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EXTENSION OF COMPLEXES OF GROUPS

by André HAEFLIGER

En hommage à Claude Godbillon et Jean Martinet,
mes compagnons de route.

The main concern of this paper is to develop homological algebra aspects of complexes of groups. To illustrate the basic notions introduced here, we show how the classical theory of extensions of groups can be generalized to the case of complexes of groups.

This paper can be read independently of our preceding one on complexes of groups (*cf.* [5]). Here we systematically work with complexes of groups $G(X)$ over ordered simplicial cell complexes X (see section 1 for a definition; they are the geometric realization of the nerve of categories without loop (see section 1 and [5]). We recall in sections 1 and 2 the basic definitions in this slightly more general setting. A complex of groups can be seen as a category verifying some conditions, and we can apply to it the general notions of fundamental group, covering, cohomology, *etc.* , as explained for instance in the first few pages of Quillen [6] (this replaces the corresponding notions for topological groupoids which were in the background of [5]).

In section 3, we study the classifying space $BG(X)$ of $G(X)$ and show its connection with complex of spaces in the sense of Scott-Wall [9], Gersten-Stallings [12] and more generally Corson [4]. A typical example of a complex of spaces is a generalized Seifert fibration $\pi : Y \to X$ (*cf.* for instance [2], p. 219) where the base space is an ordered simplicial

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cell complex well adapted to the fibration. The corresponding complex of groups $G(X)$ is a coding of the fundamental groupoid of the foliation on Y as defined in Bonatti-Haefliger [2]. When the generic fibers are aspherical, then $BG(X)$ will have the same homotopy type as Y; this will give a typical example of a Poincaré duality complex of groups; we plan to develop this notion in another paper.

In section 4, we define the notions of $G(X)$-modules and cohomology as in Quillen [6], and we prove the existence of finite free resolutions when the complex X is finite and all the vertex groups admit finite free resolutions.

In sections 5 and 6 we apply the general theory to the problem of classifying the extensions of complex of groups by abelian kernel or by locally constant kernel.

In fact, most of the homological algebra notions developed for groups (*cf.* for instance the book of Brown [3]), have their natural generalization to complexes of groups.

1. Ordered simplicial cell complexes.

Following [5], we shall work with the class of ordered simplicial cell complexes X, namely those which are geometric realizations of the nerve of a small category $C(X)$ without loops. They generalize the simplicial complexes associated to partially ordered sets, for instance the barycentric subdivision of a simplicial complex (or more generally the barycentric subdivision of a polyhedral cell complex).

1.1. *Small categories.* — Let us recall that a small category C is a category whose morphisms form a set; if σ and τ are objects of C and if α is a morphism of σ in τ, namely belongs to $\mathrm{Hom}(\sigma, \tau)$, then τ is denoted by $i(\alpha)$ and σ by $t(\alpha)$. Two morphisms α and β are composable iff $t(\beta) = i(\alpha)$. We shall often identify an object σ of C with the identity morphism 1_σ of this object.

The *nerve* of C (*cf.* Segal [10]) is the simplicial set $N(C)$ whose set of k-simplices are the sequences $\alpha_1, \ldots, \alpha_k$ of k composable morphisms of C. Its geometric realization (*cf.* Milnor [8]) will be denoted by BC; it is a CW-complex whose set of vertices corresponds to the set of objects of C and the k-cells, $k > 0$, to the sequences of k composable elements $\gamma_1, \ldots, \gamma_k$

of C none of them being identities; the corresponding k-cell in BC will be denoted by $\langle\gamma_1,\ldots,\gamma_k\rangle$.

A category C is equivalent to a category C' if there exist functors $\varphi : C \to C'$ and $\psi : C' \to C$ such that there are (natural) isomorphisms of the functors $\psi\varphi$ and $\varphi\psi$ with the identity of C and C' respectively. Recall that if φ and φ' are functors from C to C', a natural transformation of φ to φ' is a map Φ associating to each object σ of C a morphism $\Phi(\sigma)$ of C' such that for every morphism $\alpha \in C$, we have $\varphi'(\alpha)\Phi(i(\alpha)) = \Phi(t(\alpha))\varphi(\alpha)$. If each $\Phi(\sigma)$ is invertible, then Φ is called a natural isomorphism.

1.2. *Category without loop and ordered simplicial cell complexes.* A category C is without loop if, whenever α and β are composable morphisms such that $i(\beta) = t(\alpha)$, then α and β are the identity morphism of the object $i(\beta)$. In particular for every object σ, the set $\mathrm{Hom}(\sigma,\sigma)$ contains only one element (still denoted by σ). Note that a category C' in which each morphism α with $i(\alpha) = t(\alpha) = \sigma$ is the identity morphism of σ is equivalent to a category without loop unique up to isomorphism.

An *ordered simplicial cell complex* is the geometric realization of the nerve of a category without loop C. It is a cell complex $X = BC$ whose set of vertices $V(X)$ is identified to the set of objects of C and the set $E(X)$ of 1-cells to the set of morphisms of C which are not identities of objects. Each 1-cell $a \in E(X)$ has a given orientation : its initial point is the vertex $i(a)$ and its terminal point is the vertex $t(a)$. The k-cells, noted $\langle a_1,\ldots,a_k\rangle$, $k > 0$, correspond to sequences of k composable elements $a_1,\ldots,a_k$ of $E(X)$. Each k-cell is naturally isomorphic to the standard k-simplex, because its vertices are naturally ordered. Note however that many oriented edges may have the same initial vertex and the same terminal vertex. The category such that X is the geometric realization of its nerve will be denoted by $C(X)$. As a set it is the disjoint union of $E(X)$ and the set $V(X)$ of its objects identified to the set of morphisms which are identities.

A *subcomplex* X_0 of X is the geometrical realization of a subcategory of $C(X)$. It is again an ordered simplicial cell complex. Note that for instance the 1-skeleton of X, considered as a cell complex, is not a subcomplex if $\dim X > 1$.

The dual X^{op} of an ordered simplicial cell complex X is the geometric realization of the opposite of the category $C(X)$ (obtained by reversing the arrows). As a cell complex, X^{op} is canonically isomorphic to X.

For examples, see fig. 1, 1' and fig. 2, 2'.

If we start from a usual simplicial complex X, then its barycentric subdivision X' is naturally an ordered simplicial cell complex : $V(X')$ is the set of barycenters and $E(X')$ is the set of edges a of the barycentric subdivision oriented so that the initial point $i(a)$ is the barycenter of a simplex of X whose dimension is higher than the dimension of the simplex whose $t(a)$ is the barycenter.

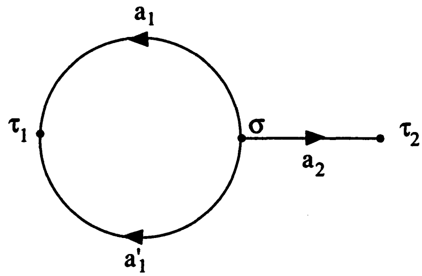
A simplicial map $f : X \to Y$ of ordered simplicial cells complexes X and Y associated to the categories $C(X)$ and $C(Y)$ is by definition the geometric realization of a functor from $C(X)$ to $C(Y)$ corresponding to f (still denoted by f).

Note that the direct product of two categories without loop is again a category without loop, so that the product of two ordered simplicial cell complexes is again an ordered simplicial cell complex.

1.3. *Dual complexes D_σ.* — For a vertex σ of X, we denote by D_σ the ordered simplicial cell complex associated to the following category $C(D_\sigma)$ without loops : the objects are the morphisms α of $C(X)$ with $t(\alpha) = \sigma$ and the morphisms are the 2-uples of composable elements (α, β) of $C(X)$ with $t(\alpha) = \sigma$; we define $i(\alpha, \beta) = \alpha\beta$ and $t(\alpha, \beta) = \alpha$. The composition of two composable morphisms $(\alpha, \beta)(\alpha', \beta')$ is $(\alpha, \beta\beta')$. There is an obvious deformation retracting linearly D_σ on σ. There is a natural projection j_σ of D_σ in X induced by the functor mapping (α, β) on β; it is injective on the union of the open simplices whose closure contains σ (see fig. 2 and 2'). For $\alpha \in C(X)$ with $i(\alpha) = \sigma$ and $t(\alpha) = \tau$, there is a natural simplicial map $j_\alpha : D_\sigma \to D_\tau$ mapping $(\alpha', \beta') \in D_\sigma$ on $(\alpha\alpha', \beta') \in D_\tau$.

By definition, Lk_σ is the full subcomplex of D_σ containing all vertices of D_σ except σ; therefore D_σ is the cone $\sigma * Lk_\sigma$ with vertex σ and basis Lk_σ.

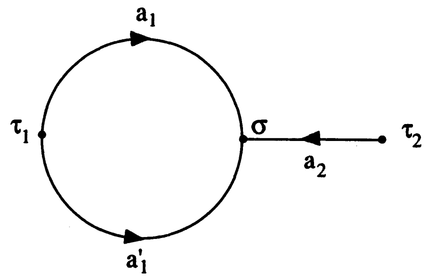
In fig. 1', Lk_σ consists of three points and in fig. 1, Lk_σ is empty.



A category C without loop

$D_\sigma = \sigma$

Figure 1



Its opposite C^{op}

$D_\sigma =$

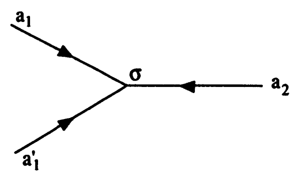
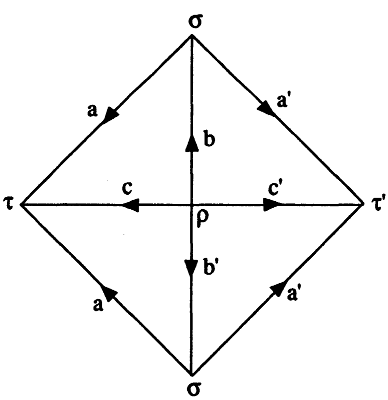
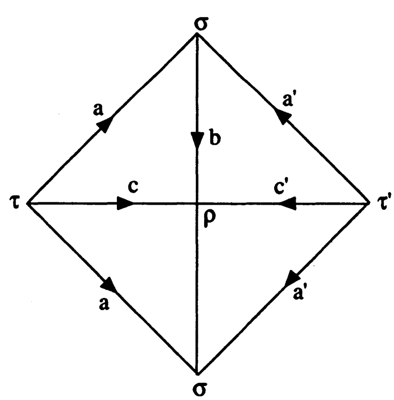


Figure 1′



Ordered simplicial cell complex X
homeomorphic to S^2
(elements with the same label
should be identified)
Figure 2



Its opposite

$D_\sigma =$

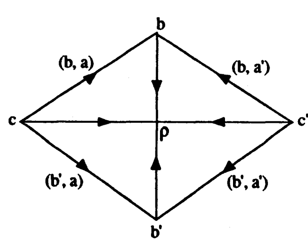


Figure 2′

Dually, we define D^σ and Lk^σ by replacing in the preceding definitions the category $C(X)$ by its opposite (reverse the arrows). For instance, if X is the barycentric subdivision of a combinatorial cell complex and if σ is the barycenter of a n-cell e, then Lk^σ is homeomorphic to a sphere of dimension $n-1$.

2. Complexes of groups on X.

2.1. *Action without inversion of a group on an ordered simplicial cell complex.* — The general definition of a complex of group will be better justified by the particular case of complexes of groups associated to an action (called developable complexes of groups).

Let $\widetilde{X}$ be an ordered simplicial cell complex which is the geometric realization of a small category without loop $C(\widetilde{X})$. An action without inversion of a group G on $\widetilde{X}$ is an action of G on $\widetilde{X}$ through simplicial homeomorphisms of $\widetilde{X}$ such that, if an element of g fixes a vertex σ, then it also fixes every edge a of $\widetilde{X}$ with $i(a) = \sigma$. The quotient X of $\widetilde{X}$ by the action of G is again an ordered simplicial cell complex and the natural projection p of $\widetilde{X}$ on X is simplicial; moreover p is injective on the union of open cells of $\widetilde{X}$ whose closure meets a vertex σ.

We want to give some data on the quotient X which are sufficient, when $\widetilde{X}$ is simply connected, to reconstruct the group G and its action on $\widetilde{X}$.

Choose for each vertex σ of X a vertex $\tilde{\sigma}$ of $\widetilde{X}$ projecting by p on σ. As the action is without inversion, this determines, for each edge $a \in E(X)$, an edge $\tilde{a} \in E(\widetilde{X})$ projecting by p on a and such that $i(\tilde{a}) = i(a)\tilde{}$.

Choose for each $a \in E(X)$ an element h_a of G such that $t(h(\tilde{a})) = t(a)\tilde{}$. For each $\sigma \in V(X)$, let G_σ be the subgroup of stability of $\tilde{\sigma}$ and, for each $a \in E(X)$, let $\psi_a : G_{i(a)} \to G_{t(a)}$ be the homomorphism mapping g to $h_a g h_a^{-1}$. For two composable elements a, b of $E(X)$, let $g_{a,b} = h_a h_b h_{ab}^{-1}$. The complex of groups associated to the action of G on $\widetilde{X}$ (and the choices above) is determined by the groups G_σ, the homomorphisms ψ_a and the elements $g_{a,b} \in G_{t(a)}$. We can check that the two conditions 2.2,i) and ii) below are satisfied. This justifies the following general definition.

2.2. DEFINITION. — *A complex of groups $G(X)$ on an ordered simplicial cell complex X is given by the following data :*

1) *for each $\sigma \in V(X)$, a group G_σ*

2) *for each $a \in E(X)$, an injective homomorphism $\psi_a : G_{i(a)} \to G_{t(a)}$*

3) *for each pair $a, b \in E(X)$ of composable edges, an element $g_{a,b} \in G_{t(a)}$ such that*

i) $\mathrm{Ad}(g_{a,b})\psi_{ab} = \psi_a\psi_b$ *($\mathrm{Ad}(g_{a,b})$ is the conjugation by $g_{a,b}$)*

ii) $\psi_a(g_{b,c})g_{a,bc} = g_{a,b}g_{ab,c}$ *for each triple $a, b, c \in E(X)$ of composable edges (cocycle condition).*

Graphs of groups over a graph X correspond to complexes of groups on the barycentric subdivision of this graph (cf. Serre [11]).

A complex of groups $G(X)$ associated to an action without inversion as in 2.1 is called *developable.* If $\dim X = 1$, then any complex of groups over X is developable (cf. Serre [11]). Sufficient conditions for developability in terms of non positive curvature are given in Stallings [12] and Haefliger [5]. A much better condition than in [5] is given by J. Corson [4] when $\dim X = 2$.

Consider a map associating to each edge $a \in E(X)$ an element $g_a \in G_{t(a)}$. The complex of groups $G'(X)$ on X given by $G'_\sigma = G_\sigma$, $\psi'_a = \mathrm{Ad}(g_a)\psi_a$ and $g'_{a,b} = g_a\psi_a(g_b)g_{a,b}g_{ab}^{-1}$ is said to be deduced from $G(X)$ by the coboundary of $\{g_a\}$.

2.2.1. PROPOSITION. — *Assume that the category $C(X)$ has an initial object σ; this means that for each object τ there is a unique $\alpha(\tau) \in C(X)$ such that $i(\alpha(\tau))) = \sigma$ and $t(\alpha(\tau)) = \tau$. Then any complex of groups on X can be modified by a suitable coboundary so that $g_{a,b} = 1$.*

Indeed, modify the given complex of groups $G(X)$ on X by the coboundary of $g_a = g^{-1}_{a,\alpha(i(a))}$.

2.3. *Examples.*

a) Let $G(X)$ be a complex of groups on X. Assume that τ is a sink if the group G_τ is not trivial (this means that there are no $a \in E(X)$ such that $i(a) = \tau$).

Then the map associating to $(a, b) \in E(Lk_\tau)$ the element $g_{a,b}$ of G_τ defines a functor of $C(Lk_\tau)$ in G_τ, thanks to the cocycle condition. On each connected component L of Lk_τ, this defines a homomorphism of the fundamental group of L in G_τ well defined up to conjugation. Conversely, the collection of those homomorphisms completely determines $G(X)$ up to a coboundary. In particular, if each component of Lk_τ is simply connected for every sink τ, then $G(X)$ is completely characterized up to a coboundary by the collection of the groups G_τ.

b) On the ordered simplicial cell complex X described in fig. 2, consider a complex of groups $G(X)$ such that G_τ and $G_{\tau'}$ are finite abelian groups and the other vertex groups are trivial. We can assume up to coboundary that only $g_{a,b}$ and $g_{a',b'}$ are possibly non trivial. This describes an orbifold structure on the 2-sphere X with two conical points at τ and τ' if and only if $g_{a,b}$ and $g_{a',b'}$ are generators of G_τ and $G_{\tau'}$ respectively. The complex of groups $G(X)$ is developable if and only if $g_{a,b}$ and $g_{a',b'}$ have the same order.

2.4. *The category $CG(X)$ associated to $G(X)$.* — For $\alpha \in C(X)$, define ψ_α as before if $\alpha \in E(X)$ and as the identity of G_α otherwise. For α, β composable elements of $C(X)$, define $g_{\alpha,\beta}$ as before if $\alpha, \beta \in E(X)$ and the identity element of $G_{t(\alpha)}$ otherwise.

We define a small category $CG(X)$ with $V(X)$ as set of objects; the set of morphisms is the set of pairs (g, α), with $\alpha \in C(X)$ and $g \in G_{t(\alpha)}$, and $i(g, \alpha) = i(\alpha)$, $t(g, \alpha) = t(\alpha)$ (the pair (g, α) will be denoted by $(1, \alpha)$ if g is the identity of the group $G_{t(\alpha)}$, and by g if $i(\alpha) = t(\alpha)$). The composition of two composable elements (g, α) and (h, β) is defined by :

$$(g, \alpha)(h, \beta) = (g\psi_\alpha(h)g_{\alpha,\beta}, \alpha\beta) .$$

The map $(g, \alpha) \to \alpha$ is a functor p of $CG(X)$ on $C(X)$. The map associating to $\alpha \in C(X)$ the element $(1, \alpha) \in CG(X)$ is a lifting of p; it is a functor if and only if all the elements $g_{a,b}$ are trivial. More generally, we can define a functor s from $C(X)$ in $CG(X)$ such that ps is the identity of $C(X)$ if and only if one can modify $G(X)$ by a coboundary so that all the elements $g_{a,b}$ become trivial. In that case one could say that $CG(X)$ is a semi-direct extension of $C(X)$.

Note that if the complex of groups $G'(X)$ is deduced from $G(X)$ by the coboundary of $\{g_a\}$, then the 1-cochain $\{g_a\}$ determines an isomorphism of $CG(X)$ on $CG'(X)$ mapping (g, a) on (gg_a^{-1}, a).

From the category $CG(X)$, we can reconstruct the category $C(X)$ as well as the complex of groups $G(X)$ up to a coboundary. The category

$C(X)$ is the quotient of $CG(X)$ by the equivalence relation which identifies γ to γ' if there is an element $h \in G_{t(\gamma)}$ such that $\gamma' = h\gamma$. This gives the functor p of $CG(X)$ on $C(X)$. The groups G_σ are the groups of elements γ with $i(\gamma) = t(\gamma) = \sigma$. For each element a of $E(X)$ choose an element $\tilde{a}$ of $CG(X)$ such that $p(\tilde{a}) = a$; then ψ_a is defined by the relation $\psi_a(h)\tilde{a} = \tilde{a}h$ for $h \in G_{i(a)}$. If a, b are composable elements of $E(X)$ with $ab = c$, then $g_{a,b}$ is defined by $g_{a,b}\tilde{c} = \tilde{a}\tilde{b}$.

The categories $C = CG(X)$ associated to complexes of groups $G(X)$ can be characterized as follows :

i) for each object σ of C, $\mathrm{Hom}(\sigma, \sigma)$ is a group G_σ,

ii) if a is a morphism with $i(a) = \sigma$ and $t(a) = \tau$, if $g, g' \in G_\tau$, and $h, h' \in G_\sigma$, then the equality $ga = g'a$ implies $g = g'$ and the equality $ah = ah'$ implies $h = h'$,

iii) for each $a \in \mathrm{Hom}(\sigma, \tau)$ and $h \in G_\sigma$, there is an element $\psi_a(h) \in G_\tau$ such that $\psi_a(h)a = ah$ ($\psi_a(h)$ is unique by ii) and $\psi_a : G_\sigma \to G_\tau$ is an injective homomorphism of groups),

iv) if $a \in C$ is invertible, then $a \in G_{i(a)}$.

Such a category has a canonical quotient C_0 which is a category without loop. Namely it has the same set of objects and two elements $a, a' \in C$ are identified if $i(a) = i(a')$, $t(a) = t(a')$ and $a' = ga$ for some g in $G_{t(a)}$.

A category satisfying i), ii) and iii) is called a precomplex of groups. It is always equivalent to a category satisfying i) – iv), well defined up to isomorphism.

2.5. *Homomorphism of complexes of groups.* — Let f be a simplicial map of X in Y (*i.e.* a functor from $C(X)$ in $C(Y)$). A homomorphism φ of a complex of groups $G(X)$ on X in a complex of groups $G(Y)$ on Y over f is a functor φ from $CG(X)$ in $CG(Y)$ projecting on f.

Explicitely φ is given by the following data : for each $\sigma \in V(X)$, one gives a homomorphism $\varphi_\sigma : G_\sigma \to G_{f(\sigma)}$ and for each $a \in E(X)$ an element $g_a \in G_{t(f(a))}$ such that

i) $\mathrm{Ad}(g_a)\psi_{f(a)}\varphi_{i(a)} = \varphi_{t(a)}\psi_a$;

and for each couple a, b of composable elements of $E(X)$

ii) $\varphi_{t(a)}(g_{a,b})g_{ab} = g_a\psi_{f(a)}(g_b)g_{f(a),f(b)}$.

The corresponding functor φ of $CG(X)$ in $CG(Y)$ maps (g,a) on $(\varphi_{t(a)}(g)g_a, f(a))$.

Let $[0,1]$ be the simplicial cell complex with two vertices 0 and 1 and one edge a with $i(a) = 0$ and $t(a) = 1$. An *homotopy* connecting two homomorphisms φ_0 and φ_1 of $G(X)$ in $G(Y)$ projecting on an homotopy $F : X \times [0,1] \to Y$ connecting f_0 to f_1 is a homomorphism $G(X) \times [0,1] \to G(Y)$ over F and whose restriction to X identified to $X \times \{i\}$ is φ_i, $i = 0,1$. This amounts to give a natural transformation of the functor φ_0 to the functor φ_1.

2.6. *Pull back, fiber product and glueing.* — If f is a simplicial map of X in Y, and if $G(Y)$ is a complex of groups on Y, then *the pull pack* $f^*G(Y)$ of $G(Y)$ by f is the complex of groups on X such $G_\sigma = G_{f(\sigma)}$, $\psi_a = \psi_{f(a)}$ and $g_{a,b} = g_{f(a),f(b)}$. There is a canonical homomorphism over f of $f^*G(Y)$ in $G(Y)$. If $G(X)$ is a complex of groups on X, a homomorphism φ of $G(X)$ in $G(Y)$ above f corresponds to a homomorphism of $G(X)$ in $f^*G(Y)$ above the identity of X.

A particular case is when f is the inclusion of a subcomplex X_0 in X; then $f^*G(X)$ is the restriction of $G(X)$ to X_0 and will be denoted by $G(X)|X_0$.

The fiber product is a generalization of those constructions. Let $\varphi : G(X) \to G(Y)$ and $\varphi' : G(X') \to G(Y)$ be two homomorphisms of complexes of groups over $f : X \to Y$ and $f' : X' \to Y$. The *fiber product* $G(X) \times_{G(Y)} G(X')$ is the complex of groups over $X \times_Y X'$ (defined up to a coboundary) associated to the subcategory of $CG(X) \times CG(X')$ consisting of pairs (γ, γ') such that $\varphi(\gamma) = \varphi'(\gamma')$. There are natural homomorphisms of $G(X) \times_{G(Y)} G(X')$ in $G(Y)$, $G(X)$ and $G(X')$.

Let X be a simplicial cell complex which is the union of two simplicial cell complexes X_1 and X_2. Let X_{12} be the intersection of X_1 and X_2. Let $G(X_1)$ and $G(X_2)$ be complexes of groups over X_1 and X_2 and let φ_{21} be an isomorphism of $G(X_1)|X_{12}$ on $G(X_2)|X_{12}$ over the identity of X_{12}. There is a complex of groups $G(X)$ over X, characterized uniquely up to isomorphism by the existence of isomorphism $\varphi_i : G(X_i) \to G(X)|X_i$ over the inclusions of X_i in X, $i = 1,2$, such that φ_{21} is equal to $\varphi_2\varphi_1^{-1}$ restricted above X_{12}. The complex of groups $G(X)$ is called a *glueing* of $G(X_1)$ and $G(X_2)$ over X_{12}.

2.7. *Fundamental group and covering theory.*

2.7.1. DEFINITION. — *The fundamental group $\pi_1(G(X),\sigma)$ of a complex of groups $G(X)$ on a ordered simplicial cell complex X based at a vertex σ of X is the fundamental group of the geometric realization $BG(X)$ of the nerve of $CG(X)$ based at σ. Hence it behaves functorially with respect to homomorphisms of complexes of groups.*

2.7.2. *Presentation.* — The usual presentation of the fundamental group of a cell complex gives in this particular case the following presentation.

Let T be a maximal tree in the 1-skeleton of X. This gives a corresponding maximal tree in the 1-skeleton of $BG(X)$. Then the fundamental group $\pi_1(G(X),T)$ has the following presentation, *cf.* Haefliger [5] (the change of signs and orders follows from the convention on the composition of paths).

The generators are all the elements of G_σ for each $\sigma \in V(X)$ and all edges $a \in E(X)$.

The relations are :

1) the relations in the G_σ's

2) for $a \in E(X)$, $h \in G_{i(a)}$, then $\psi_a(h) = a^{-1}ha$

3) for a pair (a,b) of composable edges with $ab = c$, then $c = bag_{a,b}$

4) $a = 1$ for $a \in T$.

Recall from [5] that $G(X)$ is developable if and only if, for each vertex σ, the natural homomorphism of G_σ in $\pi_1(G(X),T)$ is injective.

When X is the geometric realization of the barycentric subdivision of a combinatorial cell complex $\overline{X}$, one can give a more economical presentation of the fundamental group of $G(X)$ as follows.

Let T be a maximal tree in the 1-skeleton of $\overline{X}$. Each oriented edge e of $\overline{X}$ is the union of two edges a and a' of X with $i(a) = i(a')$, the barycenter of e. The edge e with the reversed orientation will be noted $\bar{e}$.

The generators are the elements of the groups G_τ, where τ runs over the set of vertices of $\overline{X}$, and the oriented edges e of $\overline{X}$.

There a four types of relations :

1)′ the relations in each G_τ,

2)′ for each edge e of the 1-skeleton of $\overline{X}$, union of a and a' with $i(a) = i(a') = \sigma$, with the orientation given by a, one has

$$e\psi_a(h)\bar{e} = \psi_{a'}(h) \text{ for each } h \in G_\sigma \ ,$$

3)′ for each 2-cell of $\overline{X}$ which is a polygon with k sides whose barycentric subdivision is as in fig. 3 (some sides might be identified), one has

$$e_1 g_{a_1,b_1} g^{-1}_{a'_2,b_2} \cdots e_k g_{a_k,b_k} g^{-1}_{a'_1,b_1} = 1$$

4′) $e = 1$ for $e \in T$.

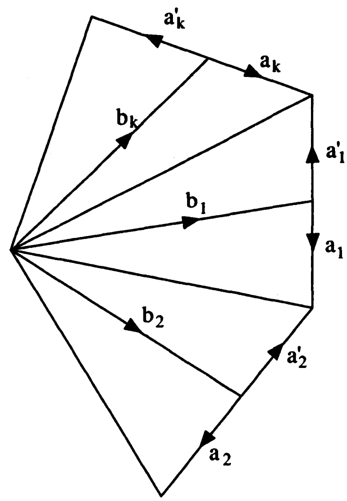


Figure 3

2.7.3. *Covering theory.* — The isomorphisms classes of coverings of a complex of groups $G(X)$ are in bijection with the isomorphism classes of the coverings of $BG(X)$, or with the equivalence classes of morphism-inverting functors of the category $CG(X)$ in the category of sets (*cf.* Quillen [6], p. 82) (or with the equivalence classes of functors of $\pi_1(G(X),\sigma)$ in the category of sets when X is connected).

Let us briefly indicate how one can associate a complex of groups to a morphism-inverting functor φ of $CG(X)$ in the category of sets (*i.e.* a functor φ such that each $\varphi(\gamma)$ is invertible). Let $\widetilde{C}$ be the category whose set of objects is the set of couples (σ, y), where $\sigma \in V(X)$ and y belongs to the set $\varphi(\sigma)$. The set of morphisms is the set of pairs (γ, y), where $\gamma \in CG(X)$ and $y \in \varphi(i(\gamma))$, and $i(\gamma, y) = (i(\gamma), y)$, $t(\gamma, y) = (t(\gamma), \varphi(\gamma)y)$. The composition of (γ, y) with (γ', y'), whenever it is defined, is equal to $(\gamma\gamma', y)$. This defines a precomplex of groups $\widetilde{C}$ (*cf.* 2.4); the corresponding covering will be a complex of groups $\widetilde{G}(X)$ such that $C\widetilde{G}(X)$ is equivalent to this precomplex of groups, and the natural projection of $\widetilde{C}$ on $CG(X)$ gives a surjective homomorphism of $\widetilde{G}(X)$ on $G(X)$. One could also characterize directly the homomorphisms $\widetilde{G}(X) \to G(X)$ which are covering maps like in Bass [1].

An explicit description of the universal covering is given in [5] (we take this opportunity to correct a misprint in [5], p. 525 : the end of line 14 should be "$(g\psi_{abc}(G_{i(c)}), a, bc)$").

3. The classifying space $BG(X)$ of a complex of groups. Complexes of spaces.

3.1. *The local retractions.* — Let $G(X)$ be a complex of groups on X. For each $\sigma \in V(X)$, let $G(D_\sigma)$ be the complex of groups on D_σ induced by the projection j_σ of D_σ in X. The elements of $CG(D_\sigma)$ are the triple (α, g, β) with $(\alpha, \beta) \in C(D_\sigma)$ and $g \in G_{t(\beta)}$. We can identify G_σ with the subcomplex of groups of $G(D_\sigma)$ restricted to the vertex $\sigma = (\sigma, \sigma)$ of D_σ. We have a deformation $R_\sigma : G(D_\sigma) \times [0,1] \to G(D_\sigma)$ connecting the identity to a retraction r_σ on G_σ and projecting on the natural deformation of D_σ on σ. Namely

$R_\sigma(\alpha, g, \beta, 0) = (\alpha, g, \beta)$,

$R_\sigma(\alpha, g, \beta, 1) = (\sigma, \psi_\alpha(g)g_{\alpha,\beta}, \sigma)$ and

$R_\sigma(\alpha, g, \beta, s) = (\sigma, \psi_a(g)g_{\alpha,\beta}, \alpha\beta)$, where s is the unique edge of $[0,1]$,

$r_\sigma(\alpha, g, \beta) = \psi_\alpha(g)g_{\alpha,\beta}$.

Note that if $a \in E(X)$, we have a natural homomorphism of $G(D_{i(a)})$ in $G(D_{t(a)})$ over the natural map j_a of $D_{i(a)}$ in $D_{t(a)}$ (*cf.* 1.3).

3.2. *The classifying space $BG(X)$.*

3.2.1. — Consider the geometric realization $BG(X)$ of the nerve of the category $CG(X)$. The functor $CG(X) \to C(X)$ gives a projection

$p : BG(X) \to X = BC(X)$. One has a canonical lifting s of the 1-skeleton $X^{(1)}$ in the 1-skeleton of $BG(X)$ mapping an edge a on the 1-cell $\langle(1, a)\rangle$ of $BG(X)$.

For each vertex σ of X, the fiber of p over σ is the Eilenberg-Mac Lane complex BG_σ and $BG(D_\sigma)$ is canonically the pull back of $BG(X)$ by the projection j_σ of D_σ in X. The homotopy R_σ gives a deformation of the identity map of $BG(D_\sigma)$ to a retraction still denoted by r_σ on the central fiber BG_σ. Each edge a with $t(a) = \sigma$ gives a well defined isomorphism of the fiber $BG_{i(a)}$ on the fiber of $BG(D_\sigma)$ above the vertex labelled a in D_σ; this isomorphism composed with r_σ gives the map of $BG_{i(a)}$ in $BG_{t(a)}$ induced by the homomorphism ψ_a of $G_{i(a)}$ in $G_{t(a)}$. Moreover an edge b of X with $t(b) = i(a)$ gives an edge (a, b) of $BG(D_\sigma)$ corresponding to the lifting $s(b)$. Its image by the retraction r_σ is the loop in BG_σ representing the element $g_{a,b}^{-1} \in \pi_1(BG_\sigma, s(\sigma))$. This shows that one can reconstruct $G(X)$ from the projection p of $BG(X)$, the partial section s and the base point preserving retractions r_σ.

Note that the modification of $G(X)$ by a coboundary amounts exactly to choose another section s.

3.2.2. *Remark.* — If $G(X)$ is locally constant, *i.e.* the groups G_σ are all isomorphic to a group G and all the homomorphisms ψ_a are isomorphisms, then $p : BG(X) \to X$ is a locally trivial bundle with fiber BG.

3.2.3. PROPOSITION. — *Assume that the complex of groups $G(X)$ is associated to an action without inversion of a group G on a ordered simplicial cell complex $\widetilde{X}$. Then $BG(X)$ as the same homotopy type as the Borel homotopy quotient $\overline{BG} \times_G \widetilde{X}$, where $\overline{BG}$ is the universal covering of BG.*

3.2.4. COROLLARY. — *If $\widetilde{X}$ is contractible, then $BG(X)$ is an Eilenberg-Mac Lane complex $K(G, 1)$, where G is the fundamental group of $G(X)$.*

Proof. — Let $C(\widetilde{X})$ be the category whose classifying space is the ordered simplicial cell complex $\widetilde{X}$. We consider the small category $G \ltimes C(\widetilde{X})$ whose set of objects is the set of objects $V(\widetilde{X})$ of $C(\widetilde{X})$ and the set of morphisms the set of pairs $(g, \alpha) \in G \times C(\widetilde{X})$ with projections i and t on $V(\widetilde{X})$ defined by $i(g, \alpha) = i(\alpha)$ and $t(g, \alpha) = gt(\alpha)$.

If $i(g,\alpha) = t(g',\alpha')$, then the composition $(g,\alpha)(g',\alpha')$ is defined by $(gg', g'^{-1}(\alpha)\alpha')$.

Let $G(X)$ be the complex of groups on $X = G\backslash\widetilde{X}$ associated to the action of G on $\widetilde{X}$ as in 2.1 (we keep the same notations). We have chosen for each vertex $\sigma \in V(X)$ a representative $\tilde{a} \in V(\widetilde{X})$. Then $CG(X)$ is in fact the full subcategory of $G \ltimes C(\widetilde{X})$ with set of objects the union of the $\tilde{\sigma}$'s and the inclusion of $CG(X)$ in $G \ltimes C(\widetilde{X})$ is clearly an equivalence, hence induces an homotopy equivalence on the classifying spaces. Therefore the proposition with follow from the following lemma.

We first introduce some notations. Given a group G, we denote by $\overline{G}$ the category whose set of objects is G and the set of morphisms the set $G \times G$, the maps i and t of $G \times G$ on G being defined by $i(g,g') = g'$ and $t(g,g') = gg'$. The composition of two composable morphisms (g_1, g_1') and (g_2, g_2') is (g_1g_2, g_2'). The classifying space $B\overline{G}$ is contractible, G acts freely on it, and BG is the quotient of $B\overline{G}$ by this action (*cf.* Segal [10]).

LEMMA. — *The classifying space $B(G \ltimes C(\widetilde{X}))$ is naturally homeomorphic to the quotient of $B\overline{G} \times \widetilde{X}$ by the diagonal action of G.*

Proof. — Consider the bisimplicial set whose set of simplices of type (p,q) is the set of commutative diagrams

$$\begin{array}{ccccccc}
\sigma_{0,q} & \xleftarrow{g_1} & \sigma_{1,q} & \xleftarrow{g_2} & \cdots & \xleftarrow{g_p} & \sigma_{p,q} \\
{\scriptstyle \alpha_{0,q}}\downarrow & & \downarrow & & & & \downarrow{\scriptstyle \alpha_{p,q}} \\
\sigma_{0,q-1} & \xleftarrow{g_1} & \sigma_{1,q-1} & \xleftarrow{g_2} & \cdots & \xleftarrow{g_p} & \sigma_{p,q-1} \\
{\scriptstyle \alpha_{0,q-1}}\downarrow & & \downarrow & & & & \downarrow{\scriptstyle \alpha_{p,q-1}} \\
\vdots & & \vdots & & & & \vdots \\
{\scriptstyle \alpha_{0,1}}\downarrow & & \downarrow & & & & \downarrow{\scriptstyle \alpha_{p,1}} \\
\sigma_{0,0} & \xleftarrow{g_1} & \sigma_{1,0} & \xleftarrow{g_2} & \cdots & \xleftarrow{g_p} & \sigma_{p,0}
\end{array}$$

where $\sigma_{r,s} \in V(\widetilde{X})$, $\alpha_{r,s} \in C(\widetilde{X})$ with $t(\alpha_{r,s}) = \sigma_{r,s-1}$; $i(\alpha_{r,s}) = \sigma_{r,s}$, $g_r \in G$ and $g_r(\alpha_{r,s}) = \alpha_{r-1,s}$.

The i-th face operator in the p-(resp. q-)direction deletes the i-th column (resp. line) and compose the morphisms if necessary.

The diagonal simplicial set is the nerve of the category $G \ltimes C(\widetilde{X})$. To construct its geometric realization (*cf.* Quillen [6]), one can first construct the geometric realization of the bisimplicial set in the q-direction, which is the nerve of the simplicial topological category $G \ltimes \widetilde{X}$, and take the geometric realization of the nerve of this category, which is precisely $B\overline{G} \times_G \tilde{X}$.

3.3. *Complex of spaces.*

3.3.1. — Before giving the general definition, inspired by Scott-Wall [9], Stallings [12] and more generally J. Corson [4], we introduce some notations. Let Y be a topological space with a continuous projection π on the ordered simplicial cell complex X. For each vertex σ of X, let $Y(\sigma)$ be $\pi^{-1}(\sigma)$; denote by $Y(D_\sigma)$ the subspace of $D_\sigma \times Y$ of pairs (x, y) with $j_\sigma(x) = \pi(y)$, by $\pi_\sigma : Y(D_\sigma) \to D_\sigma$ the projection $(x, y) \to x$ and $Y(j_\sigma) : Y(D_\sigma) \to Y$ the projection $(x, y) \to y$. We identify the fiber $Y(\sigma) = \pi^{-1}(\sigma)$ with the fiber of π_σ above the vertex of D_σ mapped by j_σ on σ. If $a \in E(X)$ is an edge with $i(a) = \sigma$ and $t(a) = \tau$, then the natural map $j_a : D_\sigma \to D_\tau$ (cf. 1.3) lifts to a map $Y(j_a) : Y(D_\sigma) \to Y(D_\tau)$ sending (x, y) on $(j_a(x), y)$. Suppose that a section s of π over the 1-skeleton $X^{(1)}$ of X is given. In particular each fiber $Y(\sigma)$ has a base point $s(\sigma)$. For each $\sigma \in V(X)$, this induces a section s_σ of π_σ over the 1-skeleton of D_σ.

3.3.2. DEFINITION (Compare with Corson [4]). — *A complex of space over X is a topological space Y with a continuous projection π on X and a section $s : X^{(1)} \to Y$ of π over the 1-skeleton of X such that with the above notations :*

i) *for each $\sigma \in V(X)$, the fiber $Y(\sigma)$ is connected and there is a retraction $r_\sigma : Y(D_\sigma) \to Y(\sigma) \subset Y(D_\sigma)$ which is homotopic to the identity relatively to $Y(\sigma)$ and which is compatible with s, namely $r_\sigma s_\sigma(x) = s(\sigma)$ for $x \in D_\sigma^{(1)}$,*

ii) *if $G_\sigma = \pi_1(Y(\sigma), s(\sigma))$ and if $a \in E(X)$ with $i(a) = \sigma$, $t(a) = \tau$, then the homomorphism $\psi_a : G_\sigma \to G_\tau$, induced by the base point preserving map $Y(\sigma) \to Y(\tau)$ defined by $y \to r_\tau Y(j_a)(y)$, is injective.*

For two composable edges a, b of $E(X)$ with $t(b) = \tau$, let $g_{a,b}^{-1}$ be the element of $\pi_1(Y(\tau), s(\tau)) = G_\tau$ which is the homotopy class of the loop image of the edge (a, b) of D_τ by the map $r_\tau s_\tau$.

3.3.3. PROPOSITION. — *The groups G_σ, the homomorphisms $\psi_a : G_{i(a)} \to G_{t(a)}$ and the elements $g_{a,b}$ determine a complex of groups $G(X)$ on X.*

Proof. — Let us check the cocycle condition. Let a, b, c be a triple of composable edges, with $t(a) =$ and $i(a) = \sigma = t(b)$. Consider in $Y(D_\sigma)$ the loop ℓ obtained by following first the path $s_\sigma(\sigma, bc)$ in the opposite direction, then the path $s_\sigma(b,c)$ and the path $s_\sigma(\sigma, b)$. The homotopy class of the image of ℓ by the retraction r_σ is the element $g_{b,c}^{-1} \in G_\sigma$. As r_σ is homotopic to the identify of $Y(D_\sigma)$, the maps $r_\tau Y(j_a)$ and $r_\tau Y(j_a) r_\sigma$ are homotopic. The homotopy class of ℓ is sent by the first map on $g_{a,bc} g_{ab,c}^{-1} g_{a,b}^{-1}$ and on $\psi_a(g_{b,c}^{-1})$ by the second one.

3.3.4. DEFINITION. — *We say that the complex of groups $G(X)$ is associated to the complex of spaces $\pi : X \to Y$ (together with the partial section $s : X^{(1)} \to Y$), and that Y is a* topological realization *of $G(X)$. If moreover Y is a cell complex and π a cellular map such each cell of Y is mapped on a cell of X, then we say that Y is a* cellular realization *of $G(X)$. In case each Y_σ is an Eilenberg-Mac Lane complex, we say that Y is a* cellular aspherical realization *of $G(X)$.*

The cell complex $BG(X)$ is an example of a cellular aspherical realization of $G(X)$; it is functorial with respect to the homomorphisms of complexes of groups.

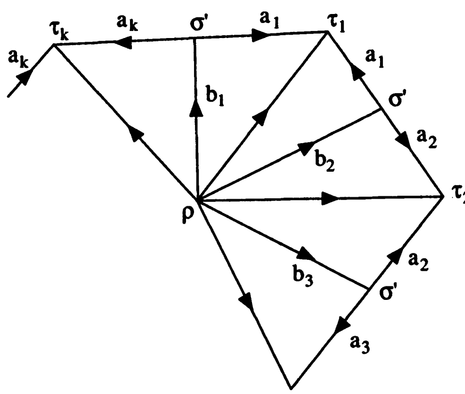
3.3.5. *Example.* — Let $\pi : Y \to X$ be an oriented Seifert fibration with k exceptional fibers above the points $\tau_1, \ldots, \tau_k$ of the compact oriented surface X of genus g. We can assume that X is realized as an ordered simplicial cell complex such that each τ_i is a vertex which is a sink. We choose a section $s : X^{(1)} \to Y$ above the 1-skeleton of X. The orientation of the fibers gives a canonical isomorphism of the fundamental group of each fiber with $\mathbf{Z}$.

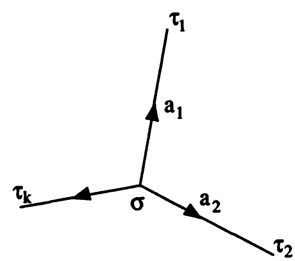
To each vertex σ of X is associated the group $G_\sigma = \pi_1(Y(\sigma), s(\sigma)) = \mathbf{Z}$. For each sink τ, the complex D_τ is a cone over the barycentric subdivision of a cell subdivision of $\partial D_\tau = Lk_\tau$ which is a circle. We can find a retraction $r_\tau : Y(D_\tau) \to Y(\tau)$ such that $r_\tau s_\tau$ maps an edge (τ, a) of D_τ on the base point $s(\tau)$. The map $r_\tau s_\tau$ maps an edge of Lk_τ on a loop in the fiber Y_τ whose homotopy class will be denoted by $-g_{a,b} \in \pi_1(Y_\tau, s(\tau)) \in \mathbf{Z}$. We get in this way a complex of groups $G(X)$ on X with each group $G_\sigma = \mathbf{Z}$, all the homomorphisms ψ_a are the identity, except if $t(a) = \tau_i$; in that case it is the multiplication by a positive integer α_i. The degree of the map $r_{\tau_i} s_{\tau_i}$ restricted to Lk_{τ_i} is an integer $-\beta_i$, and the pair (α_i, β_i) is the Seifert invariant associated to the singular fiber $Y(\tau_i)$ (see for instance Jankins-Neumann, Lectures on Seifert manifolds, Brandeis Lecture Notes 2, 1983). The integer β_i is up to sign the sum of $\pm g_{a_j,b_j}$, where (a_j, b_j) runs over the edges of Lk_{τ_i}, with the sign + or − according to the concordance of the orientation of the edge (a_j, b_j) with the orientation of Lk_{τ_i} or not. Note that such a complex of groups comes from a Seifert fibration if and only if the integers α_i, β_i are prime to each other. The complex of groups $\overline{G}(X)$ on X associated to the base space considered as an orbifold is defined by $\overline{G}_\sigma = 1$ if $\sigma \neq \tau_i$ and equal to $\mathbf{Z}/\alpha_i\mathbf{Z}$ if $\sigma = \tau_i$; if $t(a) = \tau_i$, then $\bar{g}_{a,b}$ is the reduction mod α_i of $g_{a,b}$.

For instance if $g = 0$, we can assume that X is the simplicial cell complex associated to the category described in figure 4, where the vertices or the edges with the same label have to be identified. Up to coboundary (*i.e.* with a good choice of the section s), we can assume that only the elements g_{a_i,b_i} may be non trivial. Let h be the canonical generator of $G_\sigma = \mathbf{Z}$ and h_i the canonical generator of $G_{\tau_i} = \mathbf{Z}$. Then $\psi_{a_i}(h) = h_i^{\alpha_i}$ (we use the multiplicative notation) and $g_{a_i,b_i} = h_i^{-\beta_i}$. According to 2.8.2, we get the following presentation

$$\langle h, h_1, \ldots, h_k; h = h_1^{\alpha_1}, \ldots, h = h_k^{\alpha_k}, h_1^{-\beta_1}, \ldots, h_k^{-\beta_k} = 1\rangle$$

of $\pi_1(G(X), \sigma) = \pi_1(Y, s(\sigma))$.





$X \approx S^2$

(Elements with same label
should be identified
Figure 4

3.4. *Construction of aspherical realizations.* — The following is a generalization of previous constructions of Scott-Wall [9] and J. Corson [4]. J. Corson has informed me that he has also obtained an analogous statement which should appear in a forthcoming paper.

3.4.1. THEOREM. — *Let $G(X)$ be a complex of groups over X. Choose for each $\sigma \in V(X)$ a cell complex $Y(\sigma)$ with base point $s(\sigma)$ which is an Eilenberg-Mac Lane complex with fundamental group G_σ. Then one can construct a cellular aspherical realization $\pi : Y \to X$ of $G(X)$ such that each $Y(\sigma)$ is the given cell complex. Moreover, for each k-simplex e of X with initial vertex σ, the cells of dimension $k+r$ of Y projecting on e are in bijection with the r-cells of Y_σ.*

Proof. — We begin with a few remarks. Let (X, x_0) and (Y, y_0) be two Eilenberg-Mac Lane cell complexes with base points and let $G = \pi_1(X, x_0)$ and $H = \pi_1(Y, y_0)$. For every homomorphism $\varphi : G \to H$, one can construct a base point preserving cell map $f : X \to Y$ inducing the homomorphism φ on the fundamental groups; this map is unique up to a base point preserving homotopy.

Let I^k be the cube $[0,1]^k$ with base point $(0,\dots,0)$ and let ∂I^k be its boundary. Let $f : X \times \partial I^k \to Y$ be a base point preserving cell map. We discuss the possible cell maps extensions $F : X \times I^k \to Y$ of f.

If $k = 1$, let $f_0(x) = f(x, 0)$ and $f_1(x) = f(x, 1)$. The homotopy class of an extension F, relative to $X \times \partial I$, is completely characterized by the homotopy class $h \in H$ of the loop $F(\{x_0\} \times I)$, and the homomorphisms $\varphi_0, \varphi_1 : G \to H$ induced by f_0 and f_1 are conjugated by h. Conversely, if φ_0 and φ_1 are conjugated by h, then there exists a cell map extension F such that h is the homotopy class of the loop $F(\{x_0\} \times I)$.

If $k = 2$ the extension F exists if and only if the homotopy class of the loop $f(\{x_0\} \times \partial I^2)$ is trivial. If $k \geq 3$, the extension F always exists and for $k \geq 2$, the homotopy class of F relative to $X \times \partial I^k$ is unique.

To construct Y, we first choose for each $a \in E(X)$ a base point preserving cell map $G(a) : Y(i(a)) \to Y(t(a))$ such that the induced homomorphism on the fundamental groups is ψ_a. For two composable edges $a, b \in E(X)$, we choose a cell map $G(a,b) : Y(i(b)) \times I \to Y(t(a))$ such that $G(a,b)(y,0) = G(ab)(y)$, $G(a,b)(y,1) = G(a)G(b)(y)$ and such that the homotopy class of the loop $t \to G(a,b)(s(i(a)), t)$ is $g_{a,b}^{-1}$.

We shall construct successively, for $k = 0, 1, \dots$, an increasing sequence of cell complexes $Y^{(k)}$ with a cell projection $\pi^{(k)}$ of $Y^{(k)}$ on the k-skeleton $X^{(k)}$ of X.

Case $k = 0, 1$.

$Y^{(0)}$ will be the disjoint union of the $Y(\sigma)$'s for $\sigma \in V(X)$ with the obvious projection $\pi^{(0)}$ on $X^{(0)}$. Then $Y^{(1)}$ will be the quotient of the disjoint union of $Y^{(0)}$ and the products $Y(i(a)) \times I$, $a \in E(X)$, by the equivalence relation which identifies $(y, 0) \in Y(i(a)) \times I$ with $y \in Y(i(a))$ and $(y, 1)$ with $G(a)(y) \in Y(t(a))$. We have a natural cell map $F(a) : Y(i(a)) \times I \to Y^{(1)}$ for the product structure). The composition $\pi^{(1)}F(a)$ restricted to $\{y\} \times I$ will be the natural simplicial map on the edge a of $X^{(1)}$. We also have a natural section $s : X^{(1)} \to Y^{(1)}$.

Case $k = 2$.

For each pair $a_2, a_1 \in E(X)$ of composable edges, let $f(a_2, a_1) : Y(i(a_1)) \times \partial I^2 \to Y^{(1)}$ be the cell map sending (z, t_1, t_2) on

$$\begin{array}{ll} F(a_1)(z, t_1) & \text{for } t_2 = 0 \\ F(a_2a_1)(z, t_2) & \text{for } t_1 = 0 \\ F(a_2)(F(a_1)(z, 1), t_2) & \text{for } t_1 = 1 \\ G(a_2, a_1)(z, t_1) & \text{for } t_2 = 1 \ . \end{array}$$

One checks easily the compatibility of those definitions.

Let $Y(a_2, a_1) = Y((a_1)) \times I^2$. Then $Y^{(2)}$ will be the quotient of the union of $Y^{(1)}$ with the disjoint union of the $Y(a_2, a_1)$'s (where (a_2, a_1) runs over the sequence of composable elements of $E(X)$) by the equivalence relation identifying $(z, t_1, t_2) \in Y(a_2, a_1)$ with $f(a_2, a_1)(z, t_1, t_2) \in Y^{(1)}$ for $(t_1, t_2) \in \partial I^2$. We shall denote by $F(a_2, a_1)$ the natural map of $Y(a_2, a_1) = Y(i(a_1)) \times I^2$ in $Y^{(2)}$.

Case $k = 3$.

Let $\partial' I^3$ be the boundary of the cube I^3 minus the face $t_3 = 1$. For each sequence (a_3, a_2, a_1) of three composable elements of $E(X)$, consider the map of $Y((a_1)) \times \partial' I^3 \to Y^{(2)}$ sending (z, t_1, t_2, t_3) on

$$\begin{array}{ll} F(a_2, a_1)(z, t_1, t_2) & \text{for } t_3 = 0 \\ F(a_3a_2, a_1)(z, t_1, t_3) & \text{for } t_2 = 0 \\ F(a_3, a_2a_1)(z, t_2, t_3) & \text{for } t_1 = 0 \\ F(a_3, a_2)(F(a_1)(z, t_1), t_2, t_3) & \text{for } t_1 = 1 \ . \\ F(a_3)(F(a_2, a_1)(z, t_1, t_2), t_3) & \text{for } t_2 = 1 \ . \end{array}$$

Again one can check the compatibility conditions on the common faces. The restriction of this map on the boundary of the square $t_3 = 1$ is the map in $Y(t(a_3))$ given by

$$\begin{array}{ll} G(a_3a_2, a_1)(z, t_1) & \text{for } t_2 = 0 \\ G(a_3, a_2)(G(a_1)(z), t_2) & \text{for } t_1 = 1 \\ G(a_3, a_2a_1)(z, t_2) & \text{for } t_1 = 0 \\ G(a_3)(G(a_2, a_1)(z, t_1) & \text{for } t_2 = 1 \ . \end{array}$$

By the cocycle condition and the remarks at the beginning of this proof, this map can be extended to the square $\{t_3 = 1\} \subset \partial I^3$ as a map in $Y(t(a_3)) \subset Y^{(2)}$. We obtain in this way a map $f(a_3, a_2, a_1) : Y(i(a_1)) \times \partial I^3 \to Y^{(2)}$.

Let $Y(a_3, a_2, a_1) = Y(i(a_1)) \times I^3$. As above we define $Y^{(3)}$ as the quotient of the union of $Y^{(2)}$ and the disjoint union of the $Y(a_3, a_2, a_1)$ (where (a_3, a_2, a_1) runs over the sequence of three composable edges of $E(X)$) by the equivalence relation identifying $(z, t_1, t_2, t_3) \in Y(a_3, a_2, a_1)$ with its image by $f(a_3, a_2, a_1)$ for $(t_1, t_2, t_3) \in \partial I^3$. We shall denote by $F(a_3, a_2, a_1)$ the natural map of $Y(a_3, a_2, a_1)$ in the quotient $Y^{(3)}$.

Case $k \geq 3$.

We argue by induction. Assume $Y^{(i)}$ and the cell maps

$$F(a_i, \ldots, a_1) : Y(a_i, \ldots, a_1) = Y(i(a_1)) \times I^i \longrightarrow Y^{(i-1)}$$

already constructed for $3 \leq i \leq k-1$. For each sequence $(a_k, \ldots, a_1)$ of k composable elements of $E(X)$, let $f(a_k, \ldots, a_1) : Y(i(a_1)) \times \partial I^k \to Y^{(k-1)}$ be the cell map sending $(z, t_1, \ldots, t_k)$ on

$$
\begin{array}{ll}
F(a_{k-1}, \ldots, a_1)(z, t_1, \ldots, t_{k-1}) & \text{for } t_k = 0 \\
F(a_k, \ldots, a_{i+1}a_i, \ldots, a_1)(z, t_1, \ldots, \hat{t}_i, \ldots, t_k) & \text{for } t_i = 0 \text{ and } 1 \leq i < k \\
F(a_k, \ldots, a_{i+1})(F(a_i, \ldots, a_1)(z, t_1, \ldots, t_i), t_{i+1}, \ldots, t_k)) & \\
 & \text{for } t_i = 1 \text{ and } 1 \leq i < k
\end{array}
$$

by any cell map extension in $Y(i(a_k)) \subset Y^{(k-1)}$ compatible with the above maps for $t_k = 1$.

One has to check that those formulas agree on the intersection of two faces. Then $Y^{(k)}$ will be the quotient of the union of $Y^{(k-1)}$ with the disjoint union of the $Y(a_k, \ldots, a_1) = Y(i(a_1)) \times I^k$ (where $(a_k, \ldots, a_1)$ runs over the sequences of k composable elements of $E(X)$ by the equivalence relation which identifies $(z, t_1, \ldots, t_k) \in Y(a_k, \ldots, a_1)$ with $f(a_k, \ldots, a_1)(z, t_1, \ldots, t_k) \in Y^{(k-1)}$ for $(t_1, \ldots, t_k) \in \partial I^k$. Finally $F(a_k, \ldots, a_1)$ will be the natural map of $Y(a_k, \ldots, a_1) = Y(i(a_1)) \times I^k$ in $Y^{(k)}$.

We still have to define by induction on k the projection $\pi^{(k)} : Y^{(k)} \to X^{(k)}$. Its restriction to $Y^{(k-1)}$ will be $\pi^{(k-1)}$. The composition of $F(a_k, \ldots, a_1)$ with $\pi^{(k)}$ will be the composition of the projection on the second factor I^k with a simplicial map r_k of I^k on the k-simplex $\langle a_k, \ldots, a_1 \rangle$ of $X^{(k)}$ described as follows. This k-simplex is naturally isomorphic to the standard ordered k-simplex Δ^k whose vertices will be denoted by $0, 1, \ldots, k$

(the edge between $i-1$ and i corresponding to a_i). We consider I^k as the ordered simplicial set which is the k-fold product of the complex I. Then r^k is the simplicial map sending $(0,\dots,0)$ on 0, and $(t_1,\dots,t_{i-1},1,0,\dots,0)$ on the vertex i, for $i=1,\dots,k$.

To construct the retraction $r_\tau : Y(D_\tau) \to Y(\tau)$, we procede as follows. Each k-cell of D_τ is labelled by a sequence $(\alpha, a_k,\dots,a_1)$ of composable elements of $C(X)$, with $t(\alpha)=\tau$ and $a_i \in E(X)$. We have a natural map $F_\alpha(a_k,\dots,a_1) : Y(i(a_1)) \times I^k \to Y(D_\tau)$ whose composition with $Y(j_\tau)$ is $F(a_k,\dots,a_1)$ The retraction r_τ will map $F_\alpha(a_k,\dots,a_1)(z,t_1,\dots,t_k)$ on

$$\begin{array}{ll} F(a_k,\dots,a_1)(z,t_1,\dots,t_{k-1}) & \text{for } \alpha=\tau \\ F(a,a_k,\dots,a_1)(z,t_1,\dots,t_k,1) & \text{for } \alpha=a\in E(X)\ . \end{array}$$

3.5. *Universal property of aspherical realizations.*

3.5.1. DEFINITION. — *Let $\pi : Y \to X$ be a cellular realization of a complex of groups $G(X)$ on X and let $\pi' : Y' \to X'$ be an aspherical realization of a complex of groups $G'(X')$ on X'. Let $s : X^{(1)} \to Y$ and $s' : X'^{(1)} \to Y'$ be the given sections. Let $\varphi : G(X) \to G'(X')$ be a homomorphism of complex of groups over a simplicial map $f : X \to X'$ (we use the notations of 2.6).*

A realization *of φ is a cell map $F : Y \to Y'$ such that*

i) *for each cell e of X, then F maps $\pi^{-1}(e)$ in $\pi'^{-1}(f(e))$;*

ii) *for each vertex σ of X then $F(s(\sigma)) = s'(f(\sigma))$ and the homomorphism $\pi_1(Y(\sigma),s(\sigma)) \to \pi_1(Y'(f(\sigma)),\ s'f(\sigma))$ induced by F is φ_σ;*

iii) *for each edge a of X with $t(a)=\sigma$ and $\varphi(1,a)=(g_a,f(a)) \in CG(X')$, the loop in $Y'(f(\sigma))$ obtained by restricting $r_{f(\sigma)}F$ to the oriented edge a represents the element $g_a^{-1} \in G_{f(\sigma)} = \pi_1(Y'(f(\sigma)),s'(\sigma))$.*

3.5.2. THEOREM. — *With the above notations, every homomorphism φ admits a realization; it is unique up to homotopy.*

The proof of this theorem uses the considerations made at the beginning of the proof of theorem 3.4.1 and the construction of F is made successively on the $Y^{(k)}$'s. We leave the details to the reader.

It follows in particular that any aspherical realization of $G(X)$ has the same homotopy type as $BG(X)$ (this leads to another proof of 3.2.3).

3.5.3. COROLLARY. — *Let $G(X)$ be a complex of groups over a* *finite ordered simplicial cell complex X. Assume that each G_σ is the fundamental group of a finite aspherical cell complex. Then $BG(X)$ has the homotopy type of a finite complex and its Euler-Poincaré characteristic is given by*

$$\chi(BG(X)) = \sum(1 - \chi(Lk_\sigma))\chi(G_\sigma) .$$

4. Homology and cohomology.

4.1. *$G(X)$-modules.*

4.1.1. *Left and right $G(X)$-modules.* — A left $G(X)$-module is a covariant functor M from the category $CG(X)$ in the category of abelian groups : to each $\sigma \in V(X)$ is associated an abelian group $M(\sigma)$ and to each $\gamma \in CG(X)$ is associated a homomorphism $M(\gamma) : M(i(\gamma)) \to M(t(\gamma))$ of abelian groups, such that $M(\gamma)$ is the identity if γ is the identity of an object, and $M(\gamma)M(\gamma') = M(\gamma\gamma')$ for two composable morphisms γ, γ' (for $m \in M(i(\gamma))$, the element $M(\gamma)(m)$ will be often denoted by $\gamma \cdot m$). More specifically, to each $\sigma \in V(X)$ is associated a left G_σ-module $M(\sigma)$ and to each $a \in E(X)$ a ψ_a-equivariant homomorphism $M(1,a)$ of $M(i(a))$ in $M(t(a))$ such that for two composable edges a, b and $m \in M(i(b))$ we have $g_{a,b}M(1,ab)(h) = M(1,a)(M(1,b)(h))$.

A right $G(X)$-module M is defined similarly as a contravariant functor from $CG(X)$ to the category of abelian groups. If γ is an element of $CG(X)$ with $i(\gamma) = \sigma$ and $t(\gamma) = \tau$, and $m \in M(\tau)$, then $m \cdot \gamma$ will denote the image of m by the homomorphism $M(\gamma) : M(\tau) \to M(\sigma)$.

4.1.2. *Locally constant $G(X)$-modules.* — A left $G(X)$-module M is locally constant if all the maps $M(\gamma)$ are isomorphisms. When X is connected, all $M(\sigma)$ can then be identified to a fixed abelian group A and M is given by a functor of $CG(X)$ in the group $\mathrm{Aut}(A)$ of the automorphisms of the abelian group A (or up to isomorphism by a homomorphism of the fundamental group of $G(X)$ in $\mathrm{Aut}(A)$, when X is connected, *cf.* 2.8.3). Then M is called a locally constant system A. It is called constant if all $M(\gamma)$ are the identity of A. For instance $\mathbf{Z}$ will denote the constant $G(X)$-module with all $\mathbf{Z}(\sigma)$ equal to the ring of integers $\mathbf{Z}$. It can be considered as a left or a right $G(X)$-module.

4.1.3. *Pull back.* — Let $\varphi : G(Y) \to G(X)$ be a homomorphism of complexes of groups over a map $f : Y \to X$, and let M be a left $G(X)$-module. Then $\varphi^* M$ is the $G(Y)$-module associating to $\sigma \in V(Y)$ the group $M(f(\sigma))$ and to $\gamma \in CG(Y)$ the homomorphism $M(\varphi(\gamma))$.

If $f : Y \to X$ is simplicial map and M a $G(X)$-module, then f^*M will denote the $f^*G(X)$-module φ^*M, where φ is the canonical homomorphism of $f^*G(X)$ in $G(X)$ over f.

4.1.4. $\mathrm{Hom}_{G(X)}$ *and* $\otimes_{G(X)}$. — A homomorphism f of a left $G(X)$-module M in a left $G(X)$-module N is a natural transformation of the functor M in the functor N. Explicitely for each $\sigma \in V(X)$, a G_σ-module homomorphism $f_\sigma : M(\sigma) \to N(\sigma)$ is given such that, for each $a \in E(X)$, we have $N(1,a)f_{i(a)} = f_{t(a)}M(1,a)$. The set $\mathrm{Hom}_{G(X)}(M,N)$ of homomorphisms of M in N is naturally an abelian group and it is easy to check that the $G(X)$-modules form an abelian category. The functor $M \to \mathrm{Hom}_{G(X)}(M,N)$ is a contravariant functor of the category of left $G(X)$-modules in the category of abelian groups. It is left exact.

Given a right $G(X)$-module M and a left $G(X)$-module N, we define the abelian group $M \otimes_{G(X)} N$ which is the quotient of the direct sum of the $M(\sigma) \otimes N(\sigma)$ by the equivalence relation which identifies $m \cdot \gamma \otimes n$ with $m \otimes \gamma \cdot n$, where $\gamma \in CG(X)$, $m \in M(t(\gamma))$ and $n \in M(i(\gamma))$.

We define two functors $M \to M_{G(X)}$ and $M \to M^{G(X)}$ from the category of left $G(X)$-modules in the category of abelian groups.

The first one associates to M the quotient $M_{G(X)}$ of the direct sum of the $M(\sigma)$ over the $\sigma \in V(X)$ by the subgroup generated by the element of the form $m - \gamma \cdot m$, where $\gamma \in CG(X)$. One calls $M_{G(X)}$ the group of co-invariants of M. Note that $M_{G(X)} = \mathbf{Z} \otimes_{G(X)} M$, where $\mathbf{Z}$ is considered as a constant right $G(X)$-module. The functor $M \to M_{G(X)}$ is a covariant functor which is right exact.

The second one associates to M the subgroup of the product of the $M(\sigma)$ over the $\sigma \in V(X)$ made up of the families $m_\sigma \in M(\sigma)$ such that $\gamma \cdot m_{i(\gamma)} = m_{t(\gamma)}$ for all $\gamma \in CG(X)$. Note that $M^{G(X)}$ is the group $\mathrm{Hom}_{G(X)}(\mathbf{Z}, M)$, where $\mathbf{Z}$ is considered as a left $G(X)$-module. The group $M^{G(X)}$ is called the group of invariants of M. The functor $M \to M^{G(X)}$ is a contravariant functor which is left exact.

4.1.5. *Free modules.* — For $\tau \in V(X)$, we define the left $G(X)$-module $\mathbf{Z}G(X)^\tau$ associating to $\sigma \in V(X)$ the free abelian group on the set of elements $\gamma \in CG(X)$ with $t(\gamma) = \sigma$ and $i(\gamma) = \tau$. The homomorphism associated to an element γ' of $CG(X)$ with $i(\gamma') = \sigma$ maps γ on $\gamma'\gamma$. By definition a free $G(X)$-module is a direct sum of modules of the type $\mathbf{Z}G(X)^{\tau}$.

A basic example is the left $G(X)$-module $\mathbf{Z}G(X)$ which is the direct sum of all the $\mathbf{Z}G(X)^{\tau}$, $\tau \in V(X)$.

Given a left $G(X)$-module M and an element $m \in M(\tau)$, there is a unique $G(X)$-homomorphism of $\mathbf{Z}G(X)^{\tau}$ in M mapping the element $\gamma \in \mathbf{Z}G(X)^{\tau}(\sigma)$ on $\gamma \cdot m$. This shows that every free module is projective and that every module is the quotient of a free module.

4.2. *Cohomology.*

4.2.1. DEFINITION USING THE STANDARD RESOLUTION (Compare with Quillen [6], p. 83). — *Given a left $G(X)$-module M we consider the complex $C^*(G(X), M)$ of cochains on $G(X)$ with coefficient in M. By definition a (normalized) k-cochain $f \in C^k(G(X), M)$ is a map associating to a sequence $\gamma_1, \dots, \gamma_k$ of k composable elements of $G(X)$ an element $f(\gamma_1, \dots, \gamma_k) \in M(t(a))$ which is zero if one of the elements γ_i is the identity of an object. The coboundary $\delta f \in C^{k+1}(G(X), M)$ is defined by*

$$\delta f(\gamma_0, \dots, \gamma_k) = \gamma_0(f(\gamma_1, \dots, \gamma_k)) - \sum_{i=0}^{k-1} (-1)^i (\gamma_0, \dots, \gamma_i\gamma_{i+1}, \dots, \gamma_k) + (-1)^k f(\gamma_0, \dots, \gamma_{k-1}) .$$

By definition $H^k(G(X), M)$ is the k-th cohomology group of the complex of normalized cochains. In particular $H^0(G(X), M) = M^{G(X)}$.

Remark. — If we drop the condition that $f(\gamma_1, \dots, \gamma_k) = 0$ when one of the γ_i is a unit element, we get a bigger cochain complex which gives the same cohomology (*cf.* Mac Lane [7], p. 236).

A $G(X)$-module homomorphism of M in a $G(X)$-module M' induces a homomorphism of $H^k(G(X), M)$ in $H^k(G(X), M')$. If $\varphi : G(Y) \to G(X)$ is a homomorphism, then it induces a homomorphism φ^* of $H^k(G(X), M)$ in $H^k(G(Y), \varphi^*(M))$. Those homomorphisms are obtained from the natural associated homomorphisms of the cochains complexes.

A locally constant $G(X)$-module M gives a locally constant system of coefficients on $BG(X)$, still denoted by M, and $H^k(G(X), M) = H^k(BG(X), M)$ (*cf.* Quillen [6], p. 83).

4.2.2. PROPOSITION. — *To an exact sequence* $0 \to M' \to M \to M'' \to 0$ *of left* $G(X)$*-modules is associated a long exact sequence of cohomology groups*

$$0 \longrightarrow H^0(G(X); M') \longrightarrow H^0(G(X); M) \longrightarrow H^0(G(X); M'') \longrightarrow H^1(G(X); M') \longrightarrow \cdots$$

The proof is standard.

4.2.3. *Example.* — Let Y be a subcomplex of X such that if an edge a of X has a terminal point in Y, then $a \in Y$. Given a left $G(X)$-module M, we define M_0 as the submodule of M defined by $M_0(\sigma) = M(\sigma)$ if σ does not belong to Y and zero otherwise. If $\gamma \in CG(X)$ is such that $i(\gamma)$ does not belong to $V(Y)$, this is also the case for $t(\gamma)$ and $M_0(\gamma) = M(\gamma)$ Denote by $G(Y)$ (resp. $M|Y$) the restriction of $G(X)$ (resp. M) to Y. Then $H^*(G(X), M/M_0)$ is naturally isomorphic to $H^*(G(Y), M|Y)$. Therefore the exact cohomology sequence takes the form

$$\cdots \longrightarrow H^k(G(X), M_0) \longrightarrow H^k(G(X), M) \longrightarrow H^k(G(Y), M|Y) \longrightarrow H^{k+1}(G(X), M_0) \longrightarrow \cdots$$

Let us apply this to see how to compute $H^*(G(X), N_0)$, where N_0 is a left $G(X)$-module with support on a vertex τ of X, *i.e.* $N_0(\sigma) = 0$ if $\sigma \neq \tau$.

First we note that the natural projection j_τ of D_τ in X induces an isomorphism of $H^*(G(X), N_0)$ on $H^*(G(D_\tau), N_0)$, because it induces an isomorphism on the level of cochains.

Let r_τ be the retraction of $G(D_\tau)$ on G_τ (see 3.1), and let $N = r_\tau^*(N_0)$ As observed before 4.2.2, N corresponds to a locally trivial system of coefficient on $BG(D_\tau)$ with stalk the abelian group $N_0(\tau)$ and the cohomology of X with coefficient in this local system is isomorphic to $H^*(G(D_\tau), N)$; this implies that the retraction r_σ induces an isomorphism of $H^*(G(D_\tau), M)$ on $H^*(G_\tau, N_0(\tau))$.

The above exact sequence gives the exact sequence

$$\cdots \longrightarrow H^k(G(X), N_0) \longrightarrow H^k(G_\tau, N_0(\tau)) \longrightarrow H^k(G(Lk(\tau)), N|Lk\tau) \longrightarrow H^{k+1}(G(X), N_0) \longrightarrow \cdots$$

Assume that $G(X)$ is the trivial complex of groups on X. Then $H^k(X, N_0)$ is isomorphic to the reduced cohomology group $\widetilde{H}^{k-1}(Lk\tau, N_0(\tau))$.

4.2.4. DEFINITION USING PROJECTIVE RESOLUTIONS. — *The cohomology can alternatively be defined using projective or free resolutions, as the left derived functor of the functor $M \to \mathrm{Hom}_{G(X)}(\mathbf{Z}, M) = M^{G(X)}$. Namely $H^*(G(X), M)$ is the cohomology of the complex $\mathrm{Hom}_{G(X)}(F_., M)$, where $F_.$ is a projective resolution of the left $G(X)$-module $\mathbf{Z}$.*

Similarly, the homology $H_.(G(X), M)$ can be defined as the homology of the complex $F_. \otimes_{G(X)} M$, where $F_.$ is a projective resolution of the constant right $G(X)$-module $\mathbf{Z}$. In particular $H_0(G(X), M) = M_{G(X)}$.

The definition 4.2.1 corresponds to the choice of the canonical free resolution of $\mathbf{Z}$, associated to the canonical aspherical cellular realization $BG(X)$ of $G(X)$.

4.2.5. *Construction of a free resolution from a cellular aspherical realization of $G(X)$.* — Let $\pi : Y \to X$, together with a lifting $s : X^{(1)} \to Y$, be a cellular aspherical realization of $G(X)$ as defined in 3.3.4. Each cell e of Y projects by π on a cell $\langle a_1, \ldots, a_k \rangle$ of X; the terminal point $t(e)$ will be the vertex $t(a_1)$ For each cell e of Y we choose a path ℓ_e contained in $\pi^{-1}\pi(e)$ and joining the base point $s(t(e))$ to an interior point of e.

4.2.6. PROPOSITION. — *To the cell structure of Y and the above data we can associate a free resolution F*

$$0 \longrightarrow \mathbf{Z} \longrightarrow F_0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow \cdots$$

of the constant left $G(X)$-module $\mathbf{Z}$. For $\sigma \in V(X)$, $F_k(\sigma)$ is the free abelian group on the pairs (γ, e), where e is a k-cell of Y and $\gamma \in CG(X)$ such that $t(\gamma) = \sigma$, $t(e) = i(\gamma)$. For $\gamma' \in CG(X)$, $F_k(\gamma')$ maps (γ, e) on $(\gamma'\gamma, e)$.

Proof. — We first show that there is a bijection between the pairs (γ, e) with $t(\gamma) = \sigma$ and the cells of the universal covering $\tilde{Y}(D_\sigma)$ of $Y(D_\sigma)$ constructed as the space of homotopy classes of paths starting from the base point $s_\sigma(\sigma)$. Suppose that $\gamma = (g, \beta)$; we use the notations of 3.3.1; to (γ, e) is associated the k-cell e' of $Y(D_\sigma)$ projecting by $Y(j_\sigma)$ on the k-cell e and by π_σ on a cell with terminal point the vertex of D_σ labelled β; we also have a path $\ell_{(\gamma,e)}$ in $Y(D_\sigma)$ joining the base point $s(\sigma)$ to an interior point of e' obtained by composing a loop in $Y(\sigma)$ representing γ^{-1}, then the edge $s(\beta)$ in the opposite direction and then the path ℓ'_e projecting on ℓ_e by $Y(j_\sigma)$. This determines a lifting of the cell e' in $\widetilde{Y}(D_\sigma)$ and it is clear that any lifting can be obtained in this way because there is a bijection between the elements of the fundamental group of $Y(D_\sigma)$ and the elements γ of $CG(X)$ with $p(\gamma) = \beta$.

Therefore $F_k(\sigma)$ can also be considered as the free abelian group over the set of k-cells of $\widetilde{Y}(D_\sigma)$, in other words as the group of cellular k-chains of $\widetilde{Y}(D_\sigma)$. If $\varepsilon : F_0(\sigma) \to \mathbf{Z}$ is the augmentation, the complex $F_.(\sigma)$ of cellular chains of $\widetilde{Y}(D_\sigma)$ is an acyclic resolution of $\mathbf{Z}$ because $\widetilde{Y}(D_\sigma)$ is contractible. One can check that for $\gamma \in CG(X)$, $F(\gamma)$ is a chain map of $F_.(i(\gamma))$ in $F_.(t(\gamma))$. Clearly F_k is a free $G(X)$-module with basis the set of k-cells of Y.

4.3. *Cohomological dimension, type FL and FP.*

4.3.1. DEFINITION. — *A complex of groups $G(X)$ on a finite dimensional ordered simplicial cell complex is of finite cohomological dimension (in brief $cdG(X) < \infty$) if the constant $G(X)$-module $\mathbf{Z}$ admits a projective resolution of finite length.*

If X is a finite complex, we say that $G(X)$ is of type FP (resp. FL) if $\mathbf{Z}$ admits a projective (resp. free) resolution of finite type.

4.3.2. COROLLARY. — *A complex of groups $G(X)$ on a finite complex X is of type FL) if and only if each vertex group G_σ is of type FL.*

Proof. — If a group G is of type FL, then there is a finite dimensional cell complex which is a $K(G,1)$ (cf. Brown [3]). Hence the first part of the corollary follows from 3.4.1 and 4.2.6.

Conversely, one observes that if P is a finite free $G(X)$-module, then $P(\sigma)$ is a finite free G_σ-module for each vertex σ of X, because X is finite.

Remark. — One should be able to construct directly a resolution of $G(X)$ using free resolutions of the G_σ's without using the geometric construction of 3.4.1.

4.4. *A spectral sequence.*

4.4.1. — Given a left $G(X)$-module M, we define a complex $C^\cdot(G(X), M)$ of right $C(X)$-modules (recall that $C(X)$ is the category such that $X = BC(X)$) :

$$0 \longrightarrow C^0(G(X), M) \longrightarrow C^1(G(X), M) \longrightarrow C^2(G(X), M) \longrightarrow \cdots$$

as follows. $C^{\cdot}(G(X),M)(\sigma)$ is the complex of cochains $C^{\cdot}(j_\sigma^*G(X),j_\sigma^*(M))$, where j_σ is the natural projection of D_σ in X. For $\alpha \in C(X)$ with $i(\alpha) = \sigma$ and $t(\alpha) = \tau$, then $C^k(G(X),M)(\alpha)$ is the homomorphism $j_\alpha^* : C^k(j_\tau^*G(X),j_\tau^*M) \longrightarrow C^k(j_\sigma^*G(X),j_\sigma^*M)$ (note that $j_\alpha^*j_\sigma^* = j_\tau^*$).

The cohomology of this complex in degree k is a complex of right $C(X)$-module denoted by $\mathcal{H}^k(G(X),M)$.

The following in an analogue of the Leray spectral sequence for the projection $BG(X) \to X$.

4.4.2. THEOREM. — *Given a left $G(X)$-module M, there is a spectral sequence converging to $H^*(G(X);M)$ and with*

$$E_2^{p,q} = H^p(X,\mathcal{H}^q(G(X),M)) .$$

Proof. — Consider the natural double complex

$$\bigoplus_{p,q} C^p(X,C^q(G(X),M)).$$

The following lemma shows that the inclusion of the complex $C^*(G(X),M)$ in the total complex associated to the double complex induces an isomorphism in cohomology.

4.4.3. LEMMA. — $H^0(X,C^q(G(X),M)) = C^q(G(X),M)$ *and* $H^p(X,C^q(G(X),M)) = 0$ *for* $p > 0$.

Proof. — We first prove it for $q = 0$. To simplify the notations, we denote $C^0(G(X),M)$ by M^0. We claim that the sequence

$$0 \longrightarrow C^0(G(X),M) \xrightarrow{\iota} C^0(X,M^0) \xrightarrow{\delta} C^1(X,M^0) \xrightarrow{\delta} C^2(X,M^0) \xrightarrow{\delta} \cdots$$

is exact, where ι maps $c \in C^0(G(X),M)$ on the 0-cochain $\iota c \in C^0(X,M^0)$ associating to σ the 0-cochain $\iota c(\sigma)$ on D_σ defined by $\iota c(\sigma)(\alpha) = c(i(\alpha))$. An element of $C^p(X,M^0)$ can be identified to a map f associating to a sequence $\alpha_1,\ldots,\alpha_{p+1}$ of composable elements of $C(X)$ an element $f(\alpha_1,\ldots,\alpha_{p+1}) \in M(i(\alpha_{p+1}))$. With this identification we have

$$\delta f(\alpha_0,\alpha_1,\ldots,\alpha_{p+1}) = f(\alpha_1,\ldots,\alpha_{p+1}) - \sum_0^p (-1)^i f(\alpha_1,\ldots,\alpha_i\alpha_{i+1},\ldots) .$$

It follows that the above sequence is exact, hence the result for $q = 0$.

Note that in the above argument, we used only the $M(\sigma)$'s and not the $G(X)$-module structure on M. Hence it applies by replacing the family

$M(\sigma)$ by the family $N(\sigma)$, where $N(\sigma)$ is the group of functions associating to a composable sequence $\alpha_0, \alpha_1, \ldots, \alpha_q$ an element of $M(i(\alpha_q))$. This shows the lemma for $q > 0$.

The $E_2^{p,q}$-term of the spectral sequence associated to the first filtration of the double complex gives the result.

4.4.4. PROPOSITION. — *Assume that the $G(X)$-module M is locally constant. Then the right $C(X)$-module $\mathcal{H}^q(X, M)$ is defined by $H^q(X, M)(\sigma) = H^q(G_\sigma, M(\sigma))$, and the homomorphism associated to $a \in E(X)$ with $i(a) = \sigma$ and $t(a) = \tau$ is the homomorphism of $H^q(G_\tau, M(\tau))$ in $H^q(G_\sigma, M(\sigma))$ induced by the homomorphisms ψ_a and $M(1, a)$* (*cf.* Brown [3], p. 79).

Proof. — The $j_\sigma^* G(X)$-module $j_\sigma^* M$ is isomorphic to $r_\sigma^* M(\sigma)$, where r_σ is the retraction of D_σ on σ (*cf.* 1.3). Using the deformation R_σ, one can show that r_σ induces an isomorphism of $H^q(G_\sigma, M(\sigma))$ on $H^q(j_\sigma^* G(X), j_\sigma^* M)$. Alternatively, the module $j_\sigma^* M$ gives a locally constant system on $B(j_\sigma^* G(X))$ which retracts by deformation on BG_σ and this retraction induces an isomorphism in cohomology (*cf.* Quillen [6]).

5. Extensions with abelian kernel.

5.1. DEFINITION. — *Assume that $\varphi : \widetilde{G}(X) \to G(X)$ is a surjective homomorphism over the identity of X with abelian kernel, i.e. each homomorphism $\varphi_\sigma : \widetilde{G}_\sigma \to G_\sigma$ is surjective and its kernel $A(\sigma)$ is abelian. Then we get a $G(X)$-module A called the kernel of φ : if $\gamma = (g, \alpha) \in CG(X)$, then $A(\gamma)$ is the injective homomorphism of $A(i(\gamma))$ in $A(t(\gamma))$ given by $A(g)(n) = \mathrm{Ad}(\tilde{g})\tilde{\psi}_\alpha(n)$, where $(\tilde{g}, \alpha)$ is a lifting of (g, α) in $C\widetilde{G}(X)$ (equivalently for $\gamma \in CG(X)$ and $n \in A(i(\gamma))$, $\gamma \cdot h$ is defined by $\psi_{s(\gamma)} \cdot n$ for a lifting $s(\gamma) \in C\widetilde{G}(X)$ of γ, cf. notation of 2.5); this homomorphism is independent of the choice of the lifting because $A(t(\gamma))$ is abelian.*

Another extension $\varphi' : \widetilde{G}'(X) \to G(X)$ with the same abelian kernel A is called isomorphic to φ if there is an isomorphism of $\widetilde{G}(X)$ on $\widetilde{G}'(X)$ projecting on the identity of $G(X)$ and which is the identity on A.

5.2. THEOREM. — *The isomorphism classes of extensions of $CG(X)$ with the abelian kernel A are in bijection with the elements of $H^2(G(X), A)$.*

Proof. — Given an extension with abelian kernel A, choose a lifting s of $CG(X)$ in $C\widetilde{G}(X)$ such that $s(1_\sigma) = 1_\sigma$. To s is associated a 2-cocycle $c \in C^2(G(X), A)$ defined by

$$c(\gamma, \gamma')s(\gamma\gamma') = s(\gamma)s(\gamma') \; .$$

If s is replaced by another lifting s' defining a 2-cocycle c' then $s'(\gamma) = b(\gamma)s(\gamma)$ where $b \in C^1(G(X), A)$ and c and c' differ by the coboundary of b.

Conversely, given a 2-cocycle c with coefficient in A, we construct an extension $C\widetilde{G}(X)$ whose morphisms projecting by φ on $\gamma \in CG(X)$ are the pairs (n, γ) with $n \in A(t, \gamma))$. The composition is defined by

$$(n, \gamma)(n', \gamma) = (n(\gamma \cdot n')c(\gamma, \gamma'), \gamma\gamma') \; .$$

We omit the details which are formally as in the classical case (*cf.* Brown, [3]).

5.3. *Example.* — Assume that $G(X)$ is the trivial complex of groups denoted by X (in that case $CG(X) = C(X)$). Then A is a functor of $C(X)$ in the category of monomorphisms of abelian groups. An extension of $C(X)$ by A is a complex of groups $G(X)$ on A with $G_\sigma = A(\sigma)$ and $\psi_a = A(a)$. The isomorphism classes of extensions correspond to the choices of the 2-cocycle $g_{a,b}$ up to coboundary. For instance if τ is a sink, *i.e.* a vertex such that no edge has its initial point at τ, and if A has its support on τ (*i.e.* $A(\sigma) = 0$ if $\sigma \neq \tau$) then $H^2(X, A)$ is isomorphic to $H^1(Lk\tau, A(\tau))$, where $A(\tau)$ denotes the constant coefficient system isomorphic to $A(\tau)$ (*cf.* 4.2.4).

As a specific example, let X be the 2-sphere which is the geometric realization of the category without loop given in fig. 2. Then up to isomorphism, the extensions of $C(X)$ by A are in bijection with the elements of $H^2(X, A)$ which is isomorphic to the quotient of $A(\tau) \oplus A(\tau')$ by the subgroup image of $A(\sigma)$ by $A(a) \oplus A(a')$.

5.4. *Central extensions by* **Z**. — Assume that A is the constant $G(X)$-module $\mathbf{Z}$ and that all groups G_σ are finite. We can use the spectral sequence 4.4 to compute $H^2(G(X), \mathbf{Z})$.

In that case $\mathcal{H}^0(G(X), \mathbf{Z}) = \mathbf{Z}$, $\mathcal{H}^1(G(X), \mathbf{Z}) = 0$ and the right $C(X)$-module $\mathcal{H}^2(G(X), \mathbf{Z})$ associates to each σ the group $H^2(G_\sigma, \mathbf{Z})$ of central extensions of G_σ by $\mathbf{Z}$. The spectral sequence gives the exact sequence

$$0 \to H^2(X, \mathbf{Z}) \to H^2(G(X), \mathbf{Z}) \to H^0(X, \mathcal{H}^2(G(X), \mathbf{Z})) \to H^3(G(X), \mathbf{Z})$$

which can be interpreted as follows in terms of extensions.

An element of $H^2(G(X), \mathbf{Z})$ comes from an element of $H^2(X, \mathbf{Z})$ if and only if the corresponding extension of $G(X)$ by $\mathbf{Z}$ induces the trivial central extension of each group G_σ. In that case this extension is the pull back by the projection $p : G(X) \to X$ of the complex of groups on X for which the vertex groups are all isomorphic to $\mathbf{Z}$ and which is determined up to coboundary by the element of $H^2(X, \mathbf{Z})$.

To give an element of $H^0(X, \mathcal{H}^2(G(X), \mathbf{Z}))$ is equivalent to give for each vertex σ a central extension of G_σ by $\mathbf{Z}$ in a compatible way; if $a \in E(X)$ with $i(a) = \sigma$ and $t(a) = \tau$, then the central extension of G_σ should be induced by ψ_a from the central extension of G_τ. In general such a data does not come always from a central extension of $G(X)$ by $\mathbf{Z}$; the obstruction is measured by an element of $H^3(X, \mathbf{Z})$ which is not trivial in general.

Here is an example. We start from an ordered simplicial cell complex X_0 which is a projective plane; it is the quotient of its universal covering $\widetilde{X}_0$ (topologically a 2-sphere) by a free action of the cyclic group $\mathbf{Z}_2$ of order 2. Let $\widetilde{X}$ be the ordered simplicial cell complex which is the suspension of $\widetilde{X}_0$: we add two suspension vertices τ and τ' to $V(\widetilde{X}_0)$ and from each vertex σ of $\widetilde{X}_0$, there are extra edges a and a' with $i(a) = i(a') = \sigma$ and $t(a) = \tau$, $t(a') = \tau'$. On $\widetilde{X}$ we consider the action without inversion of the cyclic group $\mathbf{Z}_2$ which is the suspension of its action on $\widetilde{X}_0$. On the quotient X (which is topologically the suspension of a projective plane) we get a complex of groups $G(X)$, whose fundamental group is $\mathbf{Z}_2$.

There is only one non trivial central extension of $G(X)$ by $\mathbf{Z}$. Indeed $H^2(G(X), \mathbf{Z})$ is isomorphic to $H^1(G(X), \mathbf{R}/\mathbf{Z}) = \mathrm{Hom}(\pi_1(G(X), \mathbf{R}/\mathbf{Z}) = \mathbf{Z}_2$ (consider the short exact sequence of trivial $G(X)$-modules $0 \to \mathbf{Z} \to \mathbf{R} \to \mathbf{R}/\mathbf{Z} \to 0$ and observe that $H^1(G(X), \mathbf{R}) = H^2(G(X), \mathbf{R}) = 0$). For this extension $\widetilde{G}(X)$, the groups associated to the two vertices of the suspension are both $\mathbf{Z}$, while they are $\mathbf{Z} \oplus \mathbf{Z}_2$ for the trivial extension. If we associate to one of the suspension vertex of X the group $\mathbf{Z} \oplus \mathbf{Z}_2$ and the group $\mathbf{Z}$ to all other vertices, we get a coherent system of extensions for the groups G_σ which does not come from an extension of $G(X)$.

Complexes of groups associated to Seifert fibrations are examples of central extensions by $\mathbf{Z}$. For instance if $G(X)$ is the complex of groups associated to an orbifold structure on a surface X, then the complex of groups associated to a Seifert bundle $\pi : Y \to X$ with base space the orbifold X (*cf.* 3.3.5) is a central extension of $G(X)$ by $\mathbf{Z}$. Conversely such an extension is associated to a Seifert bundle with base space the orbifold X if for each vertex τ the central extension of G_τ by $\mathbf{Z}$ is isomorphic to $\mathbf{Z}$.

6. Extension of complexes of groups with locally constant kernel.

In this paragraph, we generalize to complexes of groups the Eilenberg-Mac Lane theory of extension of groups (*cf.* Mac Lane [7] or Brown [3]).

6.1. — Let X be a connected ordered simplicial cell complex and let φ be a morphism of a complex of groups $\widetilde{G}(X)$ on a complex of groups $G(X)$ over the identity of X. We assume that all the homomorphisms φ_σ are surjective and that $\widetilde{\psi}_a$ induces an isomorphism of $\operatorname{Ker}\varphi_{i(a)}$ on $\operatorname{Ker}\varphi_{t(a)}$ for each edge a of $E(X)$. Therefore we can identify each $\operatorname{Ker}\varphi_0$ to a fixed group N and we denote by $\tilde{\mu}_a$ the automorphism of N given by $\widetilde{\psi}_a$. If α is the identity of a vertex of X, then $\tilde{\mu}_\alpha$ will denote the unit element of the group $\operatorname{Aut}(N)$ of automorphisms of N.

Let $\operatorname{Out}(N)$ be the quotient of $\operatorname{Aut}(N)$ by the subgroup $\operatorname{Int}(N)$ of inner automorphisms. We have an exact sequence

$$0 \longrightarrow C \longrightarrow N \longrightarrow \operatorname{Aut}(N) \longrightarrow \operatorname{Out}(N) \longrightarrow 1$$

where C is the center of N.

We get a functor μ of $CG(X)$ in $\operatorname{Out}(N)$ mapping (g,α) on the image of $\operatorname{Ad}(\tilde{g})\tilde{\mu}_\alpha$ in $\operatorname{Out}(N)$, where $(\tilde{g},\alpha)$ is a lifting of (g,α) in $\widetilde{CG}(X)$. The kernel of φ will be the pair (N,μ).

6.2. DEFINITION. — *A homomorphism $\varphi : \widetilde{G}(X) \to G(X)$ as above (with an identification of each kernel of φ_σ with the group N) is called an extension of $G(X)$ with kernel (N,μ). Another extension $\varphi' : \widetilde{G}'(X) \to G(X)$ with the same kernel (N,μ) is called equivalent to φ if there is an isomorphism of $\widetilde{G}'(X)$ on $\widetilde{G}(X)$ projecting on the identity of $G(X)$ and whose restriction to N is the identity.*

$\operatorname{Aut}(N)$ acts on N leaving invariant the center C of N and as $\operatorname{Int}(N)$ acts trivially on C, we get an action of $\operatorname{Out}(N)$ on C. Therefore μ defines a left $G(X)$-module which is locally constant with stalk C.

6.3. THEOREM. — *Let $G(X)$ be a complex of groups on X, let N be a group and μ be a functor of $CG(X)$ in $\mathrm{Out}(N)$. Let C be the center of N, considered as above as a locally constant $G(X)$-module. Then*

i) *there is an extension of $G(X)$ with kernel (N,μ) if and only if a certain element of $H^3(G(X),C)$ vanishes;*

ii) *given an extension of $G(X)$ with kernel (N,μ), the set of equivalence classes of such extensions is in bijection with $H^2(G(X),C)$.*

Proof. — Suppose we have an extension $\varphi : C\widetilde{G}(X) \to CG(X)$ as above with kernel (N,μ).

Choose a section $s : CG(X) \to C\widetilde{G}(X)$ of φ. This defines two things :

a) a map $\tilde{\mu} : CG(X) \to \mathrm{Aut}(N)$ defined as follows : if $(g,\alpha) \in CG(X)$ and if $s((g,\alpha)) = (\tilde{g},\alpha)$, $n \in N$, then

$$\tilde{\mu}(g,\alpha)(n) = \mathrm{Ad}(\tilde{g})\psi_\alpha(n) \ .$$

The composition of $\tilde{\mu}$ with the homomorphism $\mathrm{Aut}(N) \to \mathrm{Out}(N)$ is μ.

b) a map $\widetilde{F} : CG(X)^{(2)} \to N$, where $CG(X)^{(2)}$ is the set of pairs of composable elements of $CG(X)$, defined by the equality

$$s(\gamma_1)s(\gamma_2) = \widetilde{F}(\gamma_1,\gamma_2)s(\gamma_1\gamma_2) \ .$$

They satisfy two identities :

$$\mathrm{Ad}(\widetilde{F}(\gamma_1,\gamma_2))\tilde{\mu}(\gamma_1\gamma_2) = \tilde{\mu}(\gamma_1)\tilde{\mu}(\gamma_2) \tag{1}$$

which measures the non functoriality of $\tilde{\mu}$, and the cocycle condition

$$\widetilde{F}(\gamma_1,\gamma_2)\widetilde{F}(\gamma_1\gamma_2,\gamma_3) = \tilde{\mu}(\gamma_1)(\widetilde{F}(\gamma_1,\gamma_2))\widetilde{F}(\gamma_1,\gamma_2\gamma_3) \tag{2}$$

for all triples $\gamma_1,\gamma_2,\gamma_3$ of composable elements of $CG(X)$ (this follows from the associativity in $C\widetilde{G}(X)$).

Given $\tilde{\mu}$ and $\widetilde{F}$ verifying (1) and (2), one can reconstruct an extension $C\widetilde{G}(X)$ as follows. Its morphisms are the couples (n,γ) where $\gamma \in CG(X)$, $n \in N$. The composition $(n,\gamma)(n',\gamma')$ is defined if $\gamma\gamma'$ is defined and is equal to $(n\tilde{\mu}(\gamma)(n'),\gamma\gamma')$. The homomorphism φ maps (n,γ) on γ. The identification with the given $C\widetilde{G}(X)$ maps (n,γ) on $(n,t(\gamma))s(\gamma)$.

Conversely, given the functor $\mu : CG(X) \to \mathrm{Out}(N)$, we try to construct maps $\tilde{\mu}$ and $\widetilde{F}$ satisfying (1) and (2). We choose any lifting

$\tilde{\mu} : CG(X) \to \mathrm{Aut}(N)$ of μ mapping the identity of any object on the unit element of $\mathrm{Aut}(N)$. Then a map $f : CG(X)^{(2)} \to \mathrm{Int}(N)$ is defined uniquely by the equation

$$f(\gamma_1, \gamma_2)\tilde{\mu}(\gamma_1\gamma_2) = \tilde{\mu}(\gamma_1)\tilde{\mu}(\gamma_2) .$$

Note that $f(\gamma_1, \gamma_2) = 1$ if γ_1 or γ_2 is an identity.

If $F : CG(X) \to N$ is any lifting of f with respect to the homomorphism $\mathrm{Ad} : N \to \mathrm{Int}(N)$ such that $F(\gamma_1, \gamma_2) = 1$ if γ_1 or γ_2 is an identity, this equality can be written as

$$(1)' \qquad \mathrm{Ad}(F(\gamma_1, \gamma_2))\tilde{\mu}(\gamma_1\gamma_2) = \tilde{\mu}(\gamma_1)\tilde{\mu}(\gamma_2) .$$

Using the associativity property in $CG(X)$, one can define a unique map $c : CG(X)^{(3)} \to C$ by

$$(2)' \quad \mathrm{Ad}(\tilde{\mu}(\gamma_1))(F(\gamma_2, \gamma_3))F(\gamma_1, \gamma_2\gamma_3) = c(\gamma_1, \gamma_2, \gamma_3)F(\gamma_1, \gamma_2)F(\gamma_1\gamma_2, \gamma_3) .$$

It is easy to check that c is a normalized 3-cocycle on $CG(X)$ with coefficient in the locally constant $G(X)$-module C determined by μ. This means that we have, for any sequence $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ of composable elements of $CG(X)$:

$$(3)' \quad c(\gamma_1\gamma_2, \gamma_3, \gamma_4)c(\gamma_1, \gamma_2, \gamma_3\gamma_4) = \mu(\gamma_1)(c(\gamma_2, \gamma_3, \gamma_4))c(\gamma_1, \gamma_2, \gamma_3) \\ c(\gamma_1, \gamma_2\gamma_3, \gamma_4) .$$

The cohomology class of c is independent of the choice of the liftings $\tilde{\mu}$ and F. More precisely, if $\widetilde{F}$ is another lifting of f, then there is a unique map $b : CG(X)^{(2)} \to C$ such that $\widetilde{F}(\gamma_1, \gamma_2) = F(\gamma_1, \gamma_2)b(\gamma_1, \gamma_2)$. Then the 3-cocycle $\tilde{c}$ associated to $\widetilde{F}$ is the 3-cocycle c modified by the coboundary of the 2-chain b with coefficient in the local system C, namely

$$\tilde{c}(\gamma_1, \gamma_2, \gamma_3) = \tilde{\mu}(\gamma_1)(b(\gamma_2, \gamma_3))b(\gamma_1, \gamma_2, \gamma_3)^{-1}b(\gamma_1, \gamma_2, \gamma_3)b(\gamma_1, \gamma_2)^{-1} \\ c(\gamma_1, \gamma_2, \gamma_3) .$$

Therefore, if the cohomology class of c is trivial, we can choose b such that $\tilde{c} = 0$. Then the identities (1) and (2) are satisfied and we can construct as above the extension $\widetilde{C}G(X)$. This proves the first part of the theorem.

Suppose $\varphi' : \widetilde{G}(X) \to G(X)$ is another extension with kernel (N, μ). We can always choose the section $s' : G(X) \to \widetilde{G}'(X)$ so that the map $\tilde{\mu} : CG(X) \to \mathrm{Aut}(N)$ determined by the analogue of a) is equal to $\tilde{\mu}$. The analogue of b) gives a map $\widetilde{F}' : CG(X)^{(2)} \to N$, and there is a 2-cochain $d \in C^2(G(X), C)$ characterized by

$$\widetilde{F}'(\gamma_1, \gamma_2) = \widetilde{F}(\gamma_1, \gamma_2) d(\gamma_1, \gamma_2) .$$

It is easy to check that d is a 2-cocycle whose cohomology class is independent on the choice of $\widetilde{F}'$. Its vanishing implies the existence of an equivalence between the extensions φ and φ'. Conversely given a 2-cocycle $d(\gamma_1, \gamma_2)$, the above formula defines a map $\widetilde{F}'$ verifying the identity (2); as observed before, this gives an extension φ' with kernel (N, μ). This proves the second part of the theorem.

6.4. COROLLARY. — *If the center C of N is trivial, then there is a unique extension with kernel (N, μ).*

This extension can be constructed as follows. Let $\mathrm{Aut}(N)$ and $\mathrm{Out}\, N$ be the constant complex of groups on X. Then $C\widetilde{G}(X)$ is the fiber product of the two homomorphisms $\mathrm{Aut}\, N \to \mathrm{Out}\, N$ and $\mu : CG(X) \to \mathrm{Out}\, N$.

6.5. *Remarks.*

1) If the functor μ lifts as a functor $\tilde{\mu} : CG(X) \to \mathrm{Aut}(N)$, then by choosing $\widetilde{F}$ to be the trivial map, one gets an extension $\widetilde{C}G(X)$ which is the semi-direct product of $CG(X)$ by N corresponding to $\tilde{\mu}$.

2) More generally the obstruction in $H^3(G(X), C)$ is the pull back by the homomorphism $\mu : G(X) \to \mathrm{Out}(N)$ of the element of $H^3(\mathrm{Out}(N, C)$ characterizing the crossed module $N \to \mathrm{Aut}(N)$ (*cf.* Brown [3]). For instance if $G(X)$ is simply connected, then the obstruction is always trivial.

3) If $G(X)$ is the trivial complex of groups on X (*i.e.* $CG(X) = C(X)$), then the isomorphism classes of extensions with locally constant kernel N are in bijection with the isomorphism classes of fiber bundles with base space X and fiber BN (*cf.* paragraph 3).

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