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SPECTRAL GEOMETRY OF SEMI-ALGEBRAIC SETS

by Mikhael GROMOV

To the memory of
Claude Godbillon and Jean Martinet

0. Introduction.

This lecture is motivated by several conversations we have had with Jeff Cheeger about the eigenvalues of the Laplace operator Δ on algebraic and semi-algebraic (sub)sets in $\mathbf{R}^N$. Our interest was triggered by a question by Paulo D. Cordaro (in a letter addressed to one of us) concerning the possible rate of decay of the first eigenvalue λ_1 of the Laplace operator on a non-singular connected level $V_t = f^{-1}(t)$, $t \in \mathbf{R}$, of a polynomial function f on the Euclidean sphere, where V_t approaches the singular variety $f^{-1}(t_{cri})$ as t converges to a critical value t_{cri} of our $f : S^{N-1} \to \mathbf{R}$. More specifically what Cordaro wanted to know was (if I remember it right) the following lower bound on $\lambda_1 = \lambda_1(V_t)$, (we choose the sign of the Laplacian such that $\lambda_i \geq 0$)

$$(*) \qquad \lambda_1 \geq c \mid t - t_{cri} \mid^{\alpha}$$

for some positive constants c and α. (Possibly, Cordaro knew the proof of $(*)$ and only wanted to check if this could be found in the literature.)

The answer to Cordaro's question follows from the following known properties of the real algebraic varieties on one hand and the standard

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bounds on $\lambda_1(V)$ in terms of the geometry of V on the other hand. Namely the real algebraic geometry tells us that

0.A$_1$. *The gradient of the polynomial f defining V_t does not decay too fast as $t \to t_{cri}$. Namely*

$$\inf_{v \in V_t} \| \operatorname{grad} f(v) \| \geq b|t - t_{cri}|^{\beta}$$

for some positive b and β, by the *the Lojasiewicz inequality.*

0.A$_2$. *The $(N-2)$-dimensional volume of V_t (recall that $N - 2 = \dim V_t$) is bounded by*

$$\text{Vol } V_t \leq \text{const}_N \deg f$$

where $\deg f$ *denotes the algebraic degree of the polynomial f defining V_t.* In fact, by *Bezout theorem* almost every great circle meets V_t at most at $\deg f$ points and the above inequality follows from *Crofton's formula* expressing the volume as the average number of the intersection points of the subvariety in question with the great circles.

Now, since the first and second derivatives of f on S^{N-1} are bounded, the Lojasiewicz inequality implies that the *sectional curvatures* K of V_t (for the induced Riemannian metric) are bounded by

$$|K(V_t)| \leq b'|t - t_{cri}|^{\beta'}$$

and the *injectivity radius* of V_t is bounded from below by

$$\text{Inj Rad}(V_t) \geq b''|t - t_{cri}|^{\beta''}.$$

These two inequalities together with the above bound on $\text{Vol}\, V_t$ imply the required bound $(*)$ on the first eigenvalue $\lambda_1(V_t)$ according to the following standard Riemannian estimate.

0.B. *Let V be a compact connected Riemannian manifold of dimension $m \geq 1$. Then the first eigenvalue $\lambda_1 = \lambda_1(V)$ admits the following lower bound in terms of the volume of V, the injectivity radius and the supremum $|K|$ of the absolute values of the sectional curvatures of V,*

$$(**) \qquad \lambda_1 \geq \text{const}_m\, \rho^{2m-2} (\text{Vol})^{-2}$$

for

$$\rho = \min(\text{Rad}, |K|^{-\frac{1}{2}}).$$

Idea of the proof. — First we rescale the metric by ρ and thus reduce the situation to the case where $|K| \leq 1$ and Rad ≥ 1. Then V can be covered by $k \leq \mathrm{const}'_m$ Vol balls of radius $\frac{1}{2}$. Each of these balls has an "almost standard" geometry which is close to the geometry of a fixed Euclidean ball. It follows that $\lambda_1 \geq \mathrm{const}''_m k^{-2}$. In fact, the worst case is where the balls are arranged along a line and V looks like a cylinder $V_0 \times [0, k]$ for some fixed manifolds V_0 (e.g. an Euclidean $(m-1)$-ball of unit size). See Fig. 1 below.

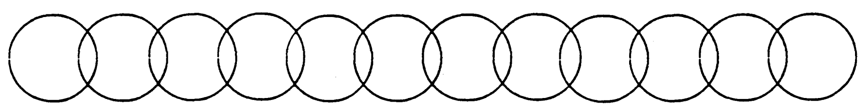

Fig. 1

Notice that for such cylinder like manifolds V the first eigenvalue is roughly proportional to $k^{-2} \approx (\mathrm{Diam})^{-2}$.

0.B′. *Remark.* — The above estimate remains valid for a significantly more general class of manifolds. Namely, if a closed connected Riemannian manifold V has Ricci $V \geq -1$ and can be covered by k unit balls, then again

$$\lambda_1(V) \geq \mathrm{const}_m k^{-2},$$

and a similar estimate holds for higher eigenvalues $\lambda_i(V)$ (see]Grol]).

0.C. *Non-local estimates on* λ_1. Now the proof of the inequality $\lambda_1(V_t) \geq c|t - t_{cri}|^\alpha$ can be summarized as follows. First, Lojasiewicz' inequality provides us with a bound on the local geometry of V_t (first of all on the *curvature* of V_1) and then the above Riemannian geometric inequality (**) applies.

At this stage we ask ourselves whether the bound on the local geometry (in particular on the curvature) is truly needed in our case. For example, may $\lambda_1(V_t)$ satisfy a universal bound $\lambda_1(V_t) \geq \epsilon > 0$ *independent* of $|t-t_{cri}|$ even though the curvature of V_t blows up for $t \to t_{cri}$? Of course,

$\lambda_1(V_t)$ does go to zero in some cases as seen in Fig. 2 below where $V_{t_{cri}}$ is reducible.

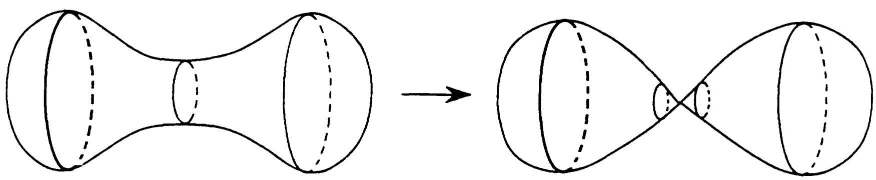

Fig. 2

On the other hand we shall see later on, that *if* $V_{t_{cri}}$ *is irreducible* (in the category of real analytic varieties) *and* $\dim V_{t_{cri}} = \dim V_t$ *then, in fact,* $\lambda_1(V_t) \geq \epsilon > 0$ *as* $t \to t_{cri}$. The proof of that can hardly be obtained by the traditional local tools of the Riemannian geometry which rely very much on curvature estimates and (or) the injectivity radius. On the other hand, the geometric theory of semialgebraic sets provides us with a sufficient information to control the eigenvalues of V_t. In fact, we may even include singular varieties V in our discussions, such as $V_{t_{cri}}$, for example, and estimate certain eigenvalues in terms of some numerical invariants of V. Here is a detailed list of topics we are going to approach (in a rather speculative manner) in this lecture.

Acknowledgement: I am thankful to the referee for correcting several errors.

Contents

7. Low and high frequences.

8. Spectra of non-algebraic sets from the semi-algebraic viewpoint.

9. Geometric uses of the spectrum.

1. Spectra and related invariants for singular sets, measures and partitions.

The notion of the spectrum makes sense for an arbitrary metric space whenever we have a measure μ on this space [Gro2]). Here we concentrate our attention on "algebraically defined" measures in the Euclidean space $\mathbf{R}^N$, where the basic example is the usual m-dimensional (Hausdorff) measure $\mu = \mu_V$ an m-dimensional *semi-algebraic* subset V in $\mathbf{R}^N$. (If V consists of several irrreducible components V_i of different dimensions m_i we assign to each component the respective Hausdorff measure of dimension m_i.)

Given a measure μ on $\mathbf{R}^N$ we define the following two L_2-norms (or rather, semi-norms)

$$\| \varphi \|_\mu = \left| \int_{\mathbf{R}^N} |\varphi|^2 d\mu \right|^{\frac{1}{2}}$$

and

$$\| \operatorname{grad}\varphi \|_\mu = \left| \int_{\mathbf{R}^N} \| \operatorname{grad}\varphi \|^2 \, d\mu \right|^{\frac{1}{2}}$$

for all C^1-functions $\varphi : \mathbf{R}^N \to \mathbf{R}$. (Notice that the above measure $\mu = \mu_V$ has $\operatorname{supp}\mu = V$ and so $\int_{\mathbf{R}^N} \ldots d\mu = \int_V \ldots d\mu$.)

Now with a pair of quadratic functions (forms)

$$\varphi \mapsto \| \varphi \|_\mu^2 \quad \textit{and} \quad \varphi \mapsto \| \operatorname{grad}\varphi \|_\mu^2$$

one can define *the spectrum of* μ (and thus of V for $\mu = \mu_V$) as the spectrum of the form $\| \operatorname{grad}\varphi \|_\mu^2$ relative to $\| \varphi \|_\mu^2$. For example, if μ equals μ_V for a smooth subvariety V in $\mathbf{R}^N$ we recapture the spectrum of *the Laplace operator* on V for the induced Riemannian metric.

Observe that the smallest eigenvalue λ_0 equals zero whenever μ has finite total mass (e.g. V is compact) since the *constant* functions φ are in L_2 for mass $\mu < \infty$.

At this moment, we assume $\mu = \mu_V$ for a compact V and define the next eigenvalue λ_1 as the best constant λ in the Poincare inequality

$$(*) \qquad \| \varphi \|_\mu^2 \leq \lambda^{-1} \| \operatorname{grad}\varphi \|_\mu^2 .$$

More precisely, λ_1 equals the supremum of those λ for which $(*)$ is satisfied by all C^1-functions orthogonal to the constants, i.e. having

$$\int \varphi d\mu = 0 .$$

Similarly, one defines λ_i for all $i = 1, 2 \ldots$, as the maximal (or better to say supremal) λ for which there exist functions $\psi_0, \ldots \psi_{i-1}$, such that every φ orthogonal to ψ_j,

$$\int \varphi\psi_j d\mu = 0, \ j = 0, \ldots, i-1,$$

satisfies

$$(**) \qquad \| \varphi \|_\mu^2 \leq \lambda^{-1} \| \operatorname{grad}\varphi \|_\mu^2 .$$

(Here and below $\int \ldots d\mu$ is an abbreviation for $\int_{\mathbf{R}^N} \ldots d\mu$.)

Remarks. — (a) The inequality $(**)$ with $\lambda = \lambda_i$ becomes sharp if we take the j-th eigenfunction for ψ_j, $j = 0, 1 \ldots, i-1$. In fact, one usually defines λ_i using eigenfunctions, but we prefer to avoid any analysis (and linear algebra) at this stage.

(b) The standard geometric approach to a lower bound on λ_i on V consists in finding an appropriate cover of V by balls $B_0, \ldots, B_{i-1}$ and then proving $(**)$ with the best possible λ for functions φ satisfying

$$\int_{B_j} \varphi d\mu = 0, \ j = 0, \ldots i-1,$$

which corresponds to the orthogonality to the characteristic functions ψ_j of B_j.

(c) A more conceptual approach to the definition of λ_i appeals to the *Morse Theory* of the energy function $E : \varphi \mapsto \| \operatorname{grad}\varphi \|_\mu / \| \varphi \|_\mu$. The function E is homogeneous and so lives on a certain (infinite dimensional) projective space Φ. This space carries a non-trivial $\mathbb{Z}_2$-homology in each dimension i and λ_i may-be defined as the first value of λ for which the level $E^{-1}(-\infty, \lambda] \subset \Phi$ captures a non-trivial i-dimensional circle in Φ. This approach allows one to extend the notion of the spectrum to non-quadratic functions, such as L_p-norms for $p \neq 2$, and it also suggests a variety of other generalizations (see [Gro2]).

1.A. *Isoperimetric profile of* μ. The basic ingradient implicit in the above spectral discussion was the (linear) operator $\varphi \mapsto \operatorname{grad}\varphi$ or, more invariantly, the exterior differential on functions, $\varphi \mapsto d\varphi$. Then one can try other linear operators, among which the most interesting are the exterior differentials and the forms of degree ≥ 1 briefly looked upon in §6 (and the Dirac operator on spinors which we do not discuss at all in this lecture). Now we want to dualize d and look at the (geometric) boundary operator ∂ on chains and currents. We limit ourselves to the special case of the top-dimensional chains in $\mathbf{R}^N$ represented by domains and we want to relate the μ-mass of these to the (appropriately defined) mass of their boundaries.

1.A₁. *Mass of a hypersurface.* For an arbitrary subset $H \subset \mathbf{R}^N$ we denote by $H_{+\epsilon} \subset \mathbf{R}^N$ the ϵ-neighbourhood of H and let

$$\mu'_+(H) = \limsup_{\epsilon \to 0} \epsilon^{-1}\mu(H_{+\epsilon})$$

and

$$\mu'_-(H) = \liminf_{\epsilon \to 0} \epsilon^{-1}\mu(H_{+\epsilon}).$$

If μ and H are sufficiently regular (e.g. μ is of algebraic origin and H is a real analytic subset) then $\mu'_+(H) = \mu'_-(H)$, and in such a case we use the notation $\mu'(H)$. Moreover, if $\mu = \mu(V)$ for a semi-algebraic V and H is a sufficiently smooth (e.g. real analytic) compact hypersurface in V then

$$0 < \mu'_+(H) = \mu'_-(H) < \infty,$$

and, in fact, $\mu'(H)$ equals the usual $(m-1)$-dimensional measure of H.

1.A₂. *Isoperimetric profile.* Consider all subsets $W \subset \mathbf{R}^N$ of a given μ-mass α and let $Is_\mu(\alpha)$ denote the infimum of the masses of their topological boundaries,

$$Is(\alpha) = \inf_W \mu'(\partial W),$$

where, in order to make sense, we take the infimum of those W with $\mu(W) = \alpha$ which have $\mu'_+(\partial W) = \mu'_-(\partial W)$.

There are obvious modification of this definition such as

(i) using μ'_+ or μ'_- instead of μ' and allowing *all* W with $\mu(W) = \alpha$;

(ii) measuring ∂W by

$$\mu_\varepsilon \stackrel{\text{def}}{=} (W_{+\varepsilon}) - \mu(W)$$

for a fixed $\epsilon > 0$, next taking the infimum of μ_ϵ over W and only then passing to the limit for $\epsilon \to 0$.

(iii) Restricting to subsets W in $V = \operatorname{supp}\mu$ and (more importantly) taking the topological boundary ∂W in V rather than in $\mathbf{R}^N$.

It is not hard to see, however, that for algebraically defined measures μ all these definitions lead to the same isoperimetric profile (function) $\mathrm{Is}(\alpha) = \mathrm{Is}_\mu(\alpha)$ of μ.

1.A₃. *Cheeger's inequality. The first eigenvalue* λ_1 *admits the following lower bound in terms of the function* $\mathrm{Is}(\alpha)$ *in the interval* $[0,\beta]$ *for* $\beta = \frac{1}{2}\mu(\mathbf{R}^N)$,

$$\lambda_1 \geq \frac{1}{4}\left[\inf_{0<\alpha\leq\beta} \alpha^{-1}\,\mathrm{Is}(\alpha)\right]^2 .$$

In other words the *isoperimetric inequality*

$$\mu(W) \leq c\mu'(\partial W),$$

for the subsets $W \subset \mathbf{R}^N$ of mass $\mu(W) \leq \dfrac{1}{2}\mu(\mathbf{R}^N)$ implies the Poincare inequality

$$\int \varphi^2 d\mu \leq \frac{1}{4}\, c^2 \int \|\operatorname{grad}\varphi\|^2\, d\mu$$

for the functions φ orthogonal to the constants.

1.A₄. *Remarks.* — (a) Cheeger proved his inequality for smooth Riemannian manifolds (see [Che1]), but his argument immediately extends to all sufficiently regular measures μ in $\mathbf{R}^N$, e.g. to $\mu = \mu_V$ for compact semi-algebraic $V \subset \mathbf{R}^N$.

(b) One can define further isoperimetric profiles using i-tuples of subsets $\{W_1,\dots,W_i\}$ of prescribed masses $\alpha_j = \mu(W_j)$, $j = 1,\dots,i$, and satisfying some extra conditions such as

(i) W_j cover $\mathbf{R}^N$ (or at least the support of μ);

(ii) W_j cover $\operatorname{supp}\mu$ with a given multiplicity;

(iii) W_j are mutually disjoint.

Then one can somehow measure the total size of the boundaries of W_i, say by

$$\mu' = \max_{j=1,\dots,i} \mu'(\partial W_j)$$

or by

$$\mu'' = \sum_{j=1}^{i} \mu(\partial W_j),$$

or else by the measure μ' (or μ) of the complement to the union $\bigcup_{j=1}^{i} W_j$ in case (iii). Finally, one may take the infimum of such a μ' over all $\{W_1, \dots, W_j \dots, W_i\}$ with given $\alpha_j = \mu(W_j)$ and obtain the isoperimetric profile (function) in the variables $\alpha_1, \dots, \alpha_i$,

$$\mathrm{Is}(\alpha_1, \dots, \alpha_i) = \inf \mu'\{W_1, \dots, W_i\}.$$

(Such multivariable profiles are sometimes (implicitly) used in estimating the eigenvalues λ_i, compare [Gro1].)

1.B. *Partitions and symmetrization.* Suppose, besides a measure μ on $\mathbf{R}^N$, we are given a partition π. This defines a subspace in the space of all functions φ on $\mathbf{R}^N$, say $\Phi_\pi \subset \Phi$ which consists of the functions constant on the subsets, called *slices*, into which $\mathbf{R}^N$ is partitioned. One also may think of Φ_π as the space of functions on the quotient space $W = \mathbf{R}^N/\pi$ and, in fact, the partitions in our discussion are usually associated to semi-algebraic maps $p : \mathbf{R}^N \to X$ which partition $\mathbf{R}^N$ into the pull-backs $p^{-1}(x)$, $x \in X$. The norms $\| \varphi \|_\mu$ and $\| \operatorname{grad} \varphi \|_\mu$, when restricted to Φ_π , define *the spectrum of* μ *relative* to π by an obvious generalization of the previous discussion. This can also be done in terms of $p : \mathbf{R}^N \to X$ by using the pushforward measure $p_*(\mu)$ on X along with a certain pushforward of the Riemannian metric.

Examples. — (a) Let $p : \mathbf{R}^N \to \mathbf{R}^{N-k}$ is an orthogonal projection. Then, obviously, the spectrum of an arbitrary μ relative to π (corresponding to p) equals the spectrum of $p_*(\mu)$ on $\mathbf{R}^{N-k}$. (Here one should not bother pushing forward the metric because p is a *Riemannian submersion.*)

Notice that the eigenvalues λ_i of μ and $\overline{\lambda}_i$ of $p_*(\mu)$ are related by the obvious inequality $\lambda_i \le \overline{\lambda}_i$. This allows one to obtain nontrivial lower bounds on the spectra of certain (singular) measures in $\mathbf{R}^{N-k}$ by representing them as pushforwards of some standard measures $\mathbf{R}^N$. One obtains an interesting example of that kind by taking a round sphere S^m

in $\mathbf{R}^N$ with the standard (isometry invariant) measure and projecting it down to $\mathbf{R}^{N-k}$. In fact, already the projection $S^m \to \mathbf{R}$ gives rise to a meaningful inequality on λ_i for the pushforward measure.

(b) Let $\mathbf{R}^N$ be acted upon by a group G and π is the partition into the orbits. Then Φ_π consists of G-invariant functions. Furthermore, if G is compact and the action is isometric and μ-preserving, then there is a natural projection $\Phi \to \Phi_\pi$ corresponding to averaging over G. This is a typical instance of *symmetrization* that is an operation on functions φ on $\mathbf{R}^N$ making them constant on the slices of a given partition. A similar symmetrization for a general *measurable* partition π consists in integrating φ over the slices of π relative to the natural measure induced by μ on almost all slices.

One can often use such symmetrization to estimate the spectrum of a measure in terms of the spectrum relative to π and the spectra of the slices (see §6). The simplest instance of that is the relation

$$\mathrm{Spec}(\mu' \times \mu'') = \mathrm{Spec}\,\mu' + \mathrm{Spec}\,\mu''$$

where $\mu' \times \mu''$ is the product measure on $\mathbf{R}^{N'} \times \mathbf{R}^{N''}$ and the sum on the right hand side stands for the sum of the subsets $\{\lambda_i'\}$ and $\{\lambda_i''\}$, i.e.

$$\{\lambda_i' + \lambda_j''\}_{i,j=0,1,\dots}.$$

Remark. — The above discussion indicates that one should enlarge the class of "algebraically defined" measures μ by allowing the push-forwards of μ under semi-algebraic maps but we do not attempt to be systematic in this matter.

2. Lower bounds on λ_i.

In this section I state and explain without giving the detailed proof the main property of the spectrum we have proved so far with Cheeger.

2.A. Uniform discreteness. Let V be a semi-algebraic subset in $\mathbf{R}^N$ defined by the relations

$$V = \{v \in \mathbf{R}^N | p_j(v) = 0,\ q_k(v) \geq 0\},$$

where p_j , $j = 1 \dots r$ and q_k , $k = 1 \dots s$ are polynomials on $\mathbf{R}^N$. We denote by $\deg V$ the sum of the degrees of the polynomials p_j and q_k.

2.A₁. THEOREM (*). — *The number Σ_λ of the eigenvalues λ_i of V (i.e. of the measure μ_V) contained in a given interval $[0, \lambda]$ satisfies*

$$(+) \qquad \Sigma_\lambda \le C(\operatorname{Diam} V)\lambda^{\frac{m}{2}} + \sigma_0$$

where $\operatorname{Diam} V$ *denotes the Euclidean diameter of* V, $m = \dim V$, *and* C *and* σ_0 *are some universal constants depending only on* m *and* $\deg V$.

2.A₂. *Remarks.* — (a) Let V be a smooth subvariety in $\mathbf{R}^N$. Then the estimate $(+)$ is standard if one allows C and σ_0 to depend on V. In our case we may take an algebraic family V_t of smooth algebraic subvarieties converging to a singular one $V_t \to V_s$ for $t \to s$, but yet $(+)$ remains valid with C and σ_0 *independent of* t.

(b) The dependence of σ_0 on $\deg V$ and $\dim V$ is essentially the same as that for the Betti numbers in the estimates by Milnor and Thom (see [Mil], [Tho]). In fact, the number Σ_ϵ for $\epsilon \to 0$ can be thought of as a kind of zero Betti number.

2.B₁. *Idea of the proof of 2.A₁.* — The estimate $(+)$ for smooth manifolds V trivially follows from the fact that for every $\delta > 0$ V can be divided into k "standard" pieces V_j, $j = 1, \dots, k$, of diameter δ for k about $\operatorname{Vol} V/\delta^m$, $m = \dim V$. In fact if a function ψ on V is "orthogonal" to V_j , i.e.

$$\int_{V_j} \varphi = 0, \ j = 1, \dots, k,$$

then the L_2-norm of ψ on V satisfies

$$\|\varphi\|_V^2 \int_V \varphi^2 \underset{\text{def}}{=} \sum_{i=1}^{k} \|\varphi\|_{V_j}^2 \le \sum_{j=1}^{k} \lambda_{1,j}^{-1} \|\operatorname{grad} \varphi\|_{V_j}^2 ,$$

where $\lambda_{1,j}$ denotes the first eigenvalue of V_j. (Compare 1.). Now, since each V_j is "standard", its first eigenvalue is about δ^{-2} and so $\|\varphi\|_V$ is bounded by a constant δ^2. Furthermore, if $\operatorname{Vol} V$ is bounded by a constant times

(*) It would be prudent at this stage to claim the proof of the theorem only for $\dim V = 2$.

$(\mathrm{Diam}\, V)^m$ (which is, in fact, the case for semialgebraic V), and if we take $\delta \approx (\mathrm{Vol}\, V/k)^{\frac{1}{m}}$, then we obtain the bound

$$\|\varphi\|_V \leq (\mathrm{Diam}\, V)^2 k^{-\frac{2}{m}},$$

which tells us (see §1) that the k-th eigenvalue of V satisfies

$$\lambda_k \geq (\mathrm{Diam}\, V)^{-2}\ k^{\frac{2}{m}}.$$

This implies (+) with constants depending on V in question.

Now, in the semialgebraic case we use the standard triangulation of V into simplices of size δ. (See [Gro3], for instance.) These simplices are not "standard", (i.e. they are not mutually Lipschitz equivalent with a fixed Lipschitz constant) but still these simplices (for an appropriate triangulation) have rather simple shapes. For example, if dim $= 2$, they can be made uniformly bilipschitz to the simplices Δ_a bounded by graphs of monotone algebraic functions like t^a on the interval $[0, \delta]$, see Fig. 3 below.

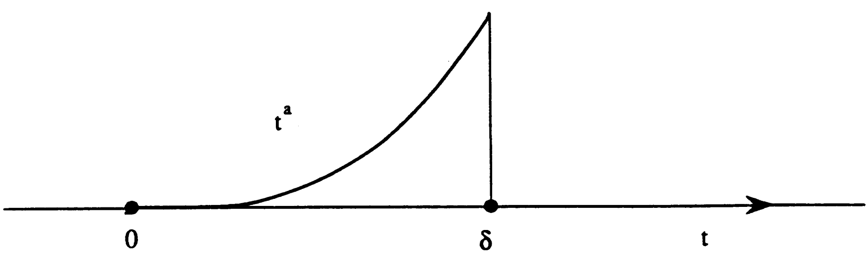

Fig. 3

Despite the presence of a "cusp" at zero, the simplices Δ_a for $a > 1$ have roughly the same spectral behaviour as the linear simplicies (corresponding to $a = 1$). For example, one can bound the first eigenvalue of Δ_a from below by Cheeger's inequality (see $1.A_3$). In fact, since the function t^a is monotone, the isometric profile $\mathrm{Is}(\alpha)$ of Δ_a satisfies

$$\mathrm{Is}(\alpha) \leq 10\delta\alpha$$

for $\alpha \leq \frac{1}{2}\, \mathrm{area}\, \Delta_a$, as an elementary argument shows.

Another way to see that the presence of a cusp for $a > 1$ does not make $\lambda_1 \to 0$ is to show first that this λ_1 can be bounded in terms of the eigenvalue $\overline{\lambda}_1$ of the projection of Δ to the t-axes. The projection of this Δ, or rather of the corresponding (Lebesgue) measure μ_{Δ_a}, is the measure on the segment $[0, \delta]$ with the density t^a. Essentially the same measure $t^a dt$ on $[0, \delta]$ can be obtained by projecting to $[0, \delta]$ the rotational body B_m in $\mathbf{R}^m$ obtained by rotating Δ_b for $b = a/m - 1$ around the t-axis. This B_m for $m \geq a + 1$ is convex and thus its spectrum is well under control. Then one can control the spectrum of the projection measure on $[0, \delta]$ as well.

Remarks. — The above lift to $\mathbf{R}^m$ agrees with the following (easy to prove) property of $\mathrm{Is}(\alpha)$ of Δ_a,

$$\mathrm{Is}(\alpha) \approx \alpha^{\frac{a}{a+1}}$$

for small α , which is just how $\mathrm{Is}(\alpha)$ behaves for convex subsets in $\mathbf{R}^{a+1}$ in case that a is an integer. In fact, one can give a qualitative description of the isoperimetric profiles $\mathrm{Is}(\alpha)$ and $\mathrm{Is}(\alpha_1, \ldots, \alpha_i)$ for general semialgebraic sets where one sees a similar phenomenon of the degree appearing together with the dimension. One also observes this picture in the non-linear spectral problems for the energies

$$E : \varphi \mapsto \|\mathrm{grad}\, \varphi\|_{L_p} \,/\, \|\varphi\|_{L_q} \qquad \textit{for} \qquad 1 \leq p, q \leq \infty.$$

(We hope to return to this discussion at another opportunity.)

3. Algebraic approximation and localization of eigenfunctions; computability of λ_i.

The gradient as well as the higher order derivatives of an eigenfunction ψ_i on a *smooth* variety V can be estimated by the L_2-norm of ψ_i and the corresponding λ_i, where the implied constant depends on the geometry of V. This shows that ψ_i can be ϵ-approximated by polynomials of degree d for $\epsilon = \epsilon(d) \to 0$ for $d \to \infty$. In particular, one can approximately evaluate λ_i by restricting the energy $\varphi \mapsto \|\mathrm{grad}\, \varphi\| \,/\, \|\varphi\|$ to the (projectivized) space of polynomials φ of degree d, where the error of the approximation is well under control in terms of d and the geometry of V.

Now let us see what happens for singular varieties and also for families of nonsingular ones as they approach a singularity

3.A. *Example.* — Consider the plane regions bounded by the graph of the function $x^2 + \epsilon$, for $\epsilon \geq 0$, over the segment $[-1, 1]$, as sketched below :

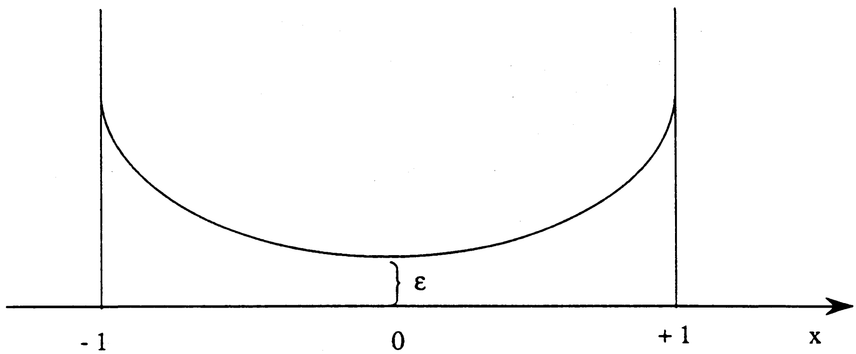

Fig. 4

It is more convenient to project this picture to the x-axes and pass to the measure $\mu_\epsilon = (x^2+\epsilon)dt$ on $[-1,1]$. Then we see that the corresponding $\lambda_1 = \lambda_1(\epsilon)$ goes to zero as $\epsilon \to 0$ and therefore the gradient of ψ_1 goes to infinity. In fact, let us evaluate the energy

$$E_\epsilon(\varphi) \underset{\text{def}}{=} \|\text{grad}\,\varphi\|_{\mu_\epsilon} \;/\; \|\varphi\|_{\mu_\epsilon}$$

at the following function

$$\varphi_\epsilon(x) = \begin{cases} \epsilon^{-\frac{1}{2}}x & \text{for} \quad |x| \le \sqrt{\epsilon} \\ 1 & \text{for} \quad x > \sqrt{\epsilon} \\ -1 & \text{for} \quad x < \sqrt{\epsilon} \end{cases}$$

Here obviously

$$\|\varphi_\epsilon\|_{\mu_\epsilon} \approx 2/3 \quad \text{for} \quad \epsilon \to 0$$

and

$$\|\text{grad}\,\varphi_\epsilon\|_{\mu_\epsilon} \approx \sqrt{\epsilon},$$

while

$$\int_{-1}^{1} \varphi_\epsilon d\mu_\epsilon = 0,$$

and so $\lambda_1(\epsilon)$ goes to zero at least as fast as ϵ.

Notice that in this example the convergence $\lambda_1(\epsilon) \to 0$ for $\epsilon \to 0$ is established with a *semialgebraic* test function, namely $\varphi_\epsilon(x)$, where a function is called semialgebraic if its graph is a semialgebraic subset. Also notice, that the same "semialgebraic test" shows that $\lambda_1(0) = 0$. Now, let $\mu = pdx$ be an arbitrary measure on the unit interval $[0,1]$ with the density function $p(x) \geq 0$. The vanishing of λ_1 is certainly due to the vanishing of $p(x)$ at some points x. In fact, a simple argument shows that

$$\lambda_1(\mu) = 0 \Leftrightarrow \int_0^1 p^{-1} = \infty.$$

If p is a semialgebraic function the condition $\int_0^1 p^{-1} = \infty$ can be obviously expressed in infinitesimal terms at the zero points of $p(x)$. Thus the condition $\lambda_1(\mu) = 0$ *is semialgebraic*, i.e. the set of μ satisfying this condition constitute a semialgebraic subset in the set T of all μ with semialgebraic density functions. (The set of the semialgebraic functions of a fixed degree d constitutes a semialgebraic set and so one may speak of semialgebraic subsets.)

This suggests that in all dimensions the eigenvalues $\lambda_1(V_t)$ and $\lambda_1(\mu_t)$ look in a first approximation as semialgebraic (or rather semianalytic if not Pfaff) functions in the parameter $t \in T$. Furthermore, a similar property can be conjectured for the eigenfunctions $\psi_{i,t}(v)$. Let us make these conjectures precise.

3.B. *The equations $\lambda_1(V) = 0$ and $\lambda_1(\mu) = 0$ define semialgebraic subsets in the space of all V's and μ's respectively.*

In fact, this looks quite easy at least for measures μ in domains Ω in $\mathbf{R}^N$ with, say, polynomial densities p. The condition on p equivalent to $\lambda_1(\mu) = 0$ seems to be

$$\int_L p^{-1} = \infty,$$

where L is a *generic* curve in Ω.

In any case the problem can be localized to a certain singularity Σ of V (or μ) and then one can probably express the vanishing $\lambda_1 = 0$ in terms of the infinitesimal geometry around Σ .

3.C. The i-th eigenvalue λ_i of V (or μ) is defined (see §1) as the solution to a certain variational problem on the space Φ of all functions on V. Now, let us take the subset $\Phi_d \subset \Phi$ of the semialgebraic functions of degrees $\leq d$ and define $\lambda_i(d)$ as the solution to our variational problem restricted to Φ_d. We conjecture that the rate of convergence $\lambda_i(d) \to \lambda_i$ for $d \to \infty$ depends only on deg(V). This would insure a principal possibility to compute λ_i with a given degree ϵ of precision, with the length of the computation depending only on ϵ, i and $\deg \Phi$.

A stronger conjecture states that the i-th eigenfunction ψ_i on V can be ϵ-approximated (in a variety of suitable norms) by semialgebraic functions of degree $d = d(\epsilon, i, \deg V)$. Here one may also ask if ψ_i look similar to a semialgebraic function of certain degree $D = D(i, \deg V)$. In fact, this is known to be true in certain cases thanks to Khovanskii's theory of Pfaff varieties (see [Kho]), and one may seek a generalization of this theory to certain non-holonomic elliptic P.D.E.

3.C_1. The above discussion (and the conjectures) apply to more general (non-quadratic) functions defined in §1. For example, in the case of the isoperimetric profile it appears geometrically obvious that one can see $\mathrm{Is}(\alpha)$ with a "semialgebraic eye" up to a given error. In particular if V can be divided into two pieces of equal mass by a small hypersurface H, then this can also be obtained with a semialgebraic hypersurface H_d of degree d bounded in terms of $\deg V$.

4. Spectra of semialgebraic families; continuity and analyticity of $\lambda_i(V_t)$ and $\lambda_i(\mu_t)$.

We continue here the discussion on analytic properties of the dependence of the spectrum on auxiliary (semi)algebraic parameters. If we look at the simplest case where V_t are smooth algebraic varieties smoothly algebraically depending on t, then $\lambda_i(V_t)$ are known to be analytic in t in-so-far as one properly takes into account the multiplicity (or, equivalently, the ordering) of λ_i. The same appears to be true (and, probably, is not hard to prove) if V_t is a *smooth* algebraic family of (possibly) *singular* varieties. For instance, one may take a family of plane n-gones for a fixed n. Another example is a family of hyperplane sections $V_t = H_t \cap V$ for a fixed singular V, which gives us a smooth family V_t for *generic* t.

On the other hand, at the points t where a family V_t is non-smooth, e.g. for the levels $V_t = p^{-1}(t)$ at the critical values t_{cri} of a polynomial p on $\mathbf{R}^N$, the eigenvalues may be not analytic anymore, nor even continuous at t_{cri}. In fact, even the (normalized) measure μ_t associated to t may become discontinuous.

4.A. *Example.* — Consider the plane regions V_t bounded by the graphs of the functions $t_1x^2+t_2$ over the segment $[-1,+1] \ni x$ for positive $t=(t_1,t_2)$. If $t \to 0$, then the non-normalized measure μ_t associated to V_t converges to zero, which is not very interesting from the spectral view point. On the other hand, if we normalize μ_t by dividing μ_t by the total mass of V_t (which, of course, has no effect on the spectrum) then we come to a more meaningful picture. Namely, the limit measure on $[-1,1]$ (which is the limit of V_t) may be any (normalized) measure with a density function of the form $ax^2+\frac{1}{2}-\frac{1}{3}a$, $a>0$. Furthermore, if we choose an arbitrary (semi)algebraic curve $t_1(s)$, $t_2(s)$ in the parameter space $T=\{t_1>0,\ t_2>0\}$ with $t_1(s)$, $t_2(s) \to 0$ for $s \to s_0$, then the measures $\mu_{t(s)}$ will converge to some measure μ on $[-1,1]$ (and the eigenvalues λ_i of $V(t(s))$ converge as well).

4.B. In general, the analytic nature of the correspondence $t \mapsto \mu_{V_t}$ is, in principle, understood. Namely, the total mass function $t \mapsto \mu(V_t)$ is obtained by integrating the algebraic functions defining V_t. (This is quite clear if V_t has full dimension N in the ambient space $\mathbf{R}^N$. If $\dim V = m < N$, one should use the Grassmann manifolds G of $(N-m)$-dimensional affine subspaces in $\mathbf{R}^N$ and pass from V to the integer valued function on G which assigns to every subspace $g \subset \mathbf{R}^N$ the geometric intersection number $\sharp\ (g \cap V)$.) Similarly, one may express the mass of $V_t \cap W$ for every (semialgebraic) domains $W \subset \mathbf{R}^N$ and the totality of these masses gives us the full information on $t \mapsto \mu_{V_t}$. The best understood case is that of the mass of the level sets of a polynomial p on $\mathbf{R}^N$, i.e. $\mu(p^{-1}[t_1,t_2])$, which is traditionally studied by means of the so-called *ζ-functions* associated to p and smooth functions φ with compact supports on $\mathbf{R}^N$,

$$\zeta(s) = \int_{\mathbf{R}^N} p^s \varphi d\mu$$

(see [At], [Ber], [BerGel], [Bj]). Apparently with the present state of knowledge one can satisfactorily address the following problems (a) and (b) concerning the function

$$t \mapsto \mu_{V_t}$$

which is essentially amounted to the study of the total mass

$$t \mapsto \mu(V_t)$$

on the parameter space T.

(a) *The analytic continuation of the function μ_{V_t} to the complexified space* $\mathbf{C}T$. Notice that the function $\mu(V_t)$ (and hence μ_{V_t}) is real analytic for generic t and so the analytic continuation makes sense. Of course, the continuation of $\mu(V_t)$ to the complex domain is a multivalued function and one of the major problems is that of the monodromy around the singularities. This is closely related to the second question :

(b) *Asymptotic expansion of* $\mu(V_t)$ *at the singular points.*

The existence of a nice asymptotic expansion in the case $V_t = p^{-1}([t_1, t_2])$, follows from the meromorphic continuation of the corresponding ζ-function to the complex domain $\mathbb{C} \ni s$. Probably, the general case falls along the same lines and, in particular, the normalized measure $t \mapsto \mu_{V_t}/\mu(V_t)$ is continuous on algebraic curves. In other words if we take an algebraic curve in T parametrized by $s \in]0,1]$, such that μ_{V_s} is analytic in s, then for every finite system of semialgebraic domains $W_1, \ldots, W_k$ in $\mathbf{R}^N$ the map from $]0,1]$ to the projective space P^k defined by

$$s \mapsto (\mu(V_s), \mu(V_s \cap W_1), \ldots, \mu(V_s \cap W_k))$$

continuously extends to $s = 0$. (Notice that such extension property suggests looking at the closure $\overline{\Gamma}$ of the graph of the map from T into the space of probability measures,

$$t \mapsto \mu_{V_t}/\mu(V_t)$$

as a kind of resolution $\overline{\Gamma} \to T$).

4.C. Now we turn to more difficult questions concerning the dependence of the eigenvalue λ_i on μ_t. Here one thing is obvious, that is the semicontinuity of λ_i if

$$\mu_t \to \mu_0$$

then

$$\liminf \lambda_i(\mu_t) \geq \lambda_i(\mu_0).$$

It is also clear, that λ_i are continuous if the degeneration of V_t to V_0 is not too severe. For example, if an *algebraic* family V_t is smooth at generic points in V_0 then λ_i are continuous. In other words the discontinuity of λ_i is due to two phenomena

(1) Discontinuity of the dimension of V_t (or some of the components of V_t).

(2) Non-trivial multiplicity of the limit fiber V_0, i.e. where the "normal projection" of nearby fibers V_t to V_0 is not one-to-one at generic points.

Notice, that in the *semi*-algebraic case the discontinuity may have another reason. For example, take V_t equal to the union of two unit segments with distance t,

$$V_t = [-1, 0] \cup [t, t+1], \quad t \leq 0.$$

Notice that whenever we have the continuity of the spectrum of V_t over a *compact* semialgebraic base T with irreducible fibers V_t , then we have (by 2.A$_1$) a uniform lower bound on λ_1,

$$\lambda_1(V_t) \geq \epsilon > 0$$

over T, as was claimed in §0.

Even in those cases where λ_1 are discontinuous one expects continuity along algebraic curves $t(s)$ as in the case of μ_t. In fact, this appears geometrically obvious. On the other hand, the analytic continuation of $\lambda_1(V_t)$ to complex t and the behavior near singular points probably needs a deeper analytic study. The problem here appears somewhat similar to the analytic continuation of eigenfunctions ψ_i from V to $\mathbb{C}V$, which does not seem easy even if V is a nonsingular algebraic variety. (All these problems are significantly simpler for $\dim V = 1$ where we deal with O.D.E. rather than P.D.E..)

5. Basic geometric means for estimating λ_i : product inequalities, decomposition relations and measures in the spaces of curves; monotonicity and quasiconvexity.

The standard (semi)algebraic tools for the study of a (semi)algebraic variety V are as follows :

(i) *Fibrations.* V may be mapped onto (fibered over) another variety X of lower dimension, $p : V \to X$, and then one looks at V as a collection of fibers $W_x = p^{-1}(x)$ parametrized by $x \in X$. (Sometimes one uses only partially defined fibrations, as in the case of *pencils* of hyperplane sections.) An obviously utility of fibrations is a possibility to apply induction on dimension, as $\dim V > \max(\dim W_x, \dim X)$.

(ii) *Submanifolds.* Submanifolds of positive codimensions, in particular algebraic curves in V carry a non-trivial information about V.

(iii) In the *semi*-algebraic world one is allowed to cut varieties into pieces. For example, one may triangulate V with geometrically controlled simplices and also one may cover V by subsets with a certain bound on geometry (see the discussion in [Gro]$_3$ around Yomdin's lemma).

(iv) If V is singular, one customarily *stratifies* V and thus divides the study into separate treatments of the singular and nonsingular parts. Near the singularity one may think of V as a cone over something of lower dimension. Also one may blow up the singularity by the Hironaka theorem which reveals much hidden geometry.

Now let us look at available geometric methods for the study of λ_i in the frameworks (i) - (iv). We are concerned here first of all with the lower bounds on λ_i and especially on λ_1.

5.A. The simplest case here is where V is the metric product, $V = W \times X$. Then, as we have already mentioned in §1 the spectrum of V is the sum of those of W and X, because the *heat kernel* is multiplicative under metric Cartesian products. Furthermore, if $p : V \to X$ is a *Riemannian fibration* with fiber W (i.e. for every geodesic segment $\sigma \subset X$ the pullback $p^{-1}(\sigma)$ is isometric to $W \times \sigma$), then the trace of the heat kernel on V is bounded by that on the product $X \times W$. This immediately follows from the Kac-Feynman formula (and was explained to me many years ago by J. Fröhlich). Since the trace of the heat kernel is related to the eigenvalues by

$$\operatorname{trace}\exp -\tau\Delta = \sum_{i=0}^{\infty} e^{-\tau\lambda_i} \ , \ \tau > 0$$

any upper bound on this trace gives a lower bound on λ_i. Thus one can bound $\lambda_i(V)$ from below in terms of $\lambda_i(B)$. Notice that such an estimate can also be obtained by more geometric means, namely, by a symmetrization argument which also applies to non-linear spectra and the isoperimetric constants.

5.A$_1$. *Example.* — Consider a domain Ω in V and let us bound $\mathrm{Vol}_m \Omega$ (for $m = \dim V = \dim \Omega$) in term of $\mathrm{Vol}_{m-1} \partial\Omega$ by "symmetrizing" the slices $\Omega_x = \Omega \cap W_x$ for $W_x = p^{-1}(x)$, $x \in X$ as follows. We assume we know the isoperimetric profile of the fiber and so we have the inequality

$$(*) \qquad \mathrm{Vol}_{k-1} \partial\Omega_x \geq I(\mathrm{Vol}_k \Omega_x) \ ,$$

for $k = \dim W$ and a certain function $I(\alpha) = I_W(\alpha)$. Next we look at the following three functions on X :

(1) $\varphi(x) = \mathrm{Vol}_k \Omega_x$. Notice that since p is a Riemannian fibration $\varphi(x)$ equals the density of the push-forward of the measure of Ω to X.

(2) $\psi(x) = \mathrm{Vol}_{k-1} \partial\Omega_x$.

(3) $\overline{\psi}(x) =$ the density of the push-forward of the measure on $\partial\Omega$ to X. That is $\int_Y \overline{\psi} = \mathrm{Vol}_{m-1}(p^{-1}(Y) \cap \partial\Omega)$ for all domains $Y \subset X$.

The three functions φ, ψ and $\overline{\psi}$ are related by the following (easy) inequality

$$\overline{\psi}^2 \geq \psi^2 + \|\mathrm{grad}\,\varphi\|^2 \ . \qquad (**)$$

Then we combine $(*)$ and $(**)$ and conclude to the following

Symmetrization inequality.

$$\overline{\psi}^2 \geq (I(\varphi))^2 + \|\mathrm{grad}\,\varphi\|^2 \ . \qquad (+)$$

This implies that

$$\mathrm{Vol}\,\partial\Omega = \int_X \overline{\psi} dx \geq \int_X \sqrt{(I(\varphi))^2 + \|\mathrm{grad}\,\varphi\|^2} dx \ .$$

In order to use that for a lower bound on $\mathrm{Vol}\,\partial\Omega$ in terms of $\mathrm{Vol}\,\Omega = \int_X \varphi dx$ we need a lower bound on the integral of the above $\sqrt{}$-expression in term of $\int \varphi$. Such a bound should be thought of as a Sobolev kind inequality and it can be reduced to the isoperimetric inequality in X (as was observed by Mazia about 30 years ago).

5.A$_2$. *Remarks.* — (a) The above symmetrization method may look rather crude but, in fact, it is sharp in many cases. For example if V is the Riemannian fibration with the base $X = \mathbf{R}^N$ and also with the Euclidean fiber, $W = \mathbf{R}^k$, then the domains Ω in V satisfy the $(n+k)$-dimensional Euclidean inequality. In other words, the isoperimetric profile of V satisfies

$$\mathrm{Is}_V(\alpha) \geq \mathrm{Is}_{\mathbf{R}^{n+k}}(\alpha) \ .$$

In the general case one can evaluate Is_V in terms of Is_X and Is_W but there is no simple explicit formula even for $\mathrm{Is}_X(\alpha) = a_1\alpha^{b_1}$ and $\mathrm{Is}_W(\alpha) = a_2\alpha^{b_2}$, with general a_i and b_i.

(b) If $V \to X$ is a *non-Riemannian* fibration then one should introduce certain correction terms into the above formulae, and then again one can relate the spectral (and isoperimetric) invariants of V to those of W and X. Notice that the relevant correction terms fall, roughly speaking, into two categories. The more important terms measure the deviation of $p : V \to X$ from a *Riemannian submersion*, where the situation becomes especially unpleasant whenever the differential of p on a horizontal tangent space (i.e. normal to the fibers) strongly deviate from an isometry, e.g. if this differential vanishes. The second group of correction terms concerns the deviation of the (horizontal) holonomies (mapping fibers to fibers along paths in X) from isometries. These terms may sometimes be fully incorporated into symmetrization inequalities, as it happens for the distance functions (and horofunctions) $p : V \to X = \mathbf{R}$ on the spaces of constant curvature (and up to some extend on general symmetric spaces of rank one). In fact these corrections may even improve our inequalities.

5.B. Instead of using the fibers W_x in V one may look at more general subvarieties, such as the family of all straight segments in a domain $V \subset \mathbf{R}^N$, or the family of all (semi)algebraic curves of a fixed degree d in V. In fact, many classical "isoperimetric" arguments giving a lower bound to λ_1 (and similar invariants) proceed as follows. First, the function φ in question is restricted to a curve ℓ in V from some family L. Then one proves a relevant inequality for $\varphi|\ell$, which is then transported to V by some integration over L. An especially convenient method of this kind (applicable to our semialgebraic discussion) is axiomatized on p. 166 in [Gro2].

Notice in this respect that in the (conformal) geometry one measures the sizes of families of curves (and higher dimensional submanifolds) by certain invariants called *moduli* (which are the duals of the *capacity* invariants). These invariants, when applied to the family of all curves (possibly, with certain mild restrictions) lead to characteristics of V similar to λ_1. The above discussion says, in effect, that one achieves a reasonably good approximation to these invariants if one uses the family of curves of degree d for d depending only on $\deg V$. This indicates a linkage of the present discussion with that in §3 concerning algebraic approximation of λ_i and ψ_i.

5.C. We have already seen in §2 that a division (e.g. triangulation) of V into "sufficiently simple" pieces (simplices) leads to a lower bound on λ_i. This approach gives rise to the following two complementary questions.

(1) What are the simplest possible pieces into which one can decompose a semialgebraic set. This is related to Yomdin's lemma as we have mentioned earlier.

(2) What are natural classes of "simple" spaces where one exercises good control over the spectrum? For example, all convex domains are quite good in this respect and a "little non-convexity" (properly understood) does not bring any problems. Intuitively, if a "reasonable" space has small λ_1 this should be seen by a naked eye : there must be an obvious narrow spot somewhere in V. This principle suggests a variety of geometric criteria for $\lambda_1 \geq \epsilon > 0$ which, in particular, apply to domains bounded by the graph of monotone funcitons (with an appropriate definition in the many variable case).

In order to use (1) and (2) together one needs a notion of simplicity which is general enough to be satisfied by the simplices of suitable semialgebraic triangulations, and yet sufficiently powerful to keep λ_i away from zero. Such a notion indeed exists (the definition I have in mind is too technical to be explained in this lecture) and leads to our lower bound on λ_i (see $2A_1$). However, this does not close the discussion as one needs more refined (and more general) notions of simplicity for other geometric problems.

5.D. Whenever one has a parametrix for Δ one obtains a control over the spectrum. A parametrix can often be produced by a local construction and then the local geometry of V is translated into an information on the spectrum. In particular, infinitesimal data of some stratification of V may be used via a parametrix to bound the spectrum from below.

A closely related approach consists in using some smoothing operators which commute (as the heat flow does) or almost commute (as the residual part of a parametrix) with Δ. Once we have such an operator then again we can bound λ_i from below.

6. The spectrum of the Laplace operator on forms.

The problems which arise here are quite similar but more difficult than those for the ordinary Laplace operator Δ on functions. For example, even a lower bound on λ_i for piecewise flat spaces V in terms of the number of simplices (i.e. degree) requires a certain thought above dimension zero. (This also leads to a more invariant and interesting questions of estimating λ_i for the above V in purely geometric terms independent of a given triangulation).

The difficulty which appears for higher degree forms when one tries to bound λ_i from below is similar to the corresponding problems in the Riemannian geometry. One can easily bound λ_i on functions in terms of the lower bound on the sectional (even Ricci) curvature (see [Gro1) but this does not work (so far) for higher degree forms appart from $\lambda_i = 0$ which corresponds to bounding Betti numbers (see [Gro4]). In both cases (real algebraic and $K \leq$ const) one needs a geometrically uniform local contractibility of the space V in question. It seems that this can be produced for semialgebraic sets on the basis of available information but we (J.C. and M.G.) have not checked the proof in detail yet.

7. Low and high frequences.

Our speculations on the spectrum presented in the earlier sections were centered on the lower bounds for λ_i which is an essentially non-local geometric problem. One enters a different domain when one turns to the asymptotic study of λ_i for $i \to \infty$. The basic means here are the asymptotics of the heat operator $\exp -\tau\Delta$ for $\tau \to 0$ and that of the *wave operator* $\cos \tau\Delta$. Here are obvious questions which come to one's mind.

Does the ζ-function $\sum_{i=0}^{\infty} \lambda_i^s$ admit a meromorphic continuation for *singular* semialgebraic varieties V? How the singularities of V are reflected in the poles of $\zeta_V(s)$? What is the dependence of $\zeta_{V_t}(s)$ on $t \in T$ as V_t algebraically varies over a parameter space T? (See [Che2] and [Che3] for a study of spaces with special singularities.)

8. Spectra of non-algebraic sets from the semialgebraic viewpoint.

The geometry of semialgebraic sets was used by I. Yomdin in his study of smooth maps. For example, Yomdin gives in [Yom] a bound on the average Betti number of a fiber of a smooth map f. Namely, he takes $f : V \to X$ and evaluates the integral

$$(*) \qquad \int_{X_0} \Big(\sum_i b_i(f^{-1}(x))\Big)^p dx\ ,$$

where X_0 is the non-critical set for f, in terms of p and the C^k-norm of f. Since the number N_ϵ of eigenvalues below a given $\epsilon > 0$ can be thought of as a geometric (or qualitative) counterpart of the Betti number (of the corresponding dimension) one expects an interesting bound similar to (*) with $N_\epsilon = N_\epsilon(i)$ (where i refers to the Laplace operator on i-forms) instead of b_i.

9. Geometric uses of the spectrum.

Why one wants to know such an invariant of V as Spec(V) which appears so farfetched and unnatural to a geometric eye? The first reason (apart from purely sociological ones) is the desire to use Spec as a test for our understanding of the overall geometry of V. A more serious reason originates from a hope that the study of the spectrum may eventually reveal a non-trivial geometrical (e.g. topological) information about V. This hope is well justified in those cases where apart from geometric estimates on λ_i one has in his possession an independent (analytical) source of knowledge about Spec. For example, if V is a locally homogeneous space, one can say something about the spectrum using the representation theory and related means. Also, if V has special holonomy (e.g. V is Kähler) then again there is a geometrically non-trivial input from the analysis. Finally, the index theorem and related results provide a valuable source of information which can not be recaptured by a pure geometry. However, for general Riemannian manifolds V the study of the spectrum appears a one way route : the geometry is used to study Spec but there is no feedback from analysis to geometry. Thus one wants to find classes of manifolds V where an interplay between geometry and analysis becomes real. One may hope this happens for (real) algebraic varieties V, or at least for the most distinguished among them, for example, for certain moduli spaces. Yet the ideas presented in this lecture fall short of the realization of this hope so far.

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