ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 43, nº 3 (1993), p. 865-892

<http://www.numdam.org/item?id=AIF_1993__43_3_865_0>

Ann. Inst. Fourier, Grenoble
43, 3 (1993), 865-892

PREPARATION THEOREMS FOR MATRIX VALUED FUNCTIONS

by Nils DENCKER

1. Introduction.

The Malgrange preparation theorem is a useful tool in analysis. It is a generalization of the Weierstrass preparation theorem to C^∞ functions as follows : if $f(t,x) \in C^\infty(\mathbf{R} \times \mathbf{R}^d)$ satisfies

$$(1.1) \quad 0 = f(0,0) = \partial_t f(0,0) = \ldots = \partial_t^{n-1} f(0,0) \quad \text{and} \quad \partial_t^n f(0,0) \neq 0,$$

then we can factor

$$(1.2) \qquad f(t,x) = c(t,x)(t^n + a_{n-1}(x)t^{n-1} + \ldots + a_1(x)t + a_0(x))$$

near (0,0), where $c(t,x)$, $a_j(x) \in C^\infty$, $c(0,0) \neq 0$ and $a_j(0) = 0$, $0 \leq j < n$. A possible generalization of this result to matrix valued functions, is to replace (1.1) by

(1.3)
$$0 = F(0,0) = \partial_t F(0,0) = \ldots = \partial_t^{n-1} F(0,0) \quad \text{and} \quad |\partial_t^n F(0,0)| \neq 0,$$

where $F(t,x) \in C^\infty$ is $N \times N$ matrix valued, and $|F|$ is the determinant. Then we should obtain (1.2) for $F(t,x)$, with matrix valued C^∞ functions $c(t,x)$ and $a_j(x)$, satisfying $|c(t,x)| \neq 0$ and $a_j(0) = 0$, $\forall j$. In the case when $n = 1$ in (1.3), this was proved in [1]. But clearly condition (1.3) is too restrictive, since it does not cover the case when $F(t,x) = (f_j(t,x)\delta_{jk})_{jk}$ is diagonal, with diagonal elements f_j satisfying (1.1) with different n (in

Key words : Malgrange preparation theorem – Matrix valued functions – Normal forms – Elementary divisors.
A.M.S. Classification : 58C27 – 26E10 – 26B40 – 35G05.

which case we can use the Malgrange preparation theorem). More generally, we assume that

$$\partial_t^m(\det F)(0,0) \neq 0, \tag{1.4}$$

for some m, thus the determinant does not vanish of infinite order at $t = 0$. Then we prove that there exists $n \geq 0$ such that

$$F(t,x) = C(t,x)\left(\sum_{t=0}^{n} t^j A_j(x)\right) \tag{1.5}$$

near (0,0), where $C(t,x)$ and $A_j(x)$ are $N \times N$ matrix valued C^∞ functions, and $|C(0,0)| \neq 0$. We also have

$$A_j(x)A_k(0) \equiv 0 \qquad j > k, \tag{1.6}$$

and $A_j(0)$ is a bijection $V_j \mapsto V_j$, where $\{V_j\}$ are linear subspaces such that

$$\mathbf{C}^N = \bigoplus_{j=0}^{n} V_j. \tag{1.7}$$

This preparation is essentially unique, up to terms vanishing of infinite order at $x = 0$, under additional conditions on $A_j(0)$. See Theorem 5.3 for the precise results.

In the special case when

$$\mathbf{C}^N = \bigoplus_{j=0}^{n} \operatorname{Im} \partial_t^j F(0,0)\Big|_{E_{j-1}} \tag{1.8}$$

where $E_k = \bigcap_{0 \leq j \leq k} \operatorname{Ker} \partial_t^j F(0,0)$, we may obtain that $A_j(0)$ in (1.5) is the orthogonal projection on $V_j = E_j^\perp \cap E_{j-1}$ by Theorem 4.3. Observe that condition (1.8) is invariant under *left* multiplication of F by invertible systems according to Proposition 4.1.

By duality, we obtain the corresponding results for *right* preparation of F, i.e. left preparation of F^*, in Theorem 6.2. We also prove the generalizations of the Malgrange division theorem in Theorems 5.9 and 6.3. The method of proof will in part follow Mather [6], with the improvements of Hörmander [2, Section 7.5]. This method also gives C^∞ bounds on the matrices $C(t,x)$ and $A_j(x)$ in (1.5). We also include the analytic versions of the results, which generalize the Weierstrass preparation and division theorems.

The preparation theorems can be useful when studying systems of partial differential equations, particularly when reducing the symbol of the system to a normal form. See the proof of Proposition 3.1 in [1] for an application to first order systems.

The plan of the paper is as follows. In section 2, we divide analytic $N \times N$ systems with matrix valued polynomials on an open set in $\mathbf{C}$. In section 3, we divide matrix valued functions in $\mathcal{S}(\mathbf{R})$ by matrix valued polynomials globally on $\mathbf{R}$. The special preparation theorem is proved in section 4, using the implicit function theorem. The general preparation theorem is proved in section 5, by a careful reduction to the special case. Finally, in section 6 we prove the dual results for right preparation.

2. Analytic division.

In what follows, let π_j be (complex) orthogonal projections in $\mathbf{C}^N$, $0 \le j \le n$, such that $\sum_{j=0}^{n} \pi_j = \mathrm{Id}_N$ and $\pi_j \pi_k = \delta_{jk}\pi_k$. Put

$$P(t, \mathbb{A}) = \sum_{0\le j\le n} t^j \pi_j + \sum_{0 \le j < n} t^j A_j, \tag{2.1}$$

with $\mathbb{A} = (A_0, \dots, A_{n-1})$, where $A_j \in \mathcal{L}_N = \mathcal{L}(\mathbf{C}^N, \mathbf{C}^N)$ is a complex $N \times N$ matrix satisfying

$$A_j \pi_k = 0 \quad \text{when} \quad j \ge k. \tag{2.2}$$

Let $|A_j| = \det A_j$ be the determinant of A_j, and let $\|\mathbb{A}\| = \sum_j \|A_j\|$, where $\|A_j\|$ is the matrix operator norm. We are going to divide matrix valued analytic functions with such matrix valued polynomials. Let ω be an open set in $\mathbf{C}$, let $G(t)$ be analytic in $\overline{\omega}$ with values in $\mathcal{L}_N$, and assume that $|P(t, \mathbb{A})| \ne 0$ on $\partial\omega \in C^1$. Then

$$G(t) = Q(t)P(t, \mathbb{A}) + R(t) \quad \text{for} \quad t \in \omega, \tag{2.3}$$

where

$$Q(t) = (2\pi i)^{-1} \int_{\partial\omega} G(s)P(s, \mathbb{A})^{-1}(s-t)^{-1}\, ds \tag{2.4}$$

is analytic in ω, and

$$R(t) = (2\pi i)^{-1} \int_{\partial\omega} G(s)P(s, \mathbb{A})^{-1}(P(s, \mathbb{A}) - P(t, \mathbb{A}))(s-t)^{-1}\, ds \tag{2.5}$$

is a polynomial of degree $n-1$ in t according to (2.6). When $P(t,\mathbb{A})^{-1}$ is analytic in ω, we find $R(t)\equiv 0$.

PROPOSITION 2.1. — *The remainder $R(t)$ in (2.3) is uniquely determined by*

$$R(t)\pi_k \quad \text{is a polynomial of degree } < k, \quad 0\le k\le n \tag{2.6}$$

$$R(t)P(t,\mathbb{A})^{-1} \quad \text{is analytic when} \quad t\notin\omega. \tag{2.7}$$

Proof. — We find from (2.5) and (2.2) that

$$R(t)\pi_k = (2\pi i)^{-1}\int_{\partial\omega} G(s)P(s,\mathbb{A})^{-1}\big((s^k-t^k)\pi_k + \sum_{j<k}(s^j-t^j)A_j\pi_k\big)(s-t)^{-1}\,ds \tag{2.8}$$

is a polynomial of degree $< k$ in t, which gives (2.6). In order to prove (2.7) we observe that (2.5) gives

$$R(t)P(t,\mathbb{A})^{-1} = -(2\pi i)^{-1}\int_{\partial\omega} G(s)P(s,\mathbb{A})^{-1}(s-t)^{-1}\,ds \qquad t\notin\overline{\omega}, \tag{2.9}$$

which is analytic in $\complement\overline{\omega}$. Since $|P(t,\mathbb{A})|\neq 0$ near $\partial\omega$, we find $R(t)P(t,\mathbb{A})^{-1}$ analytic there.

To prove the uniqueness of $R(t)$, we observe that (2.3) implies that

$$Q(t) = (2\pi i)^{-1}\int_{\partial\omega} G(s)P(s,\mathbb{A})^{-1}(s-t)^{-1}\,ds - (2\pi i)^{-1}\int_{\partial\omega} R(s)P(s,\mathbb{A})^{-1}(s-t)^{-1}\,ds$$

when $t\in\omega$. If $R(t)P(t,\mathbb{A})^{-1}$ is analytic in $\complement\omega$ we may push the integration contour to infinity in the second integral. Since we have $R(t)\pi_k = O(|t|^{k-1})$ and $R(t)P(t,\mathbb{A})^{-1} = \sum_k R(t)\pi_k^2 P(t,\mathbb{A})^{-1}$, we find from Lemma 2.2 below that the integrand is $O(|s|^{-2})$. Thus Q is equal to (2.4), which gives the uniqueness. □

LEMMA 2.2. — *If $P(t,\mathbb{A})$ is given by (2.1), where A_j satisfies (2.2), then we find*

$$\pi_k P(t,\mathbb{A})^{-1} = O(|t|^{-k}) \quad \text{as} \quad |t|\to\infty. \tag{2.10}$$

Proof. — To prove (2.10), we observe that $P(t,\mathbb{A})\pi_k \equiv P_k(t,\mathbb{A})\pi_k$, where $P_k(t,\mathbb{A}) = t^k \mathrm{Id}_N + \sum_{j<k} t^j A_j$. Now, if we assume $t \in \mathbf{C}$, such that $|P(t,\mathbb{A})| \neq 0$, then we have $\pi_k P(t,\mathbb{A})^{-1} P(t,\mathbb{A})\pi_j = \delta_{jk}\pi_k$, and $\mathbf{C}^N = \bigoplus_{0\le j\le n} V_j(t)$, where

$$V_k(t) = \operatorname{Im} P(t,\mathbb{A})\pi_k = \operatorname{Im} P_k(t,\mathbb{A})\pi_k. \tag{2.11}$$

Observe that $V_k(t) \to \operatorname{Im}\pi_k$, which are orthogonal, as $t \to \infty$. We obtain that

$$\pi_k P(t,\mathbb{A})^{-1}\Big|_{V_j(t)} \equiv 0 \quad \text{when} \quad j \neq k. \tag{2.12}$$

If also $|P_k(t,\mathbb{A})| \neq 0$, we find

$$\pi_k P(t,\mathbb{A})^{-1} \equiv \pi_k P_k(t,\mathbb{A})^{-1} \quad \text{on} \quad V_k(t). \tag{2.13}$$

It is clear that $P_k(t,\mathbb{A})^{-1} = O(|t|^{-k})$ as $|t| \to \infty$, so we obtain (2.10) from (2.12)–(2.13). □

3. Polynomial division on R.

Now, we want to make the division (2.3) when $G \in \mathcal{S}(\mathbf{R})$, depending C^∞ on the parameters $x \in \mathbf{R}^d$. We shall also obtain C^∞ bounds on Q and R. As before, we let $\{\pi_k\}$ be fixed orthogonal projections satisfying $\sum_{j=0}^{n} \pi_j = \mathrm{Id}_N$ and $\pi_i\pi_j = \delta_{ij}\pi_j$, thus $\sum_j \operatorname{Rank}\pi_j = N$. Assume $P(t,\mathbb{A})$ given by (2.1), where $\mathbb{A} = (A_0,\dots,A_{n-1})$ satisfies (2.2). Let $m_k = \operatorname{Rank}\pi_k$ and $m = \sum_{1\le j\le n} j\cdot m_j$, and let $\mathbf{V} \subseteq \bigoplus_{j=0}^{n-1} \mathcal{L}_N$ be the set of $\mathbb{A} = (A_0,\dots,A_{n-1})$ satisfying (2.2). Since $A_k = \sum_{j>k} A_k\pi_j$, A_k lies in a subspace of (complex) dimension $\sum_{j>k} m_j N$ of $\mathcal{L}_N$. This implies that $\mathbf{V} \cong \mathbf{C}^{mN}$, since we have

$$\sum_{0\le k<j\le n} m_j = \sum_{j=1}^{n} j\cdot m_j = m.$$

Lemma 3.1. — *Assume that $P(t,\mathbb{A})$ is given by (2.1), where $\mathbb{A} = (A_0,\dots,A_{n-1}) \in \mathbf{V}$ satisfies (2.2). Let $m = \sum_{j=1}^{n} j.\operatorname{Rank}\pi_j$ and $p(t) =$*

det $P(t,\mathbb{A})$, *then it follows that*

$$\partial_t^m p(0) \neq 0, \tag{3.1}$$

and $\partial_t^{m+1}p \equiv 0$. *If* $\mathbb{A} \in \mathbf{V}$ *satisfies* $\|\mathbb{A}\| = \sum \|A_j\| < \delta \leq 1$, *then we find that*

$$p(t) = 0 \Longrightarrow |t| < \delta^{1/n}. \tag{3.2}$$

Observe that it follows that the determinant of $P(t,\mathbb{A})$ is a polynomial of exactly degree $m = \sum\limits_{j=1}^{n} j \cdot \operatorname{Rank} \pi_j$.

Proof. — First we note that by a (constant) orthogonal base change, we may assume that

$$\operatorname{Im} \pi_k = \Big\{ (z_1, \dots, z_N) : z_j \neq 0 \\ \Rightarrow \sum_{i=0}^{k-1} \operatorname{Rank} \pi_i < j \leq \sum_{i=0}^{k} \operatorname{Rank} \pi_i \Big\}, \quad 0 \leq k \leq n. \tag{3.3}$$

Since $P(t,\mathbb{A})\pi_k = \Big(t^k + \sum\limits_{j<k} t^j A_j\Big)\pi_k$ we find that

$$\partial_t^m p(0) = \left|\sum_{k=0}^{n} k!\pi_k\right| \neq 0,$$

which proves (3.1). Similarly, we obtain that $\partial_t^{m+1}p \equiv 0$.

Assume that $P(t,\mathbb{A})w = 0$, where $t \neq 0$, $w = \sum w_j \neq 0$ with $w_j = \pi_j w$. If we put $v_k = t^k w_k$, we find

$$v = \sum_k v_k = -\sum_k \Big(\sum_{0 \leq j < k} A_j t^{j-k} \Big) v_k = -\Big(\sum_{0 \leq j < k \leq n} A_j \pi_k t^{j-k} \Big) v, \tag{3.4}$$

since $\sum\limits_j t^j \pi_j w = -\sum\limits_j t^j A_j w$. If $|t| \geq \delta^{1/n}$ and $\|\mathbb{A}\| < \delta \leq 1$, we find

$$\Big\| \sum_{0 \leq j < k \leq n} A_j \pi_k t^{j-k} \Big\| = \Big\| \sum_{j=0}^{n} A_j \Big(\sum_{j < k \leq n} \pi_k t^{j-k} \Big) \Big\| \leq \sum_j \|A_j\| \delta^{-1} < 1,$$

thus (3.4) implies that $v = w = 0$. This proves (3.2). $\square$

PROPOSITION 3.2. — *Let $G(t) \in \mathcal{S}(\mathbf{R})$ have values in $\mathcal{L}_N$. Then we can find $Q(t,\mathbb{A}) \in C^\infty(\mathbf{R}\times\mathbf{V})$ and $R_j(\mathbb{A}) \in C^\infty(\mathbf{V})$, $0 \le j < n$, with values in $\mathcal{L}_N$ and depending linearly on $G(t)$, such that $R_j(\mathbb{A})\pi_k \equiv 0$ when $j \ge k$, and*

(3.5)

$$G(t) = Q(t,\mathbb{A})P(t,\mathbb{A}) + \sum_{j=0}^{n-1} t^j R_j(\mathbb{A}) \quad \text{when} \quad \|\mathbb{A}\| < C \quad \text{and} \quad t \in \mathbf{R}.$$

Also, we have the following estimates

(3.6)

$$\|\partial_t^\alpha \partial_{\mathbb{A}}^\beta Q(t,\mathbb{A})\| \le C_{\alpha\beta} \textstyle\int (\|G\| + \|G^{(k)}\|)\,dt, \quad k = 3 + \alpha + m(|\beta|+1)$$

$$\|\partial_{\mathbb{A}}^\beta R_j(\mathbb{A})\| \le C_\beta \textstyle\int (\|G\| + \|G^{(k)}\|)\,dt, \quad k = 2 + m(|\beta|+1),$$

for $|t| < c$ and $0 \le j < n$, with $m = \sum_j j \cdot \mathrm{Rank}\pi_j$. Here $\partial_{\mathbb{A}}$ means differentiation with respect to the components of $\mathbb{A} \in \mathbf{V}$.

Remark 3.3. — If $G(t,x) \in \mathcal{S}(\mathbf{R}\times\mathbf{R}^d)$ depends on parameters x, then $Q(t,\mathbb{A}) \in C^\infty(\mathbf{R}\times\mathbf{V}\times\mathbf{R}^d)$ and $R_j(\mathbb{A}) \in C^\infty(\mathbf{V}\times\mathbf{R}^d)$, $\forall j$. In fact, by linearity and continuity, we may differentiate directly on G. Observe that by a dilation in t, we may choose any constant C in (3.5) (see the proof below).

Proposition 3.2 and Remark 3.3, without the linear dependence on G and the estimates (3.6), can also be obtained from (1.2) in [7]. By adapting the proof of [2, Theorem 7.5.4], to the matrix case, as in the proof of [1, Proposition A.2], we obtain $Q(t,\mathbb{A})$ satisfying the estimates in (3.6) uniformly for all $t \in \mathbf{R}$.

Proof. — We shall first divide by $p(t) = \det P(t,\mathbb{A})$, when $\mathbb{A} \in \mathbf{V}$. By [2, Theorem 7.5.4], we may find C^∞ functions $Q(t,\mathbb{A})$ and $R_j(\mathbb{A})$ with values in $\mathcal{L}_N$, depending linearly on $G(t)$, so that

$$G(t) = Q(t,\mathbb{A})p(t) + \sum_{j=0}^{m-1} t^j R_j(\mathbb{A}) \quad \text{when} \quad \|\mathbb{A}\| < c \quad \text{and} \quad t \in \mathbf{R} \tag{3.7}$$

for some $c > 0$, since the degree of $p(t)$ is equal to m by Lemma 3.1. We also get (3.6) for all $t \in \mathbf{R}$ by using (7.5.14) in [2], since the coefficients in $p(t)$ are algebraic functions of the elements of $\mathbb{A}$. From the proof of [2, Theorem 7.5.4], it is clear that (3.6) also holds for bigger k. By first making a dilation $s = \delta t$ for small enough δ and using that $A_j = \sum_{j<k} A_j\pi_k$, we obtain (3.7)

when $\|\mathbb{A}\| < C$. Since $p(t) = {}^tP(t,\mathbb{A})^{co}P(t,\mathbb{A})$, it only remains to divide $R(t,\mathbb{A}) = \sum_{j=0}^{m-1} t^j R_j(\mathbb{A})$ with $P(t,\mathbb{A})$ in order to get the wanted remainder terms. For this purpose, we use (2.4)–(2.5) with $G(t) = R(t,\mathbb{A})$ and $\omega \subseteq \mathbf{C}$ containing t and all the zeros of $\det P(t,\mathbb{A})$, and with C^1 boundary. This gives

$$R(t,\mathbb{A}) = Q_0(t,\mathbb{A})P(t,\mathbb{A}) + R_0(t,\mathbb{A}) \quad \text{when} \quad \|\mathbb{A}\| < C \quad \text{and} \quad t \in \mathbf{R}.$$

Since $P^{-1}(t,\mathbb{A})$ is analytic in $\complement\omega$, we obtain from Proposition 2.1 that the remainder $R_0(t,\mathbb{A})$ is unique, satisfying (2.6). The derivatives of $Q_0(t,\mathbb{A})$ and $R_0(t,\mathbb{A})$ can be estimated by derivatives of $R(t,\mathbb{A})$, which in turn can be estimated by (3.6). $\square$

4. Left preparation.

Now, let $F(t)$ be a C^∞ function on $\mathbf{R}$ with values in $\mathcal{L}_N$. Put $E_{-1} = \mathbf{C}^N$ and

$$E_k = \bigcap_{0\le j\le k} \operatorname{Ker} \partial_t^j F(0), \quad k \ge 0. \tag{4.1}$$

These spaces are invariant under left multiplication of $F(t)$ by invertible systems, according to the following

PROPOSITION 4.1. — *If*

$$\mathbf{C}^N = \bigoplus_{j=0}^{n} Im\, \partial_t^j F(0)\Big|_{E_{j-1}}, \tag{4.2}$$

then it follows that $E_n = \{0\}$. We find that the spaces E_k, $0 \le k \le n$, and condition (4.2) are invariant under left multiplication of $F(t)$ by invertible systems.

Proof. — Assume that $C(t)$ is an invertible system, then $\operatorname{Ker} CF(0) = \operatorname{Ker} F(0)$. Now, we have by Leibniz' rule

$$\partial_t^k(CF)(0) = \sum_{j=0}^{k} \binom{k}{j} \partial_t^{k-j} C(0) \partial_t^j F(0),$$

so by induction we obtain

$$\bigcap_{0\le j\le k} \operatorname{Ker} \partial_t^j(CF)(0) = \Big(\bigcap_{0\le j<k} \operatorname{Ker} \partial_t^j F(0)\Big) \cap \operatorname{Ker} \partial_t^k (CF)(0)$$
$$= \bigcap_{0\le j\le k} \operatorname{Ker} \partial_t^j F(0),$$

which gives the invariance of E_k, $\forall\, k$. We also obtain that

$$\operatorname{Im} \partial_t^k (CF)(0)\Big|_{E_{k-1}} = C(0) \operatorname{Im} \partial_t^k F(0)\Big|_{E_{k-1}}. \tag{4.3}$$

Since $|C(0)| \neq 0$, this gives the invariance of condition (4.2).

It remains to prove that $\dim E_n = 0$. Let $m_k = \dim E_k$, so that $m_{-1} = N$. Then, we find

$$\dim\Big(\operatorname{Im} \partial_t^k F(0)\Big|_{E_{k-1}}\Big) = \dim E_{k-1} - \dim\Big(\operatorname{Ker} \partial_t^k F(0)\Big|_{E_{k-1}}\Big) = m_{k-1} - m_k.$$

Thus, we obtain from (4.2) that

$$N \le \sum_{j=0}^{n} (m_{k-1} - m_k) = N - m_n.$$

This means that $m_n \le 0$, which proves the result. □

Observe that the proof of Proposition 4.1 also works if we have

$$\mathbf{C}^N = \sum_{j=0}^{n} \operatorname{Im} \partial_t^j F(0)\Big|_{E_{j-1}},$$

in fact, this condition implies (4.2). The spaces E_k will be used to construct the orthogonal projections π_k in the preparation. Let $E_k^\perp$ be the orthogonal complement of E_k.

PROPOSITION 4.2. — *Let $\mathbf{C}^N = E_{-1} \supseteq E_0 \supseteq \ldots \supseteq E_n = \{0\}$, and let π_k be the orthogonal projection on $E_k^\perp \bigcap E_{k-1}$ for $0 \le k \le n$. Then it follows that $\pi_j \pi_k = \delta_{jk} \pi_k$, and*

$$\bigoplus_{j=0}^{k} \operatorname{Im} \pi_j = E_k^\perp, \quad 0 \le k \le n. \tag{4.4}$$

In particular, we obtain $\bigoplus_{0\le j\le n} \operatorname{Im} \pi_j = \mathbf{C}^N$, which implies $\sum_{j=0}^{n} \pi_j = \mathrm{Id}_N$.

Proof. — Clearly, $\operatorname{Ker}\pi_k = (\operatorname{Im}\pi_k)^\perp = E_k \bigoplus E_{k-1}^\perp$, implying

$$\operatorname{Im}\pi_j \subseteq E_{j-1} \subseteq E_k \subseteq \operatorname{Ker}\pi_k \quad \text{if} \quad j > k.$$

Thus, $\pi_k\pi_j \equiv 0$ when $j > k$, and by taking adjoints we obtain this when $j < k$, which implies $\pi_j\pi_k = \delta_{jk}\pi_k$.

By taking orthogonal complements, we find that (4.4) is equivalent to

$$\bigcap_{j=0}^{k} \operatorname{Ker}\pi_j = E_k, \quad 0 \le k \le n. \tag{4.5}$$

We find $\operatorname{Ker}\pi_0 = E_0 \oplus E_{-1}^\perp = E_0$. Assume by induction that (4.5) holds for some $k \ge 0$. Then we find that

$$\bigcap_{0\le j\le k+1} \operatorname{Ker}\pi_j = E_k \bigcap (E_{k+1} \oplus E_k^\perp) = E_{k+1},$$

since $E_{k+1} \subseteq E_k$, thus by induction we obtain (4.5) for all k. Since $E_n = \{0\}$ we find that $\sum_{j=0}^{n} \pi_j$ is bijective, and since $\left(\sum_{j=0}^{n} \pi_j\right)^2 = \sum_{j=0}^{n} \pi_j$, it is equal to the identity. $\square$

When $E_k = \bigcap_{j\le k} \operatorname{Ker}\partial_t^j F(0)$ we find that π_k is an orthogonal projection into E_{k-1}, such that $\operatorname{Ker}\pi_k\Big|_{E_{k-1}} = \operatorname{Ker}\partial_t^k F(0)\Big|_{E_{k-1}} = E_k$. Now, we can prove the following generalization of the Malgrange preparation theorem.

THEOREM 4.3. — *Let $F(t,x)$ be a C^∞ function of (t,x) in a neighborhood of the origin of $\mathbf{R} \times \mathbf{R}^d$, with values in $\mathcal{L}_N$, and assume that*

$$\mathbf{C}^N = \bigoplus_{j=0}^{n} \operatorname{Im}\partial_t^j F(0,0)\Big|_{E_{j-1}}, \tag{4.6}$$

where $E_{-1} = \mathbf{C}^N$ and $E_k = \bigcap_{0\le j\le k} \operatorname{Ker}\partial_t^j F(0,0)$. Let π_k be the orthogonal projection on $E_k^\perp \bigcap E_{k-1}$ for $0 \le k \le n$. Then, we may factor

$$F(t,x) = C(t,x)\Big(\sum_{j=0}^{n} t^j\pi_j + \sum_{j=0}^{n-1} t^j A_j(x)\Big) = C(t,x)P(t,\mathbb{A}(x)) \tag{4.7}$$

near $(0,0)$, where $C(t,x)$ and $A_j(x)$ are C^∞ functions with values in $\mathcal{L}_N$, satisfying $A_j(x)\pi_k \equiv 0$, $j \ge k$. We also find $|C(0,0)| \ne 0$, and $A_j(0) = 0$

for $0 \le j < n$. If $F(t,x)$ is analytic in a neighborhood of the origin, then we may choose unique analytic $C(t,x)$ and $A_j(x)$ satisfying (4.7). If $F(t,x)$ is real (matrix) valued, then the projections π_k are real (matrix) valued and we may choose $C(t,x)$ and $A_j(x)$ real (matrix) valued.

The projections π_j are chosen orthogonal in order to get uniqueness. As in the scalar case, we find that $C(t,x)$ and $A_j(x)$ are not uniquely determined, but the proof gives C^∞ bounds on these functions, depending on $F(t,x)$. Since (4.7) implies

$$F(t,0) = C(t,0)\sum_{j=0}^{n} t^j \pi_j,$$

we obtain from Proposition 4.1 that condition (4.6) is necessary for the preparation (4.7). It is not hard to prove that $\partial_x^\alpha A_j(0)$ and $\partial_x^\alpha C(t,0)$ are uniquely determined by (4.7), $\forall\,\alpha$.

Proof. — By Proposition 4.1, we find that $E_n = \{0\}$. Since π_k is the orthogonal projection on $E_k^\perp \bigcap E_{k-1}$, we obtain from Proposition 4.2 that $\pi_j\pi_k = \delta_{jk}\pi_k$ and $\sum_{j=0}^{n} \pi_j = \mathrm{Id}_N$. Let

$$P(t,\mathbb{A}) = \sum_{0\le j\le n} t^j \pi_j + \sum_{0\le j<n} t^j A_j,$$

where $\mathbb{A} = (A_0,\dots,A_{n-1})$ satisfies $A_j\pi_k = 0$ when $j \ge k$, i.e. $\mathbb{A} \in \mathbf{V} \cong \mathbf{C}^{mN}$. Here $m = \sum_{j=1}^{n} j\cdot m_j$ and $m_j = \mathrm{Rank}\,\pi_j$. Since the result is local, we may assume $F \in C_0^\infty$. By using Proposition 3.2 and Remark 3.3, we get

$$F(t,x) = Q(t,x,\mathbb{A})P(t,\mathbb{A}) + \sum_{j=0}^{n-1} t^j R_j(x,\mathbb{A}) \tag{4.8}$$

near $(0,0,0) \in \mathbf{R}\times\mathbf{R}^d\times\mathbf{V}$. Here Q and R_j are C^∞ functions, satisfying $R_j(x,\mathbb{A})\pi_k \equiv 0$ when $j \ge k$, thus $(R_j(x,\mathbb{A}))_j$ has values in $\mathbf{V}$. Now we need the following

LEMMA 4.4. — *Assume that $F(t,x)$ satisfies (4.6) and (4.8), where Q and R_j are C^∞ functions, satisfying $R_j\pi_k \equiv 0$ when $j \ge k$. Then, we obtain*

$$|Q(0,0,0)| \ne 0 \tag{4.9}$$

$$R_j(0,0) = 0, \quad 0 \le j < n. \tag{4.10}$$

End of proof of Theorem 4.3. — Differentiation of (4.8) with respect to the components of $\mathbb{A}$, when $\mathbb{A}=0$ and $x=0$, gives

$$0 = d_{\mathbb{A}}Q(t,0,0)\sum_{j=0}^{n} t^j\pi_j + Q(t,0,0)\sum_{j=0}^{n-1} t^j dA_j + \sum_{j=0}^{n-1} t^j d_{\mathbb{A}}R_j(0,0). \tag{4.11}$$

By composition with π_k, we obtain from (4.11) that

$$Q(t,0,0)\sum_{j<k} t^j dA_j\pi_k = -d_{\mathbb{A}}Q(t,0,0)t^k\pi_k - \sum_{j<k} t^j d_{\mathbb{A}}R_j(0,0)\pi_k.$$

Since Q is invertible in a neighborhood of the origin by Lemma 4.4, we find that

$$Q(t,0,0)\sum_{j<k} t^j B_j = O(|t|^k) \qquad t\to 0,$$

implies $B_j=0$, $0\le j<k$. Thus, the differential of the mapping

$$\mathbf{V}\ni\mathbb{A}\mapsto\mathbb{R}=(R_0,\dots,R_{n-1})\in\mathbf{V}\cong\mathbf{C}^{mN} \tag{4.12}$$

is bijective at $(0,0)$. By the implicit function theorem and Lemma 4.4, we find that the equation

$$\mathbb{R}(x,\mathbb{A})\equiv 0$$

defines a unique C^∞ function $\mathbb{A}(x)$ of x in a neighborhood of the origin of $\mathbf{R}^d$, with values in $\mathbf{V}$, such that $\mathbb{A}(0)=0$. Naturally, the unique function $\mathbb{A}(x)$ depends on the choice of $Q(t,x,\mathbb{A})$ in (4.8). Since

$$F(t,x)\equiv Q(t,x,\mathbb{A}(x))P(t,\mathbb{A}(x)),$$

we obtain (4.7) with $C(t,x)=Q(t,x,\mathbb{A}(x))$. When $F(t,x)$ is real, we find that π_j is real, $0\le j\le n$. Then, we may take Q and R_j real, and use the implicit function theorem with $\mathbb{A}\in\operatorname{Re}\mathbf{V}\cong\mathbf{R}^{mN}$.

In the case when F is analytic near the origin, we choose $\varepsilon>0$ so that F is analytic in a neighborhood of $\{\,|t|\le\varepsilon\wedge|x|\le\varepsilon\,\}$. By using (2.4)–(2.5) with $\partial\omega=\{\,|t|=\varepsilon\,\}$, we get (4.8) when $\|\mathbb{A}\|<\varepsilon^n$, $|t|<\varepsilon$ and $|x|\le\varepsilon$, by Lemma 3.1. By Proposition 2.1 we find that R_j and Q are uniquely determined, since $P^{-1}(t,\mathbb{A})$ is analytic in $\complement\omega$. Since Q and R_j depend linearly on $F(t,x)$, they are analytic in x too. By using the implicit function theorem in the analytic case, we obtain unique analytic $\mathbb{A}(x)$ and $C(t,x)=Q(t,x,\mathbb{A}(x))$ in (4.7), with the required properties. $\square$

Proof of Lemma 4.4. — By taking $x = 0$ and $\mathbb{A} = 0$ in (4.8), we obtain

$$F(t,0) = Q(t,0,0)\sum_{j=0}^{n} t^j\pi_j + \sum_{j=0}^{n-1} t^j R_j(0,0). \tag{4.13}$$

Differentiation with respect to t gives
(4.14)

$$\partial_t^k F(0,0) = \sum_{j=0}^{k} \binom{k}{j} \partial_t^{k-j} Q(0,0,0) j!\pi_j + k! R_k(0,0) \quad \text{when} \quad k \leq n,$$

implying

$$\operatorname{Im} \partial_t^k F(0,0)\pi_k = \operatorname{Im} Q(0,0,0)\pi_k \quad \text{when} \quad k \leq n, \tag{4.15}$$

since $R_k\pi_k \equiv 0$. We also find from (4.14) that $R_k(0,0)\pi_j = \partial_t^k F(0,0)\pi_j/k! = 0$ for $j > k$, since $\operatorname{Im}\pi_j \subseteq E_{j-1} \subseteq \operatorname{Ker} \partial_t^k F(0,0)$ then. Since $R_k\pi_j \equiv 0$ when $j \leq k$, we find $R_k(0,0) = 0$, $\forall\, k$.

Now, condition (4.6) is equivalent to

$$\mathbf{C}^N = \bigoplus_{j=0}^{n} \operatorname{Im} \partial_t^j F(0,0)\pi_j, \tag{4.16}$$

since $\operatorname{Im}\pi_j = E_j^{\perp} \bigcap E_{j-1}$ and $\partial_t^j F(0,0)\Big|_{E_j} \equiv 0$. We find from (4.15)–(4.16), that

$$\mathbf{C}^N = \bigoplus_{j=0}^{n} \operatorname{Im} Q(0,0,0)\pi_j,$$

thus $Q(0,0,0)$ is bijective. □

Example 4.5. — Let $F(t,x)$ be a C^∞ function with values in $\mathcal{L}_N$, and assume that

$$|\partial_t^n F(0,0)| \neq 0 \quad \text{and} \quad \partial_t^j F(0,0) \equiv 0, \quad 0 \leq j < n. \tag{4.17}$$

Then we obtain from Theorem 4.3

$$F(t,x) = C(t,x)\Big(t^n \mathrm{Id}_N + \sum_{0\leq j<n} t^j A_j(x)\Big), \tag{4.18}$$

where $C(t,x)$ and $A_j(x)$ are C^∞ functions with values in $\mathcal{L}_N$, satisfying $|C(0,0)| \neq 0$ and $A_j(0) = 0$ for $0 \leq j < n$. (The case when $n = 1$ was proved in [1, Theorem A.3].)

5. The preparation theorem.

Now, condition (4.6) in Theorem 4.3 is still too restrictive. In fact, the systems $P(t, \mathbb{A}(x))$ in (4.7) do not satisfy condition (4.6) when $\mathbb{A}(0) \neq 0$ satisfies (5.6), but will be acceptable normal forms when $\mathbb{A}(x) \in \mathbf{V}$, i.e. $A_j(x)\pi_k \equiv 0$ for $j \geq k$. As before, we assume that π_j is orthogonal projection in $\mathbf{C}^N$ for $0 \leq j \leq n$, such that $\sum_{j=0}^{n} \pi_j = \mathrm{Id}_N$ and $\pi_i \pi_j = \delta_{ij}\pi_j$. First, we consider the necessary conditions for such a preparation.

Proposition 5.1. — *Let $F(t) \in C^\infty(\mathbf{R})$ with values in $\mathcal{L}_N$, and assume that*

$$F(t) = C(t)\Big(\sum_{j=0}^{n} t^j \pi_j + \sum_{j=0}^{n-1} t^j A_j\Big) = C(t)P(t, \mathbb{A}) \tag{5.1}$$

where $|C(0)| \neq 0$ and $A_j \pi_k = 0$ when $j \geq k$. Then it follows that

$$\partial_t^m (\det F)(0) \neq 0, \tag{5.2}$$

for some m. We also find

$$E_n = \bigcap_{0 \leq k \leq n} \operatorname{Ker} \partial_t^k F(0) = \{0\}. \tag{5.3}$$

Proof. — Since the spaces E_k are invariant under multiplication from left by invertible systems by Proposition 4.1, we may replace $F(t)$ by $P(t, \mathbb{A})$ in (5.3). Now $\partial_t^k P(0, \mathbb{A}) = k!(\pi_k + A_k)$, where $A_k = \sum_{k<j} A_k \pi_j$. Thus, we find that $\operatorname{Ker} \partial_t^n P(0, \mathbb{A}) = \operatorname{Ker} \pi_n$. By induction we have

$$\bigcap_{j=k}^{n} \operatorname{Ker}(\pi_j + A_j) = \Big(\bigcap_{j=k+1}^{n} \operatorname{Ker} \pi_j\Big) \cap \operatorname{Ker}(\pi_k + A_k) = \bigcap_{j=k}^{n} \operatorname{Ker} \pi_j,$$

for $0 \leq k \leq n$, which proves (5.3). It is also clear that condition (5.2) is invariant under multiplication by invertible systems. Thus, it follows from Lemma 3.1 (but not necessarily with the same m as in (3.1)). ☐

The factorization (5.1) is not unique, according to the following

Example 5.2. — Let

$$P_1(t) = \begin{pmatrix} t & 1 \\ 0 & t \end{pmatrix} = t\,\mathrm{Id}_2 + A_0$$
$$P_2(t) = \begin{pmatrix} t^2 & 0 \\ t & 1 \end{pmatrix} = \pi_0 + t^2\pi_2 + tB_1$$

and $Q(t) = \begin{pmatrix} t & -1 \\ 1 & 0 \end{pmatrix}$. Then we have $Q(t)P_1(t) = P_2(t)$, and $|Q(t)| \equiv 1$. Since $B_1\pi_0 = 0$, it is clear that $P_1(t)$ and $P_2(t)$ both are on the form (2.1)–(2.2). Observe that $P_2(t)$ has the property that $B_1 = \pi_0 B_1 \pi_2$.

Now we are ready to prove the main preparation theorem.

THEOREM 5.3. — *Let $F(t,x)$ be a C^∞ function of (t,x) in a neighborhood of the origin of $\mathbf{R}\times\mathbf{R}^d$ with values in $\mathcal{L}_N$, and assume that*

$$\partial_t^m(\det F)(0,0) \neq 0 \quad \text{and} \quad \partial_t^k(\det F)(0,0) = 0, \quad 0 \le k < m. \tag{5.4}$$

Then we may factor

$$F(t,x) = C(t,x)\Big(\sum_{j=0}^{n} t^j\pi_j + \sum_{j=0}^{n-1} t^j B_j(x)\Big) = C(t,x)P(t,\mathbb{B}(x)) \tag{5.5}$$

near $(0,0)$, where π_j is orthogonal projection in $\mathbf{C}^N$, $0\le j\le n$, such that $\pi_j\pi_k = \delta_{jk}\pi_k$ and $\sum_{j=0}^{n}\pi_j = \mathrm{Id}_N$. Here $C(t,x)$ and $B_j(x)$ are C^∞ functions with values in $\mathcal{L}_N$, satisfying $|C(0,0)|\neq 0$, $B_j(x)\pi_k \equiv 0$ when $j\ge k$ and

$$B_j(0) = \sum_{i<j<k} \pi_i B_j(0)\pi_k \qquad \forall j, \tag{5.6}$$

which implies $B_0(0) = 0$. The projections π_k and matrices $B_j(0)$ are uniquely determined by condition (5.6), and it follows that $m = \sum_j j\cdot \mathrm{Rank}\,\pi_j$ in (5.4). If $F(t,x)$ is analytic in a neighborhood of the origin, then we may choose unique analytic $C(t,x)$ and $B_j(x)$ satisfying (5.5) and (5.6). If $F(t,x)$ is real (matrix) valued, then π_k is real (matrix) valued and we may choose $C(t,x)$ and $B_j(x)$ real (matrix) valued.

Remark 5.4. — The rank of the projections π_k are determined by the elementary divisors of the Taylor expansion of $F(t,0)$ at $t=0$. In fact, let d_k be the determinant factors for $1\le k\le N$, i.e. the greatest common divisor of the minors of order k of the Taylor expansion. Then $e_k = d_k/d_{k-1}$

are the elementary divisors, and $\operatorname{Rank}\pi_j$ is the number of k such that e_k is divisible by t^j but not by t^{j+1} (see [9, § 85]). The projections π_j are harder to compute, except for $j = 0$, 1, since in these cases $\operatorname{Ker}\pi_0 = \operatorname{Ker} F(0,0)$, and

$$\operatorname{Ker}\pi_0 \cap \operatorname{Ker}\pi_1 = \operatorname{Ker}\,[\partial_t F(0,0) : \operatorname{Ker} F(0,0) \\ \mapsto \mathbf{C}^N/\operatorname{Im} F(0,0) = \operatorname{Coker} F(0,0)]\,.$$

In Example 5.2 we find that $P_2(t)$ but not $P_1(t)$ satisfies (5.6). Observe that the projection $\pi_2 \not\equiv 0$ but $\partial_t^2 P_1(t) \equiv 0$. In general, the projections can be computed by the procedure in the proof of Lemma 5.5. It follows from the proof of Theorem 5.3 that $\partial_x^\alpha B_j(0)$ and $\partial_x^\alpha C(t,0)$ are uniquely determined by (5.5), $\forall\,\alpha$. As before, $C(t,x)$ and $B_j(x)$ are not uniquely determined, but the proof of Theorem 5.3 gives C^∞ bounds on these functions.

The proof of Theorem 5.3 relies on some simple preparatory Lemmas. First we shall compute the projections $\{\pi_k\}$.

LEMMA 5.5. — *Assume that $F(t) \in C^\infty(\mathbf{R})$ with values in $\mathcal{L}_N$, such that* $\det F(t)$ *doesn't vanish of infinite order at $t = 0$. Then we may write*

$$F(t) = C(t)\Big(\sum_{j=0}^{n} t^j \pi_j + R(t)\Big) \quad \textit{near} \quad 0, \tag{5.7}$$

where $|C(0)| \neq 0$ and π_j is orthogonal projection in $\mathbf{C}^N$, $0 \le j \le n$, such that $\sum_{j=0}^{n} \pi_j = \mathrm{Id}_N$ and $\pi_i\pi_j = \delta_{ij}\pi_j$. The error term $R(t)$ satisfies

$$\begin{cases} \pi_i R(t)\pi_j \equiv 0 \quad \textit{when} \quad i \ge j \\ \pi_j R(t) = O(|t|^{j+1}) \quad \textit{as} \quad t \to 0, \quad \forall\, j, \end{cases} \tag{5.8}$$

thus $R(0) = 0$. If $F(t)$ is real (matrix) valued, then we may choose $C(t)$, π_k and $R(t)$ real (matrix) valued.

Condition (5.8) means that $R(t)$ is nilpotent and satisfies first part of (5.6). In fact, $R(t)$ maps $\operatorname{Im}\pi_k$ into $\bigoplus_{j<k} \operatorname{Im}\pi_j$, $\forall\, t$, and if $R(t) = \sum_j t^j R_j$ then $\pi_i R_j = 0$ when $i \ge j$.

Proof. — Let $E_0 = \operatorname{Ker} F(0)$, then

$$F(0) : \quad E_0^\perp \longmapsto \operatorname{Im} F(0)$$

is a bijection. By multiplying from left by a constant, invertible system, we may assume that $F(0) = \pi_0$, where π_0 is the orthogonal projection along

$\operatorname{Ker} F(0)$. If $F(0)$ is real valued, this can be done by multiplication with a real, invertible system, giving real π_0. We find

$$F(t) = \pi_0 + tF_0(t)$$

near 0, where we may assume $F_0(t)\pi_0 \equiv 0$. In fact, we can multiply with $C(t) = \big(\mathrm{Id}_N + tF_0(t)\big)^{-1}$ from left to obtain

$$\begin{aligned} C(t)F(t) = C(t)\big((\mathrm{Id}_N + tF_0(t))\pi_0 + tF_0(t)(\mathrm{Id}_N - \pi_0)\big)& \\ = \pi_0 + tC(t)F_0(t)(\mathrm{Id}_N - \pi_0)& \end{aligned}$$

near 0. When $F(t)$ is real, we find that $C(t)$ is real.

Now assume by induction, that we have found orthogonal projections π_j, $0 \le j \le k$, and invertible system $C_k(t)$, such that $\pi_i\pi_j = \delta_{ij}\pi_j$ and

$$C_k(t)F(t) = \sum_{j=0}^{k} t^j\pi_j + Q_k(t) + R_k(t) \tag{5.9}$$

near 0. We assume that $R_k(t)$ satisfies condition (5.8) when i, $j \le k$, and that

$$R_k(t) \equiv \Pi_k R_k(t), \tag{5.10}$$

where $\Pi_k = \sum\limits_{j\le k} \pi_j$. We also assume that

$$Q_k(t) \equiv (\mathrm{Id}_N - \Pi_k)Q_k(t)(\mathrm{Id}_N - \Pi_k) = O(|t|^{k+1}). \tag{5.11}$$

When $k = 0$ we obtain this, with

$$\begin{cases} R_0(t) = \pi_0 tF_0(t)(\mathrm{Id}_N - \pi_0) \\ Q_0(t) = (\mathrm{Id}_N - \pi_0)tF_0(t)(\mathrm{Id}_N - \pi_0). \end{cases}$$

When $\Pi_n = \mathrm{Id}_N$ we get the result, since $Q_n \equiv 0$.

Assume $\Pi_k \ne \mathrm{Id}_N$, and put $V_k = \operatorname{Im}(\mathrm{Id}_N - \Pi_k)$. Since $(\mathrm{Id}_N - \Pi_k)C_kF(t) \equiv Q_k(t)$, we find that $Q_k(t) \equiv Q_k(t)(\mathrm{Id}_N - \Pi_k)$ cannot vanish of infinite order on V_k at $t = 0$. Thus, we may assume that $Q_k(t) = t^\nu A_k(t)$, where $\nu > k$ and $A_k(0) \ne 0$. By using the argument above with $F(t)$ and $\mathbf{C}^N$ replaced by $A_k(t)$ and V_k, and multiplying from left with an invertible system $C_\nu(t)$ on V_k, we obtain that

$$C_\nu(t)Q_k(t) = t^\nu\pi_\nu + Q_\nu(t).$$

Here $0 \neq \pi_\nu$ is orthogonal projection on a subspace of V_k, $Q_\nu(t) = O(|t|^{\nu+1})$, $t \to 0$, and $Q_\nu \pi_\nu \equiv 0$. When $Q_k(t)$ is real, we may choose $C_\nu(t)$, π_ν and $Q_\nu(t)$ real valued. By extending trivially to $\mathbf{C}^N$, we obtain (5.9) with k replaced by ν, $R_k(t)$ by $R_k(t) + \pi_\nu Q_\nu(t)$ and $Q_k(t)$ by $(\mathrm{Id}_N - \pi_\nu)Q_\nu(t)$. Since we increase the rank of Π_k in every step, we obtain the result in finitely many steps. □

In order to obtain a system satisfying the conditions in Theorem 4.3, we must multiply from the right also. Then, we have to be careful in order to preserve the normal forms.

LEMMA 5.6. — *Assume that $F(t) \in C^\infty(\mathbf{R})$, with values in $\mathcal{L}_N$, is on the form*

$$F(t) = \sum_{j=0}^{n} t^j \pi_j + R(t) \quad \textit{near} \quad 0, \tag{5.12}$$

where π_j is orthogonal projection in $\mathbf{C}^N$ for $0 \leq j \leq n$, such that $\sum_{j=0}^{n} \pi_j = \mathrm{Id}_N$ and $\pi_i \pi_j = \delta_{ij}\pi_j$, and $R(t)$ satisfies (5.8). Then, we can write

$$F(t) = \Big(\sum_{j=0}^{n} t^j \pi_j + G(t)\Big)(Id_N + S(t)) \tag{5.13}$$

near 0, where $G(t)$ also satisfies (5.8) and

$$G(t)\pi_k = O(|t|^k) \quad \textit{as} \quad t \to 0. \tag{5.14}$$

We also find that $S(0) = 0$, and

$$\pi_j S(t) \pi_k \quad \textit{is a polynomial of degree} < k - j \textit{ in } t. \tag{5.15}$$

If $R(t)\pi_k$ is a polynomial of degree $< k$, $\forall\, k$, then we obtain that $G(t) \equiv 0$. When $F(t)$ is real valued, we may choose $G(t)$ and $S(t)$ real valued.

Observe that it follows from (5.15) that $\sum_j t^j \pi_j S(t)$ satisfies condition (5.6). The error term $G(t)$ will be eliminated in Remark 5.7.

Proof. — First we observe that if $S(t)$ satisfies (5.15), then $\pi_j S(t) \pi_k \equiv 0$ when $j \geq k$, so $S(t)$ is nilpotent. Since the matrices with property (5.15) are closed under addition and multiplication, the corresponding matrices $\mathrm{Id}_N + S(t)$ form a multiplicative subgroup of $\mathbf{SL}(N, \mathbf{C}[t])$.

Now assume by induction that we have obtained (5.13) with $G(t) = \sum_{j<k} G_{jk}(t)$, where $G_{jk}(t) = \pi_j G(t)\pi_k$ satisfies

$$(5.16) \qquad \begin{cases} G_{jk}(t) = O(|t|^k) & \text{when} \quad j \ge k-\mu, \\ G_{jk}(t) = O(|t|^{j+1}) & \forall j, \end{cases} \qquad t \to 0$$

for some $\mu \ge 0$. When $R(t)\pi_k$ is a polynomial of degree $< k$, $\forall\, k$, we assume that $G(t)\pi_k$ also is a polynomial of degree $< k$. Clearly, (5.16) holds with $G(t) \equiv R(t)$ when $\mu = 0$, and will give the result when $\mu = n$, since non-zero polynomials of degree $< k$ cannot satisfy (5.14). Take the Taylor expansion

$$G_{jk}(t) = \sum_{i=j+1}^{k-1} t^i G^i_{jk} + t^k Q_{jk}(t)$$

(observe that $G^i_{jk} = 0$ when $j \ge k-\mu$) and let

$$S_\mu(t) = \sum_{\substack{j<i<k\le n \\ 0\le j<k-\mu}} t^{i-j} G^i_{jk}.$$

Then we obtain that $S_\mu(0) = 0$, $S_\mu(t)$ satisfies (5.15), and

$$(5.17) \quad F(t)(\mathrm{Id}_N + S(t))^{-1}(\mathrm{Id}_N - S_\mu(t)) = \sum_{j=0}^{n} t^j \pi_j + \sum_{j<k} t^k Q_{jk}(t) + R_\mu(t),$$

where

$$R_\mu(t) = -\sum_{\substack{i<j<k<l \\ j<l-\mu}} G_{ij}(t) G^k_{jl} t^{k-j}.$$

Thus, we find that $\pi_j R_\mu \pi_k \equiv 0$ unless $j < k - (\mu+1)$, and $\pi_j R_\mu(t) = O(|t|^{j+2})$. In the case when $G(t)\pi_k$ is a polynomial of degree $< k$, $\forall\, k$, we obtain that $Q_{jk}(t) \equiv 0$ and $R_\mu(t)\pi_k$ is also a polynomial of degree $< k$. This proves the induction step. When $F(t)$ is real valued, we obtain recursively that $G_{jk}(t)$, $Q_{jk}(t)$, $S_\mu(t)$ and $R_\mu(t)$ are real valued. $\square$

Remark 5.7. — If $F(t)$ is on the form (5.13) with $G(t)$ satisfying (5.8) and (5.14), then by multiplication from left by an invertible system, we may obtain $G(t) \equiv 0$. In fact, assume by induction that $G_{jk}(t) = \pi_j G(t)\pi_k \equiv 0$ when $j \ge k - \mu$ for some $\mu \ge 0$, which is true for $\mu = 0$. Take $R_{jk}(t) = t^{-k} G_{jk}(t)$ for $j < k - \mu$ with $G_{jk}(t) \in C^\infty$, then

$$\Big(\mathrm{Id}_N - \sum_{j<k-\mu} R_{jk}(t)\Big) F(t) \big(\mathrm{Id}_N + S(t)\big)^{-1} = \sum_j t^j \pi_j - \sum_{j<k-\mu<l-2\mu} R_{jk}(t) G_{kl}(t),$$

which proves the induction step, since $\mu \geq 0$. When $\mu = n$ we obtain $G(t) \equiv 0$.

Proof of Theorem 5.3. — By Lemmas 5.5–5.6 and Remark 5.7, we may write

$$C(t)^{-1}F(t,0)(\mathrm{Id}_N + S(t))^{-1} = \sum_{j=0}^{n} t^j \pi_j, \tag{5.18}$$

where $|C(0)| \neq 0$, and π_j is orthogonal projection, such that $\sum_{j=0}^{n} \pi_j = \mathrm{Id}_N$ and $\pi_i \pi_j = \delta_{ij}\pi_j$. It is clear that (5.18) satisfies condition (4.6). By the left invariance, we find that $F(t,x)(\mathrm{Id}_N + S(t))^{-1}$ also satisfies (4.6). Thus by Theorem 4.3, we can factor

(5.19)
$$F(t,x)(\mathrm{Id}_N+S(t))^{-1} = C_0(t,x)\Big(\sum_{j=0}^{n} t^j\pi_j + \sum_{j=0}^{n-1} t^j A_j(x)\Big) = C_0(t,x)P(t,\mathbb{A}(x)),$$

where $|C_0(0,0)| \neq 0$, $\mathbb{A}(0) \equiv 0$, and $A_j(x)\pi_k \equiv 0$ when $j \geq k$. Here, the projections π_k are the same as in (5.18). Since $S(t)$ is a polynomial, we get analytic $C_0(t,x)$ and $\mathbb{A}(x)$ in (5.19), when $F(t,x)$ is analytic near the origin. Now, we obtain (5.5) with

$$P(t,\mathbb{B}(x)) = \sum_{j=0}^{n} t^j \pi_j + \sum_{j=0}^{n-1} t^j B_j(x) = P(t,\mathbb{A}(x))(\mathrm{Id}_N + S(t)), \tag{5.20}$$

which means that

(5.21)
$$\sum_{0\leq j<n} t^j B_j(x) = \sum_{0\leq j<n} t^j A_j(x) + \sum_{0\leq j\leq n} t^j \pi_j S(t) + \sum_{0\leq j<n} t^j A_j(x) S(t).$$

Composing (5.21) with π_k from right gives

$$\sum_{0\leq j<k} t^j A_j(x)\pi_k + \sum_{0\leq j<k} t^j S_{jk}(t) + \sum_{0\leq i<j<k} t^i A_i(x) S_{jk}(t),$$

which is a polynomial in t of degree $< k$, since $S_{jk}(t) = \pi_j S(t)\pi_k$ is a polynomial of degree $< k-j$. Since $\mathbb{A}(0) \equiv 0$ and $S(0) = 0$, we find from (5.21) that

$$\pi_j \sum_{0\leq i<n} t^i B_i(0)\pi_k \equiv t^j S_{jk}(t)$$

only contains terms of order greater than j and less than k, which gives (5.6). By choosing orthogonal basis in $\mathbf{C}^N$ such that (3.3) holds, we obtain $P(t, \mathbb{B}(0))$ on upper diagonal form. Then, we obtain that $m = \sum_j j \cdot \operatorname{Rank} \pi_j$ in (5.4). When $F(t,x)$ is real valued, we may choose π_k, $C(t)$ and $S(t)$ real valued in (5.18). By Theorem 4.3, we may choose $C_0(t,x)$ and $A_j(x)$ real valued, which gives real valued $B_j(x)$.

To prove uniqueness of the projections and the matrices $B_j(0)$, we assume that

$$(5.22)\quad F(t,0) = C_1(t)\Big(\sum_j t^j \pi_j + \sum_j t^j A_j\Big) = C_2(t)\Big(\sum_j t^j \widetilde{\pi}_j + \sum_j t^j B_j\Big),$$

where $|C_k(0)| \neq 0$ for $k = 1, 2$, $A_j = \sum_{i<j<k} \pi_i A_j \pi_k$ and $B_j = \sum_{i<j<k} \widetilde{\pi}_i B_j \widetilde{\pi}_k$, $\forall\, j$, where $\widetilde{\pi}_j$ is orthogonal projection satisfying $\widetilde{\pi}_i \widetilde{\pi}_j = \delta_{ij}\widetilde{\pi}_j$. By using Lemma 5.6, we obtain

$$C_1(t) \sum_j t^j \pi_j (\mathrm{Id}_N + S_1(t)) = C_2(t) \sum_j t^j \widetilde{\pi}_j (\mathrm{Id}_N + S_2(t)),$$

where $S_k(t)$ satisfies (5.15), and $S_k(0) = 0$ for $k = 1, 2$. Thus, we find

$$(5.23)\qquad Q(t) = C(t) \sum_j t^j \pi_j = \sum_j t^j \widetilde{\pi}_j (\mathrm{Id}_N + S(t)),$$

where $|C(0)| \neq 0$ and $S(0) = 0$. We find from (5.23) and Proposition 4.1 that

$$(5.24)\qquad \bigcap_{0 \le j \le k} \operatorname{Ker} \partial_t^j Q(0) = \bigcap_{0 \le j \le k} \operatorname{Ker} \pi_j, \quad \forall\, k,$$

and (4.3) gives

$$(5.25)\qquad \operatorname{Im} \partial_t^k Q(0)\Big|_{E_{k-1}} = C(0) \operatorname{Im} \pi_k, \quad \forall\, k,$$

where $E_k = \bigcap_{j \le k} \operatorname{Ker} \partial_t^j Q(0)$. Similarly, we obtain from Proposition 6.1 that

$$(5.26)\qquad \sum_{0 \le j \le k} \operatorname{Im} \partial_t^j Q(0) = \bigoplus_{0 \le j \le k} \operatorname{Im} \widetilde{\pi}_j, \quad \forall\, k,$$

and (6.8) gives

$$(5.27)\qquad \operatorname{Ker} \partial_t^k Q(0) \Big(\operatorname{mod} \sum_{j<k} \operatorname{Im} \partial_t^j Q(0)\Big) = \operatorname{Ker} \widetilde{\pi}_k, \quad \forall\, k,$$

since $S(0) = 0$. From (5.25)–(5.26) we obtain

$$C(0) \bigoplus_{0 \le j \le k} \operatorname{Im} \pi_j \subseteq \bigoplus_{0 \le j \le k} \operatorname{Im} \widetilde{\pi}_j, \quad \forall\, k, \tag{5.28}$$

and (5.24), (5.27) gives

$$\bigcap_{0 \le j \le k} \operatorname{Ker} \pi_j \subseteq \bigcap_{0 \le j \le k} \operatorname{Ker} \widetilde{\pi}_j, \quad \forall\, k. \tag{5.29}$$

By combining (5.28) and the orthogonal complement of (5.29), we obtain

$$C(0) \bigoplus_{0 \le j \le k} \operatorname{Im} \pi_j \subseteq \bigoplus_{0 \le j \le k} \operatorname{Im} \widetilde{\pi}_j \subseteq \bigoplus_{0 \le j \le k} \operatorname{Im} \pi_j, \quad \forall\, k. \tag{5.30}$$

Since $|C(0)| \neq 0$, the spaces in (5.30) all have the same dimension, thus they are equal. Since the projections are orthogonal, we find that $\pi_k = \widetilde{\pi}_k$, $\forall k$.

We also have to prove that $A_j = B_j$, $\forall\, j$, in (5.22). It is clear that this holds $\iff S_1(t) \equiv S_2(t) \iff S(t) \equiv 0$ in (5.23). Since $\pi_k = \widetilde{\pi}_k$, $\forall\, k$, we find that $S(t)$ satisfies (5.15). Let $C_{jk}(t) = \pi_j C(t) \pi_k$ and $S_{jk}(t) = \pi_j S(t) \pi_k$. We obtain from (5.23) that

$$C_{jk}(t) t^k \equiv S_{jk}(t) t^j \quad \text{when} \quad j < k,$$

since $\pi_k = \widetilde{\pi}_k$, $\forall\, k$. Since the right hand side is a polynomial of degree $< k$, we obtain $C_{jk} \equiv S_{jk} \equiv 0$, when $j < k$. We also obtain $C_{jj}(t) t^j \equiv \pi_j t^j$ from (5.23), making $C_{jj}(t) \equiv \pi_j$, $\forall\, j$. Finally, we get $C_{jk}(t) \equiv 0$ when $j > k$. Thus, $C(t) \equiv \mathrm{Id}_N$ and $S(t) \equiv 0$, which proves the uniqueness of $B_j(0)$, $\forall\, j$, in (5.5). When $F(t, x)$ is analytic, we obtain unique analytic $\mathbb{B}(x)$, since $\mathbb{A}(x)$ and $S(t)$ in (5.21) are unique and analytic. □

By multiplication from right with invertible systems, we may also obtain that $\pi_k B_j(x) \equiv 0$ when $k \le j$, and $\mathbb{B}(0) = 0$ in (5.5), according to the following

Proposition 5.8. — *Assume that π_j is orthogonal projection in $\mathbf{C}^N$ for $0 \le j \le n$, such that $\pi_j \pi_k = \delta_{jk} \pi_k$ and $\sum_{j=0}^{n} \pi_j = \mathrm{Id}_N$. Let*

$$P(t, \mathbb{A}(x)) = \sum_{j=0}^{n} t^j \pi_j + \sum_{j=0}^{n-1} t^j A_j(x) \tag{5.31}$$

where $A_j(x)$ is C^∞ function with values in $\mathcal{L}_N$, satisfying $A_j(x)\pi_k \equiv 0$ for $j \geq k$. Then we may find $C(t,x) \in C^\infty$ with values in $\mathcal{L}_N$, such that $|C(t,x)| \neq 0$ and

$$P(t, \mathbb{A}(x))C(t,x) = P(t, \mathbb{B}(x)), \tag{5.32}$$

where $\pi_k B_j(x) \equiv B_j(x)\pi_k \equiv 0$ when $j \geq k$. For those x_0 satisfying $\pi_k A_j(x_0) = 0$ when $k > j$, we obtain that $B_k(x_0) = 0, \forall\, k$. When $P(t, \mathbb{A}(x))$ is real (or analytic), we may take $C(t,x)$ and $B_j(x)$ real (or analytic).

Proof. — By induction over $0 \leq \mu \leq n$, we shall prove that there exist invertible $C_\mu(t,x) \in C^\infty$ with values in $\mathcal{L}_N$, such that

$$P(t, \mathbb{A}(x))C_\mu(t,x) = P(t, \mathbb{B}(x)) + \sum_{j=0}^{n-1} t^j R_j(x), \tag{5.33}$$

where

$$R_j(x)\pi_k \equiv 0 \quad \text{when} \quad j \geq k - \mu, \tag{5.34}$$

and $\pi_k B_j(x) \equiv B_j(x)\pi_k \equiv 0$ when $j \geq k$. Clearly, this holds for $\mu = 0$ with $C_0 \equiv \mathrm{Id}_N$, $B_j \equiv 0$ and $R_j \equiv A_j$. It implies (5.32) when $\mu = n$, since $R_j(x) \equiv 0$ then.

Assume that the induction hypothesis holds for some $0 \leq \mu < n$. Put

$$E(t,x) = \mathrm{Id}_N - \sum_{0 \leq j \leq k < n} \pi_j R_k(x) t^{k-j},$$

then $E(t,x)$ is invertible. In fact, since $R_j \pi_k \equiv 0$ when $j \geq k$, we find that $E(t,x) - \mathrm{Id}_N$ maps $\mathrm{Im}\, \pi_k$ into $\bigoplus_{j<k} \mathrm{Im}\, \pi_j$, thus it is nilpotent. Let $C_{\mu+1}(t,x) = C_\mu(t,x)E(t,x)$, then we find from the induction hypothesis that

$$P(t, \mathbb{A}(x))C_{\mu+1}(t,x) = P(t, \mathbb{B}(x)) + \sum_{j=0}^{n} \Big(\sum_{i>j} \pi_i \Big) R_j(x) t^j$$
$$- \sum_{\substack{0 \leq j \leq k < n \\ i < j - \mu}} R_i(x)\pi_j R_k(x) t^{k-j+i} - \sum_{0 \leq i < j \leq k < n} B_i(x)\pi_j R_k(x) t^{k-j+i}.$$

Thus, $B_j(x)$ is replaced by $B_j(x) + \sum_{i=j+1}^{n} \pi_i R_j(x)$, and $\sum_j t^j R_j(x)$ is replaced by

$$R_0(t,x) = - \sum_{\substack{0 \leq j \leq k < n \\ i < j - \mu}} R_i(x)\pi_j R_k(x) t^{k-j+i} - \sum_{0 \leq i < j \leq k < n} B_i(x)\pi_j R_k(x) t^{k-j+i}.$$

Now, we find that

$$R_0(t,x)\pi_\nu = -\sum_{\substack{0\le j\le k<\nu-\mu\\ i<j-\mu}} R_i(x)\pi_j R_k(x)\pi_\nu t^{k-j+i} - \sum_{0\le i<j\le k<\nu-\mu} B_i(x)\pi_j R_k(x)\pi_\nu t^{k-j+i},$$

is a polynomial of degree $<\nu-\mu-1$. When $P(t,\mathbb{A}(x))$ is real valued, we may take $R_j(x)$, $C_\mu(t,x)$ and $B_k(x)$ real valued. This proves the induction step. If we start with analytic $R_j \equiv A_j$, $C_0 \equiv \mathrm{Id}_N$ and $B_j \equiv 0$, we obtain analytic R_j, C_μ and B_k in each step.

By differentiating (5.32) with respect to t when $t=0$, we obtain that

$$\sum_{i=0}^{j}(\pi_i + A_i(x))\partial_t^{j-i}C(0,x)/(j-i)! = \pi_j + B_j(x).$$

At the points x_0 where $\pi_k A_j(x_0)=0$, $k>j$, we find that $\pi_k B_j(x_0)=0$, $k>j$. Since $\pi_k B_j(x)\equiv 0$ when $k\le j$, we obtain that $B_j(x_0)=0$, $\forall\, j$. □

We also obtain the following generalization of the division theorem.

THEOREM 5.9. — *Let $F(t,x)$ satisfy the hypothesis in Theorem 5.3. If $G(t,x)$ is a C^∞ function in a neighborhood of $(0,0)$ with values in $\mathcal{L}_N$, then we can write*

$$G(t,x) = Q(t,x)F(t,x) + \sum_{j=0}^{n-1} t^j R_j(x) \tag{5.35}$$

near $(0,0)$. Here $Q(t,x)$ and $R_j(x)$ are C^∞ functions with values in $\mathcal{L}_N$, satisfying $R_j(x)\pi_k \equiv 0$ when $j\ge k$, for the projections π_k in Theorem 5.3. When $G(t,x)$ and $F(t,x)$ are analytic near the origin, we may choose unique analytic $Q(t,x)$ and $R_j(x)$.

It follows from the proof that the neighborhood in which (5.35) holds only depends on $F(t,x)$, not on $G(t,x)$. As before, $Q(t,x)$ and $R_j(x)$ are not uniquely determined in general, but the proof gives C^∞ bounds similar to (3.6) on these functions. It is not hard to prove that $\partial_x^\alpha R_j(0)$ and $\partial_x^\alpha Q(t,0)$ are uniquely determined by (5.35), $\forall\,\alpha$.

Proof. — By Theorem 5.3, we may assume that

$$F(t,x) = \sum_{j=0}^{n} t^j\pi_j + \sum_{j=0}^{n-1} t^j A_j(x) = P(t,\mathbb{A}(x)), \tag{5.36}$$

where $A_j(x)\pi_k \equiv 0$ when $j \geq k$, and $A_j(0)$ satisfies (5.6). Since it is no restriction to assume $G(t,x) \in C_0^\infty$, the first statement follows from Proposition 3.2 and Remark 3.3. (When $F(t,x)$ satisfies (4.6) also, we find that π_k is the orthogonal projection on $E_k^\perp \bigcap E_{k-1}$.) When $F(t,x)$ is analytic near the origin, we may choose unique analytic $\mathbb{A}(x)$ in (5.36). Then, we may use the analytic division (2.3)–(2.5), with $\partial\omega = \{|t| = \varepsilon\}$ for small enough ε, in order to get analytic $Q(t,x)$ and $R_j(x)$. Since $A_j(0)$ satisfies (5.6), we find that $\det P(t, \mathbb{A}(0)) = t^m$, where $m = \sum_j j \cdot \operatorname{Rank} \pi_j$. (In fact, by choosing coordinates so that (3.3) holds, we obtain that $\sum_j t^j A_j(0)$ is upper triangular.) Thus $P(t, \mathbb{A}(x))^{-1}$ is analytic when $|t| \geq \varepsilon$ and $|x| \leq \delta$, for $\delta > 0$ small enough. By Proposition 2.1, the remainder $\sum_j t^j R_j(x)$ is unique, thus $Q(t,x)$ is unique. □

6. Right preparation.

In Theorems 4.3 and 5.3, we have only done left preparation of matrix valued functions. By taking transposes we also obtain the corresponding results for right preparation. We first examine what condition we get on F, when (4.6) holds for F^*. Let $F(t)$ be a C^∞ function on $\mathbf{R}$ with values in $\mathcal{L}_N$, put $E_{-1}^* = \mathbf{C}^N$, and

$$E_k^* = \bigcap_{0 \leq j \leq k} \operatorname{Ker} \partial_t^j F^*(0), \quad k \geq 0. \tag{6.1}$$

Let F_k be the mapping

$$F_k : \quad \mathbf{C}^N \ni w \longmapsto \partial_t^k F(0) w \qquad (\text{mod } I_{k-1}) \quad \text{for} \quad k \geq 0, \tag{6.2}$$

where $I_{-1} = \{0\}$, and

$$I_k = \sum_{0 \leq j \leq k} \operatorname{Im} \partial_t^j F(0), \quad k \geq 0.$$

Proposition 6.1. — *The condition*

$$\mathbf{C}^N = \bigoplus_{k=0}^{n} \operatorname{Im} \partial_t^k F^*(0)\Big|_{E_{k-1}^*} \tag{6.3}$$

is equivalent to

$$\{0\} = \bigcap_{k=0}^{n} \operatorname{Ker} F_k, \tag{6.4}$$

where F_k is given by (6.2), and implies

$$\mathbf{C}^N = \sum_{k=0}^{n} \operatorname{Im} \partial_t^k F(0) = I_n. \tag{6.5}$$

We find that condition (6.4) and the spaces $I_k = \sum_{0 \le j \le k} \operatorname{Im} \partial_t^j F(0)$, $0 \le k \le n$, are invariant under multiplication of $F(t)$ by invertible systems from right.

Proof. — We have by duality that

$$I_k = \sum_{0 \le j \le k} \operatorname{Im} \partial_t^j F(0) = \Big(\bigcap_{0 \le j \le k} \operatorname{Ker} \partial_t^j F^*(0) \Big)^{\perp} = (E_k^*)^{\perp}. \tag{6.6}$$

Let π_k be the orthogonal projection on $I_k \bigcap I_{k-1}^{\perp} = (E_k^*)^{\perp} \bigcap E_{k-1}^*$, then we find $\operatorname{Ker} F_k = \operatorname{Ker} \pi_k \partial_t^k F(0)$ and

$$\operatorname{Im} \partial_t^k F^*(0)\Big|_{E_{k-1}^*} = \operatorname{Im} \partial_t^k F^*(0) \pi_k = \big(\operatorname{Ker} \pi_k \partial_t^k F(0)\big)^{\perp}. \tag{6.7}$$

By Proposition 4.1, condition (6.3) is invariant under multiplication of $F(t)$ by invertible systems from right, and it is equivalent to (6.4) by (6.7) and the proof of Proposition 4.1. We also obtain from Proposition 4.1 that the spaces $E_k^* = I_k^{\perp}$ are invariant under multiplication of $F(t)$ by invertible systems from right. Since condition (6.3) implies $E_n^* = \{0\}$ by Proposition 4.1, we obtain (6.5). □

Let $\widetilde{F}(t) = FC(t)$, where $C(t)$ is an invertible system. Then Leibniz'rule gives

$$\operatorname{Ker} \widetilde{F}_k = C(0)^{-1} \big(\operatorname{Ker} F_k\big), \tag{6.8}$$

as in the proof of (4.3).

Now we obtain from Theorems 4.3 and 5.3 the following result :

THEOREM 6.2. — *Let $F(t,x)$ be a C^∞ function of (t,x) in a neighborhood of the origin of $\mathbf{R} \times \mathbf{R}^d$ with values in $\mathcal{L}_N$ satisfying (5.4). Then we may factor*

$$F(t,x) = \Big(\sum_{j=0}^{n} t^j \pi_j + \sum_{j=0}^{n-1} t^j A_j(x) \Big) C(t,x) = P(t, \mathbb{A}(x)) C(t,x) \tag{6.9}$$

near $(0,0)$, *where* π_j *is orthogonal projection in* $\mathbf{C}^N$, $0 \le j \le n$, *such that* $\pi_j \pi_k = \delta_{jk}\pi_k$ *and* $\sum_{j=0}^{n} \pi_j = \mathrm{Id}_N$. *Here* $C(t,x)$ *and* $A_j(x)$ *are* C^∞ *functions with values in* $\mathcal{L}_N$, *satisfying* $|C(0,0)| \ne 0$, $\pi_k A_j(x) \equiv 0$ *when* $j \ge k$, *and*

$$A_j(0) = \sum_{i>j>k} \pi_i A_j(0) \pi_k, \tag{6.10}$$

which implies $A_0(0) = 0$. *The projections* π_k *and matrices* $A_j(0)$ *are uniquely determined by condition (6.10), and it follows that* $m = \sum_j j \cdot \mathrm{Rank}\pi_j$ *in (5.4). If also condition (6.4) is satisfied, we find that* $A_j(0) = 0$, $0 \le j < n$, *and* π_k *is the orthogonal projection on* $I_k \bigcap I_{k-1}^{\perp}$ *for* $0 \le k \le n$, *where* $I_{-1} = \{0\}$, $I_k = \sum_{0 \le j \le k} \mathrm{Im}\, \partial_t^j F(0,0)$. *If* $F(t,x)$ *is analytic in a neighborhood of the origin, then we may choose unique analytic* $C(t,x)$ *and* $A_j(x)$ *satisfying (6.9) and (6.10). If* $F(t,x)$ *is real (matrix) valued, then* π_j *is real (matrix) valued and we may choose* $C(t,x)$ *and* $A_j(x)$ *real (matrix) valued.*

It is clear that condition (5.4) is necessary for the preparation (6.9), and condition (6.4) is necessary when $\mathbb{A}(0) \equiv 0$. Observe that in the proof of the analytic case, we apply Theorem 5.3 to the transpose ${}^tF$, which is analytic. Since $({}^t\pi)^* = {}^t\pi$, we obtain unique orthogonal projections. The preparation is unique up to functions vanishing of infinite order at $\{x = 0\}$. We also get C^∞ bounds on $C(t,x)$ and $A_j(x)$. We obtain the following version of the division theorem from Theorem 5.9 by duality.

THEOREM 6.3. — *Let* $F(t,x)$ *satisfy the hypothesis in Theorem 6.2. If* $G(t,x)$ *is a* C^∞ *function in a neighborhood of* $(0,0)$ *with values in* $\mathcal{L}_N$, *then we can write*

$$G(t,x) = F(t,x)Q(t,x) + \sum_{j=0}^{n-1} t^j R_j(x) \tag{6.11}$$

near $(0,0)$. *Here* $Q(t,x)$ *and* $R_j(x)$ *are* C^∞ *functions with values in* $\mathcal{L}_N$, *satisfying* $\pi_k R_j(x) \equiv 0$ *when* $j \ge k$, *for the orthogonal projections* π_k *in Theorem 6.2. When* $G(t,x)$ *and* $F(t,x)$ *are analytic near the origin, we may choose unique analytic* $Q(t,x)$ *and* $R_j(x)$.

As before, the neighborhood in which (6.11) holds only depends on $F(t,x)$, not on $G(t,x)$. The division is unique up to functions vanishing of infinite order at $\{x = 0\}$. We also get C^∞ bounds on $Q(t,x)$ and $R_j(x)$.

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Manuscrit reçu le 21 août 1992,
révisé le 8 mars 1993.

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