi(0) = 0$, where D is a disk centered at $0 \in \mathbf{C}$. Then the multiplicity of $\mathcal{F}$ at B is by definition $i(\mathcal{F},B) =$ order of $\varphi^*(X)$ at $T = 0 \in D$([8]). Remark that $\varphi^*(X)$ is holomorphic and vanishes at 0, so $i(\mathcal{F},B) > 0$. Moreover $i(\mathcal{F},B)$ does not depend of the local representation of $\mathcal{F}$ (see [8]).

PROPOSITION. – *In the above situation we have :*

$$2 - 2g(S) = \sum_B i(\mathcal{F},B) - m(n-1) \tag{1}$$

where $g(S)$ is the topological genus of S and the sum is taken over all local branches of S passing through the singularities of $\mathcal{F}$ in S.

Proof. – Take an affine coordinate system (x,y) such that S cuts the line at infinity L_∞ transversely. Let $X = P\dfrac{\partial}{\partial x} + Q\dfrac{\partial}{\partial y}$ represents $\mathcal{F}$ in this coordinate system. The $X|_S$ is a meromorphic vector field having poles at $S \cup L_\infty$ of order $n-1$, as the reader can easily verify. Let $\pi : \tilde{S} \to S$ be a resolution of S by blowing-ups at the singular points of S. Then $\tilde{S}$ is smooth and $2 - 2g(S) = \chi(\tilde{S})$ the Euler characteristic of $\tilde{S}$. On the other hand $\pi^*(X|S) = \tilde{X}$ is a meromorphic vector field in $\tilde{S}$.

For each local branch B of S passing through a singular point p of X, we obtain a singular point $p(B)$ of $\tilde{X}$ of order $i(\mathcal{F},B)$. Since the points of $S \cap L_\infty$ are poles of $X|_S$ of order $n-1$, we obtain for each such point a pole of $\tilde{X}$ of the same order. Finally by the Poincaré-Hopf formula we obtain (1). □

COROLLARY. — *Let S be an irreducible curve in* $\mathbf{CP}(2)$ *whose singularities are all of nodal type. Let $\mathcal{F}$ be a foliation having S as a separatrix and such that all singular points of $\mathcal{F}$ on S have multiplicity* 1. *Then we have $m \leqslant n+2$, where $m =$ degree (S) and n = degree $(\mathcal{F})$.*

Remark. — The multiplicity of a singular point p of $\mathcal{F}$ represented locally by $P\dfrac{\partial}{\partial x} + Q\dfrac{\partial}{\partial y}$ is the intersection number of P and Q at p. This case occurs, for instance, when all singularities of $\mathcal{F}$ are of Poincaré type.

Proof. — Since the multiplicity of all singularities of $\mathcal{F}$ on S is 1 and S is a nodal curve we have $i(\mathcal{F},B) = 1$ for all local branches B of S through the singular points of $\mathcal{F}$ on S. Let k be the number of nodal points of S. For each nodal point we have 2 local branches of S and so the contributions of the nodal points in $\sum_B i(\mathcal{F},R)$ is precisely $2k$.

Then :

$$2 - 2g(S) = 2k + \ell - m(n-1)$$

where $\ell = \sum_B i(\mathcal{F},B) - 2k \geqslant 0$.

On the other hand the genus formula for S says that :

$$2 - 2g = 2 - 2\left(\frac{(m-1)(m-2)}{2} - k\right) = -m^2 + 3m + 2k.$$

Hence :

$$2k + \ell - m(n-1) = -m^2 + 3m + 2k$$

so we have

$$0 \leqslant \ell = m(n+2-m)$$

which implies the result. □

In chapter 2 we will improve and generalize this result for arbitrary curves S having only nodal points (reducible or irreducible) and for arbitrary foliations. In order to do this we will introduce in chapter 1 the notion of relative division which will play an important rule in § 2. In § 3 we will apply our results to study foliations of degree n having singularities of Poincaré type (we call this set $\mathcal{P}_n$) ; we will prove that the subset $\mathcal{P}_n^0 \subset \mathcal{P}_n$ of foliations having a separatrix is an algebraic set in $\mathcal{P}_n$. At the same time we will give a short proof of Jouanolou's result which says that in all degrees $\geqslant 2$ there exist foliations without a separatrix.

1. Relative division and Noether's Lemma for foliations.

Let S be a curve in $\mathbf{CP}(2)$ with homogeneous equation $f = 0$. We will suppose that f is reduced. Let $\mathcal{F}$ be a homolomorphic foliation in $\mathbf{CP}(2)$ of degree n given by $\omega = 0$, where $\omega = P\,dx + Q\,dy + R\,dz$ and having S as a separatrix. We say that ω has the *property of relative division with respect to f* (notation P.R.D.), if $\omega = gdf + f\mu$, where g is a homogeneous polynomial and μ a 1-form. We say that ω has the property of relative division at a point p, if there exist germs $g_{,p}$ and μ_{1_p} such that $\omega = g_{,p}\,df + f\mu_{1_p}$. We say that ω has P.R.D. locally with respect to f, if it has the property of relative division at all points of $\mathbf{C}^3 - \{0\}$.

PROPOSITION 1. — *If ω has P.R.D. locally with respect to f, then it has P.R.D.*

Proof. — We will use the solution of Cousin's problem in $\mathbf{C}^3 - \{0\}$ by H. Cartan. Let $(U_\alpha)_{\alpha\in A}$ be a covering of $\mathbf{C}^3 - \{0\}$ by polydisks such that $\omega | U_\alpha = g_\alpha\,df + f\mu_\alpha$, where $g_\alpha \in O(U_\alpha)$ and $\mu_\alpha \in \Lambda^1(U_\alpha)$. If $U_\alpha \cap U_\beta \neq \emptyset$ we have $(g_\alpha - g_\beta)\,df = f(\mu_\beta - \mu_\alpha)$. Since f is reduced in $U_\alpha \cap U_\beta$, we have $g_\alpha - g_\beta = h_{\alpha\beta} f$. It's clear that $(h_{\alpha\beta})_{U_\alpha \cap U_\beta \neq \emptyset}$ satisfies the cocycle condition $h_{\alpha\beta} + h_{\beta\gamma} + h_{\gamma\alpha} = 0$, $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$. By Cartan's Theorem [1], we can write $h_{\alpha\beta} = h_\beta - h_\alpha$ where $h_\alpha \in O(U_\alpha)$. This implies that $g_\alpha + h_\alpha f = g_\beta + h_\beta f$ in $U_\alpha \cap U_\beta$, and so we can define a global holomorphic function g on $\mathbf{C}^3 - \{0\}$ by $g | U_\alpha = g_\alpha + h_\alpha f$. By Hartog's extension Theorem, g can be extended to $\mathbf{C}^3$. On the other hand, $\mu_\alpha - h_\alpha\,df = \mu_\beta - h_\beta\,df$ in $U_\alpha \cap U_\beta$, and so we can define a holomorphic 1-form μ in $\mathbf{C}^3 - \{0\}$ by $\mu | U_\alpha = \mu_\alpha - h_\alpha\,df$. This form can be also extended to $\mathbf{C}^3$. It is clear that g and μ satisfy $\omega = g\,df + f\mu$. Now, since f and ω are homogeneous we can suppose that g and μ are homogeneous. $\square$

Before to state the consequences of this result that we will use, we state an analogous result, which is a kind of Noether's lemma for foliation in $\mathbf{CP}(2)$.

Let $\mathcal{F}$ and ω be as before and $\mathcal{I}(\omega) = (P,Q,R)$ be the ideal of components of ω. We will suppose that the set of singularities of ω has codimension 2.

PROPOSITION 2. — *Let g be a homogeneous polynomial in $\mathbf{C}^3$ with the following property : for each $p \in \mathbf{C}^3 - \{0\}$, the germ $g_{,p}$ belongs to the local ideal $\mathscr{I}(\omega), p$. Then $g \in \mathscr{I}(\omega)$, that is, there are homogeneous polynomials A, B, C such that $g = AP + BQ + CR$.*

Proof. — Let $(U_\alpha)_{\alpha \in A}$ be a covering of $\mathbf{C}^3 - \{0\}$ by open sets such that for each $\alpha \in A$ we have

$$g | U_\alpha = A_\alpha P + B_\alpha Q + C_\alpha R \tag{1}$$

where A_α, B_α, $C_\alpha \in \mathcal{O}(U_\alpha)$. Let μ_α be the 2-form $A_\alpha \, dy \wedge dz + B_\alpha \, dz \wedge dx + C_\alpha \, dx \wedge dy$. Then (1) is equivalent to

$$\omega \wedge \mu_\alpha = g\Omega, \qquad \text{where} \qquad \Omega = dx \wedge dy \, dz. \tag{2}$$

Let $\mu_{\alpha\beta} = \mu_\alpha - \mu_\beta$, for $U_\alpha \cap U_\beta \neq \varnothing$. Formula (2) implies that $\omega \wedge \mu_{\alpha\beta} = 0$ in $U_\alpha \cap U_\beta$. If $E = x\,\partial/\partial x + y\,\partial/\partial y + z\,\partial/\partial z$ is the radial vector field, we have $i_E(\omega) = 0$, hence $\omega \wedge i_E(\mu_{\alpha\beta}) = 0$ (i_E is the interior product). Since the singular set of ω has codimension 2, we can write $i_E(\mu_{\alpha\beta}) = h_{\alpha\beta}\omega$ (cf. [6]), where $h_{\alpha\beta} \in \mathcal{O}(U_\alpha \cap U_\beta)$ and satisfy the cocycle condition $h_{\alpha\beta} + h_{\beta\gamma} + h_{\gamma\alpha} = 0$ for $U_\alpha \cap U_\beta \cap U_\gamma \neq \varnothing$. Now, if the degree of ω is k, the integrability condition $\omega \wedge d\omega = 0$ implies that $i_E(d\omega) = (k+1)\omega$. Hence we can write $i_E(\mu_{\alpha\beta} - \tilde{h}_{\alpha\beta}\, d\omega) = 0$, where $\tilde{h}_{\alpha\beta} = h_{\alpha\beta}/(k+1)$. Since E has no singularities in $U_\alpha \cap U_\beta$, we have

$$\mu_{\alpha\beta} = \tilde{h}_{\alpha\beta}\, d\omega + \ell_{\alpha\beta} i_E \Omega, \qquad \ell_{\alpha\beta} \in \mathcal{O}(U_\alpha \cap U_\beta), \qquad U_\alpha \cap U_\beta \neq \varnothing. \tag{3}$$

Since $\mu_{\alpha\beta}$ and $\tilde{h}_{\alpha\beta}$ satisfy the cocycle condition, $\ell_{\alpha\beta}$ satisfy as well. Now from Cartan's Theorem we have $\tilde{h}_{\alpha\beta} = h_\beta - h_\alpha$ and $\ell_{\alpha\beta} = \ell_\beta - \ell_\alpha$ where h_α, $\ell_\alpha \in \mathcal{O}(U_\alpha)$. This implies that we can define a holomorphic 2-form μ in $\mathbf{C}^3 - \{0\}$, and so in $\mathbf{C}^3$, by $\mu | U_\alpha = \mu_\alpha + h_\alpha \, d\omega + \ell_\alpha i_E(\Omega)$. Since $\omega \wedge d\omega = 0$ and $\omega \wedge i_E(\Omega) = 0$, we have $\omega \wedge \mu = g\Omega$. As in the proof of proposition 1 we can suppose that μ is homogeneous. □

Let us see some easy consequences.

PROPOSITION 3. — *Let S be a smooth curve in $\mathbf{CP}(2)$ given in homogeneous coordinates by an irreducible polynomial f of degree m. Let $\mathscr{F}$ be a foliation of degree n given by ω (of degree $n+1$) having S as a separatrix. Then ω has P.R.D. with respect to f and $m \leqslant n + 1$. If $m = n + 1$ then $\mathscr{F}$ coincides with a foliation given by the level curves of a rational function of type f/g^m, where the degree of g in 1. In particular we can choose an affine coordinate system $\mathbf{C}^2 \subset \mathbf{CP}(2)$ such that $\mathscr{F}/\mathbf{C}^2$ is given by the level curves of a polynomial.*

Proof. — By proposition 1 we have $\omega = g\,df + f\mu$. Note that $i_E\omega = 0$ implies $\mu \not\equiv 0$. Therefore $m \leqslant n + 1$.

If $m = n + 1$, then the degree of g is 1 and the degree of μ is 0. Since $i_E(\omega) = 0$, we have from Euler's identity that, $i_E(\mu) = -mg$. On the other hand, this implies that $\mu = -m\,dg$, because the degree of μ is 0, and so $\omega = g\,df - mf\,dg = g^{m-1}d(f/g^m)$. Hence the foliation $\mathcal{F}$ coincides with the foliation given by the level curves of f/g^m. If we take an affine coordinate system $\mathbf{C}^2 \subset \mathbf{CP}(2)$ such that $g = 0$ is the line at infinity then we have the last assertion. □

Remark. — It is possible to obtain Proposition 3 with the results of [6].

PROPOSITION 4. — *Let $\mathcal{F}$ be a foliation of degree n given by a 1-form ω. Suppose that all singularities of $\mathcal{F}$ are nondegenerated, in the sense that both eigenvalues are non zero (this is equivalent to the fact that $\mathcal{F}$ has n^2+n+1 distinct singularities). Let g be a homogeneous polynomial in $\mathbf{C}^3$ which vanishes on the singular set of ω. Then $g \in \mathcal{I}(\omega)$.*

Proof. — The assumptions on ω and g, imply that they satisfy the hypothesis of Proposition 2. □

2. Local relative division on $\mathbf{C}^2$.

In order to apply Proposition 1 in concrete cases it is necessary to know how the foliation looks like locally. In this section we will give some sufficient conditions to obtain the local relative division. Since the local case is essentially in dimension 2, we consider a germ of curve at $0 \in \mathbf{C}^2$, given by $f = 0$, f reduced. We denote by $\Lambda_0(f)$ the set of germs of analytic 1-forms at $0 \in \mathbf{C}^2$ having f as a separatrix. A form $\omega \in \Lambda_0(f)$ if, and only if, $\omega \wedge df = f\theta$, where θ is a germ of 2-form. Observe that $\Lambda_0(f)$ is a $\mathcal{O}_2$-module. Let $\Lambda_1(f)$ be the set of germs of analytic 1-forms at $0 \in \mathbf{C}^2$, which have P.R.D. locally with respect to f. It is easy to see that $\Lambda_1(f)$ is a sub-module of $\Lambda_0(f)$.

PROPOSITION 5. — $\dim_{\mathbf{C}}\left(\dfrac{\Lambda_0(f)}{\Lambda_1(f)}\right) < +\infty$.

Proof. — Since f is reduced the quotient $\dfrac{\Lambda_0(f)}{\Lambda_1(f)}$ has support zero. This implies the result. □

Now we give an explicit basis for the quotient when f is quasi-homogeneous, that is, when $f \in \mathcal{I}(f_x, f_y)$, the Jacobian ideal of f ([7]). It is well known that in this case it is possible to find a coordinate system (x,y) in a neighbourhood of 0, and a 1-form $\omega_0 = -kx\,dy + \ell y\,dx$, where k, ℓ are positive rational numbers and $\omega_0 \wedge df = f\,dx \wedge dy$. Observe that $\omega_0 \in \Lambda_0(f)$.

LEMMA 1. — *If $\omega \in \Lambda_0(f)$ then there exist g, $h \in \mathcal{O}_2$ such that $\omega = g\,df + h\omega_0$.*

Proof. — Since ω and ω_0 have $f = 0$ as separatrix and f is reduced we can write $\omega_0 \wedge \omega = fg\,dx \wedge dy$, where $g \in \mathcal{O}_2$. Let $\mu = \omega - g\,df$. Then $\mu \wedge \omega_0 = 0$. Hence we have $\mu = h\omega_0$, $h \in \mathcal{O}_2$. Therefore $\omega = g\,df + h\omega_0$. □

LEMMA 2. — *Let $h \in \mathcal{O}_2$. Then $h\omega_0 \in \Lambda_1(f)$ iff $h \in \mathcal{I}(f_x, f_y)$.*

Proof. — Suppose first that $h\omega_0 \in \Lambda_1(f)$. Then we can write $h\omega_0 = g\,df + f\mu$. Hence

$$hf\,dx \wedge dy = h\omega_0 \wedge df = (g\,df + f\mu) \wedge df = f\mu \wedge df,$$

and so $h\,dx \wedge dy = \mu \wedge df$, which implies that $h \in \mathcal{I}(f_x, f_y)$. Conversely, if $h \in \mathcal{I}(f_x, f_y)$, then $h = af_x + bf_y$, or $h\,dx \wedge dy = \mu \wedge df$ where $\mu = b\,dx - a\,dy$. This implies that $(h\omega_0 - f\mu) \wedge df = 0$. Since f is reduced we have $h\omega_0 - f\mu = dgf$, $g \in \mathcal{O}_2$. □

PROPOSITION 6. — *In the above situation,*

$$\Lambda_0(f) = \Lambda_1(f) \oplus \mathbf{C}f_1\omega_0 \oplus \cdots \oplus \mathbf{C}f_\mu\omega_0$$

where $f_1, \ldots, f_\mu$ is a basis of $\mathcal{O}_2/\mathcal{I}(f_x, f_y)$.

Proof. — It is a direct consequence of the two preceding lemmas.

COROLLARY. — *In the above situation $\mathcal{I}(f_x,f_y).\Lambda_0 \subset \Lambda_1$. In particular, if f has only two smooth components which are transverse, then $m \cdot \Lambda_0 \subset \Lambda_1$, where m is the maximal ideal of $\mathcal{O}_2$.*

Remark. — When $f = 0$ has two smooth transverse branches through 0, and ω is a germ of 1-form having $f = 0$ as a separatrix then ω has P.R.D. with respect to f if, and only if, $d\omega(0) = 0$. In fact, if $\omega = dgf + f\mu$ then $d\omega = dg \wedge df + df \wedge \mu + f\,d\mu$ and $d\omega(0) = 0$ because $df(0) = 0$. The converse is a direct application of the above results.

3. Relating degree of separatrices and foliations.

Let S be a projective nodal curve, that is all its singularities are of normal crossing type, with reduced homogeneous equation $f = 0$, of degree m.

THEOREM 1. — *Let $\mathcal{F}$ be a foliation in* $\mathbf{CP}(2)$ *of degree n, having S as separatrix. Then $m \leqslant n + 2$. Moreover if $m = n + 2$ then f is reducible and $\mathcal{F}$ is of logarithmic type, that is given by a rational closed form $\Sigma\lambda_i \dfrac{df_i}{f_i}$, where $\lambda_i \in \mathbf{C}$, and the f_i are homogeneous polynomials.*

Remark. — If S is irreducible then $m \leqslant n + 1$. Later on we will give other bounds for m in terms of the number of nodal points of S.

Proof of Theorem 1. — Let ω be an integrable 1-form in $\mathbf{C}^3$ which represents $\mathcal{F}$. Since S is a separatrix of $\mathcal{F}$ and f is reduced, we can write,

$$\omega \wedge df + f\, d\omega = f\mu \tag{4}$$

where μ is a homogeneous of degree n (ω has degree $n+1$).

Assertion. — If p is a singularity of f, then $\mu(p) = 0$.

Proof. — Let (x,y,z) be a local coordinate system such that $f = xy$. Since f is a separatrix of ω, we can write ω in this coordinate system as

$$\omega = \alpha y\, dx + \beta x\, dy + xy\gamma\, dz \tag{5}$$

where α, β, γ are holomorphic functions. Now, the 2-jet of the first member of (4) is zero, as it can be easily verified. This implies that μ vanishes at $x = y = 0$.

1[st] *case :* $\mu \equiv 0$. Observe that $\mu \equiv 0$ implies $d\left(\dfrac{\omega}{f}\right) = 0$. In this case, if $f = f_1 \dots f_k$ is the decomposition of f, then there are numbers $\lambda_1, \dots, \lambda_k$, such that $\dfrac{\omega}{f} = \sum\limits_{j=1}^{k} \lambda_j \dfrac{df_j}{f_j}$ (cf. [2]). If d_j = degree of f_j, then $i_E(\omega) = 0$ implies that $\sum\limits_{j=1}^{k} \lambda_j d_j = 0$. Hence $k \geqslant 2$ and the coefficients of ω have degree $m - 1$, and so $n = m - 2$.

2[nd] *case* : $\mu \not\equiv 0$. In this case, let g be a component of μ, $g \not\equiv 0$. Let $V = \{h; h$ homogeneous polynomial of degree n and h vanishes in the singular set of $f\}$. We have $V \neq \{0\}$, because $g \in V$.

Let $h \in V - \{0\}$. From proposition 1 and lemma 2 we can write $h\omega = a\, df + f \cdot \eta$, where a is homogenous of degree $2n + 2 - m$. Suppose first that we can write $h\omega$ as above in another way, say $h\omega = a_1\, df + f\eta_1$, where $a \neq a_1$. In this case we have $(a - a_1)\, df = f(\eta_1 - \eta)$. Since f is reduced, then f divides $a - a_1$, hence $m \leqslant 2n + 2 - m$, and so $m \leqslant n + 1$. On the other hand, if for each $h \in V - \{0\}$, a is unique, then the correspondence $h \mapsto a$ is a well defined linear map from V to the space of polynomials of degree $2n + 2 - m$. Let us denote this map by $a(h)$. We can suppose that it is injective. In fact, if it is not injective then f divides a polynomial of degree n and so $m \leqslant n$. We want to prove that $m \leqslant n + 2$ or equivalently that $n \leqslant 2n + 2 - m$. Suppose by contradiction that $n > 2n + 2 - m$. We observe that for any $h \in V$, the polynomial $a = a(h)$ vanishes in the singular set of f. In fact, if we take a local coordinate system (x,y,z) in a neighbourhood of a singular point of f, such that $f = x \cdot y$ and ω is as in (5), then the first jet of $h\omega - a\, df - f\eta = 0$, can be written as $-a(0)(x\, dy + y\, dx) = 0$, and so $a(0) = 0$. Let V_1 be the space of homogeneous polynomials of degree $2n + 2 - m$ which vanishes on the singular set of f. Then the map $a: V \to V_1$ is an injective linear map. But this is not possible because if A is any homogeneous polynomial of degree $m - n - 2 > 0$, we can construct an injective map $i_A : V_1 \to V$ by $i_A(h_1) = A \cdot h_1$. This would imply that $\dim V = \dim V_1$ and so the map i_A is an isomorphism which can not happen (it is not possible that all elements in V are divisible by all A's).

Now suppose that $m = n + 2$. We will prove that ω is of logarithmic type by induction on the degree $k = n + 1$ of ω. In the case $k = 1$, we have $\omega = L\, dM - M\, dL$, where M and L are homogeneous of degree 1. To see this, remark that since $d\omega$ is constant and $d\omega \wedge d\omega = 0$, we must have $d\omega = 2\, dL \wedge dM$ with L and M linear. Hence

$$\omega = \frac{1}{2} i_E\, d\omega = L\, dM - M\, dL.$$

Suppose we have proved the assertion for all $k \leqslant k_0 - 1$, where $k_0 \geqslant 2$, and let us prove it for $k = k_0$. In this case we have $n + 1 = k_0 = m - 1$, and so $2n + 2 - m = n$, which implies that $V_1 = V$. We can suppose that the map $a : V \to V_1 = V$ is well defined and injective. Let h be an eigenvector of a. We can write

$$h\omega = \lambda h\, df + f\eta, \qquad \lambda \in \mathbf{C}. \tag{6}$$

Observe that relation (6) implies that if h_1 is an irreducible factor of h, then h_1 divides f or η. Since degree $(h) = n < n + 2 =$ degree (f), after performing all possible divisions, we will get

$$\omega = \lambda\, df + f_1\eta_1, \tag{6'}$$

where f_1 divides f and f_1 is not a constant. This implies that $dg(\eta_1) < dg(\omega)$. Let us write $f = f_1 \cdot f_2$. Since $i_E(\omega) = 0$, we get from (6′) and Euler's identity:

$$f_1 i_E(\eta_1) + \lambda m f = 0.$$

Hence:

$$i_E(\eta_1) = -\lambda m f_2.$$

Now, let $m_1 = dg(f_1)$, $m_2 = dg(f_2)$ and $k_1 = m_2 - 1 = dg(\eta_1)$. If L_E is the lie derivative in the direction of E, we have:

$$(k_1+1)\eta_1 = L_E(\eta_1) = i_E(d\eta_1) + d(i_E\eta_1) = (k_1+1)\omega_1 - \lambda m\, df_2$$

where $dg(\omega_1) = k_1 < k$. Therefore we can write

$$\begin{aligned} \omega &= \lambda\, df + f_1\omega_1 - \lambda \frac{m}{m_2} f_1\, df_2 \\ &= \frac{\lambda}{m_2}[m_2 f_2\, df_1 - m_1 f_1\, df_2] + f_1\omega_1. \end{aligned} \tag{6''}$$

Since $i_E(m_2 f_2\, df_1 - m_1 f_1\, df_2) = 0$, we get $i_E(\omega_1) = 0$. On the other hand, it is easy to see that $f_2 = 0$ is a separatrix of ω_1 and $m_2 = k_1 + 1$, so we can apply the induction hypothesis to f_2 and ω_1, to obtain that $\omega_1 = f_2 \sum_{j=1}^{r} \lambda_j\, dh_j/h_j$, where $h_1, \ldots, h_r$ are the components of f_2. This implies the last part of the Theorem. □

Let us consider now the case where f is irreducible of degree m. We will give some estimations on the degree of the foliations which have f as a separatrix and which do not satisfy P.R.D. with respect to f. These estimations will be in terms of the number of nodal points of f and *assymptotically* are better than the previous $(m \leqslant n+1)$. In order to see the main idea, let us consider first the case where f has just one nodal point.

THEOREM 2. — *Let f be as above. If $\mathcal{F}$ is a foliation of degree n having f as a separatrix and which does not have P.R.D. with respect to f, then $m \leqslant \frac{n}{2} + 2$. Moreover this is the best estimation.*

Proof. — Let ω be a homogeneous 1-form of degree $n+1$ in $\mathbf{C}^3$, which represents $\mathcal{F}$. Let us suppose that the nodal point is $(0:0:1)$. Take the linear functions x and y. By proposition 1 and the corollary of proposition 6 the 1-forms $x.\omega$ and $y.\omega$ satisfy P.R.D. with respect to f. So we can write:

$$(7) \qquad x\cdot\omega = g_1\, df + f\mu_1, \qquad y\cdot\omega = g_2\, df + f\mu_2$$

where $dg(g_1) = dg(g_2) = n+2-(m-1) = n-m+3$. Note that if $g_1 \equiv 0$ then x divides μ_1, which is not possible. Similarly $g_2 \not\equiv 0$. Eliminating the term in ω, we obtain $(xg_2 - yg_1)\, df = f(y\mu_1 - x\mu_2)$ which implies that either $xg_2 - yg_1 \equiv 0$, or f divides $xg_2 - yg_1$. If $xg_2 - yg_1 \equiv 0$, it is clear that x divides g_1, and divides μ_1, from (7). Hence ω has P.R.D. with respect to f. On the other hand, if f divides $xg_2 - yg_1 \not\equiv 0$, then $m \leqslant n - m + 4 = dg(xg_2 - yg_1)$, which proves the first assertion.

In order to finish the proof we will construct an explicit example having degree $n = 2m-4$. This example will be constructed in affine coordinates. We can suppose that the nodal point of f is at $(0,0)$, and can choose a coordinate system such that

$$f = x^2 - y^2 + \cdots = x^2(1+\alpha) - y^2(1+\beta),$$

where $dg(1+\alpha) = dg(1+\beta) = m-2$. Let $\mathcal{F}$ be the foliation in $\mathbf{CP}(2)$, which in this affine system is given by the level curves of the meromorphic function $x^2(1+\alpha)/y^2(1+\beta) = f_1/f_2$. This foliation can be represented by the 1-form

$$(8) \quad \omega_1 = xy(1+\alpha)(1+\beta)\left[2\frac{dx}{x} - 2\frac{dy}{y} + \frac{d\alpha}{1+\alpha} - \frac{d\beta}{1+\beta}\right] = \frac{f_2\, df_1 - f_1\, df_2}{xy}.$$

Let ω_0 the homogeneous 1-form in $\mathbf{C}^3$ obtained from the above after homogenization. It is not difficult to see that $dg(\omega_0) \leqslant 2m-3$, and so $dg(\mathcal{F}) \leqslant 2m-4$. On the other hand ω_0 does not satisfy P.R.D. with respect to f.

In fact, if it would satisfy, we would have in affine coordinates $\omega_1 = g\,df + f\mu$, and so $d\omega_1(0,0) = 0$, because $(0,0)$ is a nodal point for f. But, if we compute $d\omega_1(0,0)$ from (8), we see that $d\omega_1(0,0) = -4\,dx \wedge dy$. Hence $dg(\mathscr{F}) \leqslant 2m - 4$. Since $dg(\mathscr{F}) \geqslant 2m - 4$, as we have proved at the begining, we have $dg(\mathscr{F}) = 2m - 4$. □

COROLLARY. — *Let f, ω_0 be as above and $\mathscr{F}$ be a foliation of degree n having f as a separatrix. Let ω represents $\mathscr{F}$ in homogeneous coordinates. Then, either $m + 1 \leqslant n < 2m - 4$ and ω has P.R.D. with respect to f, or $n \geqslant 2m - 4$ and there exists a homogeneous polynomial h of degree $n - 2m + 4$ such that $\omega - h\omega_0$ has P.R.D. with respect to f. Moreover, if p is the nodal point of f, then $h(p) = 0$ if, and only if, ω has P.R.D. with respect to f.*

Remark. — If we call $\Lambda_0(f,n) = \{\omega; dg(\omega) = n+1, i_E\omega = 0$ and $\omega \wedge df = f\theta\}$ and $\Lambda_1(f,n) = (\{\omega \in \Lambda_0(f,n)$; ω has P.R.D. with respect to $f\}$, then the above result implies that

$$\dim_{\mathbf{C}} \frac{\Lambda_0(f,n)}{\Lambda_1(f,n)} = \begin{cases} 0 & \text{if} \quad n < 2m - 4 \\ 1 & \text{if} \quad n \geqslant 2m - 4. \end{cases}$$

Remark. — Let $f = 0$ be a curve with one nodal point and $f_t = 0$ be a 1-parameter deformation of f with $f_0 = 0$ and such that f_t is smooth for $t \neq 0$. Let ω be a 1-form having $f = 0$ as a separatrix. If ω has P.R.D. with respect to f, then it is possible to deform ω in a 1-parameter family ω_t such that $\omega_0 = \omega$ and ω_t has $f_t = 0$ as a separatrix. On the other hand, if ω has not P.R.D. with respect to f, then this is not possible. This follows from the fact that ω_t must have P.R.D. with respect to f_t, if f_t is smooth.

Now we consider the case where $f = 0$ has $k \geqslant 2$ nodal points in general position, that is, if $k > 2$ then there are k distinct straight lines $L_1, \ldots, L_k$ such that for all $j \in \{1, \ldots, k\}$ the nodal points of f are contained in $\bigcup_{i \neq j} L_i$.

THEOREM 3. — *In the above situation, if $\mathscr{F}$ is a foliation in* $\mathbf{CP}(2)$ *of degree n, having $f = 0$ as a separatrix, then either $\mathscr{F}$ has P.R.D. with respect to f and $m \leqslant n + 1$, or :*

1) *If $k \geqslant 3$, then $2m \leqslant n + k + 2$*

2) *If $k \leqslant 2$, then $2m \leqslant n + k + 3$.*

Remark. – If all k nodes belong to just one straight line then it is possible to prove that $2m \leqslant n + k + 3$.

Proof. – Let ω be a homogeneous 1-form of degree $n + 1$ which represents $\mathscr{F}$. We will suppose that ω has not P.R.D. with respect to f. Let us consider the case $k \geqslant 3$. Let $L_1, \ldots, L_k$ be straight lines such that for all $j \in \{1, \ldots, k\}$, the nodal points of f are contained in $L_1 \ldots L_k/L_j = 0$. Set $\hat{L}_j = L_1 \ldots L_k/L_j$. Observe that,

$$\text{(9)} \qquad \hat{L}_j\omega = h_j\, df + f\mu_j, \qquad \text{for all} \qquad j \in \{1, \ldots, k\}.$$

The above relations imply that for all $i \neq j$, we have

$$(L_ih_i - L_jh_j)\, df = f(L_j\mu_j - L_i\mu_i).$$

Now we have two possibilities: either $L_ih_i - L_jh_j \equiv 0$ for all $i \neq j$, or $L_ih_i - L_jh_j \not\equiv 0$ for some $i \neq j$. In the first case L_i divides h_j for all $i \neq j$ and this implies that ω has P.R.D. with respect to f, by (9). In the second case f divides $L_ih_i - L_jh_j \not\equiv 0$, and so $m \leqslant n - m + k + 2 = dg(L_ih_i - L_jh_j)$.

Suppose now that $k \leqslant 2$, or more generally that all nodal points belong to the same line L. Take a point $p \notin L$ and denote by $L_1, \ldots, L_k$ the straigh lines joining p to the nodal points of f. We have:

$$\text{(10)} \qquad L\omega = h\, df + f\mu \qquad \text{and} \qquad L_1 \ldots L_k\omega = h_1\, df + f\mu_1$$

where $dg(h) = n - m + 3$ and $dg(h_1) = n + k - m + 2$. Now (10) implies that either $Lh_1 - L_1 \ldots L_kh \equiv 0$, and in this case we have P.R.D., or f divides

$$Lh_1 - L_1 \ldots L_kh \not\equiv 0 \qquad \text{and} \qquad m \leqslant n + k - m + 3.$$

$\square$

Remark. – Denote by k' the minimum number of straigh lines $L_1, \ldots, L_{k'}$ such that for all $j \in \{1, \ldots, k'\}$ the nodal points of f are contained in $L_1 \ldots L_{k'}/L_j = 0$. With the same argument of the proof of Theorem 3, it is possible to prove that, if $\mathscr{F}$ has not P.R.D. with respect to f, then $2m \leqslant n + k' + 3$.

Now we will consider the case where $f = 0$ has $k \geqslant 2$ nodes and the degree of the foliation is $n \geqslant 2m - 4$. First of all we will construct

k 1-forms $\omega_1, \ldots, \omega_k$ of degree $n+1 = 2m-3$ with the following property:

(11) if $p_1, \ldots, p_k$ are the nodes of $f=0$ then $d\omega_j(p_i) = 0$ if $i \neq j$, and $d\omega_j(p_j) \neq 0$.

Remark that the forms w_i are extremal for the inequality $n \geqslant 2m-4$. It is enough to construct ω_1. As we have done before, we will construct ω, in affine coordinates. Let (x,y) be an affine coordinate system in $\mathbf{CP}(2)$ with the following properties:

(i) $p_1 = (0,0)$.

(ii) $p_2, \ldots, p_k$ are not contained in the line at infinity.

(iii) For all $j \geqslant 2$, $p_j \notin \{x=0\}$ and $p_j \notin \{y=0\}$.

(iv) The second jet of f at $(0,0)$ is $x^2 - y^2$.

We leave it to the reader the proof that it is possible to obtain such an affine coordinate system. We can write,

$$f(x,y) = x^2 - y^2 + x^2\alpha - y^2\beta = x^2(1+\alpha) - y^2(1+\beta)$$

where $\max\{dg(1+\alpha), dg(1+\beta)\} = m-2$. We can suppose that:

(v) $(1+\alpha)(p_j) \neq 0$ for all $j \in \{2, \ldots, k\}$.

In fact, observe that $dg(f) = m \geqslant 4$, because $k \geqslant 2$. On the other hand,

$$f = x^2(1+\alpha) - y^2(1+\beta) = x^2(1+\alpha+\lambda y^2) - y^2(1+\beta+\lambda x^2),$$

and it is clear that we can choose $\lambda \in \mathbf{C}$ such that $(1+\alpha+\lambda y^2)(p_j) \neq 0$ for all $j \in \{2, \ldots, k\}$, because $y^2(p_j) \neq 0$ for all j. Let us consider, as before:

$$\begin{aligned}\tilde{\omega}_1 &= xy(1+\alpha)(1+\beta)\left(2\frac{dx}{x} - 2\frac{dy}{y} + \frac{d\alpha}{1+\alpha} - \frac{d\beta}{1+\beta}\right)\\ &= 2y(1+\alpha)(1+\beta)\,dx - 2x(1+\alpha)(1+\beta)\,dy + xy(1+\beta)\,d\alpha - xy(1+\alpha)\,d\beta.\end{aligned}$$

Since $d\tilde{\omega}_1(0,0) = -4\,dx \wedge dy \neq 0$, it is enough to verify that $d\tilde{\omega}_1(p_j) = 0$ for $j \geqslant 2$. Let us fix $j \in \{2, \ldots, k\}$ and put $f = f_1 - f_2$, where $f_1 = x^2(1+\alpha)$ and $f_2 = y^2(1+\beta)$. Note that $\tilde{\omega}_1$ can be written as

$$\tilde{\omega}_1 = \frac{y^3(1+\beta)^2}{x}\,d\left(\frac{f_1}{f_2}\right) = g\,d\left(\frac{f_1}{f_2}\right) \Rightarrow d\tilde{\omega}_1 = dg \wedge d\left(\frac{f_1}{f_2}\right).$$

Now, observe that $g(p_j) \neq 0$ and $f_2(p_j) \neq 0$ by (iii) and (v). This implies that

$$d\tilde{\omega}_1(p_j) = dg(p_j) \wedge (f_2(p_j)\, df_1(p_j) - f_1(p_j)\, df_2(p_j))/(f_2(p_j))^2 .$$

Since $f(p_j) = 0$ and p_j is a nodal point of f, we must have $f_1(p_j) = f_2(p_j) \neq 0$ and $df_1(p_j) = df_2(p_j)$. This implies that $d\tilde{\omega}_1(p_j) = 0$ as we wished. The degree of $\tilde{\omega}_1$ is clearly $\leqslant 2m - 3$. Moreover the homogeneous part of higher degree of $\tilde{\omega}_1$ is of the form

$$\theta = 2yab\, dx - 2xab\, dy + xyb\, da - xya\, db ,$$

where a and b are homogeneous parts of higher degrees of α and β respectively. If a and b are of the same degree $m - 2$, we have $i_R\theta = 0$, where $R = x\partial/\partial x + y\partial/\partial y$. This implies that when we homogenize $\tilde{\omega}_1$ we obtain a 1-form ω_1 in $\mathbf{C}^3$ whose degree is at most $2m - 3$. If its degree is less than $2m - 3$ we can of course multiply it by a convenient homogenous polynomial in order to get another of degree $2m - 3$ and with the same properties.

Let $\Lambda_0(f,n)$ and $\Lambda_1(f,n)$ be as in the remark after the Corollary of Theorem 2.

THEOREM 4. — *If* $n \geqslant 2m - 4$ *then* $\dim_{\mathbf{C}} \dfrac{\Lambda_0(f,n)}{\Lambda_1(f,n)} = k$, *the number of nodal points of* f.

Proof. — Let $\omega_1, \ldots, \omega_k$ be as in the above construction. Fix $n \geqslant 2m - 4$ and consider homogeneous polynomious $h_1, \ldots, h_k$ of degree $n - 2m + 4$ such that $h_j(p_j) = 1$ for all $j \in \{1, \ldots, k\}$. Observe that $d(h_j\omega_j)(p_i) = 0$ if $i \neq j$, and $d(h_j\omega_j)(p_j) = d\omega_j(p_j) \neq 0$, for all $j \in \{1, \ldots, k\}$. We assert that for any 1-form $\omega \in \Lambda_0(f,n)$, there are $\lambda_1, \ldots, \lambda_k \in \mathbf{C}$ such that $\omega - \sum_{j=1}^{k} \lambda_j h_j \omega_j \in \Lambda_1(f,u)$.

In fact, take $\lambda_1, \ldots, \lambda_k$ such that $d\omega(p_j) = \lambda_j\, d\omega_j(p_j)$. From the construction we have

$$d\omega(p_i) - \sum_{j=1}^{k} \lambda_j d(h_j\omega_j)(p_i) = d\omega(p_i) - \lambda_i\, d\omega_i(p_i) = 0 .$$

Therefore, from the remark after the corollary of Proposition 6 and from Proposition 1 we get that $\omega = \sum_{j=1}^{k} \lambda_j h_j \omega_j \in \Lambda_1(f,n)$. This implies that $\dim_{\mathbf{C}} \dfrac{\Lambda_0(f,n)}{\Lambda_1(f,n)} \leqslant k$. On the other hand, the forms $h_1\omega_1, \ldots, h_k\omega_k$ are linearly independent, because $d(h_j\omega_j)(p_i) = 0$ if $i \neq 0$ and $d(h_j\omega_j)(p_j) \neq 0$, $j \in \{1, \ldots, k\}$. This implies that $h_1\omega_1, \ldots, h_k\omega_k$ is in fact a basis of the quotient space, and so $\dim_{\mathbf{C}} \dfrac{\Lambda_0(f,n)}{\Lambda_1(f,n)} = k$. □

Remark. – If $k \geqslant 2$ it is possible to construct foliations $\mathscr{F} \in \Lambda_0(f,2m-5)$ which are not in $\Lambda_1(f,2m-5)$, but we don't know how to construct a basis of $\Lambda_0(f,2m-5)/\Lambda_1(f,2m-5)$. The idea is to decompose $f = f_1 - f_2$ in such a way that $f_1 = L^2M^2(1+\alpha)$, $f_2 = N^2(1+\beta)$, L, M, N are of degree 1, $1+\alpha$ is of degree $\leqslant m-4$ and $1+\beta$ of degree $\leqslant m-2$. It is not difficult to see that the foliation $\mathscr{F}$ given by the level curves of the meromorphic function f_1/f_2 is of degree $\leqslant 2m-5$, f is a separatrix of $\mathscr{F}$, but $\mathscr{F}$ has not P.R.D. with respect to f. Observe that $N = 0$ is a line joining two singularities of f. In fact it is possible to write $f = f_1 - f_2$ as above, if $N = 0$ is not tangent to some branch of $f = 0$ at one of the singular points that $N = 0$ joins. On the other hand, if $N = 0$ is tangent to some of these branches, it is possible to write $f = f_1 - f_2$, where $f_1 = L^3M^2(1+\alpha)$, $f_2 = N(1+\beta)$, where L and M have degree 1, $1+\alpha$ degree $\leqslant m-5$ and $1+\beta$ degree $\leqslant m-1$. In this case the foliation given by $f_1/f_2 =$ has also degree $\leqslant 2m-5$.

4. Some remarks for foliations whose singularities are of Poincaré type.

Let $\mathscr{F}$ be a holomorphic foliation on $\mathbf{CP}(2)$ and p a singular point of $\mathscr{F}$; we say that $\mathscr{F}$ is of Poincaré type at p if the ratio of the eigenvalues of the linear part of X at p is not a positive real number, where $x = P\dfrac{\partial}{\partial x} + Q\dfrac{\partial}{\partial y}$ defines $\mathscr{F}$ in an affine chart $\mathbf{C}^2$ containing p. Recall that if $\mathscr{F}$ is of Poincaré type at p then $\mathscr{F}$ has two local separatrices (and only two) at p; moreover these two separatrices are smooth and transverse. Denote by $\Lambda_n^1(\mathbf{CP}(2))$ the linear space:

$$\Lambda_n^1(\mathbf{CP}(2)) = \{\omega = A\,dx + B\,dy + C\,dz, xA + yB + zC = 0, \quad dg(A) = dg(B) = dg(C) = n+1\}.$$

Denote by $\mathscr{P}_n \subset \mathbf{P}(\Lambda_n^1(\mathbf{CP}(2)))$ the space of foliation $\mathscr{F}$ of degree n

whose singularities are of Poincaré type ; $\mathscr{P}_n$ is a real-Zariski open set in a projective space. Let $\mathscr{P}_n^0 \subset \mathscr{P}_n$ be the subset of $\mathscr{P}_n$ whose elements are the foliations having an algebraic separatrix.

THEOREM 5. — *If $n \geqslant 2$ then $\mathscr{P}_n^0$ is a proper algebraic subset of $\mathscr{P}_n$, that is, there exist homogeneous polynomials $P_1, \dots, P_m$ defined on $\Lambda_n^1(\mathbf{CP}(2))$ such that :*

$$\mathscr{P}_n^0 = \mathscr{P}_n \cap \{\mathscr{F} \text{ defined by } \omega, \text{ such that } P_1(\omega) = \cdots = P_m(\omega) = 0\}.$$

Proof. — Let $\mathscr{F} \in \mathscr{P}_n$ be given by ω, having an irreducible separatrix $S = (f=0)$. We have,

$$\omega \wedge df = f \cdot \theta, \qquad \theta \text{ a 2-form.}$$

Since the singularities of $\mathscr{F}$ are of Poincaré type then either S is smooth or is a nodal curve. As a consequence we have $m = dg(f) \leqslant n+2$. Denote by $\Sigma(m,n)$ the set

$$\{(\omega, f, \theta) ; \omega \wedge df = f \cdot \theta, dg(\omega) = n+1, dg(f) = m, dg(\theta) = n\}.$$

Note that $\Sigma(m,n)$ is an intersection of quadrics in a $\mathbf{C}^M$ and we can easily see that :

$$\mathscr{P}_n^0 = \mathscr{P}_n \cap \mathbf{P}(pr_1 \Sigma(m,n))$$

where pr_1 denote the first projection $(\omega, f, \theta) \mapsto \omega$. This implies that $\mathscr{P}_n^0$ is algebraic in $\mathscr{P}_n$; to see that $\mathscr{P}_n^0$ is proper we use the following result of Jouanolou [3] :

THEOREM. — *For $n \geqslant 2$ there exists an element in $\mathscr{P}_n$ without a separatrix.*

In fact Jouanolou proves that for $n \geqslant 2$ the foliation given by

$$\tilde{\omega}_n = (x^n - y^{n+1})\, dx - (1 - xy^n)\, dy$$

has no separatrix. We give here a short proof of this result. (*See also* [4].)

Proof of Jouanolou Theorem. — Put $N = n^2 + n + 1$ and $\lambda = e^{\frac{2i\pi}{N}}$; consider the linear periodic map $\sigma(x,y) = (\lambda x, \lambda^{n+1} y)$. Remark that $\sigma^* \tilde{\omega}_n = \lambda^{n+1} \cdot \tilde{\omega}_n$, so the foliation $\mathscr{F}_n$ given by $\tilde{\omega}_n$ is equivariant under the action of σ. Remark also that the singularities of $\mathscr{F}_n$ are the $\sigma^i(1,1)$, $i = 0, \ldots, N-1$ and that $\mathscr{F}_n \in \mathscr{P}_n$. Suppose that S_0 is a separatrix of $\mathscr{F}_n$. Then

$$S = \bigcup_{i=0}^{N} \sigma^i(S_0)$$

is a σ equivariant separatrix of $\mathscr{F}_n$. We know that $dgS \leqslant n+2$.

If S has a nodal point this point is a singularity of $\mathscr{F}_n$ and by σ equivariance S has n^2+n+1 nodal points which is impossible for a curve of degree $\leqslant n+2$ (except for $n=1$). So S is a smooth curve, hence irreducible, of degree $m \leqslant n+1$. From proposition 3 we can write:

$$\omega_{n+1} = a\, df + f\eta$$

where ω_{n+1} and f are homogeneous equations of $\mathscr{F}_n$ and S. We claim that S contains some sigularities of $\mathscr{F}_n$; if not a would be a constant and $\eta = 0$ which is not possible because $i_E\omega_{n+1} = 0$. So all the singularities of $\mathscr{F}$ are on S and are given by $(a=f=0)$; by Bezout's Theorem we have:

$$n^2+n+1 \leqslant m \cdot dg(a) = m(n+2-m) \leqslant (n+1)(n+2-m)$$

and $1 \leqslant (2-m)(n+1)$; so $m = 1$ and the singularities are on a line. $\square$

Remark. – For $n = 2$ it's possible to prove that $\mathscr{P}_2^0 \neq \mathscr{P}_2$ by computing precisely all components of $\mathscr{P}_2^0$; this is possible because we know all the configurations of curves of degree $\leqslant 4$ (reducible for degree 4) which are either smooth or nodal curves.

Consider now an element of $\mathscr{P}_1 = \mathscr{P}_1^0$; such a foliation has 3 lines as separatrices and the singularities are precisely the crossings of the lines; so the following question seems to be natural:

Is it possible for an element in $\mathscr{P}_n^0$ to have all the singularities on a separatrix?

THEOREM 6. – *Suppose that $\mathscr{F} \in \mathscr{P}_n^0$ and that all singularities of $\mathscr{F}$ are contained in a separatrix S; then $n = 1$.*

Proof. – Let ω_{n+1} and f be homogeneous equation for $\mathscr{F}_n$ and S, $m = dg(f) \leqslant n+2$. If all singularities of $\mathscr{F}_n$ are in S by Noether's lemma for foliations f is in $\mathscr{I}(\omega_{n+1})$ and as a consequence

$$n+1 \leqslant m \leqslant n+2.$$

If $m = n + 2$, then ω_{n+1} is of logarithmic type that is $f = f_1 \dots f_p$ and

$$\omega_{n+1} = f\Sigma\lambda_i \frac{df_i}{f_i}, \qquad \Sigma\lambda_i m_i = 0, \qquad m_i = dg(f_i).$$

In this case it is easy to see that the singularities of $\mathcal{F}$ on S are precisely the singularities of S; so we have:

$$n^2 + n + 1 = \#\text{ nodes of } S \leqslant \frac{(n+2)(n+1)}{2}$$

which is possible only for $n = 1$.

Now if $f \in \mathcal{I}(\omega_{n+1})$ is of degree $n + 1$ there exists a constant two form $\eta_0 = \alpha\, dy \wedge dz + \beta\, dz \wedge dx + \gamma\, dx \wedge dy$ such that

$$(*) \qquad \omega_{n+1} \wedge \eta_0 = f \cdot dx \wedge dy \wedge dz.$$

By changing linearly the coordinates it is possible to suppose that $\eta_0 = dy \wedge dz$. Write ω_{n+1} as:

$$\omega_{n+1} = A\, dx + B\, dy + C\, dz$$

then (*) implies that $A = f$, and so $A = 0$ is a separatrix of ω_{n+1}; as a consequence the constant vector field $\dfrac{\partial}{\partial x}$ is tangent to $A = 0$, so $A(x,y,z) = A(y,z)$ doesn't depend of x. Hence $A = 0$ consists of planes passing through the x axis. But, since the singularities are of Poincaré type, there are at most two planes in $A = 0$, and $n + 1 \leqslant 2$. □

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