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SUPERHARMONIC EXTENSION AND HARMONIC APPROXIMATION

by Stephen J. GARDINER

0. Introduction.

Let Ω be an open set in $\mathbb{R}^n$ and E be a relatively closed subset of Ω.

This paper solves the following problems. Find necessary and sufficient conditions on (Ω, E) so that :

(i) for each superharmonic function u on E, there is a superharmonic function $\overline{u}$ on Ω such that $\overline{u} = u$ on E (or on an open set which contains E);

(ii) for each harmonic (resp. superharmonic) function u on E and each positive number ε, there is a harmonic (resp. superharmonic) function v on Ω such that $u - \varepsilon \leq v \leq u + \varepsilon$ on E;

(iii) for each function h which is continuous on E and harmonic on E°, and for each positive number ε, there is a harmonic function H on Ω such that $|H - h| < \varepsilon$ on E;

(iv) for each harmonic (resp. superharmonic) function u on E, there is a harmonic (resp. superharmonic) function v on Ω and a positive number a such that $u - a \leq v \leq u + a$ on E.

Tangential harmonic and superharmonic approximation are also discussed.

Key words : Superharmonic functions – Extension theorem – Harmonic measure – Harmonic approximation – Thin set.
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1. Superharmonic extension.

Let Ω be an open set in Euclidean space $\mathbb{R}^n$ $(n \geq 2)$ and suppose that $E \subseteq \Omega$. A function u will be called *superharmonic* (resp. *harmonic*) *on* E if u is defined and superharmonic (resp. harmonic) on an open set which contains E. We call (Ω, E) an *extension pair for superharmonic functions* if, for each superharmonic function u on E, there is a superharmonic function $\overline{u}$ on Ω such that $\overline{u} = u$ on E. Further, (Ω, E) will be called a *strong extension pair for superharmonic functions* if it can be arranged that $\overline{u} = u$ on an open set which contains E. In the latter case we preserve not only the values of u on E, but also the associated Riesz measure on an open set which contains E. It can be observed immediately that, for either of the above extension properties to hold, E must be closed relative to Ω. For, if u is the fundamental subharmonic function with pole at some point X_0 of $(\overline{E}\backslash E) \cap \Omega$, then u is harmonic on E, but any function $\overline{u}$ on Ω which satisfies $\overline{u} = u$ on E is not bounded below near X_0, and so cannot be superharmonic.

We will use Ω^* to denote the Alexandroff one-point compactification of Ω, and $\mathcal{A}$ to denote the ideal point. However, $\overline{A}$, A° and ∂A will always represent the Euclidean closure, interior and boundary (respectively) of a subset A of $\mathbb{R}^n$. A subset A of Ω will be called Ω-*bounded* if $\overline{A}$ is a compact subset of Ω. Recall that a topological space is called *locally connected* if, for each point X in the space and each neighbourhood ω of X, there is a connected neighbourhood ω' of X such that $\omega' \subseteq \omega$. In the following result the set $\Omega^*\backslash E$ can fail to satisfy this condition only in the case where $X = \mathcal{A}$.

THEOREM 1. — *Let Ω be an open set in $\mathbb{R}^n$ and E be a relatively closed subset of Ω. Then (Ω, E) is a strong extension pair for superharmonic functions if and only if $\Omega^*\backslash E$ is both connected and locally connected.*

The condition that $\Omega^*\backslash E$ be connected is clearly equivalent to saying that $\Omega\backslash E$ has no Ω-bounded (connected) components. The particular case of Theorem 1 where E is compact (and so the local connectedness condition on $\Omega^*\backslash E$ is redundant) is closely related to several known results : see, for example, [8, Lemma 2.3 and §7], [2, Theorem 1] and [16, Theorem 2.4]. It appears that only Armitage [2] has previously considered the non-compact case (but see also the final note of this paper). Theorem 2 of [2] gives conditions on an open set ω which are sufficient to ensure that, for each superharmonic function u on ω, there is a superharmonic function $\overline{u}$ on $\mathbb{R}^n$ satisfying $\overline{u} = u$ on the set $\{X : \operatorname{dist}(X, \mathbb{R}^n \backslash \omega) > a\}$, where a is a fixed positive number. A question raised by [2] (see the last two lines of p. 216) corresponds to asking if $(\mathbb{R}^2, E)$ is a strong extension pair for superharmonic functions, where

$$E = \{(x_1, x_2) : x_2 \geq 0\} \backslash \bigcup_{k=1}^{\infty} \{(x_1, x_2) : 2k < x_1 < 2k+1 \text{ and } x_2 < 5k\}.$$

Theorem 1 supplies an affirmative answer. Below we give an example of a pair (Ω, E) such that $\Omega^* \backslash E$ is connected but not locally connected.

Example 1. — If

$$S = \bigcup_{k=1}^{\infty} \left\{ (x_1, x_2) : \frac{1}{2k+1} < x_1 < \frac{1}{2k} \text{ and } x_2 < k \right\}, \tag{1}$$

then $(\mathbb{R}^2)^* \backslash \partial S$ is connected. However, $(\mathbb{R}^2)^* \backslash \partial S$ is not locally connected : the set $(\mathbb{R}^2)^* \backslash (\partial S \cup K)$, where $K = [0,1] \times \{0\}$, is a neighbourhood of $\mathcal{A}$ in $(\mathbb{R}^2)^* \backslash \partial S$ which does not contain any connected neighbourhood of $\mathcal{A}$. It follows from Theorem 1 that $(\mathbb{R}^2, \partial S)$ is not a strong extension pair for superharmonic functions. (In fact, more can be said : see Example 3(b) below.)

The condition that $\Omega^* \backslash E$ be both connected and locally connected has arisen in the theory of holomorphic and harmonic approximation (see, for example, Arakeljan [1] and Theorem A below), but we do not make use of such results in proving Theorem 1. Our proof is based, in part, on ideas contained in [2].

If E is a relatively closed subset of Ω, then $\widehat{E}$ will denote the union of E with the Ω-bounded components of $\Omega \backslash E$. In the case where E is compact, we note that $\operatorname{dist}(\widehat{E}, \mathbb{R}^n \backslash \Omega) > 0$, and so $\mathbb{R}^n \backslash \widehat{E}$ has finitely many components. If V is an open set such that $\mathbb{R}^n \backslash V$ is not polar, then we use $\mu_{V,X}$ to denote harmonic measure for V and a point X in V. (For an account of the Dirichlet problem and related concepts, see Helms [13] or Doob [9].) The collection of all Borel subsets of $\mathbb{R}^n$ will be denoted by $\mathcal{B}$. Before presenting a complete characterization of extension pairs for superharmonic functions (see Theorem 3) we give below a special case of the solution which has a simpler formulation.

THEOREM 2. — *Let Ω be an open set in $\mathbb{R}^n$ and E be a compact subset of Ω such that each point of $\partial\widehat{E}$ is regular for the Dirichlet problem on $\mathbb{R}^n\backslash\widehat{E}$. Then (Ω, E) is an extension pair for superharmonic functions if and only if each Ω-bounded component V_0 of $\Omega\backslash E$ satisfies the following conditions :*

(i) *V_0 is regular for the Dirichlet problem, and*

(ii) *given X_k in V_k $(k = 0, \ldots, m)$, where $V_1, \ldots, V_m$ denote the components of $\mathbb{R}^n\backslash\widehat{E}$, there are positive constants $c_1, \ldots, c_m$ such that*

$$\mu_{V_0,X_0}(A) \leq \sum_{k=1}^{m} c_k \mu_{V_k,X_k}(A) \qquad (A \in \mathcal{B}). \tag{2}$$

It is clear from Harnack's inequalities that, if there exist constants $c_1, \ldots, c_m$ such that (2) holds for a given choice of $X_0, \ldots, X_m$ then corresponding constants can be found for any other choice of $X_0, \ldots, X_m$. Also, conditions (i) and (ii) above together imply that $\partial V_0 \subseteq \partial\widehat{E}$. (To see this, we note that $\partial V_0\backslash\widehat{E}$ is a relatively open subset of ∂V_0 which has zero harmonic measure, by (ii). Thus every point of $\partial V_0\backslash\partial\widehat{E}$ is irregular, and (i) now shows that $\partial V_0 \subseteq \partial\widehat{E}$.) Theorem 2 will be illustrated below by means of pairs $(\mathbb{R}^2, E)$, where E is a union of finitely many line segments. It is straightforward to write down corresponding examples in higher dimensions. Our assertions are based on the elementary observation that, if

$$S_\alpha = \{re^{i\theta} : 0 < \theta < \alpha \text{ and } 0 < r < 2\}, \text{ and } z_\alpha = e^{i\alpha/2} \quad (0 < \alpha \leq 2\pi)$$

(identifying $\mathbb{R}^2$ with $\mathbb{C}$ in the usual manner), then the restriction of μ_{S_α,z_α} to the interval (0,2) is absolutely continuous with respect to one-dimensional Lebesgue measure λ, and there are positive constants $k_1(\alpha)$, $k_2(\alpha)$ such that

$$k_1(\alpha)t^{\pi/\alpha-1} \leq (d\mu_{S_\alpha,z_\alpha}/d\lambda)(t) \leq k_2(\alpha)t^{\pi/\alpha-1} \qquad (0 < t \leq 1).$$

Examples 2. — (a) Let P denote an open polygon in $\mathbb{R}^2$. Then $(\mathbb{R}^2, \partial P)$ is an extension pair for superharmonic functions if and only if P is convex.

(b) Let

$$F_1 = [0,2]^2 \cup ([2,4] \times \{2\}), \qquad F_2 = [0,2]^2 \cup ([2,4] \times \{1\}),$$
$$F_3 = ([0,1] \cup [2,3])^2 \cup [1,2]^2, \quad F_4 = ([0,1) \cup (1,2]) \times [0,1],$$
$$F_5 = [0,2]^2\backslash\{(1,1)\}.$$

Then $(\mathbb{R}^2, \partial F_1)$ and $(\mathbb{R}^2, \partial F_3)$ are extension pairs for superharmonic functions. However $(\mathbb{R}^2, \partial F_2)$ and $(\mathbb{R}^2, \partial F_4)$ violate condition (ii) of Theorem 2, and $(\mathbb{R}^2, \partial F_5)$ violates condition (i), so these are not extension pairs.

We come now to the question of characterizing extension pairs (Ω, E) in the absence of any special conditions on E. If W is an open set which satisfies $\widehat{E} \subseteq W \subseteq \Omega$, then we define a class of superharmonic functions on W by

$$\mathcal{S}_W = \{v : v \text{ is positive and superharmonic on } W, v = 1 \text{ on } \widehat{E}\}.$$

Also, the Riesz measure associated with a superharmonic function v is denoted by ν_v. By a countable set we mean one which is either finite or countably infinite.

THEOREM 3. — *Let Ω be an open set in $\mathbb{R}^n$ and E be a relatively closed subset of Ω. Then (Ω, E) is an extension pair for superharmonic functions if and only if :*

(i) *each Ω-bounded component of $\Omega \backslash E$ is regular for the Dirichlet problem,*

(ii) *$\Omega^* \backslash \widehat{E}$ is locally connected, and*

(iii) *for each countable collection $\{(X_k, c_k) : k \in I\}$: of pairs from $(\widehat{E} \backslash E) \times (0, \infty)$ such that the points X_k are distinct and have no limit point in Ω, there exist an open set W satisfying $\widehat{E} \subseteq W \subseteq \Omega$ and a function v in $\mathcal{S}_W$ such that*

$$\sum_{k \in I} c_k \mu_{(\widehat{E} \backslash E), X_k}(A) \leq \nu_v(A) \qquad (A \in \mathcal{B}). \tag{3}$$

As in the case of Theorem 2, we observe that conditions (i) and (iii) above together imply that $\partial V \subseteq \partial \widehat{E}$ for each Ω-bounded component V of $\Omega \backslash E$. Condition (iii) is similar in nature to condition (ii) of Theorem 2, but it also implies that a given compact subset of Ω cannot intersect "arbitrarily large" Ω-bounded components of $\Omega \backslash E$. This is made precise below.

LEMMA 1. — *Let Ω be an open set in $\mathbb{R}^n$, let E be a relatively closed subset of Ω, and suppose that condition (iii) of Theorem 3 holds. Then,* *for each compact subset K of Ω, there is a compact subset L of Ω that contains every Ω-bounded component of $\Omega\backslash E$ which intersects K.*

Examples 3. — (a) Let (P_k) be a sequence of open polygons in $\mathbb{R}^2$ such that the closures $\overline{P}_k$ are pairwise disjoint and only a finite number of the polygons intersect any given compact set. Then $\left(\mathbb{R}^2, \bigcup_k \partial P_k\right)$ is an extension pair for superharmonic functions if and only if each of the polygons is convex. The "if" part of this assertion can be checked by choosing W to be $\bigcup_k Q_k$ in condition (iii) of Theorem 3, where (Q_k) is a suitable sequence of pairwise disjoint open sets such that $\overline{P}_k \subset Q_k$ for each k. The "only if" part follows from Example 2(a).

(b) Let S be as in (1). Then $(\mathbb{R}^2, \partial S)$ is not an extension pair, because condition (ii) of Theorem 3 is violated. Also, $(\mathbb{R}^2, \partial S \cup ([0,1] \times \{0\}))$ is not an extension pair because (iii) fails, by Lemma 1.

Theorems 1-3 are established in §4-6, following some preparatory material in §3. Lemma 1 is proved in §3.3.

2. Harmonic approximation.

We call (Ω, E) a *Runge pair for harmonic* (resp. *superharmonic*) *functions* if, for each harmonic (resp. superharmonic) function u on E and each positive number ε, there is a harmonic (resp. superharmonic) function v on Ω such that $u - \varepsilon \leq v \leq u + \varepsilon$ on E. Further, inspired by the main result of [1], we call (Ω, E) an *Arakeljan pair for harmonic functions* if, for each function h which is continuous on E and harmonic on E°, and for each positive number ε, there is a harmonic function H on Ω such that $|H - h| < \varepsilon$ on E. Reasoning as in the opening paragraph of §1, it is clear that these approximation properties also require E to be closed relative to Ω. The following important result is due to Gauthier, Goldstein and Ow (see [10, Theorem 3] when $n = 2$, and [11, Theorem 1] when $n \geq 3$).

THEOREM A. — *Let Ω be an open set in $\mathbb{R}^n$ and E be a relatively closed subset of Ω. If $\Omega^*\backslash E$ is both connected and locally connected, then (Ω, E) is a Runge pair for harmonic functions.*

The next result shows that the hypotheses of Theorem A can be considerably weakened.

THEOREM 4. — *Let Ω be an open set in $\mathbb{R}^n$ and E be a relatively closed subset of Ω. The following are equivalent :*

(a) (Ω, E) *is a Runge pair for superharmonic functions;*

(b) (Ω, E) *is a Runge pair for harmonic functions;*

(c) (Ω, E) *satisfies the conditions below :*

(i) $\Omega\backslash\widehat{E}$ *and* $\Omega\backslash E$ *are thin at the same points of* E, *and*

(ii) *for each compact subset* K *of* Ω, *there is a compact subset* L *of* Ω *which contains every* Ω*-bounded component of* $\Omega\backslash(E\cup K)$ *whose closure intersects* K.

Theorem 4 appears to be new even in the case where E is compact (and hence condition (c)(ii) is redundant). It is clear from this result and Examples 2 that every extension pair for superharmonic functions is a Runge pair for harmonic functions, but not conversely. Condition (c)(ii) of Theorem 4 implies that $\Omega^*\backslash\widehat{E}$ is locally connected, and also that the conclusion of Lemma 1 holds. This condition is presented in [10, Theorem 2] as necessary for (Ω, E) to be a Runge pair for harmonic functions when $n = 2$, but the proof given there is defective : see [5], where it is shown that $\Omega^*\backslash\widehat{E}$ must be locally connected. Condition (c)(i) implies that each Ω-bounded component V of $\Omega\backslash E$ is regular for the Dirichlet problem and satisfies $\partial V \subseteq \partial\widehat{E}$ (see the second paragraph of §7.1). However, the converse of this statement is false when $n \geq 3$, as the following example shows. Let $\phi_n : [0, +\infty) \to \mathbb{R}\cup\{+\infty\}$ be the function defined by $\phi_2(t) = \log(1/t)$ or $\phi_n(t) = t^{2-n}$ if $n \geq 3$. (We interpret $\phi_n(0)$ as $+\infty$ in either case.)

Example 4. — Let $n \geq 3$, let $\{Y_k' : k \in \mathbb{N}\}$ be a dense subset of $[0,1]^{n-1}$, and define

$$u(X') = \sum_{k=1}^{\infty} 2^{-k}\phi_{n-1}(|X' - Y_k'|) \qquad (X' \in \mathbb{R}^{n-1}).$$

Further, let

$$E = \partial([0,1]^{n-1}\times[-1,0])\cup\{(X', x_n) \in [0,1]^{n-1}\times(0,1] : u(X') \leq \phi_n(x_n)\}.$$

Then the only bounded component of $\mathbb{R}^n\backslash E$ is given by $V = (0,1)^{n-1}\times(-1,0)$, which is regular for the Dirichlet problem and satisfies $\partial V \subseteq \partial\widehat{E}$. However, $\mathbb{R}^n\backslash\widehat{E}$ is thin at each point of $(0,1)^{n-1}\times\{0\}$, whereas $\mathbb{R}^n\backslash E$ is not. (See §12 for details.) Thus $(\mathbb{R}^n, E)$ is not a Runge pair for harmonic functions.

Theorem 4 can be combined with known results to obtain the following.

THEOREM 5. — *Let Ω be an open set in $\mathbb{R}^n$ and E be a relatively closed subset of Ω. Then (Ω, E) is an Arakeljan pair for harmonic functions if and only if :*

(i) *$\Omega\backslash\widehat{E}$ and $\Omega\backslash E^\circ$ are thin at the same points of E; and*

(ii) *for each compact subset K of Ω, there is a compact subset L of Ω which contains every Ω-bounded component of $\Omega\backslash(E\cup K)$ whose closure intersects K.*

Example 5. — Let $n \geq 3$, let E be as in Example 4, and let $E_1 = \widehat{E}$. Then $(\mathbb{R}^n, E_1)$ (trivially) satisfies conditions (c)(i)-(ii) of Theorem 4, but not condition (i) of Theorem 5 (see §12). Hence $(\mathbb{R}^n, E_1)$ is a Runge pair for harmonic functions, but not an Arakeljan pair for harmonic functions. (Another such example may be found in [5, p. 21].)

The next result shows that the situation described in Example 5 cannot arise when $n = 2$.

THEOREM 6. — *Let Ω be an open set in $\mathbb{R}^2$ and E be a relatively closed subset of Ω. The following are equivalent :*

(a) *(Ω, E) is a Runge pair for superharmonic functions;*

(b) *(Ω, E) is a Runge pair for harmonic functions;*

(c) *(Ω, E) is an Arakeljan pair for harmonic functions;*

(d) (i)*$\partial E = \partial\widehat{E}$, and*

(ii) *for each compact subset K of Ω, there is a compact subset L of Ω which contains every Ω-bounded component of $\Omega\backslash(E\cup K)$ whose closure intersects K.*

Theorems 5 and 6 solve [6, Problem 9.10], posed by M. Goldstein. Necessary and sufficient conditions for (Ω, E) to be an Arakeljan pair for harmonic functions have recently been given also in [5, Theorem 3.3] and [12, Theorem 1], but the conditions given there are not as explicit as those above.

Now suppose that Ω has a Green function $G_\Omega(.,.)$, fix X_0 in Ω, and define

$$g(X) = \min\{1, G_\Omega(X_0, X)\} \qquad (X \in \Omega).$$

In this case we can add the following equivalent conditions to Theorem 4 :

(d) (resp. (e)) for each harmonic (resp. superharmonic) function u on E and each positive number ε, there is a harmonic (resp. superharmonic) function v on Ω such that $u - \varepsilon g \leq v \leq u + \varepsilon g$ on E.

Also, conditions (i)-(ii) of Theorem 5 are equivalent to the following : *for each function h which is continuous on E and harmonic on E°, and for each positive number ε, there is a harmonic function H on Ω such that $|H - h| < \varepsilon g$ on E.* This leads to three additional equivalent conditions in Theorem 6. These assertions have essentially the same proofs as Theorems 4-6 except that, in place of Theorem A above and Theorem B of §8.1, we appeal to corresponding recent results of Armitage and Goldstein [3] concerning tangential harmonic approximation. Saginyan [17, Theorem 1] has announced a result for tangential harmonic approximation which is similar in nature to the modified form of Theorem 5 described above, but no proof has yet appeared.

We call (Ω, E) a *weak Runge pair for harmonic* (resp. *superharmonic*) *functions* if, for each harmonic (resp. superharmonic) function u on E, there is a harmonic (resp. superharmonic) function v on Ω and a positive number a such that $u - a \leq v \leq u + a$ on E.

THEOREM 7. — *Let Ω be an open set in $\mathbb{R}^n$ $(n \geq 3)$ and E be a relatively closed subset of Ω. The following are equivalent :*

(a) *(Ω, E) is a weak Runge pair for superharmonic functions;*

(b) *(Ω, E) is a weak Runge pair for harmonic functions;*

(c) *(Ω, E) satisfies the conditions below :*

(i) *there is a compact subset C of Ω such that $\Omega\backslash\widehat{E}$ and $\Omega\backslash E$ are thin at the same points of $E\backslash C$, and*

(ii) *for each compact subset K of Ω, there is a compact subset L of Ω which contains every Ω-bounded component of $\Omega\backslash(E \cup K)$ whose closure intersects K.*

THEOREM 8. — *Let Ω be an open set in $\mathbb{R}^2$ and E be a relatively closed proper subset of Ω. The following are equivalent :*

(a) (Ω, E) *is a weak Runge pair for superharmonic functions;*

(b) (Ω, E) *is a weak Runge pair for harmonic functions;*

(c) (i) *there is a compact subset* C *of* Ω *such that* $\partial E \backslash C = \partial \widehat{E} \backslash C$,

(ii) *either* $\widehat{E} \neq \Omega$ *or* $\mathbb{R}^2 \backslash \Omega$ *is non-polar, and*

(iii) *for each compact subset* K *of* Ω, *there is a compact subset* L *of* Ω *which contains every* Ω*-bounded component of* $\Omega \backslash (E \cup K)$ *whose closure intersects* K.

Theorems 4-8 are proved in §7-11. They rely on Theorems 1 and A.

3. Preparatory material.

3.1. The following is a variant of [8, Lemma 2.3 and §7] suitable for our present purposes.

LEMMA 2. — *Let* ω_0 *be an open set in* $\mathbb{R}^n$ *and* E *be a compact subset of* ω_0. *Further, let* $\omega_1, \ldots, \omega_l$ *denote the bounded components of* $\mathbb{R}^n \backslash E$ *which are not subsets of* ω_0, *and let* $X_k \in \omega_k$ $(k = 1, \ldots, l)$. *If* u *is a superharmonic function on* ω_0, *then there exist a superharmonic function* v *on* $\mathbb{R}^n$ *and a non-negative constant* c *such that*

$$u(X) = v(X) - c \sum_{k=1}^{l} \phi_n(|X - X_k|)$$

on some open set which contains E.

To prove this, let F_0 be a compact set such that

$$E \subset F_0^\circ \subset F_0 \subset \bigcup_{k=0}^{l} \omega_k$$

and $\mathbb{R}^n \backslash F_0$ is connected. Also, for each k in $\{1, \ldots, l\}$, let F_k be a compact set with connected interior such that

$$\{X_k\} \cup (\omega_k \backslash \omega_0) \subset F_k^\circ \subset F_k \subset \omega_k,$$

and define

$$F = F_0 \backslash \left(\bigcup_{k=1}^{l} F_k^\circ \right).$$

Now let U, W be bounded open sets such that

$$E \subset U \subset \overline{U} \subset F^\circ \subset F \subset W \subset \overline{W} \subset \omega_0 \backslash \{X_1, \dots, X_l\},$$

and define

$$w(X) = \begin{cases} H_u^{W \backslash \overline{U}}(X) & (X \in W \backslash \overline{U}) \\ u(X) & (\text{elsewhere in } \omega_0), \end{cases}$$

where H_f^Ω denotes the PWB solution of the Dirichlet problem on Ω with boundary function f. The lower regularization w^*, of w, is superharmonic on ω_0, and equals u on U. Next we define h on $\mathbb{R}^n \backslash F$ as follows. On F_k° $(k = 1, \dots, l)$ let h be the Green function for F_k° with pole at X_k. On $\mathbb{R}^n \backslash F_0$ let h be the Green function for $(\mathbb{R}^2)^* \backslash F_0$ with pole at $\mathcal{A}$ if $n = 2$, or the solution to the Dirichlet problem for $\mathbb{R}^n \backslash F_0$ with boundary data 0 on ∂F_0 and 1 at the Alexandroff point for $\mathbb{R}^n$ if $n \geq 3$.

Now let

$$M > \sup \{w^*(X) : X \in \partial F\},$$
$$m < \inf \left(\{(w^*(X) - M)/h(X) : X \in \partial W\} \cup \{0\}\right),$$

and

$$s(X) = \begin{cases} w^*(X) & (X \in F) \\ \min \{M + mh(X),\ w^*(X)\} & (X \in W \backslash F) \\ M + mh(X) & (X \in \mathbb{R}^n \backslash W). \end{cases}$$

It is straightforward to check that s is superharmonic on $\mathbb{R}^n \backslash (\partial F \cup \{X_1, \dots, X_l\})$, and also on an open set T which contains the regular boundary points of $\mathbb{R}^n \backslash F$. Since $\partial F \backslash T$ is polar, s^* is superharmonic on $\mathbb{R}^n \backslash \{X_1, \dots, X_l\}$. Clearly $s^* = u$ on U and the function v defined by

$$v(X) = s^*(X) + (-m) \sum_{k=1}^{l} \phi_n(|X - X_k|) \qquad (X \notin \{X_1, \dots, X_l\})$$

has a superharmonic extension to all of $\mathbb{R}^n$. This completes the proof of the lemma.

3.2. LEMMA 3. — *Let Ω be an open set in $\mathbb{R}^n$, let E be a relatively closed subset of Ω, and suppose that, for each harmonic function h on E, there exists a superharmonic function u on Ω such that $u \geq h$ on E. Then, for each compact subset K of Ω, there is a compact subset L of Ω which contains every Ω-bounded component of $\Omega \backslash (E \cup K)$ whose closure intersects K.*

To prove this, suppose that the conclusion of the lemma fails to hold. Then there exist a compact subset K of Ω, a sequence (V_k) of distinct Ω-bounded components of $\Omega\backslash(E \cup K)$, and two sequences $(X_k), (Y_k)$ of points, such that $X_k, Y_k \in V_k$ for each k, and such that $X_k \to \mathcal{A}$ and (Y_k) converges to some point Y_0 in K. Now let U be an Ω-bounded open set which contains K and let U_0 be the component of U which contains Y_0. By deleting the first few members of the sequence (V_k) we can arrange that $V_k \cap U_0 \neq \varnothing$ for each k. We define

$$a_k = \mu_{V_k,X_k}(U_0 \cap \partial V_k) \qquad (k \in \mathbb{N}). \tag{4}$$

If $a_k = 0$, then (see [9, 1.VIII.5(b)]) there is a superharmonic function v_1 on V_k with limit $+\infty$ at each point of $U_0 \cap \partial V_k$. Hence the function

$$v_2(X) = \begin{cases} v_1(X) & (X \in U_0 \cap V_k) \\ +\infty & (X \in U_0 \backslash V_k) \end{cases}$$

is lower semicontinuous and super-meanvalued on U_0. This is impossible, since $U_0 \cap V_{k+1}$ is a non-polar subset of $U_0 \backslash V_k$. Hence $a_k > 0$ for each k.

Now let h be a harmonic function on the set $\Omega_1 = \Omega\backslash\{X_k : k \in \mathbb{N}\}$, such that, for each k, the function

$$h(X) + a_k^{-1}\phi_n(|X - X_k|)$$

has a harmonic extension to $\Omega_1 \cup \{X_k\}$. (Such a function exists by [11, Lemma 2], for example.) By hypothesis there exists a superharmonic function u on Ω such that $u \geq h$ on E. Also, since $u - h$ is superharmonic on Ω, we can define b to be a negative lower bound for $u - h$ on $\overline{U}$, and then define the open set

$$W = \{X \in \Omega : u(X) - h(X) > b - 1\}.$$

It follows from the minimum principle, and the fact that $K \subset U$, that $u - h \geq b$ on each $\overline{V}_k$, and so $\bigcup_k \overline{V}_k \subseteq W$. Also, $\overline{U} \subset W$. Clearly the function v defined by $v(X) = u(X) - h(X) - b + 1$ is positive and superharmonic on W and satisfies $\nu_v(\{X_k\}) \geq a_k^{-1}$ for each k. It follows from the Riesz decomposition theorem that a potential on W is defined by

$$w(X) = \sum_k a_k^{-1} G_W(X_k, X) \qquad (X \in W),$$

where $G_W(.,.)$ denotes the Green function for W. Let $T = \bigcup_k V_k$. Then the restriction of w to $\partial T \cap W$ is $\mu_{T,X}$-integrable when $X \in T$. However, (4) yields

$$\sum_k a_k^{-1}\mu_{T,X_k}(U_0\cap\partial T)=\sum_k a_k^{-1}\mu_{V_k,X_k}(U_0\cap\partial V_k)=+\infty.$$

Also, by monotone convergence, we have

$$\begin{aligned}\int_{\partial T\cap W} w(Y)d\mu_{T,X}(Y)&=\sum_k a_k^{-1}\int_{\partial T\cap W} G_W(X_k,Y)d\mu_{T,X}(Y)\\&=\sum_k a_k^{-1}\int_{\partial T\cap W} G_W(X,Y)d\mu_{T,X_k}(Y).\end{aligned}$$

Hence the Riesz measure associated with the superharmonic function $X\longmapsto\int_{\partial T\cap W} wd\mu_{T,X}$ is infinite on the compact set $\overline{U}_0\cap\partial T$, a contradiction. Therefore the conclusion of the lemma must hold.

3.3. Lemma 1 is now straightforward to prove. Suppose that condition (iii) of Theorem 3 holds, and that the conclusion of the lemma fails. Then there exist a compact subset K of Ω, a sequence (V_k) of distinct Ω-bounded components of $\Omega\backslash E$, and sequences $(X_k),(Y_k)$ of points, such that $X_k,Y_k\in V_k$ for each k, and such that $X_k\to\mathcal{A}$ and (Y_k) converges to some point Y_0 in K. Now let U be an Ω-bounded open set which contains K and let U_0 be the component of U which contains Y_0, define a_k as in (4), and let $c_k=a_k^{-1}$. (We know from §3.2 that $a_k>0$.) Inequality (3) now implies that $\nu_v(\overline{U}\cap E)=+\infty$. This is impossible, since $\overline{U}\cap E$ is a compact subset of W. Hence Lemma 1 is proved.

4. Proof of Theorem 1.

4.1. We begin with the "if" part of the proof. Let u be a superharmonic function on E, fix X_0 in E, let $A_1=\{X_0\}$, and let (A_k) be a sequence of compact subsets of Ω such that $A_k\subset A_{k+1}^{\circ}$ for each k and also $\bigcup_k A_k=\Omega$. A subset A of Ω will be called *Ω-solid* if $\Omega^*\backslash A$ is connected.

We will now inductively define a new sequence (C_k) of compact subsets of Ω which satisfy $C_k\subset C_{k+1}^{\circ}$ for each k and also

(I) $A_k\subseteq C_k$, (II) C_k is Ω-solid, (III) $C_k\cup E$ is Ω-solid.

Let $C_1=A_1$. Then (I)-(III) hold when $k=1$. Given C_k, we choose a compact subset F_1 of Ω which satisfies $A_{k+1}\cup C_k\subseteq F_1^{\circ}$. Since $\Omega^*\backslash E$ is locally connected, there is a compact set F_2 such that $F_1 \subseteq F_2 \subset \Omega$ and $\Omega^* \backslash (F_2 \cup E)$ is connected; that is, $F_2 \cup E$ is Ω-solid. We now define C_{k+1} to be the union of F_2 with all the Ω-bounded components of $\Omega \backslash F_2$, and observe that $C_{k+1} \cup E = F_2 \cup E$. It is clear that $C_k \subset C_{k+1}^\circ$ and that (I)-(III) hold when k is replaced by $k+1$.

Secondly, we inductively define a sequence (u_k) of functions such that

(a) u_k is superharmonic on $C_k \cup E$,

(b) $u_k = u$ on an open set U_k which contains E,

and such that $u_{k+1} = u_k$ on C_k for each k. If we define $u_1 = u$, then (a) and (b) hold when $k = 1$. Given u_k, we construct u_{k+1} as follows. We know that u_k is superharmonic on an open set ω (where $\omega \subseteq \Omega$) which contains $C_k \cup E$, and so also contains the compact set E_1 defined by $E_1 = C_{k+2} \cap (C_k \cup E)$. Since C_{k+2} and $C_k \cup E$ are Ω-solid by (II) and (III) above, it follows that E_1 is Ω-solid. Thus $\mathbb{R}^n \backslash E_1$ has finitely many bounded components $\omega_1, \ldots, \omega_l$, and we can choose X_j in $\omega_j \backslash \Omega$ for each j in $\{1, \ldots, l\}$. Lemma 2 can now be applied (with ω_0, E, u replaced by ω, E_1, u_k respectively) to obtain a superharmonic function $\overline{u}_k$ on $\mathbb{R}^n \backslash \{X_1, \ldots, X_l\}$, and hence on Ω, such that $\overline{u}_k = u_k$ on an open set ω' which contains E_1. We define $V = (\omega \backslash C_{k+2}) \cup \omega'$ and

$$u_{k+1}(X) = \begin{cases} \overline{u}_k(X) & (X \in C_{k+2}^\circ) \\ u_k(X) & (X \in V). \end{cases}$$

This function is well-defined, and hence superharmonic, on the open set $C_{k+2}^\circ \cup V$, because the two parts of the definition agree on the region of overlap, namely $C_{k+2}^\circ \cap \omega'$. We know that

$$E \backslash C_{k+2} \subseteq \omega \backslash C_{k+2} \qquad \text{and} \qquad E \cap C_{k+2} \subseteq E_1 \subseteq \omega',$$

so $E \subseteq V$ and

$$C_{k+1} \cup E \subseteq C_{k+2}^\circ \cup V.$$

It follows that u_{k+1} is superharmonic on $C_{k+1} \cup E$, that $u_{k+1} = u_k = u$ on the open set $U_{k+1} = U_k \cap V$ which contains E, and that $u_{k+1} = u_k$ on C_k (since $C_k \subseteq E_1 \subseteq \omega' \subseteq V$).

The final step of the argument is to define $\overline{u}(X) = \lim_{k \to \infty} u_k(X)$ for each X in Ω. Given Y_0 in Ω, there exists k_0 such that $Y_0 \in A_{k_0}^\circ \subseteq C_{k_0}^\circ$, and $u_k = u_{k_0}$ on C_{k_0} when $k \geq k_0$. It follows that $\overline{u}$ is superharmonic on a neighbourhood of Y_0. Thus $\overline{u}$ is superharmonic on Ω. From property (b) above, and the fact that $\overline{u} = u_k$ on C_k, it is clear that $\overline{u} = u$ on the open set $\bigcup_k (U_k \cap C_k^\circ)$ which contains E. Hence (Ω, E) is a strong extension pair for superharmonic functions.

4.2. Conversely, suppose that (Ω, E) is a strong extension pair for superharmonic functions. If $\Omega^*\backslash E$ is not connected, then there is an Ω-bounded component V of $\Omega\backslash E$. We fix X_0 in V, define $u(X) = -\phi_n(|X - X_0|)$ and conclude, by hypothesis, that there is a superharmonic function $\overline{u}$ on Ω such that $\overline{u} = u$ on an open set ω which contains E. Now let W be an Ω-bounded connected open set such that $\overline{V} \subset W$ and $\overline{W} \subset \omega \cup V$. Since u is subharmonic on Ω, we know that $H_u^V \leq H_u^W$ on V. Since $\overline{u}$ is superharmonic on Ω, it is also true that $H_{\overline{u}}^V \geq H_{\overline{u}}^W$ on V. Observing that $\overline{u} = u$ on ∂V and ∂W, it follows that $H_u^V = H_u^W$ on V. Hence $H_u^W - u$, which is a positive superharmonic function on W, takes the value 0 at every regular boundary point of ∂V : a contradiction. Thus $\Omega^*\backslash E$ must be connected.

Since any strong extension pair satisfies the hypotheses of Lemma 3, we deduce that $\Omega^*\backslash\widehat{E}$ is locally connected. The connectedness of $\Omega^*\backslash E$, shown above, means that $E = \widehat{E}$, so $\Omega^*\backslash E$ is locally connected. The proof of Theorem 1 is now complete.

5. Proof of Theorem 2.

5.1. Let (Ω, E) be as in the first sentence of Theorem 2, suppose that each Ω-bounded component V_0 of $\Omega\backslash E$ satisfies conditions (i) and (ii) of the theorem, and let u be a superharmonic function on some open set ω (where $\omega \subseteq \Omega$) which contains E. Further, let $U_1, \ldots, U_l$ be the bounded components of $\mathbb{R}^n\backslash E$ which are not subsets of Ω, and let $W_1, \ldots, W_p$ be the remaining bounded components of $\mathbb{R}^n\backslash E$ which are not subsets of ω. We choose Y_k in $U_k\backslash\Omega$ for each k in $\{1, \ldots, l\}$, and Z_k in W_k for each k in $\{1, \ldots, p\}$. It follows from Lemma 2 that there is a non-negative constant c and a superharmonic function v on $\mathbb{R}^n$ such that

$$u(X) = v(X) - c\left\{\sum_{k=1}^{l} \phi_n(|X - Y_k|) + \sum_{k=1}^{p} \phi_n(|X - Z_k|)\right\} \quad (X \in E).$$

In particular, there is a superharmonic function v_1 on Ω such that

$$u(X) = v_1(X) - c\sum_{k=1}^{p} \phi_n(|X - Z_k|) \qquad (X \in E).$$

Now let $V_0 = W_1$ and $X_0 = Z_1$. Further, let $V_1, \dots, V_m$ denote the components of $\mathbb{R}^n \backslash \widehat{E}$ and let $X_k \in V_k$ $(k = 1, \dots, m)$. By hypothesis (ii) there are non-negative constants $c_1, \dots, c_m$ such that (2) holds. We now define

$$(5) \qquad s(X) = \phi_n(|X - X_0|) - \int \phi_n(|X-Y|) d\mu_{V_0,X_0}(Y) \\ + \sum_{k=1}^{m} c_k \left\{ \int \phi_n(|X-Y|) d\mu_{V_k,X_k}(Y) - \phi_n(|X - X_k|) \right\}.$$

Inequality (2) shows that the function $s(X) - \phi_n(|X - Z_1|)$ is superharmonic on $\mathbb{R}^n \backslash \{X_1, \dots, X_m\}$, in view of the fact that $X_0 = Z_1$. Further, the regularity of V_0 (hypothesis (i)) and of the (finite) boundary points of $\mathbb{R}^n \backslash \widehat{E}$ ensure that $s = 0$ on E. To see this when $n \geq 3$ we note that, if $X \in E$, then $\Omega \backslash V_k$ is not thin at X, and so

$$\phi_n(|X - X_k|) = R^{\mathbb{R}^n \backslash V_k}_{\phi_n(|X - \cdot|)}(X_k) \\ = \int \phi_n(|X-Y|) d\mu_{V_k,X_k}(Y) \qquad (k = 0, \dots, m),$$

where R_f^F denotes the reduced function (réduite) of f relative to a set F in $\mathbb{R}^n$. A modified form of this argument applies also when $n = 2$. Hence, if we define

$$v_2(X) = v_1(X) + c\{s(X) - \phi_n(|X - Z_1|)\} \qquad (X \in \Omega),$$

we obtain a superharmonic function v_2 on $\Omega \backslash \{X_1, \dots, X_m\}$ such that

$$u(X) = v_2(X) - c \sum_{k=2}^{p} \phi_n(|X - Z_k|) \qquad (X \in E).$$

If we repeat the argument of the previous paragraph with $V_0 = W_k$ $(k = 2, \dots, p)$, it follows that there is a superharmonic function v_{p+1} on $\Omega \backslash \{X_1, \dots, X_m\}$ such that $u = v_{p+1}$ on E. Since $\Omega^* \backslash \widehat{E}$ is connected, we can apply Theorem 1 to the pair $(\Omega, \widehat{E})$ to conclude that there is a superharmonic function $\overline{u}$ on Ω such that $\overline{u} = v_{p+1}$ on $\widehat{E}$, and hence $\overline{u} = u$ on E. It follows that (Ω, E) is an extension pair for superharmonic functions.

5.2. Conversely, suppose that (Ω, E) is an extension pair for superharmonic functions, let V_0 be an Ω-bounded component of $\Omega \backslash E$, let $X_0 \in V_0$, and define $u(X) = -\phi_n(|X - X_0|)$. By hypothesis there is a superharmonic function $\overline{u}$ on Ω such that $\overline{u} = u$ on E. Thus the function $v = \overline{u} - u$ is superharmonic on Ω and vanishes on ∂V_0. In particular, v is a positive superharmonic function on V_0 which vanishes on ∂V_0, so V_0 is regular for the Dirichlet problem.

Now let $V_1, \ldots, V_m$ be the components of $\mathbb{R}^n \backslash \widehat{E}$, and let $G_k(.,.)$ be the Green function for V_k for each k in $\{0, \ldots, m\}$. Further, let W be an Ω-bounded open set which contains $\widehat{E}$ and let $X_k \in V_k \backslash \overline{W}$ for each k in $\{1, \ldots, m\}$. For each k in the latter set we can find a positive constant c_k such that

$$-c_k G_k(X_k, X) < v(X) \qquad (X \in \partial W \cap V_k). \tag{6}$$

It follows from the minimum principle that inequality (6) remains true for all X in $W \cap V_k$. It is also clear that $v \geq G_0(X_0, .)$ on V_0 and that $v \geq 0$ on $\widehat{E}$. Hence the function s defined on $\mathbb{R}^n$ by

$$s(X) = \begin{cases} G_0(X_0, X) & (X \in V_0) \\ -c_k G_k(X_k, X) & (X \in V_k; k \in \{1, \ldots, m\}) \\ 0 & (X \in \widehat{E} \backslash V_0) \end{cases}$$

is superharmonic on $\mathbb{R}^n \backslash \{X_1, \ldots, X_m\}$. The function s can be written as in (5). Since $\Delta s \leq 0$ on $\mathbb{R}^n \backslash \{X_1, \ldots, X_m\}$ in the sense of distributions, we conclude that (2) holds. Thus Theorem 2 is established.

6. Proof of Theorem 3.

6.1. Suppose that conditions (i)-(iii) of the theorem hold and let u be a superharmonic function on some open set ω (where $\omega \subseteq \Omega$) which contains E. We denote by $\{V_k : k \in I\}$ the collection of Ω-bounded components of $\Omega \backslash E$ which are not subsets of ω, choose X_k in V_k for each k in I, and let

$$\omega_1 = \omega \cup \left(\bigcup_{k \in I} (V_k \backslash \{X_k\}) \right).$$

Let K be a compact subset of Ω, and define

$$S = \bigcup_{k \in J} V_k, \text{ where } J = \{k \in I : \overline{V}_k \cap K \neq \varnothing\}.$$

It follows from (iii) and Lemma 1 that S is Ω-bounded. Since $\mathrm{dist}\,(\overline{S} \cap E, \mathbb{R}^n \backslash \omega) > 0$, it is clear that J is a finite set. Thus $\{X_k : k \in I\}$ has no limit point in Ω. Next, for each k in I, we apply Lemma 2 with ∂V_k in place of E. This allows us to construct a superharmonic function s on ω_1 such that $s = u$ on an open set which contains E, and such that the function

$$s(X) + c_k\phi_n(|X - X_k|)$$

has a superharmonic extension to $\omega_1 \cup \{X_k\}$ for a suitable choice of non-negative constant c_k.

By condition (iii) there exist an open set W satisfying $\widehat{E} \subseteq W \subseteq \Omega$ and a function v in $\mathcal{S}_W$ such that (3) holds. Let

$$\begin{aligned} w(X) = \sum_{k\in I} c_k &\left\{ \phi_n(|X - X_k|) - \int \phi_n(|X - Y|)d\mu_{V_k,X_k}(Y) \right\} \\ &+ s(X) + v(X) - 1 \qquad (X \in W \cap \omega_1). \end{aligned}$$

Inequality (3) ensures that w is superharmonic on $W \cap \omega_1$, and we have arranged s in such a way that w has a superharmonic extension to $W \cap \left(\omega \cup \left(\bigcup_k V_k\right)\right)$, which contains $\widehat{E}$. Since $\Omega^* \backslash \widehat{E}$ is connected (by the definition of $\widehat{E}$) and locally connected (by (ii)), we can apply Theorem 1 to the pair $(\Omega, \widehat{E})$ to obtain a superharmonic function $\overline{u}$ on Ω such that $\overline{u} = w$ on $\widehat{E}$. Also, $w = s = u$ on E by condition (i) and the definition of $\mathcal{S}_W$. Hence $\overline{u} = u$ on E. It follows that (Ω, E) is an extension pair for superharmonic functions.

6.2. Conversely, suppose that (Ω, E) is an extension pair for superharmonic functions. It follows as in §5.2 that (i) holds, and Lemma 3 shows that (ii) also holds.

It remains to establish (iii). Let $\{(X_k, c_k) : k \in I\}$ be a countable collection of pairs from $(\widehat{E}\backslash E) \times (0, \infty)$ such that the points X_k are distinct and have no limit point in Ω. As in the proof of Lemma 3 we can choose u to be a harmonic function on $\Omega_1 = \Omega\backslash\{X_k : k \in I\}$ such that $u(X) + c_k\phi_n(|X - X_k|)$ has a harmonic extension to $\Omega_1 \cup \{X_k\}$ for each k in I. By hypothesis there is a superharmonic function $\overline{u}$ on Ω such that $\overline{u} = u$ on E. Since $\overline{u} - u$ is superharmonic on Ω, it follows from the minimum principle that $\overline{u} - u \geq 0$ on $\widehat{E}$. For each k in I let V_k be the component of $\widehat{E}\backslash E$ to which X_k belongs. We know from Lemma 3 that any given compact subset of Ω intersects only finitely many of the sets V_k. Also, let $G_k(.,.)$ be the Green function for V_k, and define $G_k(.,.) = 0$ outside $V_k \times V_k$. Clearly $\overline{u} - u \geq \sum_k c_k G_k(X_k, .)$ on $\bigcup_k V_k$.

Let $W = \{X \in \Omega : \overline{u}(X) - u(X) > -1\}$ and

$$v(X) = 1 + \min\{\overline{u}(X) - u(X),\ 0\} \qquad (X \in W).$$

Clearly W is an open set satisfying $\widehat{E} \subseteq W \subseteq \Omega$, and also $v \in \mathcal{S}_W$. Further, the function s defined by

$$s(X) = v(X) - 1 + \sum_{k \in I} c_k G_k(X_k, X) \qquad (X \in W)$$

is also superharmonic on W. We can rewrite s as

$$s(X) = v(X) - 1 + \sum_{k \in I} c_k \{\phi_n(|X - X_k|) - \int \phi_n(|X - Y|) d\mu_{V_k, X_k}(Y)\} \qquad (X \in W),$$

and so (3) must hold. This completes the proof of Theorem 3.

7. Proof of Theorem 4.

7.1. Suppose that (Ω, E) satisfies conditions (c)(i)-(ii) of the theorem, let $\varepsilon > 0$, and let u be a superharmonic function on an open set ω (where $\omega \subseteq \Omega$) which contains E. Further, let V_k, X_k, and ω_1 be as in §6.1. Following the reasoning given there we can construct a superharmonic function s on ω_1 such that $s = u$ on an open set which contains E, and such that $s(X) + c_k\phi_n(|X - X_k|)$ has a superharmonic extension to $\omega_1 \cup \{X_k\}$ for a suitable choice of positive constant c_k. Using (c)(ii) in place of Lemma 1, we can also choose $\{U_k : k \in I\}$ to be a collection of Ω-bounded open sets such that $\overline{V}_k \subset U_k$ for each k, and such that any given compact subset of Ω intersects only finitely many of the sets $\overline{U}_k$.

Next we observe two consequences of condition (c)(i). Let V be any Ω-bounded component of $\Omega \backslash E$. If Y_0 is an irregular boundary point of V, then $\Omega \backslash V$, and hence $\Omega \backslash \widehat{E}$, is thin at Y_0. However $\Omega \backslash E$ contains V, and so is non-thin at Y_0. This contradicts (c)(i), and so V must be regular for the Dirichlet problem. Secondly, we note that $\partial V \subseteq \partial \widehat{E}$. To see this, let $A = \partial V \backslash \partial \widehat{E}$. Then $A \subseteq (\widehat{E})^\circ$, so $\Omega \backslash \widehat{E}$ is certainly thin at each point of A. It follows from (c)(i) that $\Omega \backslash E$, and hence V, is thin at each point of A. Hence A is a relatively open subset of ∂V which has zero harmonic measure for V (see [9, 1.XI.13]). Each point of A is therefore irregular for the Dirichlet problem on V, and so $A = \varnothing$, as claimed.

Now fix k temporarily. For each m in $\mathbb{N}$ let

$$A_{k,m} = \{X \in \overline{U}_k : \operatorname{dist}(X, \widehat{E}) \geq 1/m\},$$

and let $g_{k,m}$ be the Green function for $\Omega \backslash A_{k,m}$ with pole at X_k. (This must exist for all sufficiently large m, even if $n = 2$, because $\partial V_k \subseteq \partial \widehat{E}$.) If we define $g_{k,m}(X) = 0$ when $X \in A_{k,m}$, then the upper regularization $g^{**}_{k,m}$ is subharmonic on $\Omega \backslash \{X_k\}$. Thus the function $g_k = \lim_{m\to\infty} g^{**}_{k,m}$, being the limit of a decreasing sequence, is subharmonic on $\Omega \backslash \{X_k\}$ and harmonic on $V_k \backslash \{X_k\}$. Further, g_k vanishes on $U_k \backslash \widehat{E}$, and so vanishes at each point of ∂V_k where $\Omega \backslash \widehat{E}$ is non-thin. If X is a point of ∂V_k at which $\Omega \backslash \widehat{E}$ is thin, then condition (c)(i) shows that $\Omega \backslash E$, and hence V_k, are also thin at X. The set of all such points X therefore has μ_{V_k, X_k}-measure zero. It follows easily that g_k coincides (on V_k) with the Green function for the regular set V_k with pole at X_k. Thus, given a positive number ε, there is a compact subset K_k of V_k such that $X_k \in K_k^\circ$ and $g_k < 2^{-k-1} c_k^{-1} \varepsilon$ on $V_k \backslash K_k^\circ$. Hence, by the monotonicity of the sequence $(g^{**}_{k,m})_{m \geq 1}$ and Dini's theorem, there exists m_k such that

$$g^{**}_{k,m_k}(X) \leq g_k(X) + 2^{-k-1} c_k^{-1} \varepsilon < 2^{-k} c_k^{-1} \varepsilon \qquad (X \in \partial K_k).$$

It follows that $g^{**}_{k,m_k}(X) \leq 2^{-k} c_k^{-1} \varepsilon$ on E.

Now let

$$W = \Omega \backslash \left(\bigcup_{k \in I} A_{k,m_k} \right).$$

This is an open set because only finitely many of the sets A_{k,m_k} intersect a given compact subset of Ω. Also, $\widehat{E} \subseteq W$. Let $G_W(.,.)$ denote the Green function for W. Then

$$G_W(X_k, X) \leq g^{**}_{k,m_k}(X) \leq 2^{-k} c_k^{-1} \varepsilon \qquad (X \in E). \tag{7}$$

We define

$$v_1(X) = \sum_{k \in I} c_k G_W(X_k, X) \qquad (X \in W).$$

It follows from (7) that v_1 defines a potential on W, and that $v_1 \leq \varepsilon$ on E. The function $v_2 = s + v_1$, suitably redefined on the set where it is the difference of two infinite values, is superharmonic on the open set $W \cap (\omega \cup (\cup_k V_k))$, which contains $\widehat{E}$. Also, $s \leq v_2 \leq s + \varepsilon$ on E. Since $\Omega^* \backslash \widehat{E}$ is connected and locally connected (by (c)(ii)) we can apply Theorem 1 to obtain a superharmonic function v on Ω such that $v = v_2$ on $\widehat{E}$. Hence $u \leq v \leq u+\varepsilon$ on E. It follows that (Ω, E) is a Runge pair for superharmonic functions.

7.2. Suppose that (Ω, E) is a Runge pair for superharmonic functions, let h be harmonic on E, and let $\varepsilon > 0$. We know that there exist superharmonic functions u, v on Ω such that $|u-h| < \varepsilon/4$ and $|v+h| < \varepsilon/4$ on E. Now let W be the open set defined by

$$W = \{X \in \Omega : u(X) + v(X) > -\varepsilon/2\}.$$

Clearly $E \subseteq W$, and the minimum principle implies that $\widehat{E} \subseteq W$. Since $-v(X) - \varepsilon/4$ is a subharmonic minorant of $u(X) + \varepsilon/4$ on W, there is a greatest harmonic minorant, h_1 say, of $u(X) + \varepsilon/4$ on W. Hence

$$-v(X) - \varepsilon/4 \leq h_1(X) \leq u(X) + \varepsilon/4 \qquad (X \in W),$$

and so $|h_1 - h| < \varepsilon/2$ on E.

The function h_1 is harmonic on $\widehat{E}$. Further, $\Omega^* \backslash \widehat{E}$ is connected and also locally connected by our hypothesis and Lemma 3. We can thus apply Theorem A to obtain a harmonic function H on Ω such that $|H - h_1| \leq \varepsilon/2$ on $\widehat{E}$. Combining this with the conclusion of the previous paragraph, it follows that $|H - h| \leq \varepsilon$ on E. Hence (Ω, E) is a Runge pair for harmonic functions.

7.3. Finally, suppose that (Ω, E) is a Runge pair for harmonic functions, let V be an Ω-bounded component of $\Omega \backslash E$, let $X_0 \in V$ and let $h(X) = \phi_n(|X - X_0|)$. For each positive number ε there is a harmonic function H_ε on Ω such that $|H_\varepsilon - h| < \varepsilon/2$ on E. We define the open set

$$W_\varepsilon = \{X \in \Omega : h(X) - H_\varepsilon(X) + \varepsilon/2 > 0\}.$$

It follows from the minimum principle that $\widehat{E} \subseteq W_\varepsilon$, and clearly

$$h(X) - H_\varepsilon(X) + \varepsilon/2 \geq G_{W_\varepsilon}(X_0, X) \qquad (X \in W_\varepsilon).$$

Hence $G_{W_\varepsilon}(X_0, .) < \varepsilon$ on E. It follows from the arbitrary nature of ε that the Green function for $(\widehat{E})^\circ$, with pole at X_0, vanishes continuously on ∂V. This implies that $\partial V \subseteq \partial \widehat{E}$ (and that V is regular for the Dirichlet problem).

In this paragraph we assume that Ω has a Green function $G_\Omega(.,.)$. If this is not the case, we could work instead with (Ω_1, E), where Ω_1 is obtained from Ω by deleting a closed ball contained in $\Omega\backslash\widehat{E}$. (Note that, if $\widehat{E}=\Omega$, then $E=\Omega$ by the above paragraph, and so there is nothing to prove). Let V, X_0 and W_ε be as in the previous paragraph. Then

$$\begin{aligned}(8)\qquad G_\Omega(X_0,X)-R^{\Omega\backslash\widehat{E}}_{G_\Omega(X_0,.)}(X)&\le G_\Omega(X_0,X)-R^{\Omega\backslash W_\varepsilon}_{G_\Omega(X_0,.)}(X)\\ &=G_{W_\varepsilon}(X_0,X)<\varepsilon\qquad (X\in\partial V).\end{aligned}$$

Since ε can be arbitrarily small, it follows that

$$G_\Omega(X_0,X)=R^{\Omega\backslash\widehat{E}}_{G_\Omega(X_0,.)}(X)=R^{\Omega\backslash\widehat{E}}_{G_\Omega(X,.)}(X_0)\qquad (X\in\Omega\backslash V).$$

Hence

$$G_\Omega(Y,X)=R^{\Omega\backslash\widehat{E}}_{G_\Omega(X,.)}(Y)\qquad (Y\in V)$$

for each X in $\Omega\backslash V$. This holds for all such components V, so

$$(9)\qquad R^{\Omega\backslash\widehat{E}}_{G_\Omega(X,\cdot)}=R^{\Omega\backslash E}_{G_\Omega(X,\cdot)}\qquad (X\in E),$$

and this proves (c)(i).

Finally, Lemma 3 shows that (c)(ii) holds.

8. Proof of Theorem 5.

8.1. We require the following result :

THEOREM B. — *Let Ω be an open set in $\mathbb{R}^n$ and E be a relatively closed subset of Ω. The following are equivalent :*

(a) *for each function h which is continuous on E and harmonic on E°, and for each positive number ε, there is a function H harmonic on E such that $|H-h|<\varepsilon$ on E;*

(b) *$\Omega\backslash E$ and $\Omega\backslash E^\circ$ are thin at the same points of E.*

Theorem B is due to Keldyš [14] and Deny [7] under the additional assumption that E is compact. For the case of general closed sets E, see either [15, Theorem 3.10] or [4, Section 8].

8.2. Suppose that conditions (i) and (ii) of Theorem 5 hold, let h be continuous on E and harmonic on E°, and let $\varepsilon>0$. Condition (i) implies that the three sets $\Omega\backslash\widehat{E}$, $\Omega\backslash E$ and $\Omega\backslash E^\circ$ are thin at the same points of E.

It follows from Theorem B that there is a harmonic function h_1 on E such that $|h_1 - h| < \varepsilon/2$ on E. By Theorem 4 there is a harmonic function H on Ω such that $|H - h_1| < \varepsilon/2$ on E, and so $|H - h| < \varepsilon$ on E. It follows that (Ω, E) is an Arakeljan pair for harmonic functions.

Conversely, if (Ω, E) is an Arakeljan pair, then Theorems 4 and B immediately show that conditions (i) and (ii) hold. Thus Theorem 5 is established.

9. Proof of Theorem 6.

Clearly (c) implies (b). Further, Theorem 4 shows that (b) is equivalent to (a), and that (a) implies (d). (It was observed in §7.1 that, if condition (c)(i) of Theorem 4 holds, then each Ω-bounded component V of $\Omega\backslash E$ satisfies $\partial V \subseteq \partial\widehat{E}$.) Now suppose that conditions (d)(i)-(ii) hold, and let X_1 be a point of E at which $\Omega\backslash\widehat{E}$ is thin. Then (because $n = 2$) there are arbitrarily small circles, centred at X_1, which are contained in $\widehat{E}$ (see [13, Theorem 10.14]). Hence $X_1 \in (\widehat{E})^\circ$. It follows from (d)(i) that $X_1 \notin \partial E$, so $X_1 \in E^\circ$, and so $\Omega\backslash E^\circ$ is certainly thin at X_1. Hence $\Omega\backslash\widehat{E}$ and $\Omega\backslash E^\circ$ are thin at the same points of E. Applying Theorem 5, it follows that (Ω, E) is an Arakeljan pair for harmonic functions, i.e. (c) holds. Theorem 6 is now proved.

10. Proof of Theorem 7.

10.1. Let $n \geq 3$, suppose that (Ω, E) satisfies conditions (c)(i)-(ii) of the theorem, let C_1 be a compact subset of Ω such that $C \subset C_1^\circ$, and let u be a superharmonic function on an open set ω (where $\omega \subseteq \Omega$) which contains E. It follows that there are only finitely many Ω-bounded components $V_1, \ldots, V_m$ of $\Omega\backslash E$ which satisfy both $\overline{V}_k \cap C_1 \neq \varnothing$ and $V_k\backslash\omega \neq \varnothing$. For each k in $\{1, \ldots, m\}$ choose X_k in $V_k\backslash\omega$. Next define $\Omega_1 = \Omega\backslash\{X_1, \ldots, X_m\}$, and define E_1 to be the union of E with the Ω-bounded components of $\Omega\backslash E$ which are contained in ω. The pair (Ω_1, E_1) satisfies conditions (c)(i)-(ii) of Theorem 4, so there exists a superharmonic function v_1 on Ω_1 such that $u - 1 \leq v_1 \leq u + 1$ on E_1, and hence on E. Further, in view of Lemma 2, it can be arranged that there are non-negative constants $c_1, \ldots, c_m$ such that $v_1(X) + c_k\phi_n(|X - X_k|)$ has a superharmonic extension to $\Omega_1 \cup V_k$. Thus, if we define

$$v(X) = v_1(X) + \sum_{k=1}^{m} c_k G_\Omega(X_k, X) \qquad (X \in \Omega),$$

we obtain a superharmonic function v on Ω such that $u - a \leq v \leq u + a$ on E, where

$$a = 1 + \sup_{X \in E} \sum_{k=1}^{m} c_k G_\Omega(X_k, X) < \infty.$$

Thus (Ω, E) is a weak Runge pair for superharmonic functions, i.e. (a) holds.

10.2. The proof that (a) implies (b) is directly analogous to the argument given in §7.2.

10.3. Suppose now that (Ω, E) is a weak Runge pair for harmonic functions. It follows from Lemma 3 that condition (c)(ii) must hold.

Next let (V_k) be a sequence of Ω-bounded components of $\Omega \backslash E$ such that only a finite number of the sets V_k intersect any given compact subset of Ω. (If no such sequence exists, then (c)(i) clearly holds.) Also, let (c_k) be a sequence of positive numbers, and let $X_k \in V_k$ for each k. We now define h to be a harmonic function on the set $\Omega_2 = \Omega \backslash \{X_k : k \in \mathbb{N}\}$ such that, for each k, the function $h(X) + c_k \phi_n(|X - X_k|)$ has a harmonic extension to $\Omega_2 \cup \{X_k\}$. By hypothesis there is a harmonic function H on Ω and a positive number a such that $|H - h| < a$ on E. We define the open set

$$W = \{X \in \Omega : H(X) - h(X) + a > 0\}.$$

It follows from the minimum principle that $\widehat{E} \subseteq W$, and clearly

$$H(X) - h(X) + a \geq c_k G_W(X_k, X) \qquad (X \in W; k \in \mathbb{N}).$$

Hence $c_k G_W(X_k, .) < 2a$ on E, and we can argue as in (8) to deduce that

$$c_k \left\{ G_\Omega(X_k, X) - R^{\Omega \backslash \widehat{E}}_{G_\Omega(X_k, \cdot)}(X) \right\} < 2a \qquad (X \in \partial V_k; k \in \mathbb{N}).$$

It follows from the arbitrary nature of the sequence (c_k), and the reasoning given in §7.3 that, for all but a finite number of the components V_k,

$$G_\Omega(X_k, X) = R^{\Omega \backslash \widehat{E}}_{G_\Omega(X_k, .)}(X) = R^{\Omega \backslash \widehat{E}}_{G_\Omega(X, .)}(X_k) \qquad (X \in \Omega \backslash V_k).$$

Arguing as in (9) we obtain condition (c)(i), and the proof of Theorem 7 is complete.

11. Proof of Theorem 8.

11.1. Let $n = 2$, let E be a relatively closed proper subset of Ω, and suppose that (Ω, E) satisfies conditions (c)(i)-(iii) of the theorem. If $\mathbb{R}^2 \backslash \Omega$ is non-polar, then Ω possesses a Green function, and the reasoning of §9 and §10.1 establishes (a). If $\mathbb{R}^2 \backslash \Omega$ is polar, then (c)(ii) implies that $\widehat{E} \neq \Omega$. In this case, let B be a closed ball in $\Omega \backslash \widehat{E}$, and let $\Omega_0 = \Omega \backslash B$. Thus Ω_0 possesses a Green function. The arguments in §9 and §10.1 show that (Ω_0, E) is a weak Runge pair for superharmonic functions. It follows, by applying Theorem 1 to the pair $(\Omega, \widehat{E})$, that (Ω, E) is also a weak Runge pair for superharmonic functions.

11.2. The proof that (a) implies (b) is directly analogous to the argument given in §7.2.

11.3. Now suppose that (b) holds. As before, condition (c)(iii) follows from Lemma 3.

Next, suppose that $\widehat{E} = \Omega$. Let V_0 be an Ω-bounded component of $\Omega \backslash E$, let $X_0 \in V_0$, and let $u(X) = \phi_2(|X - X_0|)$. It follows, by hypothesis, that there exist a harmonic function v on Ω and a positive constant a such that $|u - v| \leq a$ on E. Hence, by the maximum principle, $|u - v| \leq a$ on $\Omega \backslash V_0$. Thus $u - v + 2a$ is a non-constant positive superharmonic function on Ω, and this implies that $\mathbb{R}^2 \backslash \Omega$ is non-polar. It follows that (c)(ii) holds.

If Ω has a Green function, then we can argue as in §9 and §10.3 that (c)(i) must hold. If Ω does not possess a Green function, then we know from the previous paragraph that $\widehat{E} \neq \Omega$. We can thus reason as before with (Ω_0, E) in place of (Ω, E), where Ω_0 is obtained from Ω be deleting a closed ball contained in $\Omega \backslash \widehat{E}$. This completes the proof of Theorem 8.

12. Details of Examples 4 and 5.

12.1. Let $n \geq 3$, and let $\{Y_k' : k \in \mathbb{N}\}$, u and E be as in Example 4. The lower semicontinuity of u ensures that E is closed, and hence compact. Also, if we define $E_y = \{X' \in \mathbb{R}^{n-1} : (X', y) \notin E\}$, then $E_y \subseteq E_z$ whenever

$0 < y < z$. It follows that $\mathbb{R}^n \backslash E$ has only one bounded component, namely $V = (0,1)^{n-1} \times (-1,0)$. Clearly V is regular for the Dirichlet problem and satisfies $\partial V \subseteq \partial \widehat{E}$.

Define $v(X', x_n) = u(X')$ on $\mathbb{R}^{n-1} \times \mathbb{R}$. Then v is superharmonic. If $Y' \in (0,1)^{n-1}$, then

$$\begin{aligned} v(X', x_n) &= u(X') > \phi_n(x_n) \\ &\geq \phi_n(|(X', x_n) - (Y', 0)|) \qquad ((X', x_n) \in (0,1)^n \backslash E), \end{aligned}$$

so $\mathbb{R}^n \backslash \widehat{E}$ is thin at $(Y', 0)$, whereas $\mathbb{R}^n \backslash E$ is not. This establishes Example 4.

12.2. Let E be as above, let $E_1 = \widehat{E}$ and let $Y' \in (0,1)^{n-1}$. Then §12.1 shows that $\mathbb{R}^n \backslash E_1$ (which equals $\mathbb{R}^n \backslash \widehat{E}_1$) is thin at $(Y', 0)$. However, $\mathbb{R}^n \backslash E_1^\circ$ is not, because it contains $(0,1)^{n-1} \times [0, +\infty)$. This establishes Example 5.

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Note. — Professor Paul Gauthier has independently obtained an extension theorem for subharmonic functions in a preprint entitled "Subharmonic extensions and approximations", which is to appear in *Canad. Math. Bull.* His main result is distinct from, but related to, our Theorem 1. Connections with the theory of harmonic approximation are also discussed, as are a number of open problems. Problem 1 (concerning the characterization of extension pairs and Runge pairs for subharmonic functions) is solved by Theorems 2-4 of the present paper.

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