FERNANDO Q. GOUVÊA
BARRY MAZUR

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Annales de l'institut Fourier, tome 43, nº 2 (1993), p. 301-312
<http://www.numdam.org/item?id=AIF_1993__43_2_301_0>

Ann. Inst. Fourier, Grenoble
43, 2 (1993), 301-312

ON THE CHARACTERISTIC POWER SERIES OF THE U OPERATOR

by F.Q. GOUVÊA and B. MAZUR

Let p be a prime number, and let k be an integer. Atkin's U operator acts in a completely continuous manner on the p-adic space of overconvergent modular forms of weight k. The goal of this note is to show that the "Fredholm" characteristic power series of U varies "p-adically continuously" in the weight k, in the following sense. If $a_m(k)$ is the m-th coefficient of the characteristic power series of U acting on overconvergent forms of weight k, we show that if $k_1 \equiv k_2 \pmod{p^n(p-1)}$ then $a_m(k_1) \equiv a_m(k_2) \pmod{p^{n+1}}$ for every $m \geq 0$. We then extend this to "higher order differences" of the function $k \mapsto a_m(k)$, in the spirit of [Ser2], Thm. 14.

Our p-adic continuity result leads us to hope that there is a notion of "overconvergent p-adic modular form of weight k" not only for rational integers k, but for k in the p-adic space

$$\mathcal{X} = \varprojlim_n \mathbb{Z}/(p-1)p^n\mathbb{Z},$$

and that the U operator preserves overconvergence and is completely continuous (and therefore has a spectral theory) for all $k \in \mathcal{X}$. If so, our result would suggest that this spectral theory is uniformly continuous in k. At present, however, it is not evident to us how to define overconvergence for p-adic modular forms of general p-adic weights.

The methods we use are a direct extension of those in [Gou2], and our main result answers one of the "Further Questions" posed there. This

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The first author's research is supported in part by grant number # DMS-9203469 from the National Science Foundation.
Key words : p-adic modular forms – U Operator – Characteristic power series.
A.M.S. Classification : 11F33.

paper fits into the general research project outlined in [Gou], and we refer our readers to that paper for further discussion of motivation.

The authors would like to thank Robert Coleman and Noriko Yui for their comments and interest. Part of the research we report on was done while the first author was a Visiting Professor at Queen's University in Kingston, Ontario; he would like to thank the University for its hospitality. This note was originally prepared using $\mathcal{AMS}$-LATEX and two implementations of TEX : emTEX (running under OS/2), due to Eberhard Mattes, and DirectTEX (running on a Macintosh), due to Wilfried Ricken. We would like to thank both for their effort in producing two exemplary pieces of software.

1. Introduction.

To describe our main result precisely, let p be a prime number, and assume $p \geq 5$. Fix a "tame level" N not divisible by p; we will be working with p-adic modular forms of integral weight on $\Gamma_1(\mathrm{N})$. Let B be a p-adically complete and separated ring, and let $r \in B$. We will let $M_k(\mathrm{N}, B; r)$ denote the space of r-overconvergent p-adic modular forms of weight k on $\Gamma_1(\mathrm{N})$ defined over B (for definitions and properties of these spaces, whose importance was first realized by Dwork, we refer to the accounts in [Kat] and [Gou2]). If B is a discrete valuation ring and K is its field of fractions, we write $M_k(\mathrm{N}, K; r) = M_k(\mathrm{N}, B; r) \otimes K$; this is a p-adic Banach space over K with respect to the norm determined by making $M_k(\mathrm{N}, B; r)$ the unit ball. This space contains the classical spaces considered in [GM].

We fix a discrete valuation ring B, let K be its field of fractions, and write, for simplicity, $M_k(r) = M_k(\mathrm{N}, K; r)$. When $0 < \mathrm{ord}(r) < p/(p+1)$, the Atkin U operator is a completely continuous linear operator on the p-adic Banach space $M_k(r)$, and hence has a spectral theory. In particular we can consider the characteristic power series $P_k(t) = \det(1 - t\mathrm{U}|M_k(r))$ and, for each rational number α, the "slope α subspace" $M_{k,\alpha}$ which is spanned by all the forms $f \in M_k(r)$ such that $(\mathrm{U} - \lambda)^m(f) = 0$ for some integer $m > 0$ and some $\lambda \in \overline{K}$ such that $\mathrm{ord}(\lambda) = \alpha$. It is a basic result in the spectral theory of the U operator that the space $M_{k,\alpha}$ is finite-dimensional and independent of the choice of r (provided $0 < \mathrm{ord}(r) < p/(p+1)$).

We can now state our main result.

THEOREM 1. — *Let* $p \geq 5$ *be a prime number,* N *an integer not divisible by* p, B *a* p*-adically complete and separated discrete valuation ring, and* K *its field of fractions. Choose any* $r \in B$ *satisfying* $0 < \operatorname{ord}(r) < p/(p+1)$. *Let* $P_k(t)$ *be the characteristic power series of the* U *operator acting on the space* $M_k(\mathrm{N}, K; r)$ *of* r*-overconvergent* p*-adic modular forms of weight* k *and level* N. *Write* $P_k(t) = \sum a_m(k)t^m$. *If* k_1 *and* k_2 *are integers such that*

$$k_1 \equiv k_2 \pmod{p^n(p-1)},$$

then we have, for each m,

$$a_m(k_1) \equiv a_m(k_2) \pmod{p^{n+1}}.$$

Much of the technical complication in the proof of such a result is due to the fact that there are two natural topologies on the Banach spaces $M_k(r)$. For the first topology, recall that elements of $M_k(r)$ can be interpreted as functions of "not too supersingular" elliptic curves E defined over some p-adically complete and separated B-algebra A. The restriction on the curve E is that $\mathrm{E}_{p-1}(\mathrm{E}, \omega)$ should be a divisor of $r \in B$. (See [Kat] and [Gou2] for details.) A "test-object of level N and growth condition r" is simply such a curve together with a level structure. The first topology is just the natural topology on such "functions" : its norm $\|\cdot\|_{\mathrm{mod}}$ is characterized by

$$\|f\|_{\mathrm{mod}} \leq 1 \qquad \text{if and only if} \qquad f(\mathrm{E}/A, \omega, \iota, Y) \in A$$

for any test-object $(\mathrm{E}/A, \omega, \iota, Y)$ of level N and growth condition r. We call this topology the *modular topology*; its unit ball is precisely the space $M_k(\mathrm{N}, B; r)$. The second topology, which we call the *q-expansion topology*, is induced by the q-expansion map; its norm $\|\cdot\|_{q-\exp}$ can be described by saying that $\|f\|_{q-\exp} \leq 1$ if and only if all the coefficients of the q-expansion of f are in B (i.e., are integral). It is a basic fact that the modular topology on $M_k(1)$ (i.e., for $r = 1$) coincides with the q-expansion topology, so that the unit ball in $M_k(r)$ with respect to the q-expansion topology can also be described as the intersection $M_k(r) \cap M_k(\mathrm{N}, B; 1)$. (A proof can be found in [Kat].) This shows, in particular, that $M_k(1)$ is isomorphic to Serre's space of p-adic modular forms of weight k, which is defined in [Ser2] in terms of limits of q-expansions.(1) We have an inclusion of the "closed" unit balls

$$M_k(\mathrm{N}, B; r) \subset M_k(r) \cap M_k(\mathrm{N}, B; 1),$$

(1) In other words, given a sequence of classical forms f_i whose q-expansions $f_i(q)$ converge, coefficient-by-coefficient, to $f(q) \in B[[q]]$, there always exists a form f in

but the set on the right is *unbounded* with respect to the modular topology.

It is sometimes convenient to use the q-expansion map to identify $M_k(r)$ with its image in $K \otimes B[[q]]$. (Except in the case when r is a unit in B, the image will not be closed with respect to the "natural" topology on $K \otimes B[[q]]$.) From this point of view, the "unit ball with respect to the q-expansion topology" is just the intersection $M_k(r) \cap B[[q]]$.

2. Proof of Theorem 1.

As usual, there are Hecke operators T_ℓ for each prime number $\ell \neq p$ which act on $M_k(\mathrm{N}, B; r)$; these have the expected action on q-expansions. (See [Kat] or [Gou2] for the definitions.) For $\ell = p$, however, the relevant operator is not T_p (even though $p \not| \mathrm{N}$), but Atkin's U operator, which acts on q-expansions by the formula

$$\mathrm{U}(\sum a_n q^n) = \sum a_{np} q^n.$$

This is defined on $M_k(\mathrm{N}, K; r)$ as $1/p$ times the trace of the Frobenius operator Frob, which acts on q-expansions as

$$\mathsf{Frob}(\sum a_n q^n) = \sum a_n q^{np}.$$

The theory of these two operators is described in detail in Chapter II of [Gou2]. We will recall here only the most important points for our purposes. To begin with,

PROPOSITION 1. — *If* $\mathrm{ord}(r) < 1/(p+1)$, *then we have*

$$\mathrm{U}(M_k(\mathrm{N}, K; r)) \subset M_k(\mathrm{N}, K; r^p).$$

See [Gou2] for a proof; we refer to this result by the code phrase "U improves overconvergence." As Dwork was the first to point out, the fact that U improves overconvergence implies that U is a *completely continuous* endomorphism of $M_k(\mathrm{N}, K; r)$ for any r satisfying $0 < \mathrm{ord}(r) < p/(p+1)$. What this means is that for any integer n one can find a *finite-dimensional*

some $M_k(1)$ (here k may be a p-adic weight) whose q-expansion is $f(q)$. Conversely, any such form is obtained in this way. A form defined by such a limit *may or may not be overconvergent*, since it is an element of $M_k(1)$, which properly contains $M_k(r)$, and there seems to be no direct way of deciding if it is from the existence of such a construction.

subspace $V_n \subset M_k(\mathrm{N}, k; r)$ such that the image of the unit ball $M_k(\mathrm{N}, B; r)$ is contained in $V_n + p^n M_k(\mathrm{N}, K; r)$. In our case, one can find V_n quite explicitly : it is generated by the p-adic modular forms obtained as quotients f/E_{p-1}^i, where f is a classical modular form of level N and weight $k+i(p-1)$, for $0 \leq i < (n+1)/((p-1)\mathrm{ord}(r))$. It is straightforward to estimate that we have $\dim V_n = \mathrm{O}(n^2)$ as n tends to infinity.(2)

The fact that U is overconvergent implies that it has a spectral theory, as explained in [Ser] and [Mon] (see also the discussion in [Gou2]). In particular, we emphasize the following three facts :

(1) The U operator has a characteristic power series

$$P_k(t) = \det(1 - t\mathrm{U}|M_k(r)) \in B[[t]]$$

which is independent of r and defines a p-adic entire function whose reciprocal roots are the eigenvalues of U on $M_k(r)$ and form a sequence tending to zero in B. In particular, we can write

$$P_k(t) = \prod_i (1 - \lambda_i t)$$

with λ_i ranging through the nonzero eigenvalues of U (taken in the algebraic closure of K). We know that $\mathrm{ord}(\lambda_i) \geq 0$ and $\lambda_i \to 0$.

(2) It is possible to define the exterior powers $\bigwedge^n \mathrm{U}$ of any completely continuous operator; they are again completely continuous, hence have traces. Then, if we write

$$P_k(t) = \sum a_n(k) t^n,$$

we have

$$a_n(k) = \mathrm{trace}(\bigwedge^n \mathrm{U}).$$

See [Ser2], [Lan], Chapt. 15, §5 and [Gou2] for more information on this.

(3) Fix $\alpha \geq 0$, and define $M_{k,\alpha}$ to be the subspace of $M_k(r)$ spanned by the forms f such that we have

$$(\mathrm{U} - \lambda)^m(f) = 0$$

for some integer $m > 0$ and some $\lambda \in \overline{K}$ with $\mathrm{ord}(\lambda) = \alpha$. $M_{k,\alpha}$ is then a finite-dimensional vector space, and there exists a closed Banach subspace

(2) After a conversation with G. Stolzenberg, we have come to think of an estimate for $\dim V_n$ as giving a "modulus of complete continuity" for our operator.

$F_{k,\alpha}$ such that we have a U-equivariant decomposition of $M_k(r)$ as a direct sum :

$$M_k(\mathrm{N}, K; r) = M_{k,\alpha} \oplus F_{k,\alpha}.$$

We call $M_{k,\alpha}$ the *slope α eigenspace* for U acting on forms of weight k.

Recall that a $\mathbb{Z}_p$-lattice $D \subset V$ in a p-adic vector space V is a free $\mathbb{Z}_p$-submodule of V such that $D \otimes \mathbb{Q}_p = V$.

LEMMA 2. — *Let Φ_1 and Φ_2 be completely continuous operators on a p-adic Banach space V, and let $D \subset V$ be any $\mathbb{Z}_p$-lattice in V. If $\Phi_1(D) \subset D$, $\Phi_2(D) \subset D$ and*

$$(\Phi_1 - \Phi_2)(D) \subset p^n D,$$

then

$$P(t, \Phi_1) \equiv P(t, \Phi_2) \pmod{p^n},$$

where we understand the congruence coefficient-by-coefficient.

Proof. — Put

$$P(t, \Phi_1) = \sum a_i t^i \qquad \text{and} \qquad P(t, \Phi_2) = \sum b_i t^i.$$

We have $a_0 = b_0 = 1$, and we want to show that $a_i \equiv b_i \pmod{p^n}$ for each $i \geq 1$.

Let $\Psi = \Phi_1 - \Phi_2$. Clearly, Ψ is completely continuous, and $\Psi(D) \subset p^n D$ implies that every eigenvalue of Ψ is divisible by p^n. Hence we have

$$\mathrm{trace}(\Phi_1) - \mathrm{trace}(\Phi_2) = \mathrm{trace}(\Psi) = \sum \lambda \equiv 0 \pmod{p^n},$$

where the sum is over the eigenvalues of Ψ. Since $a_1 = \mathrm{trace}(\Phi_1)$ and $b_1 = \mathrm{trace}(\Phi_2)$, this proves the first congruence.

For the remaining congruences, recall that we have

$$a_m = \mathrm{trace}\left(\bigwedge^m \Phi_1\right) \qquad \text{and} \qquad b_m = \mathrm{trace}\left(\bigwedge^m \Phi_2\right),$$

so we need to look at $\Psi = \bigwedge^m \Phi_1 - \bigwedge^m \Phi_2$. These are operators on $\bigwedge^m V$, which contains the $\mathbb{Z}_p$-lattice $D' = \bigwedge^m D$. Then, noting that

$$\bigwedge^m \Phi_1 - \bigwedge^m \Phi_2 = \left(\bigwedge^{m-1} \Phi_1\right) \wedge (\Phi_1 - \Phi_2) + \left(\bigwedge^{m-1} \Phi_1 - \bigwedge^{m-1} \Phi_2\right) \wedge \Phi_2,$$

we prove by induction that $\Psi(D')$ is contained in $p^n D'$. Thus, $a_m \equiv b_m \pmod{p^n}$, as claimed. □

Now assume $k_1 \equiv k_2 \pmod{p^n(p-1)}$, and let $\mathcal{E} : M_{k_1}(\mathrm{N}, K; r) \to M_{k_2}(\mathrm{N}, K; r)$ denote multiplication by $\mathrm{E}_{p-1}^{(k_2-k_1)/(p-1)}$. This is easily seen to be an isomorphism of Banach spaces. (One needs only check that the inverse map preserves overconvergence; for this, note that if $f \in M_{k_2}(B, \mathrm{N}; r)$ then one sees directly from the definition that $r^{(k_2-k_1)/(p-1)}\mathcal{E}f \in M_{k_1}(B, \mathrm{N}; r)$.)

Write U_k for the U operator acting on forms of weight k. We consider the operators

$$\Phi = \mathrm{U}_{k_1} \qquad \text{and} \qquad \Psi = \mathcal{E}^{-1}\mathrm{U}_{k_2}\mathcal{E},$$

both acting on $M_{k_1}(\mathrm{N}, K; r)$. Note, first, that both are completely continuous, because both U operators are. Furthermore, our two series may be computed using them :

$$P_1(t) = \det(1 - t\mathrm{U}|M_{k_1}(\mathrm{N}, K; r)) = \det(1 - t\Phi)$$

and, since conjugate operators have the same characteristic series,

$$\begin{aligned} P_2(t) &= \det(1 - t\mathrm{U}|M_{k_2}(\mathrm{N}, K; r)) \\ &= \det(1 - t(\mathcal{E}^{-1}\mathrm{U}\mathcal{E})|M_{k_1}(\mathrm{N}, K; r)) = \det(1 - t\Psi). \end{aligned}$$

Now we are in position to invoke Lemma 2. We take

$$D = M_{k_1}(\mathrm{N}, K; r) \cap M_{k_1}(\mathrm{N}, B; 1) = \{f \in M_{k_1}(\mathrm{N}, K; r) | f(q) \in B[[q]]\}.$$

This is a lattice in $M_k(r)$, since the q-expansions of modular forms have bounded denominators. To apply the lemma, we need to see that $(\Phi - \Psi) D \subset p^n D$.

LEMMA 3. — *Let W be a vector space over K, and let L be a lattice in W. Suppose $E : W \to W$ satisfies $E = I + p^t T$, where I is the identity map and $T : W \to W$ is a linear map stabilizing L. Set $F = E^{-1}$.*

If $\Upsilon : W \to W$ is a linear operator mapping L into vL for some $v \in K$, then the linear operator $F\Upsilon E - \Upsilon$ maps L into $p^t vL$.

Proof. — Simply note that

$$F\Upsilon E - \Upsilon = F\Upsilon(E - I) + (F - I)\Upsilon,$$

that both $E - I$ and $F - I$ map L to $p^t L$, and that F preserves L. □

In our situation, we take $W = K \otimes B[[q]]$, $L = B[[q]]$, $E = \mathcal{E}$ to be multiplication by $\mathrm{E}_{p-1}^{(k_2-k_1)/(p-1)}$, and $\Upsilon = \mathrm{U}$, so that $v = 1$. Applying the lemma, we get

$$(\Phi - \Psi)B[[q]] \subset p^n B[[q]].$$

Since we already know that the operator $\Phi - \Psi$ preserves $M_{k_1}(r)$, it follows that $(\Phi - \Psi)(D) \subset p^n D$, as claimed.

Thus, the hypotheses of Lemma 2 are satisfied, and this completes the proof of the theorem.

3. Higher order differences.

Given what has just been proved, it is natural to ask whether the coefficients $a_m(k)$ are Iwasawa functions, i.e., if there exist power series $A_m \in \mathbb{Z}_p[[T]]$ such that we have $a_m(k) = A_m((1+p)^k - 1)$. We cannot yet answer this question. We can, however, move a few more steps in the direction of an answer by obtaining further congruence relations among the coefficients $a_m(k)$. In fact, Iwasawa functions can be completely characterized (as in [Ser2], Theorem 14) in terms of congruence properties; what we will show is that at least some of the congruences in Serre's characterization are indeed satisfied.

To state these congruences, let $k \mapsto a(k)$ be any function from $\mathbb{Z}$ to $\mathbb{Z}_p$. Fix an n, set $s = p^n(p-1)$, and construct difference functions as follows :

$$\begin{aligned} \delta_1(a,k) &= a(k+s) - a(k) \\ \delta_2(a,k) &= \delta_1(a,k+s) - \delta_1(a,k) \\ &= a(k+2s) - 2a(k+s) + a(k) \end{aligned}$$

and, in general, for $i > 1$,

$$\delta_i(a,k) = \delta_{i-1}(a,k+s) - \delta_{i-1}(a,k).$$

What Serre shows is that if there exists a power series $A \in \mathbb{Z}_p[[T]]$ such that $a(k) = A((1+p)^k - 1)$ for all $k \equiv k_0 \pmod{p-1}$, then we must have

$$\delta_i(a,k_0) \equiv 0 \pmod{p^{i(n+1)}}.$$

Theorem 1 is the special case of $a(k) = a_m(k)$ and $i = 1$. The basic idea of the proof, however, easily extends to handle the general case, as follows.

THEOREM 2. — *Let $p \geq 5$ be a prime number,* N *an integer not divisible by p, B a p-adically complete and separated discrete valuation ring, and K its field of fractions. Let $P_k(t)$ be the characteristic power series of the* U *operator acting on the space $M_k(\mathrm{N}, K; r)$ of r-overconvergent p-adic modular forms of weight k and level* N*. Write $P_k(t) = \sum a_m(k)t^m$. Let δ_i be as above; then we have, for each m and k,*

$$\delta_i(a_m, k) \equiv 0 \pmod{p^{i(n+1)}}.$$

Proof. — Fix an integer m, and recall that $a_m(k)$ is the trace of the exterior power $\bigwedge^m \mathrm{U}$ acting on (the m-th exterior power of) forms of weight k. We use this fact to express $\delta_i(a_m, k)$ as the trace of an operator.

Consider first the case when $i = 2$. Let E be the map $\bigwedge^m M_k(r) \to \bigwedge^m M_{k+s}(r)$ which is the m-th exterior power of the map given by multiplication by $\mathrm{E}_{p-1}^{p^n}$. Then, as we saw above,

$$\delta_1(a_m, k) = \mathrm{trace}\left(E^{-1} \circ \bigwedge^m \mathrm{U} \circ E - \bigwedge^m \mathrm{U}\right).$$

Similarly, we have

$$\delta_2(a_m, k) = \mathrm{trace}\left(E^{-2} \circ \bigwedge^m \mathrm{U} \circ E^2 - 2E^{-1} \circ \bigwedge^m \mathrm{U} \circ E + \bigwedge^m \mathrm{U}\right).$$

But since

$$\begin{aligned} &E^{-2} \circ \bigwedge^m \mathrm{U} \circ E^2 - 2E^{-1} \circ \bigwedge^m \mathrm{U} \circ E + \bigwedge^m \mathrm{U} \\ &\qquad = E^{-1} \circ \left(E^{-1} \circ \bigwedge^m \mathrm{U} \circ E - \bigwedge^m \mathrm{U}\right) \circ E - \left(E^{-1} \circ \bigwedge^m \mathrm{U} \circ E - \bigwedge^m \mathrm{U}\right), \end{aligned}$$

we can apply Lemma 3 twice : once with

$$\Upsilon = \bigwedge^m \mathrm{U} \qquad \text{and} \qquad \upsilon = 1,$$

and once with

$$\Upsilon = E^{-1} \circ \bigwedge^m \mathrm{U} \circ E - \bigwedge^m \mathrm{U} \qquad \text{and} \qquad \upsilon = p^{n+1}.$$

We conclude that $E^{-2} \circ \bigwedge^m \mathrm{U} \circ E^2 - 2E^{-1} \circ \bigwedge^m \mathrm{U} \circ E + \bigwedge^m \mathrm{U}$ maps $\bigwedge^m D$ to $p^{2(n+1)} \bigwedge^m D$, and therefore that its trace is congruent to zero modulo $p^{2(n+1)}$, as desired.

The general case follows in an analogous way, by repeated application of Lemma 3. □

4. Open questions.

What about the other congruences given by Serre in [Ser2] ? Specifically, we would like to know the answer to the following :

QUESTION. — *Let c_{ij} be defined by the equation*

$$Y(Y-1)\cdots(Y-j+1) = \sum c_{ij} Y^i,$$

and, with notations as above, let

$$\gamma_j(a, k_0) = \sum_{i=1}^{j} c_{ij} p^{-i(n+1)} \delta_i(a, k_0).$$

Is it true that we have

$$\operatorname{ord}_p(\gamma_j(a_m, k)) \geq \operatorname{ord}_p(j!)$$

for every m and k ?

The point is that, according to [Ser2], this extra series of congruences, along with the congruences already proven, would be sufficient to guarantee that the $a_m(k)$ are Iwasawa functions of k.

There is a connection between Theorem 1 and the conjectures about "p-adic families" of modular eigenforms which we proposed in [GM]. In that paper, we considered the classical spaces $\mathbf{M}_k(K, \mathrm{N}p)$ of modular forms of weight k on $\Gamma_1(\mathrm{N}) \cap \Gamma_0(p)$. On these spaces, there is an action of the U operator; thus, for each rational number $\alpha \geq 0$ we can look at the subspace $\mathbf{M}_{k,\alpha}$ spanned by the eigenforms for the U operator whose eigenvalues had valuation α. We write $d(k, \alpha)$ for the dimension of this space. In [GM], we made the following conjecture :

CONJECTURE 1. — *Let k_1 and k_2 be integers. Suppose both k_1 and k_2 are bigger than $2\alpha + 2$, and that $k_1 \equiv k_2 \pmod{p^n(p-1)}$ for some integer $n \geq \alpha$. Then $d(k_1, \alpha) = d(k_2, \alpha)$.*

In attempting to prove this conjecture, it seems natural to embed the classical spaces into the corresponding spaces of overconvergent p-adic modular forms, which should be the "correct" context for studying p-adic properties of modular forms. Recall that we have an inclusion

$$\mathbf{M}_k(K, \mathrm{N}p) \hookrightarrow M_k(B, \mathrm{N}; r) \otimes K,$$

which therefore gives an inclusion $\mathbf{M}_{k,\alpha} \hookrightarrow M_{k,\alpha}$ of the slope α subspaces. Writing $d_p(k,\alpha) = \dim M_{k,\alpha}$ for the dimension of the p-adic slope α subspace, one might then consider a p-adic variant of our conjecture :

CONJECTURE 2. — *Let k_1 and k_2 be integers such that $k_1 \equiv k_2 \pmod{p^n(p-1)}$ for some integer $n \geq \alpha$. Then $d_p(k_1,\alpha) = d_p(k_2,\alpha)$.*

Both of these conjectures are known to be true when $\alpha = 0$, by the work of Hida on ordinary modular forms.(3) In Hida's work, proving Conjecture 2 is the first step in the proof of Conjecture 1, so that it is not unreasonable to expect the two conjectures to be similarly connected in general.

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(3) The case $\alpha = 0$ of Conjecture 2 also follows from the main theorem in this note, as one can see by considering the Newton polygons of characteristic power series in weights k_1 and k_2. The dimension in question is the length of the first (horizontal) segment of the polygon, which is clearly the same when the coefficients of two power series are congruent modulo p.

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Manuscrit reçu le 18 mai 1992.

F.Q. GOUVÊA,
Department of Mathematics and Computer Science
Colby College
Waterville, ME 04901 (U.S.A.)
&
B. MAZUR,
Department of Mathematics
Harvard University
One Oxford St.
Cambridge, MA 02138 (U.S.A.).