ANNALES DE L'INSTITUT FOURIER

DAVID GABAI

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Annales de l'institut Fourier, tome 42, nº 1-2 (1992), p. 193-208
<http://www.numdam.org/item?id=AIF_1992__42_1-2_193_0>

Ann. Inst. Fourier, Grenoble
42, 1-2 (1992), 193-208

TAUT FOLIATIONS OF 3-MANIFOLDS AND SUSPENSIONS OF S^1

by David GABAI (*)

0. Introduction.

Let M be a compact oriented 3-manifold whose boundary contains a single torus P and let $\mathcal{F}$ be a taut foliation on M whose restriction to ∂M has a Reeb component. The main technical result of the paper, Operation 2.4.4 asserts that if N is obtained by Dehn filling P along any curve not parallel to the Reeb component, then N has a taut foliation.

Here are two applications :

i) If $\mathcal{F}$ is a taut foliation on M then either $\partial\mathcal{F}$ is a suspension of S^1 or every manifold $M(\alpha)$ obtained by nontrivial (*i.e.* not the Reeb direction) surgery has a taut foliation, thus $M(\alpha)$ is irreducible with infinite π_1.

ii) If $\mathcal{F}$ is a taut foliation on $M = S^3 - \overset{\circ}{N}(k)$, for k a knot in S^3, then $\partial\mathcal{F}$ is either a suspension of S^1 or has meridional Reeb components.

This paper is organized as follows. After the usual preliminaries in §1, we present in §2 several operations which allow one to modify a taut foliation (or essential lamination) on a given manifold M to create a new taut (or essential) one on either M or possibly another manifold obtained by surgery. Most of these operations are known or mild variations of known

(*) Partially supported by NSF grant DMS-8902343

Key-words : Taut foliation – Essential lamination – Dehn surgery – 3-manifold.
A.M.S. Classification : 57N10.

ones. Instead of burying them in the proof of the main result we present the main result as just another operation. Of perhaps independent interest is Operation 2.4.2 which asserts that surgering a manifold along an essential curve in a leaf of a taut foliation (or essential lamination) yields a new manifold with a taut foliation (or essential lamination, almost always). In §3 we give examples and discuss applications.

All the results in this paper relating to taut foliations were obtained in 1982–83. For the sake of completeness I have included in this paper the natural generalizations to essential laminations. I thank Larry Conlon for pushing me to exposit this work. In particular he asked whether there exist taut foliations on knot complements of S^3 with non meridional boundary Reeb components.

1. Preliminaries.

Definition 1.1. — Let $\mathcal{F}$ be a transversely orientable codimension-1 foliation on the compact orientable 3-manifold M such that $\mathcal{F}$ is transverse to ∂M. $\mathcal{F}$ is *taut* provided each leaf of $\mathcal{F}$ intersects a closed transverse curve. $\partial\mathcal{F}$ will denote the restriction of $\mathcal{F}$ to ∂M. We will assume that the leaves of $\mathcal{F}$ are smooth however the transverse structure may by C^0.

We now give the basic definitions about essential laminations. See [GO] for more detail. A *lamination* λ is a foliation of a closed subset of M, *i.e.* M is covered by open sets U of the form $\mathbf{R}^2\times\mathbf{R}$ such that $\lambda\cap U = \mathbf{R}^2\times C$ where C is a closed set, and coordinate neighborhoods of leaves are of the form $\mathbf{R}^2 \times x$, $x \in C$. A fibred neighborhood $N(\lambda)$ of λ is a codimension-0 submanifold of M containing λ with a 1-dimensional *vertical* foliation V, by closed intervals which are transverse to the lamination. Usually the ends of the vertical fibres are contained in leaves of λ. The space of leaves of V is a *branched surface* B.

A *complementary region* is a component R of $M-\lambda$. A *closed complementary region* refers to the closure in the path metric. A *boundary leaf* is a leaf corresponding to a component of ∂R, for R a closed complementary region. A boundary leaf C is *end incompressible* if every properly embedded line in A that bounds a properly embedded half space in a complementary region also bounds a properly embedded half space in C.

A lamination in a closed manifold (resp. manifold with boundary) is *essential* [GO] if

i) No leaf is a 2-sphere and no complementary region contains a 2-sphere not bounding a 3-cell (and no leaf is a D^2).

ii) Each torus and Klein bottle are π_1-injective (and each annulus and Mobius band is boundary incompressible).

iii) If W is a closed complementary region, then $\partial W \cap \lambda$ is incompressible and end incompressible in W (and $\partial W \cap \lambda$ is boundary incompressible).

Remark. — Essential laminations were developed in [GO] as a generalization of both taut foliation and incompressible. One of the main results of [GO] asserts that the leaves of an essential lamination are incompressible and transverse efficient arcs cannot be homotoped rel endpoints into a leaf. Furthermore the universal covering of M is $\mathbf{R}^3$. (As Hatcher notes, one might replace ii) and iii) above in the definition by "each leaf is π_1-injective, end incompressible and ∂-incompressible", and the above definition could then be used as a criterion to recognize essential laminations.)

It seems that there are "many more" manifolds with essential laminations than either incompressible surfaces or taut foliations. Not all aspherical 3-manifolds have them [Br]. Conjecturally manifolds with essential laminations have the same properties as those with taut foliations or incompressible surfaces.

If α is an essential simple closed curve in a torus component P of ∂M, then the manifold $M(\alpha)$ obtained by *Dehn filling* P along α is the manifold obtained by first attaching a 2-handle to P along α, then capping off the resulting 2-sphere with a 3-ball. If k is a knot in M, then $M(\alpha)$ denotes the manifold obtained by Dehn surgery along α an essential simple closed curve in $\partial N(k)$.

$N(E)$ will usually denote a regular neighborhood of E, and $|E|$ will denote the number of components of E, $\mathring{E}$ will denote the interior of E, and $\overline{E}$ will denote the closure of E.

If G is a foliation on $P = S^1 \times S^1$ which is transverse to the $x \times S^1$ factors, then in a natural way G determines a homeomorphism of S^1. Conversely a homeomorphism of S^1 determines such a foliation on P, well defined up to topological conjugacy. Such a foliation is called a *suspension*. Note that suspension foliations on P are exactly those which have no Reeb components. Suspension foliations on annuli are defined similarly.

M is *irreducible* means that every embedded S^2-bounds a 3-cell and M in *anannular* means that there every π_1-injective annulus is boundary parallel.

In this paper we will assume that all 3-manifolds are compact oriented and irreducible.

2. Modifications of taut foliations and essential laminations.

Operation 2.1.1. (Thickening leaves). — If L is a leaf of a lamination λ in M, then define $\hat{\lambda}$ to be the lamination obtained by replacing L with a smoothly embedded foliated I-bundle Y over L. This foliation is transverse to the I factors and is horizontally trivial, *i.e.* each leaf is either isotopic in Y to L or to a double covering of L, the latter occurring if L is not two sided in M. In general the resulting foliation or lamination is C^0. This well known operation essentially due to Denjoy [D] keeps taut foliations taut and essential lamination essential.

Operation 2.1.2. (Opening up leaves). — If L is a leaf of a lamination λ in M, then define $\hat{\lambda}$ to be the lamination obtained by thickening L, Operation 2.1.1 and then deleting the interior I-bundle leaves.

Operation 2.1.3. (I-bundle replacement). — This is the same operation as 2.1.1 except that we drop the requirement that the foliation be horizontally trivial.

If V is a line field transverse to a lamination $\mathcal{F}$, then a surface σ is *vertical and proper* in M, if σ is an immersed surface in M which is union of properly embedded intervals which are integral curves of V, $\mathring{\sigma}$ is injectively immersed, and for each leaf L of $\mathcal{F}$ the set $\sigma \cap L$ is properly mapped in L.

Operation 2.2. (Creating or killing holonomy). — If σ is a vertical and proper $J \times I$, $J \in \{S^1, I, [0,\infty), \mathbf{R}\}$, and $\mathcal{F}|\sigma$ is the product foliation (which will always be the case unless $J = S^1$) and $f : I \to I$ is a homeomorphism fixing the endpoints, then $\widehat{\mathcal{F}}$ is the lamination obtained by *creating f-holonomy along σ* if $\widehat{\mathcal{F}}$ is obtained by excising $N(J) \times I$, then regluing via the identity map except along $1 \times J \times I$ where the gluing is done according to the rule that $(1, j, x)$ is identified with $(1, j, f(x))$ where $N(j) = [0,1] \times J$.

Again this operation preserves the properties of being taut or essential provided no new torus or annulus leaves are created. This operation is useful in the case that $J = [0, \infty)$ and $\partial\mathcal{F} \neq \emptyset$ to kill boundary holonomy, or $J = \mathbf{R}$ to create interior holonomy of $J = S^1$ to kill compact leaves.

Operation 2.3.1. (Creating boundary Reeb components). — Let $\mathcal{F}$ be a foliation on M, tangent to ∂M. If α is a simple closed curve in ∂M with contracting holonomy, then the foliation obtained by digging out a trough near α (figure 9 [T]) has a boundary Reeb component.

Remark. — This operation due to Thurston (unpublished) was used to construct new depth 1 foliation from old ones, *i.e.* if N and M are depth one foliated manifolds, with foliations tangent to the boundary, σ and τ oriented simple closed curves on ∂N and ∂M which have contracting holonomy (the leaves spiraling in along the orientation), then the manifold W obtained by gluing N and M along neighborhoods of σ and τ has a depth 1 foliation. Simply dig out troughs in neighbordhoods of σ and τ and glue together the Reeb components using the S^1 parametrization.

Operation 2.3.2. (Creating boundary Reeb components). — Suppose that M is anannular, $\mathcal{F}$ is taut $\partial\mathcal{F}$ is transverse to P, a torus component of ∂M. Let A be an embedded annulus with one component p in P and the other component q in a leaf Q of $\mathcal{F}$. If A is π_1-injective and cannot be isotoped into Q (*i.e.* A does not cut off an annulus in Q which together with A is a boundary parallel annulus), then there exists a taut foliation $\mathcal{G}$ of M such that $\partial\mathcal{G}$ has Reeb components which are parallel to p.

Special Case. — A is transverse to $\mathcal{F}$, p is transverse to $\partial\mathcal{F}$, q is the only compact leaf of $\mathcal{F}|A$ and q has contracting holonomy also on the non A side.

Proof of Special Case. — Drill out $\mathring{N}(A)$ from M and isotope $\partial N(A)$ back to ∂M. This operation creates no new 0 Euler compact leaves unless some component of $Q - q$ was an annulus, the latter contradicting the fact that M is anannular. After recalling that $\mathcal{F}$ is taut if and only if there exists no subset of transversely oriented compact Euler characteristic 0 leaves which bound in M it follows that $\mathcal{G}$ is taut. □

Case 2. — A is as above but q does not have contracting holonomy on the non A side.

Proof of Case 2. — Use Operation 2.1.1 to replace Q by a product I-

bundle of leaves, then use Operation 2.2 to introduce contracting holonomy on the non A side of Q. □

Case 3. — $\partial\mathcal{F}$ has Reeb components.

Proof of Case 3. — If $\partial\mathcal{F}$ has Reeb components parallel to p, then there is nothing to prove. We now show that this is the only possibility. If not make p transverse to $\partial\mathcal{F}$ except where it crosses the middle of Reeb components and make A transverse to $\mathcal{F}$ near p and tilted near the tangencies of Q so that no leaf of the induced foliation on A has a tangency near p. By the procedure of [R] A can be isotoped, via an isotopy which restricts to the identity near ∂A, to eliminate the centers of $A \cap \mathcal{F}$. Now A contains a disc D whose boundary consists of two arcs r, s where $r \subset A \cap \mathcal{F}$, $s \subset P$, $\partial r \cap (\partial \text{ Reeb components }) \neq \emptyset$, and D is innermost with this properly. s contains a tangency of p with $\partial\mathcal{F}$ so a component of ∂r lies either interior to the Reeb component (in which case one finds a closed transverse curve which is null homotopic, contradicting [N]), or both components of ∂r lie in ∂(Reeb component). Using the incompressibility of leaves [N] and the irreducibility of M [Ro] it follows that $r \subset (\partial - \text{parallel annulus})$, contradicting the fact that $\mathcal{F}$ is taut. □

Case 4. — General case.

Proof of Case 4. — If $\partial\mathcal{F}$ has no Reeb components then isotope A so that p is transverse or tangent to $\partial\mathcal{F}$. Again use [R] to get rid of the center tangencies of A and $\mathcal{F}$. Let $C = \{\text{compact curves of } A \cap \mathcal{F}\}$ and $E = \{r \in C \text{ which cut off annuli in their leaves}\}$. C is closed in A and E is open in C by considering germs of leaves. By restricting the size of A we can assume that $q = C - E$. q is not in $\overline{E}$ because [Ha] implies that the set of closed leaves in a codimension-1 foliation of a compact manifold are closed and his argument extends to the set of compact leaves of a fixed topological type in the foliation with corners $\mathcal{F}|(M - \mathring{N}(A))$ where $N(A)$ is a normal product neighborhood. Therefore there exists a furthest out $e \in E$, so replacing A by the union of the annulus cut off by e and the annulus between e and q we obtain an annulus satisfyting Case 2. □

Remarks. — See Fenley's paper [F] for a related result.

If one drops the annular hypothesis, then using a similar procedure we obtain a foliation $\mathcal{G}$ with a boundary Reeb component. $\mathcal{G}$ though not necessarily taut is essential, in particular it has no torus (resp. annular) leaves bounding solid tori (boundary parallel solid tori).

The generalization of Operation 2.3.2 to essential laminations is stated below.

Operation 2.3.3. Creating boundary Reeb Components - Laminations version). — Suppose that λ is an essential lamination in the compact oriented manifold M and λ transverse to P a torus component of ∂M. Let A be an embedded annulus with one component p in P and the other component q in a leaf Q of λ. If A is π_1-injective and cannot be isotoped into Q, then there exists an essential lamination σ such that $\partial\sigma$ has a Reeb component which is parallel to p.

Two Reeb components on a surface are *parallel* if they lie in isotopic annuli via isotopies taking one foliation to the other.

Operation 2.4.1. (Surgering Reeb components). — Let M be a compact oriented 3-manifold with torus boundary component P. Let $\mathcal{F}$ be a taut foliation on M such that $\mathcal{F}|\partial M$ consists of parallel Reeb components. If α is an essential simple closed curve on P which is not isotopic to the core of a Reeb component, then $M(\alpha)$, the manifold obtained by α-filling P, contains a taut foliation $\mathcal{G}$ which is an extension of $\mathcal{F}$.

Proof. — $M(\alpha) = M \cup V$ where V is a $D^2 \times S^1$, and α is a meridian of V. $\mathcal{F}|\partial V$ is a union of parallel Reeb components whose cores are not meridional. Let $a_1, a_2, \dots, a_{2n}$ be the compact leaves of $\mathcal{F}|\partial V$ cyclically ordered with the Reeb annulus R_i between a_{2i} and a_{2i-1}. $\mathcal{F}$ extends to a foliation on all of V by attaching boundary parallel annuli A_i between a_{2i} and a_{2i-1}. In the natural way (consider the inverse to Operation 2.3.1) extend $\mathcal{F}$ to the $D^2 \times S^1$ between A_i and R_i. Call this extension $\mathcal{H}$. V-(these solid tori) is a $D^2 \times S^1$ called W. Extend $\mathcal{H}|\partial W$ to W by attaching cut off k-saddles, (a 2-saddle is a standard one, a 3-saddle is a monkey saddle, *etc.*) where k is the number of times that α crosses the Reeb components (*i.e.* crosses 1 Reeb component k/n times). Since each leaf of $\mathcal{G}$ contains a leaf of $\mathcal{F}$, $\mathcal{G}$ is taut. □

Remarks. — This operation was known at least to Fried, Ghys and Thurston in the 70's although I suspect that Tischler was aware of it when he wrote [Ti].

In an analogous way this operation takes essential laminations to essential laminations.

Operation 2.4.2. (Surgering leaves). — Let M be a compact oriented 3-manifold with taut foliation or essential lamination $\mathcal{F}$. Let β be an essential simple closed curve in a leaf L of $\mathcal{F}$. Let γ be a component of ∂A, where A is the component of $V = N(\beta) \cap L$ containing β. If $\mathcal{F}$ is a taut foliation and $\alpha \neq \gamma$, then $M(\alpha)$ possesses a taut foliation. If $\mathcal{F}$ is an essential lamination, then $M(\alpha)$ possesses an essential lamination if either α is two sided in L and $\alpha \neq \gamma$ or α is one sided in L and $|\alpha \cap \gamma| > 1$.

Proof. — A is an annulus, if β is 2-sided or a Mobius band, if β is one sided. In the former case γ is isotopic to β, while in the latter case it 2-fold covers β in V. If $\mathcal{F}$ is a taut foliation, then β is 2-sided.

If $\mathcal{F}$ is a taut foliation and $|\alpha \cap \beta| = k$, then apply Operation 2.1.2 to L and do α-surgery to β, which is now disjoint from the opened $\mathcal{F}$. The closed new complementary region W in $M(\alpha)$ which contains β is the union of a solid torus and a trivial I-bundle (not over D^2) glued together along 2 annuli. Fill W in with noncompact leaves. E.g. insert in the solid torus a S^1-stack of $2k$-saddles, then extend to the rest of W. Since the new leaves are noncompact, the new foliation on $M(\alpha)$ is taut.

Now suppose that $\mathcal{F}$ is an essential lamination. Let $\mathcal{G}$ be the lamination obtained in $M(\alpha)$ by applying Operation 2.1.2 to L and do α-surgery to β. We show that $\mathcal{G}$ is essential. The closed complementary region W containing β is the union of a solid torus and an I bundle (over a surface not D^2) glued together along 1 or 2 annuli (depending on whether β is one sided or not). These annuli wrap around the $D^2 \times S^1$, $|\alpha \cap \gamma| = k$ times.

i) Since $\mathcal{G}$ has the same leaves as $\mathcal{F}$ (with the exception of L) it has no spheres or disc leaves. Except for W it has the same complementary regions. Thus the complementary regions are irreducible.

iii) An elementary topological argument shows that ∂W is incompressible, end incompressible and boundary incompressible since $k > 1$ (or $2k > 1$, if A is an annulus). As before these properties hold for the other complementary regions since they hold for $\mathcal{F}$.

ii) By the loop theorem it suffices to show that (after possibly using operation 2.1.2 to replace a Klein bottle by a torus) there exists no embedded disc D such that ∂D lies in a leaf L of $\mathcal{G}$, and ∂D is essential in its leaf. By restricting the size of D we can assume that each leaf of $\mathcal{G} \cap \mathring{D}$ is compact and bounds a disc in its leaf. ∂D is isolated in $\mathcal{G} \cap D$

else $\mathcal{F}$ possesses a vanishing cycle hence would not be essential [GO]. Since laminations are carried by branched surfaces [GO], the other circles fall into a finite number of parallel bands which can be eliminated by rechoosing D. Thus we can assume that $\mathcal{G} \cap D = \partial D$, contradicting the fact that $\partial_h(N(\mathcal{G}))$ is incompressible. □

Remark. — In lamination theory complementary regions behave very much like leaves, e.g. they π_1-inject and the lamination analogue of the space of leaves is the quotient space obtained by identifying non boundary leaves or closed complementary regions to points. Thus the natural generalization to Operation 2.4.2 is the answer to the following.

Question 2.4.3. — If M has an essential lamination λ and if β is a simple closed curve in a complementary region, when does λ remain essential in $M(\alpha)$ a manifold obtained by surgery on β?

Case 1. — β lies in a $D^2 \times S^1$ complementary region W of an essential branched surfaces which carries λ.

This question is equivalent to the question of whether $W(\alpha)$ is irreducible and whether the horizontal boundary remains incompressible and end incompressible. The precise answer which is very involved can be found by combining the remarkable main results of [Sc] and [B].

Case 2. — General case.

Hatcher and Oertel [HO] and Wu [W] have results in the general case which say in a strong way "most of the time". The exact answer which is still unknown is likely to very complex to state.

Operation 2.4.4 (Surgering Reeb components). — Let M be a compact orientable 3-manifold, $\mathcal{F}$ a taut foliation on M and P a torus boundary component such that $\mathcal{F}|P$ has a Reeb component. If α is a simple closed curve on P which is not isotopic to the core of the Reeb component, then $M(\alpha)$ the manifold obtained by α-filling P has a taut foliation.

Proof. — Let $V = D^2 \times S^1$ be the solid torus which P bounds having α as a meridian. By thickening leaves, Operation 2.1.1 we can assume that the Reeb components on P are disjoint. The number of such is even since $\mathcal{F}$ is transversely oriented. Let $R_1, S_1, , \ldots, R_n, S_n$ be the annuli cyclically ordered so that $\mathcal{F}|R_i$ is a Reeb component and $\mathcal{F}|S_i$ is a suspension of an homeomorphism h_i of the interval. Isotope α to intersect these annuli in essential arcs. We will give the proof in the case that $n = 4$ and α intersects each annulus in a single arc, the general case being similar.

The proof simply involves extending $\mathcal{F}$ by capping off the Reeb components R_1 and R_3 as in Operation 2.4.1, extending R_2 across V to R_4, extending S_2 (resp. S_4) across V to S_3 (resp. S_1). The problem is that the Reeb component R_2 may point in the opposite direction to R_4 and that h_2 may not remotely resemble h_3.

How to extend R_2 across V to R_4. — If R_2 and R_4 are parallel, then extend across V as in Operation 2.4.1. Otherwise turn R_2 around as follows. First cap off R_2 with a solid torus as in Operation 2.4.1. Suppose that L is the leaf corresponding to the annulus (with core μ) which caps off ∂R_2. Do I-bundle replacement, Operation 2.1.3 to L, so that there exists contracting holonomy along μ (now isotoped to the boundary). Now do Operation 2.3.1 to create a boundary Reeb component which will be parallel to R_4 provided the holonomy of μ was chosen correctly. This procedure shows how to replace $\mathcal{F}$ by a new taut foliation so that all the Reeb components are parallel.

Let c_i, c_i' be the boundary components of S_i ordered via the cyclic ordering. Let L_i, L_i' be the corresponding leaves of $\mathcal{F}$. The idea is to do I-bundle replacement in $\mathcal{F}$ along these leaves, then extend the foliations on the S_i's across V. By doing a preliminary I-bundle replacement it follows that $\{L_1, L_2', L_3, L_4'\} \cap \{L_1', L_2, L_3', L_4\} = \emptyset$.

How to extend S_2 across V to S_3. — Do I-bundle replacement to L_2 and L_3 as follows. Let $J = \alpha \cap S_2$ and $K = \alpha \cap S_3$ be oriented intervals with head at R_3. The foliation on S_2 (resp. S_3) is a suspension of J (resp. K) called j (resp. k). We need to fill in the I-bundles so that the resulting suspension maps $\hat{j}$, $\hat{k}$ are conjugate. Open up leaves L_2, L_3 *i.e.* apply Operation 2.1.2. This splits open the annuli S_2, S_3, so let C_2 be the annulus corresponding to $c_2 \times I \subset L_2 \times I$ and J_1 the interval $C_2 \cap J$ and $J_2 = J - \mathring{J}_1$. Similarly $K = K_1 \cup K_2$ with K_2 corresponding to $c_3 \times I$.

If L is an oriented surface with boundary $\neq D^2$, and C is a designated boundary component, then one can foliate $L \times I$ with an arbitrary suspension homeomorphism on $C \times I$ and a foliation conjugate to the product foliation on $(\partial L - C) \times I$, provided that L is either noncompact or has positive genus. In the remaining case, when L is a compact planar surface one can foliate $L \times I$ with an arbitrary suspension homeomorphism μ on $C \times I$, μ on $E \times I$, (E another boundary component with the orientation opposite to the boundary orientation) and the product foliation on $(\partial L - \{E, C\} \times I)$.

Case 1. — One of (say L_2) L_2, L_3 is not a compact planar surface.

Case 1a. — L_3 is a compact planar surface and has some component E in J_2.

Proof of Case 1a. — In this case the foliation on $L_3 \times I$ will be chosen to be a product off of E and $c_3 \times I$. It will change k to k' on K_1 as follows. The points $L_3 \cap K_1$ will blow up to intervals on which k' will act as the identity. Off these intervals k' acts "like" k. We will fill in $L_2 \times I$ with a suspension homeomorphism μ on $c_2 \times I$ and product foliation on the other boundary components. This changes the homeomorphism k' to k'' in a similar way, except that infinite k' orbits of $L_2 \cap K_1$ will blow up to intervals which are permuted by k''. Thus define μ to be a homeomorphism of J_1 conjugate to k''.

LEMMA 2.1. — *Let f, h, σ, τ be either homeomorphisms of I fixing endpoints or maps of the empty set.*

i) *Then there exists a homeomorphism g of I conjugate to the concatenation of f, g, h.*

ii) *There exists homeomorphisms g, μ of I such that μ is conjugate to the concatenation f, g^{-1}, h and g is conjugate to the concatenation σ, μ^{-1}, τ.*

Proof.

i) View $I = [-1, 1]$. By the concatenation τ of f, g, h we mean there exists $a < b$ so that $\tau|[-1, a]$ is conjugate to f, $\tau|[a, b]$ is conjugate to g, and $\tau|[b, 1]$ is conjugate to h. The choice of $a < b$ does not effect the conjugacy class. We consider the case that f, h have non empty domain. Define g so that $g|[-1/i, -1/(1+i)]$ is conjugate to $f, g(0) = 0$ and $g|[1/(1+i), 1/i]$ to be conjugate to h.

ii) In this case define $g(0) = 0$, $g[-1, -1/2]$ to be conjugate to $\sigma, g[1/2, 1]$ to be conjugate to τ, $\mu(0) = 0$, $\mu[-1, -1/2]$ to be conjugate to f, $\mu[1/2, 1]$ to be conjugate to h, $g[-1/2, -1/3]$ to be conjugate to f^{-1}, $g[1/3, 1/2]$ to be conjugate to h^{-1}, *etc.* □

Suppose that we define the suspension map on $c_3 \times I$ to be g, then the suspension map on J_2 will be the concatenation of 3 maps h, g, f (possibly $h = \emptyset$). In that case by viewing $K_1 = [-1, 1]$ obtain the desired g by applying Lemma 2.1. Therefore the resulting suspension maps on J and K are conjugate. □

Case 1b. — L_3 is a compact planar surface and does not have a component in J_2.

Proof of Case 1b. — In this case proceed by first defining g to be conjugate to the induced map on J_2 (which is known since L_2 is not compact planar) and then define μ on J_1 conjugate to the resulting k' on K_1 □

Case 1c. — L_3 is not compact planar. □

Case 2a. — Both L_2 and L_3 are planar and some non c_2 component of ∂L_2 is disjoint from J_2 or some non c_3 component of ∂L_3 is disjoint from K_1.

Proof of Case 2a. — If ∂L_3 has another component off of K_1, then we can define g in a way that effects K_1 in a trivial way. Thus using, if necessary, Lemma 2.1, i) we first define μ, then define g. If L_3 does not have such a component, first define g and then define μ. □

Case 2b. — Both L_2 and L_3 are planar and each non c_2 component of ∂L_2 is in J_2 and each non c_3 component of ∂L_3 lies in K_1.

Proof of Case 2b. — Let d_i be another boundary component of L_i. We define the foliation on $L_i \times I$ to be trivial off of $(c_i \cup d_i) \times I$. Thus to make the resulting maps on K_i and J_i and thus on K and J conjugate it suffices to find g, μ so that μ is conjugate to the concatenation f, g^{-1}, h and g is conjugate to the concatenation σ, μ^{-1}, τ, where f, h, σ, τ are homeomorphisms of the components of $K \cup J - (c_i \cup d_i) \times I$. Now apply Lemma 2.1, ii).

In a similar way extend S_4 across V to S_1. This completes the proof of Operation 2.4.4. □

If P is a toral boundary component of M and λ is an essential lamination on M transverse to P, then say an annulus $A \subset P$ is *deft* if either $\lambda|A$ is a Reeb component or A is a closed complementary region of $\lambda|P$ and if W is the closed complementary region containing A, then $(\partial W - \partial M) \cup A$ is incompressible, ∂-incompressible and end incompressible. Two deft annuli $A, B \subset P$ are *bideft* if either one is a Reeb component or $(\partial W - \partial M) \cup A \cup B$ is incompressible, ∂-incompressible and end incompressible.

The following result is the essential laminations version of Operation 2.4.4. Its proof is similar.

Operation 2.4.5. (Surgering Reeb components). — Let M be a compact orientable 3-manifold, λ an essential lamination on M and P a torus boundary component such that $\lambda|P$ has a deft annulus A with core a. If α is a simple closed curve on P which satisfies $|a \cap \alpha| > 1$ or $|a \cap \alpha| = 1$ but there exist bideft annuli, then $M(\alpha)$ the manifold obtained by α-filling P has an essential lamination.

3. Examples and applications.

COROLLARY 3.1. — *Let M be a compact oriented 3-manifold with a torus boundary component P. If $\mathcal{F}$ is a taut foliation on M, then either $\mathcal{F}|P$ is a suspension of S^1 or every manifold $M(\alpha)$ obtained by nontrivial (not the Reeb direction) Dehn filling has a taut foliation $\mathcal{G}$.*

Since by [N] (resp. [GO]) manifolds with foliations without Reeb components (resp. essential laminations) have infinite π_1, we obtain

COROLLARY 3.2. — *If $M = S^3 - \overset{\circ}{N}(k)$, for k a knot in S^3 and $\mathcal{F}$ is a taut foliation transverse to ∂M, then $\partial\mathcal{F}$ is either a suspension of S^1 or has meridional Reeb components.*

COROLLARY 3.3. — *If $M = S^3 - \overset{\circ}{N}(k)$, for k a knot in S^3 and λ is an essential lamination transverse to ∂M, then any deft annulus crosses the meridian at most once.*

COROLLARY 3.4. — *If $M = S^3 - \overset{\circ}{N}(k)$, for k a knot in S^3 and $\mathcal{F}$ is a foliation without Reeb components transverse to ∂M, then $\partial\mathcal{F}$ is either a suspension of S^1 or has meridional Reeb components or $\partial\mathcal{F}$ has one Reeb component which crosses the meridian exactly once.*

We now give examples to show that these results are sharp.

Example 3.5. — If $k =$ figure 8 knot, then $S^3 - \mathring{N}(k)$ is the manifold obtained by deleting a neighborhood of the fixed orbit corresponding to 0, of the torus bundle over S^1 with monodromy $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. Let $\mathcal{G}$ be the transverse stable foliation on this bundle and let $\mathcal{F} = \mathcal{G}|S^3 - \mathring{N}(k)$. Then $\mathcal{F}|\partial N(k) = 2$ parallel Reeb components which are parallel to meridians.

Example 3.6. — If $k = (-2,3,7)$ pretzel knot, then $S^3 - \mathring{N}(k)$ fibres over S^1 with fibre a surface of genus $5 - \mathring{D}^2$. The pseudo-Anosov representative of the extended monodromy (*i.e.* the $\mathring{D}^2$ filled in) has an 18 prong singularity at 0 (more precisely the monodromy extended to the corresponding closed surface is isotopic to this representative via an isotopy fixing 0). Thus the stable foliation $\mathcal{F}$ of the surface bundle restricted to $\partial(S^3 - \mathring{N}(k))$ is a genuine nonsingular foliation $\mathcal{F}$ and $\mathcal{F} \cap \partial N(k)$ has a Reeb component of slope 18.

Proof. — The prong structure of the mapping was explicitly calculated using Bob Penner's computer program which computes the action of the mapping class group on measured lamination spaces. One can also obtain this result by applying [GO] Lemma 5.4 to the stable lamination knowing [FS] that 17/1, 18/1, 19/1 surgeries on k yield manifolds with finite fundamental group. □

Remark 3.7 [GM]. — These knots are not exceptional in that every hyperbolic knot k in S^3 has a lamination such that the restriction to $\partial N(k)$ has a Reeb component.

See [GK] for explicit calculations in the case of k a fibred 2-bridge knot.

COROLLARY 3.8. — *If $\mathcal{F}$ is a taut foliation on $S^3 - \mathring{N}(k)$ and A is an essential annulus with one boundary component p on $\partial N(k)$ and the other on a leaf of $\mathcal{F}$, then p is a meridian.*

Proof. — Apply Corollary 3.2 and Operation 2.3.2. □

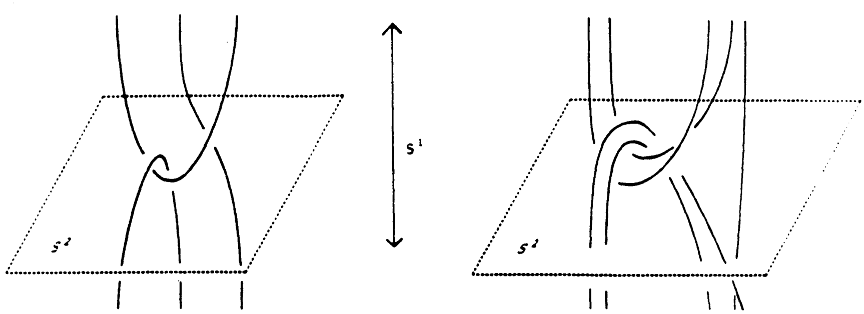
This corollary is the taut foliations version of the following incompressible surfaces result proved earlier by Menasco [M]. "If there exists two non isotopic essential annuli connecting a closed incompressible surface S in $S^3 - \mathring{N}(k)$ to $\partial N(k)$, then the annuli approach $N(k)$ along meridians".

Recall that if M is a compact oriented 3-manifold whose boundary P consists of a single torus, then the *longitude* consists of the unique isotopy class of curves in P such that some positive oriented multiple bounds a surface.

CONJECTURE 3.9. — *If k is a nontrivial knot in $S^2 \times S^1$ which is homotopic to a π_1 generator, then there exists a taut foliation $\mathcal{F}$ on $S^2 \times S^1 - \mathring{N}(k)$ transverse to the boundary so that $\mathcal{F}|\partial N(k)$ has a Reeb component parallel to the longitude.*

Remark 3.10. — This conjecture combined with Corollary 3.2 asserts that the foliated knot theory of $S^2 \times S^1$ is in a sense orthogonal to that of S^3. I verified it for any reasonable knot that I could write down, however I don't think that I was close to a general proof. To show that a given knot had such a foliation, it suffices to find a sutured manifold hierarchy $(S^2 \times S^1 - \mathring{N}(k), \partial N(k)) \longrightarrow (M_1, \gamma_1) \longrightarrow \cdots \longrightarrow (M_n, \gamma_n)$, an $m > 0$, and an essential annulus connecting the core of a boundary suture of γ_m (*i.e.* a suture coming from $\partial N(k)$ split open along circles) to a component of $R(\gamma_m)$. Upon creating the foliation from the hierarchy, this annulus give rise to an essential annulus connecting the foliation to a longitude on $\partial N(k)$. Operation 2.4.4 builds the desired foliation.

It is instructive to try this on several examples, e.g. the ones in the following figure



Knots in $S^2 \times S^1$

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David GABAI,
California Institute of Technology
Pasadena CA 91125 (USA).