ANNALES DE L'INSTITUT FOURIER

SHAI HARAN

Analytic potential theory over the p-adics

Annales de l'institut Fourier, tome 43, nº 4 (1993), p. 905-944

<http://www.numdam.org/item?id=AIF_1993__43_4_905_0>

Ann. Inst. Fourier, Grenoble
43, 4 (1993), 905–944

ANALYTIC POTENTIAL THEORY OVER THE p-ADICS

by Shai HARAN

In these notes we develop the potential theory of the p-adic analogue of the symmetric stable distributions. We do this purely analytically and in an explicit manner. In §1 we recall Weil's formulation [11] of (the local part of) Tate's thesis [3] in terms of α-homogeneous distributions, and relate these in §2 to probability. We prove that the function $|x|^\alpha$ is negative definite over Q_p for $\alpha \in (0,\infty)$, generating a semi-group of probability measures μ_t^α, explicitly given by

$$\mu_t^\alpha(dx) = \sum_{n\geq 1} \frac{(-t)^n}{n!} \frac{\zeta_p(1+n\alpha)}{\zeta_p(-n\alpha)} |x|^{-(1+n\alpha)}\, dx$$

(this formula being an analogue of a formula of Feller [4] for the α-symmetric stable distribution over the reals $\mathbf{R}$). For $\alpha \in (0,1)$, μ_t^α is transient, and its potential is the Riesz potential

$$k_t^\alpha(x) = \int_0^\infty \mu_t^\alpha(x)\, dt = \frac{\zeta_p(1-\alpha)}{\zeta_p(\alpha)} |x|^{\alpha-1}.$$

When we approach the boundary $\alpha \to \infty$ (in analogy with the real case $\alpha \to 2$) we obtain the "normal law", a very simple process which degenerates and "lives" on $\mathbf{Q}_p/\mathbf{Z}_p$. In §3 we recall the analytic properties of the Riesz potentials [10], their distributional meromorphic continuation, and the Riesz Reproduction formula. In §4 we begin to develop the potential theory of R_p^α, $\alpha \in (0,1)$. In view of §2 we can identify our potentials with probabilistic potentials and deduce the various potential theoretic principles in one blow [1], [2], [7], [9]; we prefer, nevertheless, to develop these principles purely analytically, and to grasp things along the way in a most explicit manner. We note, for example, that the Harnack inequalities

Key words : p-adic numbers – Potential theory.
A.M.S. Classification : 11K41 – 11M06 – 43A05 – 45N05 – 60B15.

become in our p-adic situation trivial equalities. We prove the principles of Descent, Dichotomy, Maximum, Regularization, and Uniqueness. We follow mostly the analogous real case as developed in [8], the p-adic setting offering many simplifications. In §5 we consider the finite energy measures leading to the proof that they form a complete positive cone in the Hilbert space of finite energy distributions. In §6 we prove the existence and various characterizations of the equilibrium measure, and show that the α-capacity satisfies the usual properties and, moreover, is given explicitly via the α-diameter :

$$\operatorname{cap}_\alpha(K) = \frac{\zeta_p(\alpha)}{\zeta_p(1-\alpha)} \lim_{N\to\infty} \left[\min_{x_1,\dots,x_N\in K} \frac{1}{n(n-1)} \sum_{i<j} |x_i-x_j|^{\alpha-1} \right]^{-1}.$$

In §7, we approach Balayage and the Green measure using the Keldish transform [8]; i.e. instead of using Cartan's method of projections in the Hilbert space of finite energy distributions (which equally works well), we use the more geometrical situation of the analysis of the $\mathrm{PGL}_2(\mathbf{Q}_p)$-action on $\mathbf{P}^1(\mathbf{Q}_p)$ and on equilibrium measures. We calculate explicitly the Green measures of balls and their complements. Throughout §4 to §7 there are various strengthenings of the uniqueness principle which demonstrate the increasing grasp we have over our potentials. In §8, we develop the concepts of α-(super)-harmonic functions. Explicitly, a function $f : \mathbf{Q}_p \to \mathbf{C}$ is α-harmonic at $x \in \mathbf{Q}_p$ if for all N sufficiently large :

$$f(x) = \frac{1}{\zeta_p(\alpha)} \int_{|y|\geq 1} f(x+p^N y) \frac{d^*y}{|y|^\alpha}.$$

We give an explicit solution to Dirichlet problems, prove the Riesz representation theorem, and prove our last principles for potentials : Domination and Harmonic minorant, concluding with a convexity property of the α-capacity. We note that with slight modifications one can carry over the whole discussion to the case of an arbitrary finite dimensional vector space over an arbitrary non-archimedean local field.

1. The local zeta function and homogeneous distributions.

We let ψ denote the "canonical" character of $\mathbf{Q}_p$, given by

$$\psi(x) = e^{-2\pi i\langle x\rangle} : \mathbf{Q}_p \twoheadrightarrow \mathbf{Q}_p/\mathbf{Z}_p \overset{\langle\,\rangle}{\hookrightarrow} \mathbf{Q}/\mathbf{Z} \overset{e^{-2\pi i x}}{\hookrightarrow} \mathbf{C}^*.$$

The self-dual Haar measure with respect to ψ is given by $dx(\mathbf{Z}_p) = 1$. We let $\mathcal{S} = \{\varphi : \mathbf{Q}_p \to \mathbf{C},\ \varphi \text{ locally constant and compactly supported}\}$, and its dual $\mathcal{S}^*$ = the space of temperate distributions.

$\mathbf{Q}_p^*$ acts on $\mathbf{Q}_p$, hence on $\mathcal{S}$ via $\pi_a\varphi(x) = \varphi(ax)$ and hence on $\mathcal{S}^*$ via $\pi_a^*\mu(\varphi) = \mu(\pi_a\varphi)$, e.g.$\pi_a^*\delta = \delta$, $\pi_a^*\ dx = |a|^{-1}\ dx$.

We let $\mathcal{S}^*[\alpha] = \{\mu \in \mathcal{S}^* \mid \pi_a^*\mu = |a|^{-\alpha}\mu,\ \forall a \in \mathbf{Q}_p^*\}$, the homogeneous distributions of order $\alpha \in \mathbf{C}/\frac{2\pi i}{\log p}\mathbf{Z}$, e.g. $\delta \in \mathcal{S}_p^*[0]$, $dx \in \mathcal{S}_p^*[1]$.

We normalize the Haar measure d^*x on $\mathbf{Q}_p^*$ by $d^*x(\mathbf{Z}_p^*) = 1$, so that $d^*x = (1-p^{-1})^{-1}\frac{dx}{|x|}$. By uniqueness of Haar measure, if $\mu \in \mathcal{S}^*[\alpha]$ then $\mu_{|\mathbf{Q}_p^*} = \text{cons}\cdot|x|^\alpha\ d^*x$. Thus, if $\mu_1, \mu_2 \in \mathcal{S}^*[\alpha]$, there exist constants $c_1, c_2 \in \mathbf{C}$ such that $c_1\mu_1 + c_2\mu_2$ has support $\{0\}$. The only distribution on $\mathbf{Q}_p$ with support $\{0\}$ is δ. For $\operatorname{Re}\alpha > 0$ we have $|x|^{\alpha-1} \in L^1_{\text{loc}}$ hence $\mathcal{M}^\alpha \overset{\text{def}}{=} |x|^\alpha\ d^*x \in \mathcal{S}^*[\alpha]$, and by the above :

$$\mathcal{S}^*[\alpha] = \mathbf{C}\cdot\mathcal{M}^\alpha\ ,\quad \operatorname{Re}\alpha > 0.$$

The "additive vacuum" is ϕ = characteristic function of $\mathbf{Z}_p$; $\phi \in \mathcal{S}$, and we have : $\mathcal{M}^\alpha(\phi) = (1-p^{-\alpha})^{-1} \overset{\text{def}}{=} \zeta(\alpha)$, $\operatorname{Re}\alpha > 0$.

We denote the Fourier transform by $\mathcal{F}\varphi(y) = \int \varphi(x)\bar{\psi}(xy)\ dx$ so that $\mathcal{F}\mathcal{F} = \pi_{-1}$, $\mathcal{F}\pi_a = |a|^{-1}\pi_{a^{-1}}\mathcal{F}$, $\mathcal{F}\phi = \phi$. On distributions we have $\mathcal{F}^*\mu(\varphi) = \mu(\mathcal{F}\varphi)$, $\mathcal{F}^*\mathcal{F}^* = \pi_{-1}^*$, $\pi_a^*\mathcal{F}^* = |a|^{-1}\mathcal{F}^*\pi_{a^{-1}}^*$; in particular, $\mathcal{F}^* : \mathcal{S}^*[\alpha] \xrightarrow{\sim} \mathcal{S}^*[1-\alpha]$, and hence

$$\mathcal{S}^*[\alpha] = \mathbf{C}\cdot\mathcal{F}^*\mathcal{M}^{1-\alpha},\quad \operatorname{Re}\alpha < 1.$$

We let $\widetilde{\mathcal{M}}^\alpha = (1-\pi_{p^{-1}}^*)\mathcal{M}^\alpha = (1-p^{-\alpha})\mathcal{M}^\alpha = \frac{1}{\zeta(a)}\mathcal{M}^\alpha$. For $\varphi \in \mathcal{S}$, $\widetilde{\mathcal{M}}^\alpha(\varphi) = \int(\varphi(x)-\varphi(p^{-1}x))|x|^\alpha\ d^*x$, where the integral converges for all α, since $\varphi(x)-\varphi(p^{-1}x) = 0$ for x near 0 (φ locally constant!). Hence, $\mathcal{M}^\alpha = \zeta(\alpha)\cdot\widetilde{\mathcal{M}}^\alpha$ has meromorphic continuation to all $\alpha \in \mathbf{C}/\frac{2\pi i}{\log p}\mathbf{Z}$, with a unique simple pole at $\alpha = 0$. On the other hand, $\widetilde{\mathcal{M}}^\alpha$ is entire, and since $\mathcal{F}^*\mathcal{M}^\alpha = \text{cons}\cdot\widetilde{\mathcal{M}}^{1-\alpha}$, $\widetilde{\mathcal{M}}^\alpha(\phi) = 1$, $\mathcal{F}\phi = \phi$, we get $\mathcal{F}^*\widetilde{\mathcal{M}}^\alpha = \widetilde{\mathcal{M}}^{1-\alpha}$ and hence $\mathcal{F}^*\mathcal{M}^\alpha = \frac{\zeta(\alpha)}{\zeta(1-\alpha)}\mathcal{M}^{1-\alpha}$. An easy check gives $\widetilde{\mathcal{M}}^0 = \delta$, and hence $\operatorname*{Res}_{\alpha=0}\mathcal{M}^\alpha = \frac{1}{\log p}\cdot\delta$.

2. Probabilistic interpretation.

Consider the "Riesz kernel" : $k^{\alpha}(x) = \dfrac{\zeta(1-\alpha)}{\zeta(\alpha)}|x|^{\alpha-1}$. For $\alpha \in (0,\infty)$, $\mathcal{L}^{\alpha}(x) = -k^{-\alpha}(x)dx = \dfrac{p^{\alpha}-1}{1-p^{-(1+\alpha)}}\dfrac{dx}{|x|^{1+\alpha}}$ is a positive measure on $\mathbf{Q}_p \setminus \{0\}$.

2.1. LEMMA.

(i) $$\int_{|x|\le p^k} \psi_p(x)\, dx = \begin{cases} p^k, & k\le 0 \\ 0, & k>0 \end{cases}$$

(ii) $$\int_{|x|=p^k} \psi_p(x)\, dx = \begin{cases} p^k(1-p^{-1}), & k\le 0 \\ -1, & k=1 \\ 0, & k>1 \end{cases}$$

(iii) $$\int_{|x|\le p^k} (1-\psi_p(xy))\, \frac{dx}{|x|^{1+\alpha}} = \begin{cases} 0, & |y|\le p^{-k} \\ p^{-\alpha}|y|^{\alpha} = p^{-\alpha k}, & |y| = p^{1-k} \\ p^{-k\alpha}(1-p^{-1}), & |y| > p^{1-k}. \end{cases}$$

Proof. — Straightforward using (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

Using this lemma we get : $|y|^{\alpha} = \displaystyle\int_{\mathbf{Q}_p\setminus\{0\}} (1-\operatorname{Re}\psi(xy))d\mathcal{L}^{\alpha}(x)$, the "Levy representation" (cf. [1], Cor. 18.20, p. 184).

2.2. COROLLARY. — *For $\alpha \in (0,\infty)$, $|y|^{\alpha}$ is a negative definite function with associated Levy measure $\mathcal{L}^{\alpha}(x)$.*

Hence we get a convolution semi-group of probability measures μ_t^{α}, (the p-adic analogue of) the "symmetric stable semi-group of order α", given by :

$$\mu_t^{\alpha} = \mathcal{F}e^{-t|y|^{\alpha}}, \quad t\in[0,\infty), \quad \alpha\in(0,\infty),$$

i.e. μ_t^{α} is a probability measure on $\mathbf{Q}_p$, $\mu_{t_1}^{\alpha} * \mu_{t_2}^{\alpha} = \mu_{t_1+t_2}^{\alpha}$, $\mu_t^{\alpha} \underset{t\to 0}{\rightsquigarrow} \delta$ (where "$\rightsquigarrow$" denotes vague convergence : $\mu_t \rightsquigarrow \mu$ iff $\mu_t(\varphi) \to \mu(\varphi)$ for all $\varphi \in C_c(\mathbf{Q}_p)$).

Explicitly, we have

$$\begin{aligned}
\mu_t^\alpha &= \int e^{-t|y|^\alpha}\psi(xy)\,dy \\
&= \sum_{m\in\mathbf{Z}} e^{-tp^{-m\alpha}} \int_{|y|=p^{-m}} \psi(xy)\,dy \qquad \text{(and using 2.1)} \\
&= (1-p^{-1})\sum_{m\geq 0} e^{-t|p^{-m}x|^{-\alpha}}|p^{-m}x|^{-1} - e^{-t|px|^{-\alpha}}|x|^{-1} \\
&= \sum_{k\geq 0}\frac{(-t)^k}{k!}|x|^{-k\alpha-1}\Big[(1-p^{-1})\sum_{m\geq 0}p^{-(k\alpha+1)m} - p^{k\alpha}\Big] \\
&= \sum_{n\geq 1}\frac{(-1)^n}{n!}k^{-n\alpha}(x),
\end{aligned}$$

i.e. we have the following "*Feller's formula*" (cf. [4] XVII, 6, Lemma 1 (6.8) p. 549) :

$$\mu_t^\alpha = \sum_{n\geq 1}\frac{(-t)^n}{n!}k^{-n\alpha}(x). \tag{2.3}$$

For $\alpha \in (0,1)$, $|x|^{-\alpha} \in L^1_{\text{loc}}$, hence μ_t^α is integrable, i.e. $\displaystyle\int_0^\infty \mu_t\,dt$ defines a potential kernel which is nothing but $k^\alpha(x)$. Thus,

$$k^\alpha(x) = \mathcal{F}(|x|^{-\alpha}\,dx) = \int_0^\infty \mu_t^\alpha(x)\,dt \tag{2.4}$$

(cf. [1] Theorem 13.23, p. 108).

Note that $\mu_t^\alpha(x)$ is "symmetric", i.e. it is invariant under $\mathbf{Z}_p^*$-action. The support of μ_t^α is all of $\mathbf{Q}_p$, i.e. $\mu_t^\alpha(x) > 0$ for all $x \in \mathbf{Q}_p$. We have $\mu_t^\alpha(\phi) = (1-p^{-1})\sum_{m\geq 0} p^{-m}e^{-tp^{-m\alpha}}$.

As $\alpha \to \infty$, $e^{-t|x|^\alpha} \to e^{-t}\phi(x) + (1-e^{-t})\phi\Big(\dfrac{1}{p}x\Big)$, pointwise, hence, upon taking Fourier transform, we get $\mu_t^\alpha \rightsquigarrow \mu_t^\infty(x) = e^{-t}\phi(x) + (1-e^{-t})p^{-1}\phi(px)$, vaguely. μ_t^∞ is a convolution semi-group of probability measures on $\mathbf{Q}_p/\mathbf{Z}_p$, (the p-adic analogue of) "the normal distribution", i.e. μ_t^∞ is a probability measure on $\mathbf{Q}_p/\mathbf{Z}_p$, $\mu_{t_1}^\infty * \mu_{t_2}^\infty = \mu_{t_1+t_2}^\infty$, and $\mu_t^\infty \underset{t\to 0}{\rightsquigarrow} \delta$ on $\mathbf{Q}_p/\mathbf{Z}_p$.

Its infinitesimal generator, (the p-adic analogue of) "the Laplacian", is

$$\frac{\partial}{\partial t}\Big|_0 \mu_t^\infty(x) = p^{-1}\phi(px) - \phi(x) = -\Delta(x).$$

Letting $\phi^*(x)$ denote the characteristic function of $\mathbf{Z}_p^*$, we can rewrite the above in the following form, remembering that $\psi(x)$ induces duality between $\mathbf{Z}_p$ and $\mathbf{Q}_p/\mathbf{Z}_p$: the Levy measure is $\mathcal{L}^\infty = \frac{1}{p} \sum_{a \bmod p} \delta_{a/p}$ on $\mathbf{Q}_p/\mathbf{Z}_p$; the associated negative definite function on $\mathbf{Z}_p$ is

$$\phi^*(x) = \int_{\mathbf{Q}_p/\mathbf{Z}_p} (1-\psi_p(xy))d\mathcal{L}^\infty(y);$$

and the associated "normal law" is given by $\mu_t^\infty = \mathcal{F}e^{-t\phi^*(x)}$, while its infinitesimal generator is $\Delta = \mathcal{F}\phi^*$.

3. The Riesz kernel.

We call the measure

$$\text{(3.1)} \qquad k^\alpha = \frac{\zeta(1-\alpha)}{\zeta(\alpha)}|x|^{\alpha-1}dx, \quad \alpha > 0, \quad \alpha \neq 1, \quad \text{the M. Riesz kernel.}$$

As a distribution, it has a meromorphic continuation to all $\alpha \not\equiv 1$ $\left(\bmod \frac{2\pi i}{\log p}\mathbf{Z}\right)$, given by

$$\text{(3.2)} \qquad k^\alpha(\varphi) = \frac{\zeta(1-\alpha)}{\zeta(1)}\varphi(0) + \frac{\zeta(1-\alpha)}{\zeta(\alpha)}\Big[\int_{|x|>1} \varphi(x)\frac{dx}{|x|^{1-\alpha}} + \int_{|x|\leq 1} (\varphi(x)-\varphi(0))\frac{dx}{|x|^{1-\alpha}}\Big].$$

In particular, for $\mathrm{Re}(\alpha) > 0$:

$$\text{(3.3)} \qquad \begin{aligned} k^\alpha(\varphi) &= \frac{\zeta(1-\alpha)}{\zeta(\alpha)}\int \varphi(x)|x|^{\alpha-1}\,dx, \quad \alpha \not\equiv 1 \left(\bmod \frac{2\pi i}{\log p}\mathbf{Z}\right) \\ k^{-\alpha}(\varphi) &= \frac{\zeta(1+\alpha)}{\zeta(-\alpha)}\int (\varphi(x)-\varphi(0))\frac{dx}{|x|^{1+\alpha}}. \end{aligned}$$

For $\alpha = 0$, $k^0(\varphi) = \varphi(0)$, i.e. $k^0 = \delta_0$.

For $\mathrm{Re}\,\alpha < 1$, $\mathcal{F}^*k^\alpha = |x|^{-\alpha}\,dx$; in particular, for $\alpha \in (0,1)$, both k^α and $\mathcal{F}k^\alpha$ are measures. Consider next k^α as an operator via convolution, namely for φ locally constant and such that $\int_{|x|>1} |\varphi(x)|\frac{dx}{|x|^{1-\mathrm{Re}\,\alpha}} < \infty$, we can form the convolution $k^\alpha * \varphi$:

$$\text{(3.4)} \qquad k^\alpha * \varphi(x_0) = \frac{\zeta(1-\alpha)}{\zeta(1)}\varphi(x_0) + \frac{\zeta(1-\alpha)}{\zeta(\alpha)}\Big[\int_{|x|>1} \varphi(x_0+x)\frac{dx}{|x|^{1-\alpha}} + \int_{|x|\leq 1} (\varphi(x_0+x)-\varphi(x_0))\frac{dx}{|x|^{1-\alpha}}\Big]$$

and again, for $\mathrm{Re}(\alpha) > 0$,

$$
(3.5)\qquad
\begin{aligned}
k^{\alpha} * \varphi(x_0) &= \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \int \varphi(x_0 + x)|x|^{\alpha-1}\, dx,\\
k^{-\alpha} * \varphi(x_0) &= \frac{\zeta(1+\alpha)}{\zeta(-\alpha)} \int (\varphi(x_0 + x)-\varphi(x_0))\frac{dx}{|x|^{1+\alpha}}.
\end{aligned}
$$

E.g., generalizing Levy's representation, $k^{-\alpha} * \psi_y(x) = |y|^{\alpha} \cdot \psi_y(x)$, i.e. $\psi_y(x) = \psi(y \cdot x)$ is a $k^{-\alpha}$-eigenvector with eigenvalue $|y|^{\alpha}$. For $\varphi \in \mathcal{S}$, $k^{\alpha} * \varphi$ is again locally constant (with the same modulus as that of φ), and moreover, $k^{\alpha} * \varphi(x) = O(|x|^{\mathrm{Re}\,\alpha-1})$, hence we can form $k^{\beta} * (k^{\alpha} * \varphi)$ for $\mathrm{Re}(\alpha+\beta) < 1$. For α, β such that $0 < \alpha, \beta$, $\alpha+\beta < 1$, the equation $k^{\alpha} * k^{\beta} = k^{\alpha+\beta}$ is immediate upon taking Fourier transforms, and since everything is holomorphic in α, β we get

3.6. M. Riesz reproduction formula. — *For $\varphi \in \mathcal{S}$, and $\alpha, \beta \not\equiv 1 \bmod \frac{2\pi i}{\log p}\mathbf{Z}$ with $\mathrm{Re}(\alpha+\beta) < 1$,*

$$k^{\alpha} * (k^{\beta} * \varphi) = k^{\alpha+\beta} * \varphi.$$

3.7. Corollary. — *For $\varphi \in \mathcal{S}$ and $\alpha \not\equiv \pm 1 \bmod \frac{2\pi i}{\log p}\mathbf{Z}$, $k^{-\alpha} * \varphi(y) = O(|y|^{-(1+\mathrm{Re}\,\alpha)})$ as $|y| \to \infty$, and we have : $k^{\alpha} * (k^{-\alpha} * \varphi) = \varphi$.*

4. Basic principles for potentials.

We let :

$$
\begin{aligned}
\mathcal{M}^{+} &= \{\text{positive measures on } \mathbf{Q}_p\},\\
\mathcal{M} &= \{\text{signed measures on } \mathbf{Q}_p\},\\
\mathcal{M}_c^{+}, \mathcal{M}_c &= \{\text{(positive) measures with compact support on } \mathbf{Q}_p\},\\
\mathcal{M}^{+}(K), \mathcal{M}(K) &= \{\text{(positive) measures with compact support on } K\}.
\end{aligned}
$$

We shall consider next k^{α} as an operator on $\mathcal{M}$ via convolution, restricting ourselves to the case $\alpha \in (0,1)$.

For $\nu \in \mathcal{M}_c^{+}$, we have

$$
\begin{aligned}
k^{\alpha} * \nu(x) &= \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \int |x-y|^{\alpha-1} d\nu(y)\\
&= \lim_{N\to\infty} \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \int_{|y|\geq p^{-N}} |x-y|^{\alpha-1} d\nu(y)
\end{aligned}
$$

an increasing limit of continuous functions, hence is a lower-semicontinuous function of x, i.e. $k^\alpha * \nu(x_0) \leq \underline{\lim}_{x\to x_0} k^\alpha * \nu(x)$; moreover $k^\alpha * \nu \in L^1_{\mathrm{loc}}$ as follows from Fubini's theorem :

$$\int_{|x|\leq p^N} k^\alpha * \nu(x)\, dx = \int_{\mathrm{sup}(\nu)} d\nu(y) \int_{|x|\leq p^N} k^\alpha(x-y)\, dx < \infty.$$

For $0 \neq \nu \in \mathcal{M}^+$, $k^\alpha * \nu(x) = \dfrac{\zeta(1-\alpha)}{\zeta(\alpha)} \displaystyle\int |x-y|^{\alpha-1}\, d\nu(y)$ is a well-defined positive function of x, although it may assume the value ∞, $0 < k^\alpha * \nu(x) \leq \infty$.

4.1. DEFINITION. — *For $\nu \in \mathcal{M}^+$, we write $\nu \in \mathcal{M}^+_\alpha$ if it satisfies one of the following equivalent conditions :*

(i) $\{x \mid k^\alpha * \nu(x) = \infty\}$ *has measure* 0 *w.r.t.* dx.

(ii) $\displaystyle\int_{|y|>p^N} |y|^{\alpha-1}\, d\nu(y) < \infty$ *for some* N.

(iii) $\displaystyle\int_{|y|>p^N} |y|^{\alpha-1}\, d\nu(y) < \infty$ *for all* N.

Indeed, if (iii) is not satisfied, then for $|x| \leq p^N$

$$\frac{\zeta(\alpha)}{\zeta(1-\alpha)} \cdot k^\alpha * \nu(x) = \int |x-y|^{\alpha-1}\, d\nu(y) \geq \int_{|y|>|x|} |y|^{\alpha-1}\, d\nu(y) = \infty,$$

and $k^\alpha * \nu(x) \equiv \infty$, showing (i) $\Rightarrow$ (iii); while since $\displaystyle\int_{|x|<p^N} |x-y|^{\alpha-1}\, dx = O(|y|^{\alpha-1})$ as $|y| \to \infty$, we have assuming (ii),

$$\begin{aligned}\int_{|x|<p^N} k^\alpha * \nu(x)\, dx &= \int d\nu(y) \int_{|x|<p^N} k^\alpha(x-y)\, dx \\ &= O\Big(\int_{|y|>p^N} |y|^{\alpha-1}\, d\nu(y)\Big) < \infty,\end{aligned}$$

showing (ii) $\Rightarrow$ (i).

We say $\nu \in \mathcal{M}_\alpha$ if $\nu = \nu^+ - \nu^-$ with $\nu^+, \nu^- \in \mathcal{M}^+_\alpha$. We have $\mathcal{M}^+_c \subseteq \mathcal{M}^+_\alpha$, $\mathcal{M}_c \subseteq \mathcal{M}_\alpha$.

We denote by $B(p^N) = \{x \mid |x| \leq p^N\}$, $\check{B}_N = \{x \mid |x| \geq p^{-N}\}$, the ball of radius p^N and its exterior.

We note that the Harnack inequalities (cf. [8] IV, §5, n°20, p. 266) become in our non-archimedean setting actually equalities. For $\nu \in$

$\mathcal{M}(B(p^N))$ and any $y \notin B(p^N)$ we have $k^\alpha * \nu(y) = \dfrac{\zeta(1-\alpha)}{\zeta(\alpha)}\nu(1) \cdot |y|^{\alpha-1}$, and so for $y_1, y_2 \notin B(p^N)$: $k^\alpha * \nu(y_1) = |y_1/y_2|^{\alpha-1} \cdot k^\alpha * \nu(y_2)$. Similarly, for $\nu \in \mathcal{M}(\check{B}_N)$ and any $y \notin \check{B}_N$, $k^\alpha * \nu(y) = \dfrac{\zeta(1-\alpha)}{\zeta(\alpha)} \displaystyle\int |x|^{\alpha-1}\, d\nu(x)$, and so for $y_1, y_2 \notin \check{B}_N : k^\alpha * \nu(y_1) = k^\alpha * \nu(y_2)$. For a general $\nu \in \mathcal{M}_\alpha$, we have that $k^\alpha * \nu$ is locally constant away from $\operatorname{supp} \nu$, and

$$\begin{aligned}\int_{|x|>1} |k^\alpha * \nu(x)| \frac{dx}{|x|^{1+\alpha}} &= \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \int d\nu(y) \int_{|x|>1} |x-y|^{\alpha-1}|x|^{-(1+\alpha)}\, dx \\ &\leq \text{cons} \cdot \int |y|^{\alpha-1}\, d\nu(y) < \infty.\end{aligned}$$

4.2. Principle of descent. — *Let $\nu_n \in \mathcal{M}_\alpha^+$, $\nu_n \rightsquigarrow \nu_\infty$, $x_n \to x_\infty$, then*

$$k^\alpha * \nu_\infty(x_\infty) \leq \underline{\lim}_{n\to\infty}\, k^\alpha * \nu_n(x_n).$$

4.3. Corollary. — *For $\nu \in \mathcal{M}_\alpha^+$, $k^\alpha * \nu$ is lower-semicontinuous.*

4.4. Corollary. — *For $\nu_n \in \mathcal{M}_\alpha^+$, $\nu_n \rightsquigarrow \nu$, $k^\alpha * \nu = \underline{\lim} k^\alpha * \nu_n$.*

4.5. Corollary. — *For $\nu_n \in \mathcal{M}_\alpha^+$, $\nu_n \rightsquigarrow \nu$, $\nu_n \leq \nu_{n+1}$, $k^\alpha * \nu = \lim k^\alpha * \nu_n$.*

Proof of 4.2. — Assume first $\operatorname{supp}(\nu_n) \subseteq \{x \mid |x| \leq p^N\}$ for all n. Let $k_\varepsilon^\alpha * \nu_n(x) = \displaystyle\int_{|x-y|>\varepsilon} k^\alpha(x-y)\, d\nu_n(y)$, it's uniformly continuous and $k_\varepsilon^\alpha * \nu(x) = \lim\limits_{n\to\infty} k_\varepsilon^\alpha * \nu_n(x)$, hence we have $k_\varepsilon^\alpha * \nu(x_\infty) = \lim\limits_{n\to\infty} k_\varepsilon^\alpha * \nu_n(x_n) \leq \underline{\lim} k^\alpha * \nu_n(x_n)$ and taking $\varepsilon \to 0$ we get the theorem in this special case. In the general case, let $\nu_n^{(N)}$, $\nu_\infty^{(N)}$ denote the restrictions of ν_n, ν_∞ to $\{x \mid |x| \leq p^N\}$. Then $\nu_n^{(N)} \underset{n\to\infty}{\rightsquigarrow} \nu_\infty^{(N)}$, and by the above $k^\alpha * \nu_\infty^{(N)}(x_\infty) \leq \underline{\lim}_{n\to\infty}\, k^\alpha * \nu_n^{(N)}(x_n)$. Taking the limit $N \to \infty$, first on the right-hand side and then on the left we conclude the theorem.

4.6. Dichotomy principle. — *Let $\nu_n \in \mathcal{M}^+$ such that $k^\alpha * \nu_n \leq k^\alpha * \nu_{n+1}$. Then either $\lim\limits_{n\to\infty} k^\alpha * \nu_n(x) \equiv \infty$ for all x, or $\lim\limits_{n\to\infty} k^\alpha * \nu_n(x) = k^\alpha * \nu(x) + c$, for some $\nu \in \mathcal{M}_\alpha^+$, $c \geq 0$, and almost all x.*

Proof. — Suppose $\lim\limits_{n\to\infty} k^\alpha * \nu_n(x) \not\equiv \infty$.

(i) For any compact K, $\nu_n(K)$ is bounded. For if $\nu_{n_i}(K) \underset{i\to\infty}{\longrightarrow} \infty$ then $\lim\limits_{i\to\infty} k^\alpha * \nu_{n_i}(x) \geq \overline{\lim\limits_{i\to\infty}} \int_K |x-y|^{\alpha-1}\, d\nu_{n_i}(y) \equiv \infty$.

(ii) For any N, $\int_{|y|>p^N} |y|^{\alpha-1}\, d\nu_n(y)$ is bounded. For if $\int_{|y|>p^N} |y|^{\alpha-1}$ $d\nu_{n_i}(y) \underset{i\to\infty}{\longrightarrow} \infty$ then

$$\lim_{i\to\infty} k^\alpha * \nu_{n_i}(x) \geq \overline{\lim_{i\to\infty}} \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \int_{|y|>|x|} |y|^{\alpha-1}\, d\nu_{n_i}(y) \equiv \infty\ .$$

From (i), (ii) we get $\nu_n \rightsquigarrow \nu$, for some $\nu \in \mathcal{M}^+$.

Let $F(x) = \lim\limits_{n\to\infty} k^\alpha * \nu_n(x)$. $F \in L^1_{\text{loc}}$ since

$$\begin{aligned}\int_{|x|<p^N} F(x)\, dx &\leq \lim_{n\to\infty} \int_{|x|<p^N} k^\alpha * \nu_n(x)\, dx \\ &\leq \lim_{n\to\infty} \int d\nu_n(y) \int_{|x|<p^N} k^\alpha(x-y)\, dx < \infty.\end{aligned}$$

Let $\varphi \in \mathcal{S}$, $\int \varphi(x)\, dx = 0$, then $|x|^{-\alpha}\mathcal{F}\varphi(x) \in \mathcal{S}$, and $k^\alpha * \varphi = \mathcal{F}^{-1}|x|\mathcal{F}\varphi \in \mathcal{S}$. We have :

$$\begin{aligned}\int F(x)\varphi(x)\, dx &= \lim_{n\to\infty} \int k^\alpha * \nu_n(x)\varphi(x)\, dx \\ &= \lim_{n\to\infty} \int k^\alpha * \varphi(x)\, d\nu_n(x) \\ &= \int k^\alpha * \varphi(x)\, d\nu(x) = \int k^\alpha * \nu(x)\varphi(x)\, dx.\end{aligned}$$

Thus, for any $\varphi \in \mathcal{S}$ orthogonal to 1 : $\int \lim\limits_{n\to\infty} k^\alpha * \nu_n(x)\varphi(x)\, dx = \int k^\alpha * \nu(x)\varphi(x)\, dx$ and hence $\lim\limits_{n\to\infty} k^\alpha * \nu_n(x) - k^\alpha * \nu(x) = c$ almost everywhere, where c is a constant, by the principle of descent $c \geq 0$.

4.7. 1^{st} MAXIMUM PRINCIPLE. — *Let $\nu \in \mathcal{M}^+$ be such that $k^\alpha * \nu(x) \leq M$, ν-almost everywhere. Then $k^\alpha * \nu(x) \leq M$ for all x.*

Proof. — Assume first $x \in \operatorname{supp}(\nu)$, then the inequality follows by the lower semi-continuity of $k^\alpha * \nu$, hence $k^\alpha * \nu(x) \leq M$ for $x \in \operatorname{supp}(\nu)$. Let $x \notin \operatorname{supp}(\nu)$, and let $x' \in \operatorname{supp}(\nu)$ be such that $|x-x'|$ is minimal. Then for $y \in \operatorname{supp}(\nu)$, $|y-x'| \leq \max(|y-x|, |x-x'|) \leq |y-x|$, hence $|y-x'|^{\alpha-1} \geq |y-x|^{\alpha-1}$, and so $k^\alpha * \nu(x) \leq k^\alpha * \nu(x') \leq M$.

4.8. COROLLARY. — *If $k^\alpha * \nu$ is continuous on* $\mathrm{supp}(\nu)$, *then it's continuous everywhere.*

Proof. — On $\mathbf{Q}_p \setminus \mathrm{supp}(\nu)$, $k^\alpha * \nu$ is locally constant, so it's enough to check continuity at the points of $\mathrm{supp}(\nu)$. Let $x_0 \in \mathrm{supp}(\nu)$, ν_ε the restriction of ν to $\{x \mid |x-x_0| < \varepsilon\}$, $\nu'_\varepsilon = \nu-\nu_\varepsilon$, so that $k^\alpha * \nu(x) = k^\alpha * \nu_\varepsilon(x) + k^\alpha * \nu'_\varepsilon(x)$, hence $|k^\alpha * \nu(x) - k^\alpha * \nu(x_0)| \leq k^\alpha * \nu_\varepsilon(x) + k^\alpha * \nu_\varepsilon(x_0) + |k^\alpha * \nu'_\varepsilon(x) - k^\alpha * \nu'_\varepsilon(x_0)|$. Since $k^\alpha * \nu'_\varepsilon$ is continuous at x_0, it suffices to show that $k^\alpha * \nu_\varepsilon(x) \underset{\varepsilon\to 0}{\longrightarrow} 0$ uniformly in x. Let $\eta > 0$. Since $k^\alpha * \nu(x_0) < \infty$ we can find ε_1 such that $k^\alpha * \nu_{\varepsilon_1}(x_0) \leq \frac{1}{2}\eta$; and since $k^\alpha * \nu_{\varepsilon_1}$ is continuous on $\mathrm{supp}(\nu)$ we can find $\varepsilon < \varepsilon_1$ such that $|k^\alpha * \nu_{\varepsilon_1}(x) - k^\alpha * \nu_{\varepsilon_1}(x_0)| \leq \frac{1}{2}\eta$ for all $x \in \mathrm{supp}(\nu)$, $|x-x_0| < \varepsilon$; hence $k^\alpha * \nu_\varepsilon(x) \leq k^\alpha * \nu_{\varepsilon_1}(x) \leq \eta$ for $x \in \mathrm{supp}(\nu_\varepsilon)$. By the maximum principle, $k^\alpha * \nu_\varepsilon(x) \leq \eta$ everywhere.

Let $\phi^{(N)}(x) = p^N\phi(p^{-N}x)$ denote the uniform distribution on $\{x \mid |x| \leq p^{-N}\}$. For any measure ν, let $\nu^{(N)} = \nu * \phi^{(N)}$ its N^{th} regularization, and let $\nu^{((N))} = \nu^{(N)} \cdot \phi(p^N x)$ denote the restriction of $\nu^{(N)}$ to $\{x \mid |x| \leq p^N\}$.

4.9. REGULARIZATION PRINCIPLE. — *Let $\nu \in \mathcal{M}^+_\alpha$, then $k^\alpha * \nu^{(N)}$ is locally constant and $k^\alpha * \nu^{(N)}(x) \underset{N\to\infty}{\longrightarrow} k^\alpha * \nu(x)$, $k^\alpha * \nu^{((N))}(x) \underset{N\to\infty}{\longrightarrow} k^\alpha * \nu(x)$ for all x.*

Proof.

$$\begin{aligned} k^\alpha * \nu^{(\nu)}(x) &= \frac{\zeta(1-\alpha)}{\zeta(\alpha)} p^N \int_{|z-x|\leq p^{-N}} dz \int_{\mathrm{supp}(\nu)} |z-y|^{\alpha-1}\, d\nu(y) \\ &= \frac{\zeta(1-\alpha)}{\zeta(\alpha)} p^N \int_{\mathrm{supp}(\nu)} |x-y|^{\alpha-1} d\nu(y) \int_{|z-x|\leq p^{-N}} \frac{|x-y|^{1-\alpha}}{|z-y|^{1-\alpha}} dz \\ &< C \cdot k^\alpha * \nu(x) \end{aligned}$$

where $C = \sup_{N,y} \left\{ p^N \int_{|z-x|\leq p^{-N}} \frac{|x-y|^{1-\alpha}}{|z-y|^{1-\alpha}}\, dz \right\} < \infty.$

If $k^\alpha * \nu(x) = \infty$ then $k^\alpha * \nu^{(N)}(x)$, $k^\alpha * \nu^{((N))}(x) \underset{N\to\infty}{\longrightarrow} \infty$ by lower-semicontinuity. So assume $k^\alpha * \nu(x) < \infty$. Given $\eta > 0$, we can find $\varepsilon > 0$ such that writing $\nu = \nu_\varepsilon + \nu'_\varepsilon$, where ν_ε is the restriction of ν to $\{z \mid |z-x| < \varepsilon\}$, we have $k^\alpha * \nu_\varepsilon(x) < \eta$. Hence by the above, $k^\alpha * (\nu_\varepsilon)^{(N)} < C \cdot \eta$. Now

$k^\alpha * \nu'_\varepsilon$ is continuous at x, so $k^\alpha * (\nu'_\varepsilon)^{(N)}(x) \underset{N\to\infty}{\longrightarrow} k^\alpha * \nu'_\varepsilon(x)$, and we get :

$$k^\alpha * \nu(x) - (C+1)\eta \le \varliminf_{N\to\infty} k^\alpha * \nu^{(N)}(x) \le \varlimsup_{N\to\infty} k^\alpha * \nu^{(N)}(x)$$
$$\le k^\alpha * \nu(x) + (C+1)\eta.$$

Letting $\eta \to 0$ we obtain the theorem for $\nu^{(N)}$.

Next, considering $\nu^{((N))}$, we have :

$$k^\alpha * \nu^{(N)}(x) - k^\alpha * \nu^{((N))}(x) = \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \int_{|y|>p^N} |x-y|^{\alpha-1}\, d\nu^{(N)}(y)$$
$$\le \frac{\zeta(1-\alpha)}{\zeta(\alpha)} p^N \int_{|z|\le p^N} dz \int_{|y|>p^N} |x-z-y|^{\alpha-1}\, d\nu(y).$$

But since $\nu \in \mathcal{M}^+_\alpha$, we know that $\lim\limits_{N\to\infty} \displaystyle\int_{|y|>p^N} |x-z-y|^{\alpha-1}\, d\nu(y) = 0$ uniformly with respect to bounded z's, which gives the theorem for $\nu^{((N))}$.

4.10. Uniqueness principle. — *Let $\nu \in \mathcal{M}_\alpha$, if $k^\alpha * \nu(x) = 0$ for a.a.x's, then $\nu = 0$ (and $k^\alpha * \nu \equiv 0$).*

Proof. — Let $\varphi \in \mathcal{S}$, $\check{\varphi}(x) = \varphi(-x)$, and apply Riesz formula to write $\check{\varphi} = k^\alpha * (k^{-\alpha} * \check{\varphi})$. Since $|k^{-\alpha} * \check{\varphi}|(x) = O(|x|^{-(1+\alpha)})$ as $|x| \to \infty$, we easily obtain $k^\alpha * |k^{-\alpha} * \check{\varphi}|(x) = O(|x|^{\alpha-1})$, and since by assumption, $\displaystyle\int_{|y|>p^N} |y|^{\alpha-1} d|\nu|(y) < \infty$, we see that the convolution $|\nu| * (k^\alpha * |k^{-\alpha} * \check{\varphi}|)$ is defined, and hence $\nu * (k^\alpha * (k^{-\alpha} * \check{\varphi})) = (k^\alpha * \nu) * (k^{-\alpha} * \check{\varphi})$. Thus, $\nu(\varphi) = \nu * \check{\varphi}(0) = \nu * (k^\alpha * (k^{-\alpha} * \check{\varphi}))(0) = (k^\alpha * \nu) * (k^{-\alpha} * \check{\varphi})(0) = 0$ since $k^\alpha * \nu(x) = 0$ for a.a.x's. Since $\varphi \in \mathcal{S}$ was arbitrary, we get $\nu \equiv 0$.

5. Measures with finite energy, $0 < \alpha < 1$.

We shall consider next signed measures $\nu = \nu^+ - \nu^- \in \mathcal{M}_\alpha$ such that

$$\iint k^\alpha(x-y)\, d\nu^+(x)\, d\nu^-(y) < \infty.$$

The "mutual α-energy" of two such measures ν_1 and ν_2 is

$$\langle \nu_1, \nu_2 \rangle_\alpha = \iint k^\alpha(x-y)\, d\nu_1(x)\, d\nu_2(y)$$

which is well defined under the condition that $\displaystyle\iint k^\alpha(x-y) d\nu_1^\pm(x) d\nu_2^\pm(y) < \infty$. Since $k^\alpha(x-y) = k^\alpha(y-x)$, we get the

5.1. SYMMETRY PRINCIPLE. — $\langle \nu_1, \nu_2 \rangle_\alpha = \langle \nu_2, \nu_1 \rangle_\alpha$.

5.2. POSITIVE DEFINITENESS. — $\langle \nu, \nu \rangle_\alpha \geq 0$ *and* $\langle \nu, \nu \rangle_\alpha = 0 \Longleftrightarrow \nu = 0$.

Proof. — Write $\nu = \nu^+ - \nu^-$, $|\nu| = \nu^+ + \nu^-$; we are assuming $\langle \nu^+, \nu^- \rangle_\alpha < \infty$, and we may assume $\langle \nu, \nu \rangle_\alpha < \infty$, which gives us $\langle |\nu|, |\nu| \rangle_\alpha < \infty$. Hence we can apply Fubini's theorem and the M. Riesz Formula to obtain :

$$\begin{aligned}
\langle \nu, \nu \rangle_\alpha &= \iint d\nu(x)\, d\nu(y) k^\alpha(x-y) \\
&= \iint d\nu(x)\, d\nu(y) \int k^{\alpha/2}(x-y-z) k^{\alpha/2}(z)\, dz \\
&= \iint d\nu(x)\, d\nu(y) \int k^{\alpha/2}(x-z) k^{\alpha/2}(z-y)\, dz \\
&= \int dz \int k^{\alpha/2}(x-z)\, d\nu(x) \int k^{\alpha/2}(z-y)\, d\nu(y) \\
&= \int |k^{\alpha/2} * \nu(z)|^2\, dz \geq 0.
\end{aligned}$$

I.e. $\langle \nu, \nu \rangle_\alpha = \|k^{\alpha/2} * \nu\|_{L^2}^2$.

Moreover, if $\langle \nu, \nu \rangle_\alpha = 0$ then $k^{\alpha/2} * \nu = 0$ for *a.a.x*'s, and by the 1^{st} uniqueness principle $\nu = 0$.

As a corollary we get,

5.3. 2^{nd} UNIQUENESS PRINCIPLE. — *Let* $\nu \in \mathcal{M}_\alpha$ *satisfy* $\langle \nu^+, \nu^- \rangle_\alpha < \infty$. *If* $k^\alpha * \nu(x) = 0$ *for* $\nu-$*a.a.x's, then* $\nu = 0$.

We can now define the "α-energy" of ν, by $\|\nu\|_\alpha = \langle \nu, \nu \rangle_\alpha^{1/2}$. We let $\mathcal{E}_\alpha = \{\nu \in \mathcal{M}_\alpha$ such that $\langle \nu_+, \nu_- \rangle_\alpha < \infty$ and $\|\nu\|_\alpha < \infty\}$ denote the space of signed measures with finite α-energy, and $\mathcal{E}_\alpha^+ = \mathcal{E}_\alpha \cap \mathcal{M}^+$:

e.g. supp(ν) compact and $k^\alpha * \nu$ bounded on supp$(\nu) \Rightarrow \nu \in \mathcal{E}_\alpha$;

e.g. $k^\alpha * \nu$ bounded of compact support $\Rightarrow \nu \in \mathcal{E}_\alpha$.

$\mathcal{E}_\alpha$ with the inner product $\langle\ ,\ \rangle_\alpha$ is a pre-Hilbert-space. In particular we have Schwartz inequality : $|\langle \nu_1, \nu_2 \rangle_\alpha| \leq \|\nu_1\|_\alpha \cdot \|\nu_2\|_\alpha$. We write $\nu_n \Rightarrow_\alpha \nu$ for strong convergence (i.e. $\|\nu - \nu_n\|_\alpha \underset{n\to\infty}{\longrightarrow} 0$), and $\nu_n \to_\alpha \nu$ for weak convergence (i.e. $\langle \nu_n, \mu \rangle_\alpha \underset{n\to\infty}{\longrightarrow} \langle \nu, \mu \rangle_\alpha$, $\forall \mu \in \mathcal{E}_\alpha$). By the Schwartz inequality, $\nu_n \Rightarrow_\alpha \nu$ implies $\nu_n \to_\alpha \nu$.

We note also the "lower-semi-continuity" of $\langle\ ,\ \rangle_\alpha$, namely : if $\nu_n, \nu_n' \in \mathcal{E}_\alpha^+$, $\nu_n \rightsquigarrow \nu$, $\nu_n' \rightsquigarrow \nu'$ then $\langle \nu, \nu' \rangle_\alpha \leq \underline{\lim}_{N\to\infty} \langle \nu_n, \nu_n' \rangle_\alpha$. In particular, $\|\nu\|_\alpha \leq \underline{\lim}_{N\to\infty} \|\nu_n\|_\alpha$.

5.4. Regularization lemma. — *If $\nu \in \mathcal{E}_\alpha$ then $\nu^{(N)} \Rightarrow_\alpha \nu$ and $\nu^{((N))} \Rightarrow_\alpha \nu$.*

Proof. — It's enough to consider $\nu \in \mathcal{E}_\alpha^+$.

$$\|\nu - \nu^{(N)}\|_\alpha^2 = \int |k^{\alpha/2} * \nu(x) - k^{\alpha/2} * \nu^{(N)}(x)|^2\, dx.$$

But

$$\begin{aligned}|k^{\alpha/2} * \nu(x) - k^{\alpha/2} * \nu^{(N)}(x)|^2 &\leq 2|k^{\alpha/2} * \nu(x)|^2 + 2|k^{\alpha/2} * \nu^{(N)}(x)|^2 \\ &\leq C_\alpha \cdot |k^{\alpha/2} * \nu(x)|^2.\end{aligned}$$

Hence, $\lim_{N\to\infty} \|\nu - \nu^{(N)}\|_\alpha^2 = \int \lim_{N\to\infty} |k^{\alpha/2} * \nu(x) - k^{\alpha/2} * \nu^{(N)}(x)| dx = 0$ by the regularization principle; similarly for $\nu^{((N))}$.

5.5. Corollary. — *$\mathcal{S} \cap \mathcal{E}_\alpha$ is dense in $\mathcal{E}_\alpha$.*

Proof. — Indeed $\nu^{((N))} \in \mathcal{S}$.

5.6. Corollary. — *$\{\nu \in \mathcal{E}_\alpha$ such that $k^\alpha * \nu \in \mathcal{S}\}$ is dense in $\mathcal{E}_\alpha$.*

Proof. — By the above it's enough to show that for any $\nu \in \mathcal{S} \cap \mathcal{E}_\alpha$, $\exists \varphi_n \in \mathcal{E}_\alpha$ such that $k^\alpha * \varphi_n \in \mathcal{S}$, $\varphi_n \Rightarrow_\alpha \nu$. Let $f_n(x) = k^\alpha * \nu(x) \cdot \phi(p^n x)$ where $n \gg 0$ so that $\nu = \nu^{((n))}$. We have $f_n \in \mathcal{S}$ hence $f_n = k^\alpha * \varphi_n$, $\varphi_n = k^{-\alpha} * f_n$. Note that $k^\alpha * \varphi_n(x) = k^\alpha * \nu(x)$ for $|x| \leq p^n$, and $\|\varphi_n\|_\alpha^2 = \int_{|x|\leq p^n} f_n(x)\varphi_n(x)\, dx < \infty$, so $\varphi_n \in \mathcal{E}_\alpha$. Recall that $\varphi_n(x) = k^{-\alpha} * f_n(x)$ is given by

$$\frac{1-p^{-1}}{1-p^{-(1+\alpha)}} f_n(x) + \frac{1-p^\alpha}{1-p^{-(1+\alpha)}} \left[\int_{|y|>1} f_n(x+y) \frac{dy}{|y|^{1+\alpha}} + \int_{|y|\leq 1} (f_n(x+y) - f_n(x)) \frac{dy}{|y|^{1+\alpha}} \right]$$

which for $|x| > p^n$ reduces to

$$\begin{aligned}\frac{1-p^\alpha}{1-p^{-(1+\alpha)}} \int_{|y|>1} f_n(x+y) \frac{dy}{|y|^{1+\alpha}} &= \frac{1-p^\alpha}{1-p^{-(1+\alpha)}} \int_{|z|\leq p^n} f_n(z) \frac{dz}{|z-x|^{1+\alpha}} \\ &= \frac{1-p^\alpha}{1-p^{-(1+\alpha)}} |x|^{-(1+\alpha)} \int_{|z|\leq p^n} k^\alpha * \nu(x) dz.\end{aligned}$$

But for $|z| > p^{n_0}$, $k^\alpha * \nu(z) = \dfrac{1-p^{-\alpha}}{1-p^{\alpha-1}}\nu(1)|z|^{\alpha-1}$, hence we get for $n \geq n_0$

$$|\varphi_n(x)| = O(|x|^{-(1+\alpha)}p^{n\alpha}), \quad |x| > p^n.$$

Now,

$$\begin{aligned}
\|\varphi_n-\nu\|_\alpha^2 &= \int_{|x|>p^n} k^\alpha * (\varphi_n-\nu)(x)(\varphi_n-\nu)(x)\, dx \\
&\qquad\qquad \text{since } k^\alpha * (\varphi_n-\nu)(x) = 0, \quad |x| \leq p^n, \\
&= -\int_{|x|>p^n} k^\alpha * \nu(x)\varphi_n(x)\, dx \\
&\qquad\qquad \text{since } k^\alpha * \varphi_n(x) = \nu(x) = 0, \quad |x| > p^n, \\
&= O\Big(\int_{|x|>p^n} (|x|^{\alpha-1}\cdot|x|^{-(1+\alpha)}p^{n\alpha}\, dx\Big) = O(p^{n(\alpha-1)}) \underset{n\to\infty}{\longrightarrow} 0 \\
&\qquad\qquad \text{since } \alpha < 1.
\end{aligned}$$

So $\varphi_n \Rightarrow_\alpha \nu$ and the theorem is proved.

5.7. COROLLARY. — *Let $\nu_n \in \mathcal{E}_\alpha^+$, $\nu_n \to_\alpha \nu$, then $\nu_n \rightsquigarrow \nu$.*

Proof. — Let $f \in \mathcal{S}$ and write $f = k^\alpha * \varphi$, $\varphi = k^{-\alpha} * f$, then $\varphi \in \mathcal{E}_\alpha$ and so $\lim_{n\to\infty} \langle\varphi, \nu_n\rangle_\alpha = \langle\varphi,\nu\rangle_\alpha$, i.e. $\lim_{n\to\infty} \int f(x)\, d\nu_n(x) = \int f(x)\, d\nu(x)$, hence $\nu_n \rightsquigarrow \nu$.

5.8. COROLLARY. — *Let $\nu_n \in \mathcal{E}_\alpha^+$, $\nu_n \rightsquigarrow \nu$, $\|\nu_n\|_\alpha < C$. Then $\nu \in \mathcal{E}_\alpha^+$ and $\nu_n \to_\alpha \nu$.*

Proof. — By lower-semi-continuity $\|\nu\|_\alpha \leq \underline{\lim}_{n\to\infty} \|\nu_n\|_\alpha < C$, so $\nu \in \mathcal{E}_\alpha^+$.

Let $\mu \in \mathcal{E}_\alpha$, $k^\alpha * \mu \in \mathcal{S}$, then

$$\langle\nu_n,\mu\rangle_\alpha = \int k^\alpha * \mu(x) d\nu_n(x) \underset{n\to\infty}{\longrightarrow} \int k^\alpha * \mu(x) d\nu(x) = \langle\nu,\mu\rangle_\alpha.$$

Given any $\xi \in \mathcal{E}_\alpha$, and $\varepsilon > 0$, we can find by Corollary 5.6 such a μ with $\|\xi-\mu\|_\alpha < \varepsilon$, and then for $n > n_0(\varepsilon)$:

$$\begin{aligned}
|\langle\nu_n,\xi\rangle_\alpha - \langle\nu,\xi\rangle_\alpha| &\leq |\langle\nu_n,\mu\rangle_\alpha - \langle\nu,\mu\rangle_\alpha| + |\langle\nu_n,\xi-\mu\rangle_\alpha| + |\langle\nu,\xi-\mu\rangle_\alpha| \\
&\leq |\langle\nu_n,\mu\rangle_\alpha - \langle\nu,\mu\rangle_\alpha| + 2C\cdot\varepsilon < (1+2C)\cdot\varepsilon.
\end{aligned}$$

Thus $\langle\nu_n,\xi\rangle_\alpha \underset{n\to\infty}{\longrightarrow} \langle\nu,\xi\rangle_\alpha$ and $\nu_n \to_\alpha \nu$.

5.9. LEMMA. — *Let $\nu_n \in \mathcal{E}_\alpha^+$ be a Cauchy sequence such that $\nu_n \rightsquigarrow \nu$. Then $\nu \in \mathcal{E}_\alpha^+$ and $\nu_n \Rightarrow_\alpha \nu$.*

Proof. — There exists a bound $\|\nu_n\|_\alpha < C$, so by the above corollary, $\nu \in \mathcal{E}_\alpha^+$ and $\nu_n \rightarrow_\alpha \nu$. Hence we get

$$\begin{aligned}\|\nu-\nu_n\|_\alpha^2 = \langle \nu-\nu_n, \nu-\nu_n\rangle_\alpha &= \lim_{m\to\infty} \langle \nu-\nu_n, \nu_m-\nu_n\rangle_\alpha \\ &\leq \lim_{m\to\infty} \|\nu-\nu_n\|_\alpha \cdot \|\nu_m-\nu_n\|_\alpha,\end{aligned}$$

so $\|\nu-\nu_n\|_\alpha \leq \lim\limits_{m\to\infty} \|\nu_m-\nu_n\|_\alpha$ and

$$\lim_{n\to\infty} \|\nu-\nu_n\|_\alpha \leq \lim_{n\to\infty} \lim_{m\to\infty} \|\nu_m-\nu_n\|_\alpha = 0,$$

or $\nu_n \Rightarrow_\alpha \nu$.

5.10. Completeness of the positive cone. — *$\mathcal{E}_\alpha^+$ is a complete metric space.*

Proof. — Let $\nu_n \in \mathcal{E}_\alpha^+$ be a Cauchy sequence. By Lemma 5.9 it suffices to show $\nu_n \rightsquigarrow \nu$ for some ν. Given any $\mu \in \mathcal{E}_\alpha$, we have $|\langle \nu_n-\nu_m, \mu\rangle_\alpha| \leq \|\nu_n-\nu_m\|_\alpha \cdot \|\mu\|_\alpha$ hence the sequence $\langle \nu_n, \mu\rangle_\alpha$ converges. Since any $\varphi \in \mathcal{S}$ can be written as $\varphi = k^\alpha * \mu$ with $\mu \in \mathcal{E}_\alpha$, and $\displaystyle\int \varphi(x)d\nu_n(x) = \langle \nu_n, \mu\rangle_\alpha$ converges, we conclude the theorem.

6. Capacity and equilibrium measure.

$K \subseteq \mathbf{Q}_p$ will denote a compact set.

$\mathcal{M}^+(K)$, resp. $\mathcal{M}^{(1)}(K)$, will be the positive, resp. probability, measures supported in K.

$\mathcal{E}_\alpha^+(K) = \mathcal{M}^+(K) \cap \mathcal{E}_\alpha$, $\mathcal{E}_\alpha^{(1)}(K) = \mathcal{M}^{(1)}(K) \cap \mathcal{E}_\alpha$.

6.1. Lemma. — *$\mathcal{M}^{(1)}(K)$ is convex and vaguely compact.*

Proof. — That $\mathcal{M}^{(1)}(K)$ is convex is immediate; we prove compactness. Let $\mathcal{S}_\mathbf{Q}$ denote the set of locally constant compactly supported functions $\varphi : \mathbf{Q}_p \to \mathbf{Q}$. $\mathcal{S}_\mathbf{Q}$ is denumerable; write $\mathcal{S}_\mathbf{Q} = \{\varphi_m\}_{m=1}^\infty$. Let $\mu_n \in \mathcal{M}^{(1)}(K)$ be an arbitrary sequence. Since $|\mu_n(\varphi)| \leq \sup \varphi$, we can find a subsequence $\mu_{n_i^{(1)}}$ such that $\{\mu_{n_i^{(1)}}(\varphi_1)\}$ converges. Proceeding by induction we find a subsequence $\mu_{n_i^{(m)}}$ such that $\{\mu_{n_i^{(m)}}(\varphi_1)\}, \ldots, \{\mu_{n_i^{(m)}}(\varphi_m)\}$ all converges. Looking at the diagonal, we have a subsequence $\{\mu_{n_m^{(m)}}\} = \{\mu_{n_j}\}$ of our original sequence $\{\mu_n\}$ such that $\{\mu_{n_j}(\varphi_m)\}$ converges for all m's. Define $\mu(\varphi_m) = \lim \mu_{n_j}(\varphi_m)$, and extend μ by linearity to all of $\mathcal{S} = \mathcal{S}_{\mathbf{Q}} \otimes \mathbf{C}$, to give a positive distribution, hence a positive measure μ. Moreover, $\mu_{n_j} \rightsquigarrow \mu$, hence $\mu \in \mathcal{M}^{(1)}(K)$.

Let $W_\alpha(K) = \inf\{\|\mu\|_\alpha^2 \mid \mu \in \mathcal{M}^{(1)}(K)\} \in (0, \infty]$. We have $W_\alpha(K) \geq \dfrac{\zeta(1-\alpha)}{\zeta(\alpha)} \operatorname{diam}(K)^{\alpha-1} > 0$, and $W_\alpha(K) = \infty$ iff $\mathcal{E}_\alpha^{(1)}(K)$ is empty. We let $\operatorname{cap}_\alpha(K) = W_\alpha(K)^{-1} \in [0, \infty)$, the α-capacity of K. We have

6.2. THEOREM. — *The following conditions are equivalent :*

(i) $\operatorname{cap}_\alpha(K) = 0$;

(ii) $\mathcal{E}_\alpha^{(1)}(K)$ *is empty;*

(iii) $\nu(K) = 0$ *for all* $\nu \in \mathcal{E}_\alpha^+$;

(iv) $\nu(K) = 0$ *for all* $\nu \in \mathcal{E}_\alpha$.

Proof. — Easy.

In particular, if $\operatorname{cap}_\alpha(K) = 0$ then K has measure zero with respect to Haar measure. For an arbitrary set $X \subseteq \mathbf{Q}_p$ we define the inner and outer capacity respectively by

$$\underline{\operatorname{cap}}_\alpha(X) = \sup\{\operatorname{cap}_\alpha(K) \mid K \subseteq X \text{ compact}\}$$

$$\overline{\operatorname{cap}}_\alpha(X) = \inf\{\underline{\operatorname{cap}}_\alpha(U) \mid X \subseteq U \text{ open}\}.$$

We have $\underline{\operatorname{cap}}_\alpha(X) \leq \overline{\operatorname{cap}}_\alpha(X)$; if equality holds we say X is capacitable and denote the common value by $\operatorname{cap}_\alpha(X)$.

We say that a property holds for α-almost-all x's ($\alpha - a.a.x$'s) if the set of x where it doesn't hold has inner capacity 0.

6.3. THEOREM. — *Assume* $\operatorname{cap}_\alpha(K) > 0$. *There exists a unique* $\lambda_K \in \mathcal{E}_\alpha^{(1)}(K)$ *such that* $\|\lambda_K\|_\alpha^2 = W_\alpha(K)$; *moreover, if* $\mu_n \in \mathcal{E}_\alpha^{(1)}(K)$ *is any sequence such that* $\|\mu_n\|_\alpha^2 \underset{n\to\infty}{\longrightarrow} W_\alpha(K)$ *then* $\mu_n \Rightarrow_\alpha \lambda_K$.

Proof. — Let μ_n be as above; by vague compactness there is a subsequence μ_{n_j} such that $\mu_{n_j} \rightsquigarrow \lambda \in \mathcal{M}^{(1)}(K)$, and by the principle of descent $\|\lambda\|_\alpha^2 \leq \underline{\lim}\|\mu_{n_j}\|_\alpha^2 = W_\alpha(K)$, so $\|\lambda\|_\alpha^2 = W_\alpha(K)$. Moreover, since $W_\alpha(K)^{1/2} \leq \|\frac{1}{2}(\mu_n + \mu_m)\|_\alpha \leq \frac{1}{2}\|\mu_n\|_\alpha + \frac{1}{2}\|\mu_m\|_\alpha$ and $\|\mu_n\|_\alpha \to$

$W_\alpha(K)^{1/2}$, we get $\|\frac{1}{2}(\mu_n+\mu_m)\|_\alpha^2 \underset{n,m\to\infty}{\longrightarrow} W_\alpha(K)$; and since we have

$$\|\mu_n-\mu_m\|_\alpha^2 = 2\|\mu_n\|_\alpha^2 + 2\|\mu_m\|_\alpha^2 - 4\|\frac{1}{2}(\mu_n+\mu_m)\|_\alpha^2$$

we see that $\|\mu_n-\mu_m\|_\alpha \underset{n,m\to\infty}{\longrightarrow} 0$, i.e. μ_n is a Cauchy sequence. By Lemma 5.9 $\mu_{n_j} \Rightarrow_\alpha \lambda$, hence also $\mu_n \Rightarrow_\alpha \lambda$. Suppose next $\lambda, \lambda' \in \mathcal{E}_\alpha^{(1)}(K)$ are such that $\|\lambda\|_\alpha^2 = \|\lambda'\|_\alpha^2 = W_\alpha(K)$, then

$$W_\alpha(K)^{1/2} \le \Big\|\frac{1}{2}(\lambda+\lambda')\Big\|_\alpha \le \frac{1}{2}\|\lambda\|_\alpha + \frac{1}{2}\|\lambda'\|_\alpha = W_\alpha(K)^{1/2}$$

so

$$\Big\|\frac{1}{2}(\lambda+\lambda')\Big\|_\alpha^2 = W_\alpha(K)$$

and since $\|\lambda-\lambda'\|_\alpha^2 = 2\cdot\|\lambda\|_\alpha^2 + 2\cdot\|\lambda'\|_\alpha^2 - 4\cdot\Big\|\frac{1}{2}(\lambda+\lambda')\Big\|_\alpha^2 = 0$ we get $\lambda = \lambda'$.

6.4. — We let $\gamma_K = \mathrm{cap}_\alpha(K)\cdot\lambda_K \in \mathcal{E}_\alpha^+(K)$ be the equilibrium measure of K. Thus $\|\gamma_K\|_\alpha^2 = \gamma_K(1) = \mathrm{cap}_\alpha(K)$.

6.5. THEOREM. — *γ_K is characterized among all $\gamma \in \mathcal{M}^+(K)$ by any of the following equivalent conditions :*

(i) $k^\alpha * \gamma(x) = 1$ *for* $\alpha - a.a.x \in K$, *and* $k^\alpha * \gamma(x) \le 1$ *for all* x*;*

(ii) $k^\alpha * \gamma(x) \le 1$ *for all* x, *and* $\gamma(1)$ *is maximal;*

(iii) $\gamma(1) = \mathrm{cap}_\alpha(K)$, *and* $\|k^\alpha * \gamma\|_{L^\infty}$ *is minimal;*

(iv) $\|\gamma\|_\alpha^2 - 2\gamma(1)$ *is minimal.*

Proof. — We first show γ_K satisfies (i), by showing

(a) $k^\alpha * \gamma_K(x) \ge 1$ for $\alpha - a.a.x \in K$,

(b) $k^\alpha * \gamma_K(x) \le 1$ for $x \in \mathrm{supp}\,\gamma_K$.

Indeed, by the maximum principle we get $k^\alpha * \gamma_K(x) \le 1$ for all x, hence (i). Assume that (a) does not hold, then we find a compact $K_0 \subseteq K$, $\mathrm{cap}_\alpha(K_0) > 0$, such that $k^\alpha * \gamma_K(x) < 1$ for $x \in K_0$. If $\nu \in \mathcal{E}_\alpha^{(1)}(K_0)$ then, on the one hand,

$$\langle \nu, \gamma_K\rangle_\alpha = \int k^\alpha * \gamma_K(x)d\nu(x) < 1 \ \text{ and so } \ \langle \nu, \lambda_K\rangle_\alpha < \|\lambda_K\|_\alpha^2.$$

On the other hand, for any $t \in [0,1]$, $t\cdot\nu + (1-t)\lambda_K \in \mathcal{E}_\alpha^{(1)}(K)$, so that

$$t^2\|\nu\|_\alpha^2 + 2t(1-t)\langle\nu,\lambda_K\rangle_\alpha + (1-t)^2\|\lambda_K\|_\alpha^2 = \|t\cdot\nu + (1-t)\lambda_K\|_\alpha^2 \ge \|\lambda_K\|_\alpha^2$$

from which we get $\langle \nu, \lambda_K \rangle_\alpha \geq \|\lambda_K\|_\alpha^2$, a contradiction. Assume that (b) does not hold, say $k^\alpha * \gamma_K(x_0) > 1$, $x_0 \in \operatorname{supp} \gamma_K$; then by lower semicontinuity $k^\alpha * \gamma_K(x) > 1$ for $x \in U$, a neighbourhood of x_0, and moreover $\gamma_K(U) > 0$. But then we get a contradiction, using (a), namely

$$\begin{aligned}\operatorname{cap}_\alpha(K) = \|\gamma_K\|_\alpha^2 &= \int_U k^\alpha * \gamma_K(x) d\gamma_K(x) + \int_{\operatorname{supp}\gamma_K \smallsetminus U} k^\alpha * \gamma_K(x) d\gamma_K(x) \\ &> \gamma_K(U) + \gamma_K(\operatorname{supp}\gamma_K \smallsetminus U) = \gamma_K(1) = \operatorname{cap}_\alpha(K).\end{aligned}$$

Next, let $\gamma \in \mathcal{M}^+(K)$ satisfy (i). We have $\|\varphi\|_\alpha^2 = \displaystyle\int k^\alpha * \varphi(x) d\gamma(x) = \gamma(1)$, but $\gamma(1)^{-1} \cdot \gamma \in \mathcal{M}^{(1)}(K)$, so

$$\operatorname{cap}_\alpha(K)^{-1} \leq \|\gamma(1)^{-1} \cdot \gamma\|_\alpha^2 = \gamma(1)^{-2} \cdot \|\gamma\|_\alpha^2 = \gamma(1)^{-1},$$

or $\gamma(1) \leq \operatorname{cap}_\alpha(K)$. Hence

$$\begin{aligned}\|\gamma - \gamma_K\|_\alpha^2 &= \|\gamma\|_\alpha^2 + \|\gamma_K\|_\alpha^2 - 2\langle \gamma, \gamma_K \rangle_\alpha \\ &= \gamma(1) + \operatorname{cap}_\alpha(K) - 2\int k^\alpha * \gamma(x) d\gamma_K(x) \\ &\leq 2 \cdot \operatorname{cap}_\alpha(K) - 2\gamma_K(1) = 0\end{aligned}$$

and we get that $\gamma = \gamma_K$.

To prove (ii), let $\gamma \in \mathcal{M}^+(K)$ be such that $k^\alpha * \gamma(x) \leq 1$. Then $\|\gamma\|_\alpha^2 = \displaystyle\int k^\alpha * \gamma(x) d\gamma(x) \leq \gamma(1)$, and since $\gamma(1)^{-1} \cdot \gamma \in \mathcal{M}^{(1)}(K)$, we get $\gamma_K(1)^{-1} = W_\alpha(K) \leq \|\gamma(1)^{-1} \cdot \gamma\|_\alpha^2 = \gamma(1)^{-2} \cdot \|\gamma\|_\alpha^2 \leq \gamma(1)^{-1}$, so $\gamma(1) \leq \gamma_K(1)$. Moreover, if equality holds, then $\|\gamma(1)^{-1} \cdot \gamma\|_\alpha^2 = W_\alpha(K)$, so $\gamma(1)^{-1} \cdot \gamma = \lambda_K$ and $\gamma = \gamma_K$.

To prove (iii), assume $\gamma \in \mathcal{M}^+(K)$ is such that $\gamma(1) = \operatorname{cap}_\alpha(K)$ and $\|k^\alpha * \gamma\|_{L^\infty} \leq \|k^\alpha * \gamma_K\|_{L^\infty} = 1$. Then $\|\gamma\|_\alpha^2 = \displaystyle\int k^\alpha * \gamma(x) d\gamma(x) \leq \gamma(1) = \operatorname{cap}_\alpha(K) = \gamma_K(1)$, and again since $\gamma(1)^{-1} \cdot \gamma \in \mathcal{M}^{(1)}(K)$ we have $W_\alpha(K) \leq \|\gamma(1)^{-1} \cdot \gamma\|^2\alpha = \gamma(1)^{-2}\|\gamma\|_\alpha^2 \leq \gamma(1)^{-1} = W_\alpha(K)$, $\gamma(1)^{-1} \cdot \gamma = \lambda_K$, $\gamma = \gamma_K$. To prove (iv), we note that for any $\gamma \in \mathcal{M}^+(K)$,

$$\|\gamma - \gamma_K\|_\alpha^2 = \|\gamma\|_\alpha^2 + \|\gamma_K\|_\alpha^2 - 2\int k^\alpha * \gamma_K(x) d\gamma(x) \leq \|\gamma\|_\alpha^2 + \operatorname{cap}_\alpha(K) - 2\gamma(1)$$

hence $\|\gamma\|_\alpha^2 - 2 \cdot \gamma(1) \geq \|\gamma - \gamma_K\|_\alpha^2 - \operatorname{cap}_\alpha(K)$ and the minimum is obtained for $\gamma = \gamma_K$ where $\|\gamma_K\|_\alpha^2 - 2 \cdot \gamma_K(1) = -\operatorname{cap}_\alpha(K)$.

Since the maximum in (ii) above is equal $\gamma_K(1) = \operatorname{cap}_\alpha(K)$ we get

6.6. De la Vallée-Poussin's definition of capacity.

$$\operatorname{cap}_\alpha(K) = \max\{\gamma(1) \mid \gamma \in \mathcal{M}^+(K),\ k^\alpha * \gamma(x) \leq 1 \textit{ for all } x\}.$$

Let $\phi^{(N)}(x) = p^N\phi(p^{-N}x)$, the uniform distribution on $B(p^{-N}) = \{x \mid |x| \le p^{-N}\}$. We have

$$k^\alpha * \phi^{(N)}(y) = \frac{\zeta(1-\alpha)}{\zeta(\alpha)} p^N \int_{|x|\le p^{-N}} |y-x|^{\alpha-1} dx$$
$$= \begin{cases} \dfrac{\zeta(1-\alpha)}{\zeta(1)} p^{N(1-\alpha)} & \text{for } y \in B(p^{-N}) \\ \dfrac{\zeta(1-\alpha)}{\zeta(\alpha)} |y|^{\alpha-1} & \text{for } |y| > p^{-N}. \end{cases}$$

Thus, $\gamma_{B(p^{-N})}(x) = \dfrac{\zeta(1)}{\zeta(1-\alpha)} p^{N(\alpha-1)} \cdot \phi^{(N)}(x) = \dfrac{1-p^{\alpha-1}}{1-p^{-1}} \cdot p^{N\alpha}\phi(p^{-N}x)$ is the equilibrium measure on $B(p^{-N})$:

$$k^\alpha * \gamma_{B(p^{-N})}(y) = \begin{cases} 1 & \text{for } y \in B(p^{-N}) \\ \dfrac{\zeta(1)}{\zeta(\alpha)} \cdot |p^{-N}y|^{\alpha-1} < \dfrac{\zeta(1)}{\zeta(\alpha)} < 1 & \text{for } |y| > p^{-N} \end{cases}$$

and $\operatorname{cap}_\alpha(B(p^{-N})) = \dfrac{\zeta(1)}{\zeta(1-\alpha)} p^{N(\alpha-1)} = \dfrac{1-p^{\alpha-1}}{1-p^{-1}} p^{N(\alpha-1)}$ (cf. [8], Appendix A.1, A.2 on p. 401).

6.7. THEOREM. — $\operatorname{cap}_\alpha$ *satisfies the following properties :*

(i) *monotoneness :* $K_1 \subseteq K_2 \Rightarrow \operatorname{cap}_\alpha(K_1) \le \operatorname{cap}_\alpha(K_2)$;

(ii) *subadditivity :* $K = \bigcup_{i=1}^{\infty} K_i \Rightarrow \operatorname{cap}_\alpha(K) \le \sum_{i=1}^{\infty} \operatorname{cap}_\alpha(K_i)$;

(iii) *continuity : given* $K, \varepsilon > 0$, *we can find an open* $U \supseteq K$ *such that for all* K',

$$K \subseteq K' \subseteq U \Rightarrow \operatorname{cap}_\alpha(K') \le \operatorname{cap}_\alpha(K) + \varepsilon \ ;$$

(iv) *translation invariance :* $\operatorname{cap}_\alpha(a+K) = \operatorname{cap}_\alpha(K)$, $\operatorname{cap}_\alpha(a.K) = |a|^{1-\alpha}\operatorname{cap}_\alpha(K)$.

Proof.

(i) follows since $\mathcal{M}^+(K_1) \subseteq \mathcal{M}^+(K_2)$, hence $W_\alpha(K_1) \ge W_\alpha(K_2)$.

(ii) follows since if $\gamma_i = \gamma_{K|K_i}$ is the restriction of γ_K to K_i, then $\gamma_i(1) \le \operatorname{cap}_\alpha(K_i)$, and so $\operatorname{cap}_\alpha(K) = \gamma_K(1) \le \sum_{i=1}^{\infty} \gamma_i(1) \le \sum_{i=1}^{\infty} \operatorname{cap}_\alpha(K_i)$.

Assume (iii) does not hold, so that we have K, $\varepsilon > 0$, and a sequence of compact sets $K_i \supseteq K_{i_n} \supseteq \cdots \supseteq K$, $\bigcap_{i=0}^{\infty} K_i = K$, $\operatorname{cap}_\alpha(K_i) \ge$

$\operatorname{cap}_\alpha(K)+\varepsilon$. The equilibrium measures γ_{K_i} are weakly bounded ($\gamma_{K_i}(1) = \operatorname{cap}_\alpha(K_i) \le \operatorname{cap}_\alpha(K_0)$), hence vaguely compact, so after passage to a subsequence we may assume $\gamma_{K_i} \rightsquigarrow \mu \in \mathcal{M}^+(K)$. By the principle of descent $k^\alpha * \mu \le \underline{\lim} k^\alpha * \gamma_{K_i} \le 1$ so $\mu(1) \le \operatorname{cap}_\alpha(K)$. On the other hand, $\mu(1) = \lim \gamma_{K_i}(1) = \lim \operatorname{cap}_\alpha(K_i) \ge \operatorname{cap}_\alpha(K) + \varepsilon$, a contradiction.

(iv) follows easily from the definition.

By Choquet's Theorem, we obtain

6.8. COROLLARY. — *Every analytic set is capacitable.*

For $x_1 \cdots x_n \in K$, put $d^\alpha(x_1 \cdots x_n) = \binom{n}{2}^{-1} \sum_{i<j} |x_i - x_j|^{\alpha-1}$. This function obtains its minimal value on the compact set K at certain points $x_i = \xi_i^{(n)}$, and we define

$$\operatorname{diam}_\alpha^{(n)}(K) = [\min_{x_1\cdots x_n \in K} d^\alpha(x_1 \cdots x_n)]^{-1} = \binom{n}{2}\Big[\sum_{i<j} |\xi_i^{(n)} - \xi_j^{(n)}|^{\alpha-1}\Big]^{-1}.$$

We have

$$\begin{aligned}
[\operatorname{diam}_\alpha^{(n)}(K)]^{-1} &= \binom{n}{2}^{-1} \cdot \frac{1}{n-2} \sum_{k=1}^{n} \sum_{\substack{i<j \\ i,j\neq k}} |\xi_i^{(n)} - \xi_j^{(n)}|^{\alpha-1} \\
&\ge \binom{n}{2}^{-1} \cdot \frac{1}{n-2} \cdot \binom{n-1}{2} [\operatorname{diam}_\alpha^{(n-1)}(K)]^{-1} \\
&= [\operatorname{diam}_\alpha^{(n-1)}(K)]^{-1}.
\end{aligned}$$

Thus $\operatorname{diam}_\alpha^{(n-1)}(K) \ge \operatorname{diam}_\alpha^{(n)}(K)$, and we have a well defined limit $\operatorname{diam}_\alpha(K) = \lim_{n\to\infty} \operatorname{diam}_\alpha^{(n)}(K)$.

6.9. THEOREM. — $\operatorname{cap}_\alpha(K) = \dfrac{\zeta(\alpha)}{\zeta(1-\alpha)} \operatorname{diam}_\alpha(K).$

Proof. — Taking the inequality $\binom{n}{2}[\operatorname{diam}_\alpha^{(n)}(K)]^{-1} \le \sum_{i<j} |x_i - x_j|^{\alpha-1}$, multiplying by $d\mu(x_i)d\mu(x_j)$, where $\mu \in \mathcal{M}^{(1)}(K)$ and integrating $\binom{n}{2}$ times with respect to $(x_i, x_j) \in K \times K$, we get

$$[\operatorname{diam}_\alpha^{(n)}(K)]^{-1} \le \iint_{K\times K} |x-y|^{\alpha-1} d\mu(x)d\mu(y) = \frac{\zeta(\alpha)}{\zeta(1-\alpha)} \|\mu\|_\alpha^2.$$

Letting $n \to \infty$, then taking the inf over all $\mu \in \mathcal{M}^{(1)}(K)$, we get $[\operatorname{diam}_\alpha(K)]^{-1} \le \dfrac{\zeta(\alpha)}{\zeta(1-\alpha)}[\operatorname{cap}_\alpha(K)]^{-1}$. On the other hand, let

$\mu^{(n)} = \frac{1}{n}\sum_{i=1}^{n}\delta_{\xi_i^{(n)}} \in \mathcal{M}^{(1)}(K)$, and consider the truncated kernel $k^\alpha_\varepsilon(x) = \begin{cases} k^\alpha(x), & |x| \geq |\varepsilon| \\ k^\alpha(\varepsilon), & |x| \leq |\varepsilon| \end{cases}$, where $\varepsilon \in \mathbf{Q}_p$ is small. We have

$$\begin{aligned}\iint_{K\times K} & k^\alpha_\varepsilon(x-y)d\mu^{(n)}(x)d\mu^{(n)}(y) \\ & \leq \frac{\zeta(1-\alpha)}{\zeta(a)}\left[\frac{1}{n^2}\sum_{i\neq j}|\xi_i^{(n)} - \xi_j^{(n)}|^{\alpha-1} + \frac{|\varepsilon|^{\alpha-1}}{n}\right] \\ & = \frac{\zeta(1-\alpha)}{\zeta(a)}\left[\frac{2}{n^2}\binom{n}{2}(\mathrm{diam}^{(n)}_\alpha(K))^{-1} + \frac{|\varepsilon|^{\alpha-1}}{n}\right].\end{aligned}$$

By vague compactness, after passage to a subsequence, we may assume $\mu^{(n)} \rightsquigarrow \mu \in \mathcal{M}^{(1)}(K)$. Since k^α_ε is continuous, we can pass to the limit $n \to \infty$, and obtain

$$\iint_{K\times K} k^\alpha_\varepsilon(x-y)d\mu(x)d\mu(y) \leq \frac{\zeta(1-\alpha)}{\zeta(\alpha)}[\mathrm{diam}_\alpha(K)]^{-1}.$$

Now letting $\varepsilon \to 0$, we get $\|\mu\|^2_\alpha \leq \frac{\zeta(1-\alpha)}{\zeta(\alpha)}[\mathrm{diam}_\alpha(K)]^{-1}$, hence $[\mathrm{cap}_\alpha(K)]^{-1} \leq \frac{\zeta(1-\alpha)}{\zeta(\alpha)}[\mathrm{diam}_\alpha(K)]^{-1}$.

Note. — By the uniqueness of the measure λ_K, we also get $\lambda_K = \mu$.

6.10. COROLLARY. — *If $f : K \to \mathbf{Q}_p$ satisfies $|f(x)-f(y)| \leq |x-y|$, then $\mathrm{cap}_\alpha(f(K)) \leq \mathrm{cap}_\alpha(K)$.*

Proof. — This property is immediate if cap_α is replaced by diam_α, hence it follows from the above theorem.

6.11. LIMIT THEOREM. — *Let $\nu_n \in \mathcal{M}^+_\alpha$, $\nu_n \rightsquigarrow \nu$, and assume that we have*

$$\lim_{R\to\infty}\int_{|x|>R}|x|^{\alpha-1}d\nu_n(x) = 0 \text{ uniformly with respect to } n.$$

Then $k^\alpha * \nu(x) = \underline{\lim}_{n\to\infty} k^\alpha * \nu_n(x)$ for α – a.a.x's. Moreover, if we set $f(x) = \underline{\lim}_{n\to\infty} k^\alpha * \nu_n(x)$ and introduce its lower semicontinuous regularization $\tilde{f}(x) = \underline{\lim}_{y\to x} f(y)$, then $k^\alpha * \nu(x) = \tilde{f}(x)$ for all x's.

Proof. — By the descent principle $k^\alpha * \nu(x) \leq \underline{\lim}_{n\to\infty} k^\alpha * \nu_n(x)$, so assume we have strict inequality on a set K of a positive α-capacity, and without loss of generality K compact. Then

$$\int_K k^\alpha * \nu(x) d\gamma_K(x) < \int_K \varliminf_{n\to\infty} k^\alpha * \nu_n(x) d\gamma_K(x)$$
$$\leq \varliminf_{n\to\infty} \int_K k^\alpha * \nu_n(x) d\gamma_K(x).$$

On the other hand,

$$\int_K k^\alpha * (\nu_n - \nu)(x) d\gamma_K(x)$$
$$= \int_K k^\alpha * \gamma_K(x) d(\nu_n - \nu)(x)$$
$$\leq \int_{|x|\leq R} d(\nu_n - \nu)(x) + O\Big[\int_{|x|>R} |x|^{\alpha-1} d(\nu_n - \nu)(x)\Big].$$

Since $\nu_n \rightsquigarrow \nu$, the first integral approaches zero $n \to \infty$, while the second can be made arbitrary small by our uniformity assumption. This gives the desired contradiction and establishes the first part of our theorem. For the second part, note that since $k^\alpha * \nu$ is lower semicontinuous and everywhere $\leq f$, we have

$$k^\alpha * \nu(x) \leq \varliminf_{y\to x} k^\alpha * \nu(y) \leq \varliminf_{y\to x} f(y) = \tilde{f}(x).$$

On the other hand, using the above and the regularization principle, we have

$$\tilde{f}(x) = \lim_{N\to\infty} \min_{|y-x|\leq p^{-N}} f(y) \leq \varliminf_{N\to\infty} f^{(N)}(x)$$
$$= \lim_{N\to\infty} k^\alpha * \nu^{(N)}(x) = k^\alpha * \nu(x).$$

7. The Green measure.

The group $\mathrm{PGL}_2(\mathbf{Q}_p)$ acts on $\mathbf{P}^1(\mathbf{Q}_p)$ by fractional linear transformations, and we get an induced action on $\mathcal{C}(\mathbf{P}^1(\mathbf{Q}_p))$, the $\mathbf{C}$-valued continuous functions on $\mathbf{P}^1(\mathbf{Q}_p)$, and an adjoint action on the (signed) measures on $\mathbf{P}^1(\mathbf{Q}_p)$. We write $\mathbf{P}^1(\mathbf{Q}_p) = \mathbf{Q}_p \cup \{\infty\}$, $\mathcal{C}(\mathbf{P}^1(\mathbf{Q}_p)) = \mathcal{C}_0(\mathbf{Q}_p) \oplus \mathbf{C}$, $\mathcal{M}(\mathbf{P}^1(\mathbf{Q}_p)) = \mathcal{M}(\mathbf{Q}_p) \oplus \mathbf{R}.\delta_\infty$. For $g \in \mathrm{PGL}_2(\mathbf{Q}_p)$ and $\nu \in \mathcal{M}(\mathbf{P}^1(\mathbf{Q}_p))$, we have $g\nu \in \mathcal{M}(\mathbf{P}^1(\mathbf{Q}_p))$ defined by the formula

$$\int \varphi(x) dg\nu(x) = \int \varphi(gx) d\nu(x) \text{ for } \varphi \in \mathcal{C}(\mathbf{P}^1(\mathbf{Q}_p)).$$

We shall be concerned only with the action of the "inversion" $I_{x_0} = \begin{bmatrix} x_0 & 1 - x_0^2 \\ 1 & -x_0 \end{bmatrix}$, $x_0 \in \mathbf{Q}_p$, where $I_{x_0}(x) = (x - x_0)^{-1} + x_0$ for $x \neq x_0, \infty$ and

$I_{x_0}(x_0) = \infty$, $I_{x_0}(\infty) = x_0$. We have : $I_{x_0} \circ I_{x_0} = \mathrm{id}$, $|I_{x_0}(x) - I_{x_0}(y)| = \dfrac{|x-y|}{|x-x_0| \cdot |y-x_0|}$.

If $K \subseteq \mathbf{Q}_p$ is compact and $x_0 \notin K$, there are positive constants $\Gamma_1 = p^{n_1}$, $\Gamma_2 = p^{n_2}$ such that for all $x, y \in K : \Gamma_1 \cdot |x-y| \leq |I_{x_0}(x) - I_{x_0}(y)| \leq \Gamma_2 \cdot |x-y|$. We obtain $\Gamma_1^{1-\alpha} \cdot \mathrm{cap}_\alpha(K) \leq \mathrm{cap}_\alpha(I_{x_0}(K)) \leq \Gamma_2^{1-\alpha} \cdot \mathrm{cap}_\alpha(K)$; in particular, $\mathrm{cap}_\alpha(K) = 0$ if and only if $\mathrm{cap}_\alpha(I_{x_0}(K)) = 0$, from which we get that I_{x_0} preserves the collection of sets of inner capacity zero.

In the following, let $K \subseteq \mathbf{P}^1(\mathbf{Q}_p)$ be an arbitrary closed set such that $x_0 \notin K$. Since $x_0 \notin K$, $\infty \notin I_{x_0}(K)$, the set $I_{x_0}(K)$ is compact, so let $\gamma_{I_{x_0}(K)}$ denote its equilibrium measure. We have a well defined measure $k^\alpha * \delta_{x_0} \in \mathcal{M}^+(I_{x_0}(K))$ given by

$$(k^\alpha * \delta_{x_0}) \cdot \gamma_{I_{x_0}(K)}(y) = \frac{\zeta(1-\alpha)}{\zeta(\alpha)} |y-x_0|^{\alpha-1} \cdot \gamma_{I_{k_0}(K)}(y).$$

We define $P_K^\alpha \delta_{x_0} = I_{x_0}(k^\alpha * \delta_{x_0} \cdot \gamma_{I_{x_0}(K)}) \in \mathcal{M}^+(K)$, explicitly :

$$\int \varphi(y) dP_K^\alpha \delta_{x_0}(y) = \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \int \varphi(y)|y-x_0|^{1-\alpha} d\gamma_{I_{x_0}(K)}((y-x_0)^{-1} + x_0).$$

Let us estimate the potential of the measure $P_K^\alpha \delta_{x_0}$. We have

$$\begin{aligned}
&k^\alpha * P_K^\alpha \delta_{x_0}(x) \\
&= \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \int_K |x-y|^{\alpha-1} dP_K^\alpha \delta_{x_0}(y) \\
&= \Big(\frac{\zeta(1-\alpha)}{\zeta(\alpha)}\Big)^2 \int_K |x-y|^{\alpha-1} |y-x_0|^{1-\alpha} d\gamma_{I_{x_0}(K)}(I_{x_0}(y))
\end{aligned}$$

and upon substituting $y := I_{x_0}(y)$ we get

$$\begin{aligned}
&= \Big(\frac{\zeta(1-\alpha)}{\zeta(\alpha)}\Big)^2 \int_{I_{x_0}(K)} \Big(\frac{|I_{x_0}(x) - y|}{|I_{x_0}(x) - x_0| \cdot |y - x_0|}\Big)^{\alpha-1} \cdot \\
&\qquad\qquad |y - x_0|^{\alpha-1} d\gamma_{I_{x_0}(K)}(y) \\
&= \frac{\zeta(1-\alpha)}{\zeta(\alpha)} |x-x_0|^{\alpha-1} \cdot \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \int_{I_{x_0}(K)} |I_{x_0}(x) - y|^{\alpha-1} d\gamma_{I_{x_0}(K)}(y) \\
&= k^\alpha * \delta_{x_0}(x) \cdot k^\alpha * \gamma_{I_{x_0}(K)}(I_{x_0}(x)).
\end{aligned}$$

Since $k^\alpha * \gamma_{I_{x_0}(K)}(x^*) \leq 1$, with equality for $\alpha - a.a. x^* \in I_{x_0}(K)$, and since I_{x_0} preserves sets of capacity zero, we get :

$$k^\alpha * P_K^\alpha \delta_{x_0}(x) \leq k^\alpha * \delta_{x_0}(x), \quad \text{with equality for } \alpha - a.a. x \in K.$$

We will say that a measure ν on $\mathbf{P}^1(\mathbf{Q}_p)$ is α-absolutely continuous if for any set E, $\underline{\mathrm{cap}}_\alpha(E) = 0$ implies $\nu(E) = 0$. Thus every measure of finite

α-energy is α-absolutely continuous, but not conversely. We summarize the above discussion of $P_K^\alpha \delta_{x_0}$ in the following

7.1. THEOREM. — *Let $K \subseteq \mathbf{P}^1(\mathbf{Q}_p)$ be a closed set, and fix $x_0 \notin K$. Then there exists a (unique) α-absolutely continuous measure $P_K^\alpha \delta_{x_0} \in \mathcal{M}^+(K)$ such that $k^\alpha * P_K^\alpha \delta_{x_0}(x) = k^\alpha * \delta_{x_0}(x)$ for $\alpha - a.a.x \in K$. Furthermore $k^\alpha * P_K^\alpha \delta_{x_0}(x) \leq k^\alpha * \delta_{x_0}(x)$ for all x, and $P_K^\alpha \delta_{x_0}(1) = k^\alpha * \gamma_{I_{x_0}(K)}(x_0) \leq 1$.*

The formula for $P_K^\alpha \delta_{x_0}(1)$ is immediate from the definition. The uniqueness of $P_K^\alpha \delta_{x_0}$ follows from :

7.2. 3^{rd} UNIQUENESS PRINCIPLE. — *Let K be a closed set, $\nu_1, \nu_2 \in \mathcal{M}_\alpha^+(K)$, ν_i α-absolute continuous. If $k^\alpha * \nu_1(x) = k^\alpha * \nu_2(x)$ for $\alpha - a.a.x \in K$, then $\nu_1 = \nu_2$.*

Proof. — Setting $\nu = \nu_1 - \nu_2$, we claim that $|k^\alpha * \nu_i(x)| < \infty$ for $\alpha - a.a.x$'s, hence a posteriori for $\nu - a.a.x$'s. Indeed, otherwise we could find a compact $K' \subseteq K$, $\text{cap}_\alpha(K') > 0$, with $k^\alpha * \nu_i(x) = \infty$ for all $x \in K'$. But then we would get a contradiction :

$$\begin{aligned}\infty &= \int_{K'} k^\alpha * \nu_i(x) d\gamma_{K'}(x) = \int k^\alpha * \gamma_{K'}(x) d\nu_i(x) \\ &\leq \int_{|x|\leq R} d\nu_i(x) + O\Big(\int_{|x|>R} |x|^{\alpha-1} d\nu_i(x)\Big) < \infty.\end{aligned}$$

We have, moreover, $k^\alpha * \nu(x) = 0$ for $\alpha - a.a.x \in K$, hence again for $\nu - a.a.x$'s, so that $\|\nu\|_\alpha^2 = \displaystyle\int_K k^\alpha * \nu(x) d\nu(x) = 0$, and we conclude that $\nu = 0$.

In the following we will let $K \subseteq \mathbf{P}(\mathbf{Q}_p)$ denote a closed set and $U = \mathbf{P}(\mathbf{Q}_p) \setminus K$ its open complement, with y's denoting points of U, x's points of K. We have the following

7.3. SYMMETRY PROPERTY. — $k^\alpha * P_K^\alpha \delta_{y_1}(y_2) = k^\alpha * P_K^\alpha \delta_{y_2}(y_1)$, $y_1, y_2 \in U$.

Proof.

$$\begin{aligned} k^\alpha * P_K\delta_{y_1}(y_2) &= \int_K k^\alpha(y_2 - x)dP_K^\alpha\delta_{y_1}(x) = \int_K k^\alpha * P_K^\alpha\delta_{y_2}(x)dP_K^\alpha\delta_{y_1}(x) \\ &= \int_K k^\alpha * P_K^\alpha\delta_{y_1}(x)dP_K^\alpha\delta_{y_2}(x) \\ &= \int_K k^\alpha(y_1-x) * P_K^\alpha\delta_{y_2}(x) = k^\alpha * dP_K^\alpha\delta_{y_2}(y_1). \end{aligned}$$

7.4. DEFINITION.

$$K_{\text{reg}} = \{x \in K \mid k^\alpha * \delta_y(x) = k^\alpha * P_K^\alpha\delta_y(x) \textit{ for all } y \in U\}$$
$$K_{\text{irr}} = \{x \in K \mid k^\alpha * \delta_y(x) > k^\alpha * P_K^\alpha\delta_y(x) \textit{ for some } y \in U\}.$$

The points of K_{reg} (resp. K_{irr}) are called regular (resp. irregular) points, and $K = K_{\text{reg}} \cup K_{\text{irr}}$. By the regularization principle and since $k^\alpha * \delta_y(x) = k^\alpha * P_K^\alpha\delta_y(x)$ for $\alpha - a.a.x \in K$, it follows that $k^\alpha * \delta_y(x) = k^\alpha * P_K^\alpha\delta_y(x)$ for an interior point $x \in \overset{\circ}{K} = K \setminus \partial K$, so $K_{\text{irr}} \subseteq \partial K$. In particular, if $\partial K = \emptyset$, $K = K_{\text{reg}}$. If $k^\alpha * \delta_y(x) > k^\alpha * P_K^\alpha\delta_y(x)$, then also $k^\alpha * \delta_{y'}(x) > k^\alpha * P_K^\alpha\delta_{y'}(x)$ for y' near y. From this we get $\text{cap}_\alpha(K_{\text{irr}}) = 0$. Regular points of the boundary ∂K are characterized by the following property of "concentration" of the Green measure,

7.5. THEOREM. — *Let $x_0 \in \partial K$; $x_0 \in K_{\text{reg}} \iff P_K^\alpha\delta_y \rightsquigarrow \delta_{x_0}$ as $y \to x_0$, $y \in U$.*

Proof. — To prove "$\Longleftarrow$" we use the regularization principle and the fact that $k^\alpha * \delta_y(x) = k^\alpha * P_K^\alpha\delta_y(x)$ for $\alpha - a.a.x \in K$, hence for $a.a.x \in K$, to write :

$$\begin{aligned} k^\alpha*\delta_{y_0}(x_0) - k^\alpha*P_K^\alpha\delta_{y_0}(x_0) &= \lim_{N\to\infty} p^N \int_{|x-x_0|\le p^{-N}} k^\alpha*(\delta_{y_0} - P_K^\alpha\delta_{y_0})(x)dx \\ &= \lim_{N\to\infty} p^N \int_{\substack{|y-x_0|\le p^{-N} \\ y\in U}} k^\alpha*(\delta_{y_0} - P_K^\alpha\delta_{y_0})(y)dy. \end{aligned}$$

Thus if $x_0 \notin K_{\text{reg}}$, we can find $y_0 \in U$ such that the left hand side is > 0, hence we can find a sequence of points $y_n \to x_0$, $y_n \in U$, such that $\underline{\lim}_{n\to\infty} k^\alpha * (\delta_{y_0} - P_K^\alpha\delta_{y_0})(y_n) > 0$. But assuming $P_K^\alpha\delta_{y_n} \rightsquigarrow \delta_{x_0}$, we get using the descent principle, and the symmetry property,

$$\begin{aligned} \lim_{n\to\infty} k^\alpha * \delta_{y_0}(y_n) &= k^\alpha * \delta_{y_0}(x_0) = k^\alpha * \delta_{x_0}(y_0) \\ &\le \underline{\lim}_{n\to\infty} k^\alpha * P_K^\alpha\delta_{y_n}(y_0) = \underline{\lim}_{n\to\infty} k^\alpha * P_K^\alpha\delta_{y_0}(y_n). \end{aligned}$$

To prove "$\Longrightarrow$" we take $0 \le \varphi \in \mathcal{S}(\mathbf{Q}_p)$, and show $P_K^\alpha \delta_y(\varphi) \longrightarrow \delta_{x_0}(\varphi) = \varphi(x_0)$. We write $\varphi = k^\alpha * \varphi'$, where $\varphi' = k^{-\alpha} * \varphi$ is locally constant and $O(|x|^{-(1+\alpha)})$ as $|x| \to \infty$. We have

$$\begin{aligned}\varphi(x_0) - P_K^\alpha \delta_y(\varphi) &= k^\alpha * \varphi'(x_0) - \int k^\alpha * \varphi'(x) dP_K^\alpha \delta_y(x) \\ &= \int (k^\alpha(x_0 - x) - k^\alpha * P_K^\alpha \delta_y(x)) \cdot \varphi'(x) dx \\ &= \int (k^\alpha(y-x) - k^\alpha * P_K^\alpha \delta_y(x)) \varphi'(x) dx \\ &\quad + \int (k^\alpha(x_0 - x) - k^\alpha(y-x)) \varphi'(x) dx.\end{aligned}$$

Letting $y \to x_0$, the second integral tends to zero, and the first integral may be rewritten

$$\begin{aligned}\Big| \int_U k^\alpha * (\delta_y - P_K^\alpha \delta_y)(\tilde{y}) \varphi'(\tilde{y}) d\tilde{y} \Big| &\le \max |\varphi'(\tilde{y})| \cdot \int_U k^\alpha * (\delta_y - P_K^\alpha \delta_y)(\tilde{y}) d\tilde{y} \\ \le \max |\varphi'| \cdot \Big[\int_{|\tilde{y} - x_0| > \delta} & k^\alpha * (\delta_y - P_K^\alpha \delta_y)(\tilde{y}) d\tilde{y} + \int_{|\tilde{y} - x_0| \le \delta} k^\alpha * \delta_y(\tilde{y}) d\tilde{y} \Big].\end{aligned}$$

Taking δ sufficiently small the second integral inside the brackets can be made arbitrarily small, while for the first we have by the symmetry property and our assumption that $x_0 \in K_{\mathrm{reg}}$,

$$\lim_{y \to x_0} \int_{|\tilde{y} - x_0| > \delta} k^\alpha * (\delta_y - P_K^\alpha \delta_y)(\tilde{y}) d\tilde{y} = \int_{|\tilde{y} - x_0| > \delta} k^\alpha * (\delta_{\tilde{y}} - P_K^\alpha \delta_{\tilde{y}})(x_0) d\tilde{y} = 0.$$

We shall next extend the Green measure $P_K^\alpha \delta_y$ to points of K. For $x \in K_{\mathrm{reg}}$ we set $P_K^\alpha \delta_x = \delta_x$.

Assuming $x \in K_{\mathrm{irr}}$, by the above theorem we can find some sequence $y_n \to x$, $y_n \in U$, such that $P_K^\alpha \delta_{y_n}$ does not converge to δ_x. By the vague compactness of $\{P_K^\alpha \delta_{y_n}\}$ we may assume that $P_K^\alpha \delta_{y_n} \rightsquigarrow \nu \neq \delta_x$. Set $m = \nu(\{x\}) \in [0, 1)$ and define

$$P_K^\alpha \delta_x = \frac{1}{1-m} (\nu - m \cdot \delta_x) \in \mathcal{M}^+(K).$$

The independence of this definition from the choice of $\{y_n\}$ follows from the 3^{rd} uniqueness principle and the following property which completely characterizes $P_K^\alpha \delta_x$:

7.6. Theorem. — *$P_K^\alpha \delta_x$ is α-absolutely continuous and $k^\alpha * P_K^\alpha \delta_x(x') = k^\alpha * \delta_x(x')$ for $\alpha - a.a.x' \in K$.*

Proof. — Using the limit theorem, we have for $\alpha - a.a.x' \in K$,

$$k^\alpha * \nu(x') = \lim_{n\to\infty} k^\alpha * P_K\delta_{y_n}(x') = \lim_{n\to\infty} k^\alpha * \delta_{y_n}(x') = k^\alpha * \delta_x(x')$$

and by the descent principle $k^\alpha * \nu(x') \le k^\alpha * \delta_x(x')$ for all x'. Since $k^\alpha * \nu$ is bounded away from x, it follows that $P_K^\alpha \delta_x = \frac{1}{1-m}\nu_{|K \setminus \{x\}}$ is α-absolutely continuous. Moreover, for $\alpha - a.a.x' \in K$,

$$\begin{aligned} k^\alpha * P_K^\alpha \delta_x(x') = \frac{1}{1-m} k^\alpha * (\nu - m \cdot \delta_x)(x') &= \frac{1}{1-m} k^\alpha * (\delta_x - m\delta_x)(x') \\ &= k^\alpha * \delta_x(x'). \end{aligned}$$

7.7. Definition. — *$G_U^\alpha(x,y) = k^\alpha * \delta_y(x) - k^\alpha * P_K^\alpha \delta_y(x)$ will be called the Green's function of U. We have $G_U^\alpha(x,y) \ge 0$ for all x, y; $G_U^\alpha(x,y) = 0$ for all $x \in K_{\mathrm{reg}}$, $y \in U$ and all $y \in K_{\mathrm{reg}}$. $G_U^\alpha(x,y) = G_U^\alpha(y,x)$ for all $x, y \notin K_{\mathrm{irr}}$.*

We consider next the problem of reconstructing the potential $k^\alpha * \nu$, $\nu \in \mathcal{M}_\alpha(K)$, from its values on K : for all $y \in U$ we have

$$\begin{aligned} \int_{K_{\mathrm{irr}}} G_U^\alpha(x,y)d\nu(x) &= \int_K G_U^\alpha(x,y)d\nu(x) \\ &= \int_K k^\alpha * \delta_y(x)d\nu(x) - \int_K k^\alpha * P_K^\alpha \delta_y(x)d\nu(x) \\ &= k^\alpha * \nu(y) - \int_K k^\alpha * \nu(x)dP_K^\alpha\delta_y(x). \end{aligned}$$

Thus, we obtain

7.8. M. Riesz reconstruction formula. — *For $\nu \in \mathcal{M}_\alpha(K)$, $y \in U = \mathbf{Q}_p \setminus K$,*

$$k^\alpha * \nu(y) = \int_K k^\alpha * \nu(x)dP_K^\alpha\delta_y(x) + \int_{K_{\mathrm{irr}}} G_U^\alpha(x,y)d\nu(x).$$

In particular, if $\nu_{|K_{\mathrm{irr}}} = 0$ then

$$k^\alpha * \nu(y) = \int_K k^\alpha * \nu(x)dP_K^\alpha\delta_y(x).$$

As a corollary we obtain

7.9. 4^{th} uniqueness theorem. — *Let $\nu_1, \nu_2 \in \mathcal{M}(K)$, $\nu_{i|K_{\mathrm{irr}}} = 0$ (or just $\nu_{1|K_{\mathrm{irr}}} = \nu_{2|K_{\mathrm{irr}}}$). If $k^\alpha * \nu_1(x) = k^\alpha * \nu_2(x)$ for $\alpha - a.a.x \in K$, then $\nu_1 = \nu_2$.*

Proof. — Setting $\nu = \nu_1 - \nu_2$, $\nu_{|K_{\mathrm{irr}}} = 0$, so $k^\alpha * \nu(y) = \int_K k^\alpha * \nu(x) dP_K^\alpha \delta_y(x) = 0$ since $P_K^\alpha \delta_y$ is α-absolutely continuous. Thus $k^\alpha * \nu(z) = 0$ for $\alpha - a.a.z$'s hence for $a.a.z$'s and our theorem follows from the 1^{st} uniqueness principle.

The family of measures $\{P_K^\alpha \delta_z\}$ is Borel, i.e. for any $\varphi \in \mathcal{S}(\mathbf{Q}_p)$, $P_K^\alpha \delta_z(\varphi)$ is a Borel measurable function of z. For any measure ν we define $P_K^\alpha \nu = \int P_K^\alpha \delta_z d\nu(z)$, i.e. $P_K^\alpha \nu(\varphi) = \int P_K^\alpha \delta_z(\varphi) d\nu(z)$. $P_K^\alpha \nu$ is a positive distribution, hence a measure : $P_K^\alpha \nu \in \mathcal{M}^+(K)$, and is such that $P_K^\alpha \nu_{|K_{\mathrm{irr}}} = 0$. Similarly, if ν is any signed measure, $P_K^\alpha \nu$ is again a signed measure. Moreover, $P_K^\alpha \nu$ solves the problem of "Balayage" or sweeping out ν onto K while preserving the potential.

7.10. BALAYAGE PRINCIPLE. — *For any (signed) measure ν there exists a unique (signed) measure $P_K^\alpha \nu$, such that :*

(i) $\operatorname{supp} P_K^\alpha \nu \subseteq K$;

(ii) $P_K^\alpha \nu_{|K_{\mathrm{irr}}} = 0$;

(iii) $k^\alpha * P_K^\alpha \nu(x) = k^\alpha * \nu(x)$ *for* $\alpha - a.a.x \in K$.

Moreover, if ν is a measure, then we have also

(iv) $k^\alpha * P_K^\alpha \nu(x) \leq k^\alpha * \nu(x)$ *for all* x.

Proof. — Properties (iii) and (iv) follow immediately from the corresponding properties of $P_K \delta_z$. Uniqueness follows from the 4^{th} uniqueness theorem.

From the uniqueness of $P_K^\alpha \nu$ we get the following transitivity property of balayage :

7.11. COROLLARY. — *Let $K_1 \supset K_2$ be closed sets, ν a signed measure, then*

$$P_{K_2}^\alpha \nu = P_{K_2}^\alpha (P_{K_1}^\alpha \nu).$$

Note also that we have, for K compact,

$$\begin{aligned} P_K^\alpha \nu(1) &= \int_K 1 dP_K^\alpha \nu(x) = \int_K k^\alpha * \gamma_K(x) dP_K^\alpha \nu(x) \\ &= \int_K k^\alpha * P_K^\alpha \nu(x) d\gamma_K(x) = \int_K k^\alpha * \nu(x) d\gamma_K(x) \\ &= \int k^\alpha * \gamma_K(x) d\nu(x) \leq \int d\nu = \nu(1). \end{aligned}$$

7.12. EXAMPLE. — For $B(p^N) = \{x \mid |x| \leq p^N\}$, $x_0 \notin B(p^N)$, we have

$$P_{B(p^N)}^\alpha \delta_{x_0} = \frac{\zeta(1)}{\zeta(\alpha)} p^{-N\alpha} |x_0|^{\alpha-1} \phi(p^N x) dx.$$

Indeed, the potential of this measure is equal to

$$\frac{\zeta(1-\alpha)}{\zeta(\alpha)} \frac{\zeta(1)}{\zeta(\alpha)} p^{-N\alpha} |x_0|^{\alpha-1} \int_{|x|\leq p^N} |y-x|^{\alpha-1} dx \ ;$$

for $y \in B(p^N)$ we can put $x := x+y$ and we get $\dfrac{\zeta(1-\alpha)}{\zeta(\alpha)} |x_0|^{\alpha-1} = k^\alpha * \delta_{x_0}(y)$. For $y \notin B(p^N)$ we obtain $\dfrac{\zeta(1-\alpha)}{\zeta(\alpha)} \dfrac{\zeta(1)}{\zeta(\alpha)} p^{N(1-\alpha)} |x_0|^{\alpha-1} |y|^{\alpha-1}$ which is $< k^\alpha * \delta_{x_0}(y)$, since $\dfrac{\zeta(1)}{\zeta(\alpha)} < 1$, and since for $|y| > p^N$ and $|x_0| > p^N$, $|x_0^{-1} - y^{-1}| < p^{-N}$, hence $p^{N(1-\alpha)} |x_0|^{\alpha-1} |y|^{\alpha-1} < |x_0 - y|^{\alpha-1}$ (cf. [8], Appendix, A.6, p. 402).

7.13. EXAMPLE. — Let $\check{B}_N = \{x \mid |x| \geq p^{-N}\}$, and set

$$\check{\delta}_{(N)}^\alpha \stackrel{\text{def}}{=} \frac{\zeta(1)}{\zeta(\alpha)} p^{-N\alpha} (1 - \phi(p^{-1-N} x)) \frac{dx}{|x|^{1+\alpha}} \in \mathcal{M}^+(\check{B}_N).$$

We have for all $x_0 \notin \check{B}_N$, $P_{\check{B}_N}^\alpha \delta_{x_0} = \check{\delta}_{(N)}^\alpha$. Indeed, for $y \in \check{B}_N$

$$\begin{aligned}
k^\alpha * \check{\delta}^\alpha_{(N)}(y) &= \frac{\zeta(1-\alpha)}{\zeta(\alpha)}\frac{\zeta(1)}{\zeta(\alpha)}p^{-N\alpha}\int_{|x|\geq p^{-N}}|y-x|^{\alpha-1}\frac{dx}{|x|^{1+\alpha}}\\
&= \frac{\zeta(1-\alpha)}{\zeta(\alpha)}\frac{\zeta(1)}{\zeta(\alpha)}p^{-N\alpha}\int_{|x|\geq p^{N}}|yx-1|^{\alpha-1}dx\\
&= \frac{\zeta(1-\alpha)}{\zeta(\alpha)}\frac{\zeta(1)}{\zeta(\alpha)}p^{-N\alpha}|y|^{-1}\int_{|x|\geq p^{N}|y|}|x-1|^{\alpha-1}dx\\
&= \frac{\zeta(1-\alpha)}{\zeta(\alpha)}\frac{\zeta(1)}{\zeta(\alpha)}p^{-N\alpha}|y|^{-1}\int_{|x|\geq p^{N}|y|}|x|^{\alpha}\frac{dx}{|x|}\\
&= \frac{\zeta(1-\alpha)}{\zeta(\alpha)}\frac{\zeta(1)}{\zeta(\alpha)}p^{-N\alpha}|y|^{-1}\frac{\zeta(\alpha)}{\zeta(1)}p^{N\alpha}|y|^{\alpha}\\
&= \frac{\zeta(1-\alpha)}{\zeta(\alpha)}|y|^{\alpha-1} = k^\alpha * \delta_{x_0}(y).
\end{aligned}$$

On the other hand for $y \notin \check{B}_N$,

$$k^\alpha * \check{\delta}^\alpha_{(N)}(y) = \frac{\zeta(1-\alpha)}{\zeta(\alpha)}\frac{\zeta(1)}{\zeta(\alpha)}p^{-N\alpha}\int_{|x|\geq p^{-N}}\frac{dx}{|x|^2} = \frac{\zeta(1-\alpha)}{\zeta(\alpha)}\frac{\zeta(1)}{\zeta(\alpha)}p^{N(1-\alpha)}.$$

Since for any $x_0 \notin \check{B}_N$, $|y-x_0| < p^{-N}$ so $|y-x_0|^{\alpha-1} > p^{N(1-\alpha)}$; and since $\frac{\zeta(1)}{\zeta(\alpha)} < 1$, we get $k^\alpha * \check{\delta}^\alpha_{(N)}(y) < \frac{\zeta(1-\alpha)}{\zeta(\alpha)}|y-x_0|^{\alpha-1} = k^\alpha * \delta_{x_0}(y)$ (cf. [8], Appendix, A.3, A4, p. 401).

Note that $\check{\delta}^\alpha_{(N)}(1) = \frac{\zeta(1)}{\zeta(\alpha)}p^{-N\alpha}\int_{|x|\geq p^{-N}}\frac{dx}{|x|^{1+\alpha}} = 1$, i.e. $\check{\delta}^\alpha_{(N)}$ is a probability measure. We note also that we have, as is easily verified,

$$\begin{gathered}
\check{\delta}^\alpha_{(N)} \rightsquigarrow \delta_0 \text{ as } N \to \infty,\\
\check{\delta}^\alpha_{(N)} \rightsquigarrow \check{\delta}^\beta_{(N)} \text{ as } \alpha \to \beta.
\end{gathered}$$

The measure $\check{\delta}^\alpha_{(N)}$ defines an operator (via convolution), applicable to any function $f \in L^1_{\mathrm{loc}}$ such that $\int_{|x|>1}|f(x)|\frac{dx}{|x|^{1+\alpha}} < \infty$; in particular, it's applicable to any potential $f(x) = k^\alpha * \mu(x)$, $\mu \in \mathcal{M}_\alpha$.

8. α-super-harmonic-functions.

The following lemma establishes a relation between $\check{\delta}^\alpha_{(N)}$ and $k^{-\alpha}$ that will motivate much of the development of this section.

8.1. LEMMA. — *Let $f \in L^1_{\mathrm{loc}}$, $\int_{|x|>1} |f(x)| \frac{dx}{|x|^{1+\alpha}} < \infty$, and assume f is locally constant near x_0. Then*

$$k^{-\alpha} * f(x_0) = \frac{\zeta(1+\alpha)}{\zeta(1)} \lim_{N\to\infty} p^{(1+N)\alpha}(\delta_0 - \check{\delta}^{\alpha}_{(N)}) * f(x_0),$$

indeed, equality holds for $N \geq N_0$.

Proof.

$$\frac{\zeta(1+\alpha)}{\zeta(1)} p^{(1+N)\alpha}(\delta_0 - \check{\delta}^{\alpha}_{(N)}) * f(x_0)$$

$$= \frac{\zeta(1+\alpha)}{\zeta(\alpha)} p^{\alpha} \int_{|x|\geq p^{-N}} (f(x_0) - f(x_0+x)) \frac{dx}{|x|^{1+\alpha}}$$

$$= \frac{\zeta(1+\alpha)}{\zeta(-\alpha)} \int_{|x|\geq p^{-N}} (f(x_0+x) - f(x_0)) \frac{dx}{|x|^{1+\alpha}}$$

since $\frac{1}{\zeta(\alpha)} p^{\alpha} = p^{\alpha} - 1 = -\frac{1}{\zeta(-\alpha)}$, and the lemma is obtained upon letting $N \to \infty$.

8.2. DEFINITION. — *A function $f : \mathbf{Q}_p \to [0,\infty]$, $f \not\equiv \infty$, is called α-superharmonic (α-s.h.) if*

(i) *f is lower semicontinuous;*

(ii) $\int_{|x|>1} f(x) \frac{dx}{|x|^{1+\alpha}} < \infty$;

(iii) *For any $x_0 \in \mathbf{Q}_p$, there exists N_0 such that for all $N \geq N_0$: $f(x_0) \geq \check{\delta}^{\alpha}_{(N)} * f(x_0)$.*

8.3. DEFINITION. — *A function $f : \mathbf{Q}_p \to \mathbf{C} \cup \{\infty\}$, $f \not\equiv \infty$, is called α-harmonic (α-h.) at x_0, if :*

(i) *f is locally constant near x_0;*

(ii) $\int_{|x|>1} |f(x)| \frac{dx}{|x|^{1+\alpha}} < \infty$;

(iii) $k^{-\alpha} * f(x_0) = 0$.

By Lemma 8.1 conditions (i) and (iii) are equivalent to $f(x_0) = \check{\delta}^{\alpha}_{(N)} * f(x_0)$ for all $N \geq N_0$.

f will be called α-harmonic in an open set U, if it is α-h. at each $x_0 \in U$. Explicitly, $f(x_0) = (1-p^{-\alpha}) \int_{|x|\geq 1} f(x_0 + p^N x)|x|^{-\alpha} d^*x$, for all $N \geq N(x_0)$.

8.4. *Example.* — The constant function $f(x) \equiv c$ is everywhere harmonic, since $\check{\delta}^{\alpha}_{(N)}(1) = 1$; for $c \geq 0$ it is α-s.h.

8.5. *Example.* — A potential $f(x) = k^{\alpha} * \nu(x)$, $\nu \in \mathcal{M}^{+}_{\alpha}$, is α-s.h. Indeed, we have

$$\begin{aligned}\check{\delta}^{\alpha}_{(N)} * f(x_0) &= \int k^{\alpha} * \nu(x_0 - x) d\check{\delta}^{\alpha}_{(N)}(x) = \int k^{\alpha} * \check{\delta}^{\alpha}_{(N)}(x_0 - x) d\nu(x)\\ &\leq \int k^{\alpha}(x_0 - x) d\nu(x) = k^{\alpha} * \nu(x_0) = f(x_0).\end{aligned}$$

Moreover, in the complement of $\mathrm{supp}(\nu)$ it's α-h. Indeed, if $x_0 + B(p^{-N_0})$ is disjoint from $\mathrm{supp}(\nu)$, then for $x \in \mathrm{supp}(\nu)$, $x_0 - x \in \check{B}_N$ for all $N \geq N_0$, and we obtain

$$\begin{aligned}\check{\delta}^{\alpha}_{(N)} * f(x_0) &= \int k^{\alpha} * \nu(x_0 - x) d\check{\delta}^{\alpha}_{(N)}(x) = \int k^{\alpha} * \check{\delta}^{\alpha}_{(N)}(x_0 - x) d\nu(x)\\ &= \int k^{\alpha}(x_0 - x) d\nu(x) = f(x_0).\end{aligned}$$

We shall establish below a converse to these examples.

8.6. LEMMA.

(i) *Let $f \in L^1(B(p^N), dx)$ be extended to all of $\mathbf{Q}_p$ by putting, for $|x_0| > p^N$, $f(x_0) = \dfrac{\zeta(1)}{\zeta(\alpha)} p^{-N\alpha} \displaystyle\int_{|x| \leq p^N} f(x) dx \cdot |x_0|^{\alpha - 1}$; then f is α-h. in $\mathbf{Q}_p \setminus B(p^N)$.*

(ii) *Similarly, let $f \in L^1\left(\check{B}_N, \dfrac{dx}{|x|^{1+\alpha}}\right)$ be extended to all of $\mathbf{Q}_p$ by giving it the constant value $\dfrac{\zeta(1)}{\zeta(\alpha)} p^{-N\alpha} \displaystyle\int_{|x| \geq p^{-N}} f(x) \frac{dx}{|x|^{1+\alpha}}$ on $\mathbf{Q}_p \setminus \check{B}_N$; then f is α-h. in $\mathbf{Q}_p \setminus \check{B}_N$.*

Proof. — An easy calculation gives for $|x_0| > p^N > p^{-M}$:

$$\frac{\zeta(1)}{\zeta(\alpha)} \int_{\substack{|x| \geq p^{-M} \\ |x - x_0| > p^N}} |x - x_0|^{\alpha - 1} |x|^{-(1+\alpha)} dx = p^{M\alpha} |x_0|^{\alpha - 1} - p^{N\alpha} |x_0|^{-\alpha - 1}.$$

To prove (i) we calculate for $|x_0| > p^N$, $M > -N$,

$$\check{\delta}^{\alpha}_{(M)} * f(x_0) = \frac{\zeta(1)}{\zeta(\alpha)} p^{-M\alpha} \int_{|x| \geq p^{-M}} f(x_0 - x) \frac{dx}{|x|^{1+\alpha}}$$

$$= \frac{\zeta(1)}{\zeta(\alpha)} p^{-M\alpha} \Big[\int_{|y| \leq p^N} f(y) dy \cdot |x_0|^{-(1+\alpha)}$$

$$+ \int_{\substack{|x| \geq p^{-M} \\ |x - x_0| > p^N}} |x_0 - x|^{\alpha-1} |x|^{-(1+\alpha)} dx \cdot \frac{\zeta(1)}{\zeta(\alpha)} p^{-N\alpha} \int_{|y| \leq p^N} f(y) dy \Big]$$

$$= \frac{\zeta(1)}{\zeta(\alpha)} p^{-N\alpha} \int_{|y| \leq p^N} f(y) dy \cdot |x_0|^{\alpha - 1} \cdot \Big[|x_0|^{-2\alpha} p^{(N-M)\alpha}$$

$$+ |x_0|^{1-\alpha} p^{-M\alpha} \frac{\zeta(1)}{\zeta(\alpha)} \int_{\substack{|x| \geq p^{-M} \\ |x - x_0| > p^N}} |x - x_0|^{\alpha-1} |x|^{-(1+\alpha)} dx \Big]$$

$$= f(x_0)$$

by the definition of $f(x_0)$ and the above claim.

The calculation for (ii) is even easier : for $|x_0| < p^N$, $M > N$,

$$\check{\delta}^{\alpha}_{(M)} * f(x_0) = \frac{\zeta(1)}{\zeta(\alpha)} p^{-M\alpha} \Big[\int_{|x| \geq p^N} f(x) \frac{dx}{|x|^{1+\alpha}}$$

$$+ \int_{p^{-M} \leq |y| < p^{-N}} \frac{dy}{|y|^{1+\alpha}} \cdot \frac{\zeta(1)}{\zeta(\alpha)} p^{-N\alpha} \int_{|x| \geq p^N} f(x) \frac{dx}{|x|^{1+\alpha}} \Big]$$

$$= \frac{\zeta(1)}{\zeta(\alpha)} p^{-N\alpha} \int_{|x| \geq p^N} f(x) \frac{dx}{|x|^{1+\alpha}} \Big[p^{(N-M)\alpha}$$

$$+ \frac{\zeta(1)}{\zeta(\alpha)} p^{-M\alpha} \int_{p^{-M} \leq |y| < p^{-N}} \frac{dy}{|y|^{1+\alpha}} \Big]$$

$$= f(x_0).$$

We note that if f is α-s.h., then since $\check{\delta}^{\alpha}_{(N)} \underset{N \to \infty}{\rightsquigarrow} \delta_0$ and f is semicontinuous, we get $\lim_{N \to \infty} \check{\delta}^{\alpha}_{(N)} * f(x) = f(x)$. In particular, if f_1, f_2 are α-s.h., and $f_1(x) = f_2(x)$ for *a.a.x*'s then $f_1 \equiv f_2$ everywhere.

Let $U \subseteq \mathbf{P}^1(\mathbf{Q}_p)$ be an open bounded set and put $K = \mathbf{P}^1(\mathbf{Q}_p) \setminus U$. Let f be a continuous function in K such that $\int_{|x|>1} |f(x)| \frac{dx}{|x|^{1+\alpha}} < \infty$. Define :

$$h_f(y) = \int_K f(x) dP^{\alpha}_K \delta_y(x).$$

8.7. SOLUTION TO THE DIRICHLET PROBLEM. — *h_f is α-h. in U, $h_f\big|_{K_{\mathrm{reg}}} = f\big|_{K_{\mathrm{reg}}}$.*

Proof. — For $x \in K_{\mathrm{reg}}$, $P_K^\alpha \delta_x = \delta_x$ by definition, so indeed $h_f(x) = f(x)$. Fix $y_0 \in U$. The integral defining $h_f(y_0)$ is finite; indeed, take M such that $\check{B}_M \subseteq K$, then as is easily seen $P_K^\alpha \delta_{y_0}\big|_{\check{B}_M} \le P_{\check{B}_M}^\alpha \delta_{y_0}$, and by our assumption on f, we get

$$\begin{aligned}\int_{\check{B}_M} |f(x)| dP_K^\alpha \delta_{y_0}(x) &\le \int_{\check{B}_M} |f(x)| dP_{\check{B}_M}^\alpha \delta_{y_0}(x) \\ &= \frac{\zeta(1)}{\zeta(\alpha)} p^{-M\alpha} \int_{|x| \ge p^{-M}} |f(x)| \frac{dx}{|x|^{1+\alpha}} < \infty.\end{aligned}$$

Next, let N_0 be such that $y_0 + B(p^{-N_0}) \subseteq U$, then for any $N \ge N_0$, $K \subseteq y_0 + \check{B}_N$, and we have

$$\begin{aligned}\check{\delta}_{(N)}^\alpha * h_f(y_0) &= \int_{y_0 + \check{B}_N} h_f(y) dP_{y_0+\check{B}_N}^\alpha \delta_{y_0}(y) \\ &= \int_{y_0+\check{B}_N} \int_K f(x) dP_K^\alpha \delta_y dP_{y_0+\check{B}_N}^\alpha \delta_{y_0}(y) \text{ (by definition of } h_f) \\ &= \int_K f(x) d \int_{y_0+\check{B}_N} dP_K^\alpha \delta_y(x) dP_{y_0+\check{B}_N}^\alpha \delta_{y_0}(y) \\ &= \int_K f(x) dP_K^\alpha \delta_{y_0}(x) \text{ (by transitivity of Balayage)} \\ &= h_f(y_0).\end{aligned}$$

Since for any $y_0' \in y_0 + B(p^{-N_0})$, $y_0' + \check{B}_{N_0} = y_0 + \check{B}_{N_0}$, we also see that $h_f(y_0') = h_f(y_0)$, i.e. h_f is locally constant in U, concluding the proof that it is α-h. in U. We note that, since for $U \ni y \to x_0 \in K_{\mathrm{reg}} \cap \partial K$, $P_K^\alpha \delta_y \rightsquigarrow \delta_{x_0}$, we have $h_f(y) \to h_f(x_0)$, i.e. h_f is also continuous in K_{reg}, and its only discontinuities can appear in K_{irr}.

8.8. PRINCIPLE OF THE HARMONIC MINORANT. — *Let f be α-s.h., and let g be α-h. in an open set U, continuous in $\bar{U}$, and assume $\infty \notin U$ (possibly $\infty \in \bar{U}$).*

If $f \ge g$ on $\mathbf{Q}_p \setminus U$, then $f \ge g$ everywhere; if $f(x_0) = g(x_0)$ for some $x_0 \in U$, then $f = g$ almost everywhere.

Proof. — Let $d = f - g$; d is lower semicontinuous on $\bar{U}$. At any point $b \in \bar{U} \setminus U$, we have $\underline{\lim}_{U \ni x \to b} d(x) = \underline{\lim}_{x \to b} f(x) - \lim_{U \ni x \to b} g(x) \ge f(b) - g(b) \ge 0.$

Consequently, if d assumes negative values, then there is a point $x_0 \in U$, where $d(x_0)$ achieves its minimum, and we have a contradiction : for $N \gg 0$, $d(x_0) < \check{\delta}^{\alpha}_{(N)} * d(x_0) = \check{\delta}^{\alpha}_{(N)} * f(x_0) - \check{\delta}^{\alpha}_{(N)} * g(x_0) \leq f(x_0) - g(x_0) = d(x_0)$.

If $f(x_0) = g(x_0)$ for some $x_0 \in U$, then again $\check{\delta}^{\alpha}_{(N)} * d(x_0) = 0$, hence $d(x) = 0$ for $a.a.x$'s.

8.9. COROLLARY. — *Let f be α-s.h.. If $f(x_0) = \inf f(x)$ for some $x_0 \in U$, then $f(x) \equiv f(x_0)$.*

E.g., we have $G^{\alpha}_U(y_1, y_2) > 0$ for all $y_1, y_2 \in U$; indeed, $G^{\alpha}_U(y_1, y_2)$ is α-s.h., and α-h. for $y_1 \neq y_2$.

We have the following converse to the above principle :

8.10. LEMMA. — *Let $f : \mathbf{Q}_p \to [0, \infty]$ ($f \not\equiv \infty$) be lower semicontinuous, $\displaystyle\int_{|x|>1} f(x) \frac{dx}{|x|^{1+\alpha}} < \infty$, and assume f satisfies the principle of the harmonic minorant. Then f is α-s.h.*

Proof. — Fix $x_0 \in \mathbf{Q}$, $N \in \mathbf{Z}$ and let g be the function constructed in lemma (ii) with respect to $f\big|_{x_0 + \check{B}_N}$. Since g is α-h. and continuous (even constant) in the open (and closed) set $\{x \mid |x - x_0| < p^{-N}\}$, and since on the complement we have $f \geq g$ (even $f = g$), we obtain by applying the principle of the harmonic minorant $\check{\delta}^{\alpha}_{(N)} * f(x_0) = g(x_0) \leq f(x_0)$.

Noting that in the above proof N was arbitrary, we get

8.11. COROLLARY. — *If f is α-s.h., then $\check{\delta}^{\alpha}_{(N)} * f \leq f$ for all N.*

8.12. THEOREM. — *The class of α-s.h. functions is closed under :*

(i) *addition and multiplication by a positive constant;*

(ii) *passage to the limit of a sequence converging uniformly;*

(iii) *passage to the limit of a monotone increasing sequence (if the limit is $\not\equiv \infty$);*

(iv) *the operation* inf, *applied to a finite number of elements;*

(v) *convolution with a measure (if the result is $\not\equiv \infty$).*

Proof. — (i), (ii) and (iv) are trivial. To prove (iii), assume f_n are α-s.h., $f_n \leq f_{n+1}$, $f(x) = \lim f_n(x) \not\equiv \infty$. Then $f(x) \geq f_n(x) \geq \check{\delta}^{\alpha}_{(N)} * f_n(x)$, and by monotonicity we can let $n \to \infty$ to obtain $f(x) \geq \check{\delta}^{\alpha}_{(N)} * f(x)$. To prove (v), let f be α-s.h., μ a positive measure such that $\mu * f \not\equiv \infty$. Then we have $\mu * f = \lim_{n\to\infty} (\mu_{|B(p^n)}) * f$ a monotone increasing limit, so by (iii) we are reduced to the case that μ has compact support, but then we can use Fubini theorem to obtain

$$\mu * f(x_0) \geq \mu * (\check{\delta}^{\alpha}_{(N)} * f)(x_0) = (\mu * f) * \check{\delta}^{\alpha}_{(N)}(x_0).$$

8.13. Corollary. — *If f is α-s.h., then $k^{-\alpha} * f$ is a positive measure.*

Proof. — Here $k^{-\alpha} * f(\varphi) \overset{\text{def}}{=} k^{-\alpha}(f * \varphi)$ for $\varphi \in \mathcal{S}(\mathbf{Q}_p)$, and we need to show that, if $\varphi \geq 0$, then $k^{-\alpha} * f(\varphi) \geq 0$. By (v), $f * \varphi$ is again α-s.h., and noting that it satisfies the conditions of Lemma 8.1, we get $k^{-\alpha}(f * \varphi) = \dfrac{\zeta(1+\alpha)}{\zeta(1)} \lim_{N\to\infty} p^{(1+N)\alpha}(\delta_0 - \check{\delta}^{\alpha}_{(N)}) * (f * \varphi) \geq 0.$

We now arrive at the following important result :

8.14. Riesz representation theorem. — *A function f is α-s.h. if and only if $f = k^{\alpha} * \nu + c$, $\nu \in \mathcal{M}^{+}_{\alpha}$, $c \geq 0$. In this representation, the measure ν and the constant c are unique. Moreover, f is α-h. at x_0 if and only if $x_0 \notin \operatorname{supp}(\nu)$.*

Proof. — The "if" part of the first statement has been established above, so let f be α-s.h.. Consider the function f_N constructed in Lemma 8.6, (i) for $f_{|B(p^N)}$, i.e.

$$f_N(x) = \begin{cases} f(x) & |x| \leq p^N \\ a_N \cdot |x|^{\alpha-1} & |x| > p^N \end{cases} \text{ where } a_N = \frac{\zeta(1)}{\zeta(\alpha)} p^{-N\alpha} \int_{|x|\leq p^N} f(x)dx.$$

Since f_N is α-h. outside $B(p^N)$, and since $f_N(x) \leq f(x)$ for $x \in B(p^N)$ and $x = \infty$, we get by applying the principle of the harmonic minorant : $f_N \leq f$ everywhere. Consequently, for $x_0 \in B(p^N)$: $\check{\delta}^{\alpha}_{(M)} * f_N(x_0) \leq \check{\delta}^{\alpha}_{(M)} * f(x_0) \leq f(x_0) = f_N(x_0)$, i.e. f_N is again α-s.h.. Moreover, since f_{N-1} is α-h. outside $B(p^{N-1})$, and since $f_{N-1}(x) \leq f_N(x)$, for $x \in B(p^{N-1})$ and $x = \infty$, we obtain by another application of the principle of the harmonic minorant : $f_{N-1} \leq f_N$ everywhere.

Since $f_N(x) - a_N|x|^{\alpha-1}$ has compact support, we easily get

$$f_N(x_0) - a_N|x_0|^{\alpha-1} = k^{\alpha} * (k^{-\alpha} * (f_N(x) - a_N|x|^{\alpha-1}))(x_0) \text{ for } a.a.x_0\text{'s}.$$

But the right hand side equals $k^\alpha * (k^{-\alpha} * f_N)(x_0) - a_N|x_0|^{\alpha-1}$, hence

$$f_N(x) = k^\alpha * (k^{-\alpha} * f_N)(x) \text{ for } a.a.x\text{'s.}$$

Since f_N is α-s.h., $\nu_N = k^{-\alpha} * f_N$ is a positive measure and $k^\alpha * \nu_N$ is again α-s.h., hence

$$f_N(x) = k^\alpha * \nu_N(x) \text{ for all } x.$$

Since the f_N monotonically increase to f, we obtain by the dichotomy principle (4.6)

$$f(x) = \lim_{N\to\infty} f_N(x) = \lim_{N\to\infty} k^\alpha * \nu_N(x) = k^\alpha * \nu(x) + c \text{ for } a.a.x\text{'s}$$

for some $\nu \in \mathcal{M}_\alpha^+$ and $c \geq 0$. Again, since both sides are α-s.h., we get

$$f(x) = k^\alpha * \nu(x) + c \text{ for all } x.$$

Furthermore, since $\displaystyle\int_{|x|>1} k^\alpha * \nu(x) \frac{dx}{|x|^{1+\alpha}} < \infty$, we have

$$\check{\delta}^\alpha_{(-N)}(k^\alpha * \nu) = \frac{\zeta(1)}{\zeta(\alpha)} \int_{|x|\geq 1} k^\alpha * \nu(p^{-N}x) \frac{dx}{|x|^{1+\alpha}} \xrightarrow[N\to\infty]{} 0$$

hence : $\lim_{N\to-\infty} \check{\delta}^\alpha_{(N)}(f) = c$, which proves the uniqueness of c, and hence of ν. The harmonicity statement is clear.

8.15. Corollary. — *A function f is a potential of a measure if and only if f is α-s.h. and $\lim_{N\to-\infty} \check{\delta}^\alpha_{(N)}(f) = 0$.*

8.16. Corollary. — *Let f be α-s.h. and $\nu \in \mathcal{M}_\alpha^+$. Then $\inf\{f, k^\alpha * \nu\}$ is a potential of a measure.*

Proof. — Since both f and $k^\alpha * \nu$ are α-s.h. so is $\inf\{f, k^\alpha * \nu\}$, but we also have

$$\lim_{N\to-\infty} \check{\delta}^\alpha_{(N)}(\inf\{f, k^\alpha * \nu\}) \leq \lim_{N\to-\infty} \check{\delta}^\alpha_{(N)}(k^\alpha * \nu) = 0.$$

As another application we have,

8.17. Principle of domination. — *Let f be α-s.h., $\nu \in \mathcal{E}_\alpha^+$; then if $k^\alpha * \nu(x) \leq f(x)$ for $\nu - a.a.x$'s, then $k^\alpha * \nu \leq f$ everywhere.*

Proof. — By the above corollary $\inf\{f, k^\alpha * \nu\} = k^\alpha * \nu'$, for some $\nu' \in \mathcal{M}_\alpha^+$. But $\|\nu'\|_\alpha = \displaystyle\int k^\alpha * \nu'(x) d\nu'(x) \leq \int k^\alpha * \nu(x) d\nu'(x) = \int k^\alpha * \nu'(x) d\nu(x) \leq \int k^\alpha * \nu(x) d\nu(x) = \|\nu\|_\alpha < \infty$, i.e. $\nu' \in \mathcal{E}_\alpha^+$. Moreover,

$$\|\nu - \nu'\|_\alpha^2 = \int (k^\alpha * \nu(x) - k^\alpha * \nu'(x))(d\nu(x) - d\nu'(x))$$

and we can restrict the integral to $\{x \mid k^\alpha * \nu(x) > k^\alpha * \nu'(x)\} = \{x \mid k^\alpha * \nu(x) > f(x)\}$ which by assumption has ν-measure 0, hence we obtain $\|\nu - \nu'\|_\alpha^2 = -\int (k^\alpha * \nu(x) - k^\alpha * \nu'(x)) d\nu'(x) \leq 0$, hence $\|\nu - \nu'\|_\alpha = 0$, and by the 2^{nd} uniqueness principle $\nu = \nu'$, so $k^\alpha * \nu = k^\alpha * \nu' = \inf\{f, k^\alpha * \nu\}$, i.e. $k^\alpha * \nu \leq f$.

8.18. *Remark.* — The condition $\nu \in \mathcal{E}_\alpha^+$ can be weakened to : $k^\alpha * \nu(x) < \infty$ for $\nu - a.a.x$'s (cf. [8], Theorem 1.29, p. 115).

Recall that for a compact set K, we have γ_K its equilibrium measure, and denote by $f_K = k^\alpha * \gamma_K$ its potential, so $f_K(x) \leq 1$, $f_K(x) = 1$ for $\alpha - a.a.x \in K$.

8.19. COROLLARY. — *If $K' \subseteq K$, then $f_{K'} \leq f_K$.*

Proof. — $f_{K'}(x) = 1 = f_K(x)$ for $\alpha - a.a.x \in K'$, so in particular $f_{K'}(x) \leq f_K(x)$ for $\gamma_{K'} - a.a.x$'s, and by the principle of domination $f_{K'} \leq f_K$ everywhere.

8.20. COROLLARY. — $f_{K_1 \cup K_2} + f_{K_1 \cap K_2} \leq f_{K_1} + f_{K_2}$.

Proof. — Since $f_{K_1 \cap K_2} \leq f_{K_2}$, and since for $\alpha - a.a.x \in K_1$, $f_{K_1 \cup K_2}(x) = 1 = f_{K_1}(x)$, we obtain $k^\alpha * (\gamma_{K_1 \cup K_2} + \gamma_{K_1 \cap K_2})(x) = f_{K_1 \cup K_2}(x) + f_{K_1 \cap K_2}(x) \leq f_{K_1}(x) + f_{K_2}(x) = k^\alpha * (\gamma_{K_1} + \gamma_{K_2})(x)$ for $\alpha - a.a.x \in K_1$. By symmetry this inequality holds also for $\alpha - a.a.x \in K_2$, hence it holds for $(\gamma_{K_1 \cup K_2} + \gamma_{K_1 \cap K_2}) - a.a.x$'s and by the principle of domination it holds everywhere.

This corollary gives us the following convexity property of cap_α.

8.21. COROLLARY.
$$\mathrm{cap}_\alpha(K_1 \cup K_2) + \mathrm{cap}_\alpha(K_1 \cap K_2) \leq \mathrm{cap}_\alpha(K_1) + \mathrm{cap}_\alpha(K_2).$$

Proof. — Set $\nu = \gamma_{K_1} + \gamma_{K_2} - \gamma_{K_1 \cup K_2} - \gamma_{K_1 \cap K_2}$, so that $k^\alpha * \nu(x) \geq 0$ for all x. We have

$$\begin{aligned}
\mathrm{cap}_\alpha(K_1) + \mathrm{cap}_\alpha(K_2) - \mathrm{cap}_\alpha(K_1 \cup K_2) - \mathrm{cap}_\alpha(K_1 \cap K_2)& \\
&= \gamma_{K_1}(1) + \gamma_{K_2}(1) - \gamma_{K_1 \cup K_2}(1) - \gamma_{K_1 \cap K_2}(1) \\
&= \nu(1) = \int k^\alpha * \gamma_{K_1 \cup K_2}(x) d\nu(x) \\
&= \int k^\alpha * \nu(x) d\gamma_{K_1 \cup K_2}(x) \geq 0.
\end{aligned}$$

BIBLIOGRAPHY

[1] C. Berg, G. Forst, Potential theory on locally compact abelian groups, Ergebn. d. Math., 87, Springer-Verlag (1975).

[2] J. Bliedtner, W. Hansen, Potential theory, Universitext, Springer-Verlag (1986).

[3] J.W.S. Cassels, A. Fröhlich, Algebraic number theory, Thompson, 1967.

[4] W. Feller, An introduction to probability theory and its applications, Vol. II, John Wiley & Sons, 1970.

[5] S. Haran, Riesz potentials and explicit sums in arithmetic, Inventiones Math., 101 (1990), 697–703.

[6] S. Haran, Index theory, potential theory and the Riemann hypothesis, in Proc. LMS Symp. on L-functions and Arithmetic, Durham, 1989.

[7] G.A. Hunt, Markoff processes and potentials I–III, Illinois J. Math., 1 (1957), 44–93 and 316–369; 2 (1958), 151–213.

[8] N.S. Landkof, Foundations of modern potential theory, Grundl. d. math. Wiss., 180, Springer-Verlag (1972).

[9] S.C. Port, C.J. Stone, Infinitely divisible processes and their potential theory I–II, Ann. Inst. Fourier, 21-2 (1971), 157–275; 21-4 (1971), 179–265.

[10] M. Taibleson, Fourier analysis over local fields, Princeton Univ. Press, 1975.

[11] A. Weil, Fonction zêta et distributions, séminaire Bourbaki (1966), 312.

Manuscrit reçu le 24 août 1992.

Shai HARAN,
Department of Mathematics
Technion Institute of Technology
Technion City
32000 Haifa (Israël).