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COMPLETE MINIMAL SURFACES IN $\mathbf{R}^3$ WITH TYPE ENNEPER END

by Nedir DO ESPIRITO-SANTO

Introduction.

Let $x = (x_1, x_2, x_3) : M \to \mathbb{R}^3$ be a complete minimal immersion. If the total curvature $\int_M K\,\mathrm{d}A$ of $x(M)$ is finite, then M is conformally equivalent to a punctured compact Riemann surface, (see [H]). Let us denote by $M \simeq \overline{M}_\gamma \setminus \{p_1, \dots, p_k\}$, $p_j \in M_\gamma$, $j = 1, \dots, k$, $k < \infty$, where γ is the genus of $\overline{M}_\gamma$. If $D_j \subset \overline{M}_\gamma$ is a small topological disk, $p_j \in D_j$ and $p_i \notin D_j$ for $i \neq j$, then $E_j = x(D_j \setminus \{p_j\})$ is an end of the immersion. Take local coordinates z in D_j with $z(p_j) = 0$, and for small ε consider the curve $\beta(t) = \varepsilon\,\mathrm{e}^{it}$. After a rotation we may assume that the limiting normal of the end E_j is vertical. The multiplicity I_j of E_j is the winding number of the curve $x_1(\beta(t)) + ix_2(\beta(t))$. We recall that E_j is of Enneper type if its multiplicity is 3. Jorge-Meeks [JM] proved that

$$(1) \qquad \int_M K\,\mathrm{d}A = 2\pi\Big(\chi(M) - \sum_{j=1}^{k} I_j\Big) \leq 2\pi\big(\chi(M) - k\big).$$

Furthermore the equality holds if and only if each end E_j is embedded, that is, $I_j = 1$ for every $j = 1, 2, \dots, k$.

Chen and Gackstatter [CG] have shown an example of a complete minimal immersion $x : M \to \mathbb{R}^3$ such that M is conformally equivalent to a torus minus one point, the end is of Enneper type and the total curvature is -8π. Furthermore, they proved that there exists a complete minimal immersion $x : M \to \mathbb{R}^3$ such that M is conformally equivalent to a compact hyperelliptic Riemann surface (i.e., a double ramified covering

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of the Riemann sphere) of genus 2 minus one point. The end is again of Enneper type and the total curvature is -12π. In the same work they asked about the existence of a complete minimal immersion into $\mathbb{R}^3$, where M is conformally equivalent to a compact Riemann surface of genus three minus one point and the end of Enneper type. The purpose of this paper is to prove the following theorem :

THEOREM A. — *There exists a complete minimal immersion $x : M \to \mathbb{R}^3$ such that M is conformally equivalent to a compact hyperelliptic Riemann surface of genus three minus one point, the end is of Enneper type and the total curvature is -16π.*

For the proof we use some basic results of algebraic curves and the normalization theorem which states that any irreducible plane algebraic curve admits a holomorphic parametric representation as a compact Riemann surface. These results can be found in [G].

Also, from [O] we have the following process of construction of complete minimal immersions. Let $M \simeq \overline{M}_\gamma \setminus \{p_1, \ldots, p_k\}$ be as above and let g and η be a meromorphic function and a meromorphic differential in $\overline{M}_\gamma$, respectively. Consider the map $x = (x_1, x_2, x_3) : M \to \mathbb{R}^3$,

$$x_j(z) = \operatorname{Re} \int_{z_0}^{z} \phi_j,$$

where $j = 1, 2, 3$, and

$$\phi_1 = \tfrac{1}{2}(1 - g^2)\eta, \quad \phi_2 = \tfrac{1}{2}(1 + g^2)\eta, \quad \phi_3 = g\eta.$$

Assume that g and η satisfy :

(2)
- (c_1) η is holomorphic in M and $p \in M$ is a pole of order m of g if and only if p is a zero of order $2m$ of η;
- (c_2) $\operatorname{Re} \int_\ell \phi_j = 0$, $j = 1, 2, 3$, for any closed path $\ell \subset \overline{M}_\gamma$;
- (c_3) every divergent path in M has infinite length.

Then x is a complete minimal immersion in $\mathbb{R}^3$ with total curvature

$$\int_M K \, \mathrm{d}A = -4m\pi,$$

where m is the degree of g.

Furthermore, $g \circ \pi^{-1}$ is the Gauss map of x, where π is the stereographic projection to $\mathbb{C} \cup \{\infty\}$. The pair (g, η) is called the *Weierstrass representation* for the immersion x.

We consider M_3 the compact Riemann surface produced by

$$P(z, \omega) = \omega^2 - z(z^2 - a^2)(z^2 - b^2)(z^2 - c^2) = 0,$$

where $z, \omega \in \mathbb{C}$ and $a, b, c \in \mathbb{R}$, $0 < a < b < c < \infty$. On $\overline{M}_3$ we define a meromorphic function

$$g(z) = \lambda \frac{\omega}{z(z^2 - b^2)}$$

and a meromorphic differential

$$\eta = f\, \mathrm{d}z, \quad f(z) = \frac{z(z^2 - b^2)}{\omega},$$

where $\lambda \in \mathbb{R}$, $0 < \lambda < \infty$. Then we show that there exists real numbers a, b, c, λ such that (g, η) is the Weierstrass representation of a complete minimal imersion as we want. The difficulty of the proof is to find a, b, c and λ such that the condition (c_2) from (2) will be satisfied. Certainly, the requirement that the differentials ϕ_j, $j = 1, 2, 3$, have no real periods on curves γ_i, $i = 1, \dots, 6$, which generate the fundamental group of M_3 is a difficult global problem. In our case, this problem is equivalent to show the existence of real numbers a, b, c such that

$$\lambda = \frac{F_1(a, b, c)}{F_2(a, b, c)} = \frac{F_3(a, b, c)}{F_4(a, b, c)} = \frac{F_5(a, b, c)}{F_6(a, b, c)}$$

where each F_i, $i = 1, 2, \dots, 6$ is a hyperelliptic integral. To solve this we fix $a_0 = 1.1632$ and we consider a compact $K = J_1 \times J_2 \subset \mathbb{R}^2$. Then we define the functions $F, G : K \to \mathbb{R}$, where

$$F(b, c) = \frac{F_2(a_0, b, c)}{F_1(a_0, b, c)} - \frac{F_4(a_0, b, c)}{F_3(a_0, b, c)},$$

$$G(b, c) = \frac{F_2(a_0, b, c)}{F_1(a_0, b, c)} - \frac{F_6(a_0, b, c)}{F_5(a_0, b, c)}.$$

Thus, we reduce the problem to prove that there are $(b_0, c_0) \in K$ such that $F(b_0, c_0) = G(b_0, c_0) = 0$. For that we prove in Lemma 1 that there are sequences of functions $\{F_n\}$, $\{G_n\}$ in K which converge uniformly to F and G, respectively. Furthermore there is $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$,

F_n and G_n have respective curves of zeroes, α_n and β_n in K, such that $\alpha_n \cap \beta_n \neq \emptyset$. From uniform convergence and from compactness of K we obtain the existence of $(b_0, c_0) \in K$ such that the condition (c_2) is satisfied.

The total curvature -16π follows from the fact that $(\overline{M}_3, g)$ is a ramified cover of degree four of the Riemann sphere. Furthermore, from (1) we conclude that the end is of Enneper type and the total curvature -16π is sharp under our topological assumptions.

1. Proof of theorem A.

Consider

$$D = \left\{q = (a,b,c) \in \mathbb{R}^3;\ \tfrac{1}{2} < a < b - \tfrac{1}{2} < c - 1 < \infty\right\} \text{ and } I = [0,1]. \tag{3}$$

We define the functions

$$\begin{cases} F_1(q) = a \displaystyle\int_I f_1(q,x)\,\mathrm{d}x, & F_2(q) = a \displaystyle\int_I \frac{1}{f_1(q,x)}\,\mathrm{d}x, \\ F_3(q) = (b-a) \displaystyle\int_I f_3(q,x)\,\mathrm{d}x, & F_4(q) = (b-a) \displaystyle\int_I \frac{1}{f_3(q,x)}\,\mathrm{d}x, \\ F_5(q) = (c-b) \displaystyle\int_I f_5(q,x)\,\mathrm{d}x, & F_6(q) = (c-b) \displaystyle\int_I \frac{1}{f_5(q,x)}\,\mathrm{d}x, \end{cases} \tag{4}$$

where

$$\begin{cases} f_1(q,x) = h_1(q,x)\sqrt{\dfrac{x}{1-x}}, & h_1(q,x) = \left[\dfrac{p(q,x)}{a(1+x)\tilde{p}(q,x)}\right]^{1/2}, \\ f_3(q,x) = h_3(q,x)\sqrt{\dfrac{1-x}{x}}, & h_3(q,x) = \left[\dfrac{u(q,x)}{v(q,x)\omega(q,x)}\right]^{1/2}, \\ f_5(q,x) = h_5(q,x)\sqrt{\dfrac{x}{1-x}}, & h_5(q,x) = \left[\dfrac{\tilde{u}(q,x)}{\tilde{v}(q,x)\widetilde{\omega}(q,x)}\right]^{1/2}, \end{cases} \tag{4'}$$

and

$$\begin{cases} p(q,x) = b^2 - a^2x^2, \\ u(q,x) = \big[(b-a)x + a\big]\big[(b-a)x + b + a\big], \\ v(q,x) = (b-a)x + 2a, \\ \omega(q,x) = c^2 - a^2 - 2a(b-a)x - (b-a)^2x^2, \\ \tilde{p}(q,x) = c^2 - a^2x^2, \\ \tilde{u}(q,x) = \big[(c-b)x + b\big]\big[(c-b)x + 2b\big], \\ \tilde{v}(q,x) = (c-b)x + c + b, \\ \widetilde{\omega}(q,x) = (c-b)^2x^2 + 2b(c-b)x + b^2 - a^2. \end{cases} \tag{4''}$$

The following propositions give some results about the behavior of the functions defined above.

PROPOSITION 1.1. — *Let $h_j : D \times I \to \mathbb{R}$, $j = 1,3,5$, be the functions defined in (4′). Then :*

(a) *h_1 and h_5 are strictly decreasing functions on I, for all $q \in D$, and strictly positive functions on $D \times I$,*

(b) *h_3 is a strictly increasing function in I for all $q \in D$ and a strictly positive function on $D \times I$.*

The proof of this result is elementary and it is left to the reader.

PROPOSITION 1.2. — *The functions $F_i : D \to \mathbb{R}$, $i = 1,2,\ldots,6$ defined in (4) are continuous on D and there is $\delta > 0$ such that $F_i(q) > \delta$, $i = 1,2,\ldots,6$, for all $q \in D$.*

Proof. — Observe that the functions h_i and $1/h_j$, $j = 1,3,5$, are continuous in $D \times I$, since they are roots of quotients of continuous and strictly positive functions in $D \times I$.

Let $\{q_n = (a_n, b_n, c_n)\}_{n\in\mathbb{N}}$ be a sequence in D which converges to $q_0 = (a_0, b_0, c_0) \in D$ and $\varepsilon > 0$.

Since $(c-b)h_5$ is continuous function on $D \times I$, then for each $x \in I$, there is $n(x) \in \mathbb{N}$ such that, for all $n \geq n(x)$

$$\left|(c_n - b_n)h_5(q_n, x) - (c_0 - b_0)h_5(q_0, x)\right| < \frac{2}{\pi}\varepsilon.$$

Let $n_0 = \sup_{x\in I}\{n(x)\}$. Note that $n_0 < \infty$, since I is compact. Then for all $n \geq n_0$ and for all $x \in I$:

$$\begin{aligned}
&\left|F_5(q_n) - F_5(q_0)\right| \\
&\quad = \left|(c_n - b_n)\int_I h_5(q,x)\sqrt{\frac{x}{1-x}}\,\mathrm{d}x - (c_0 - b_0)\int_I h_5(q_0,x)\sqrt{\frac{x}{1-x}}\,\mathrm{d}x\right| \\
&\quad \leq \int_I \sqrt{\frac{x}{1-x}}\left|(c_n - b_n)h_5(q_n,x) - (c_0 - b_0)h_5(q_0,x)\right|\mathrm{d}x \\
&\quad < \frac{\pi}{2}\cdot\frac{2}{\pi}\varepsilon = \varepsilon.
\end{aligned}$$

Then F_5 is continuous in D. Analogously we show the continuity of F_i for $i = 1,2,3,4,6$.

Since (see definitions of D in (3)),

$$b > a + \tfrac{1}{2}, \quad c > b + \tfrac{1}{2}, \quad c > a + 1,$$

then

$$b^2 > b^2 - a^2 > a + \tfrac{1}{4}, \quad c^2 > c^2 - b^2 > b + \tfrac{1}{4}, \quad c^2 > c^2 - a^2 > 2a + 1,$$

and from Proposition 1.1 we have for all $(q, x) \in D \times I$

$$\begin{aligned}
F_1(q) &= a \int_I h_1 \sqrt{\frac{x}{1-x}}\, \mathrm{d}x > \tfrac{1}{2}\pi a h_1(q,1) = \delta_1 > 0,\\
F_2(q) &= a \int_I \frac{1}{h_1} \sqrt{\frac{1-x}{x}}\, \mathrm{d}x > \tfrac{1}{2}\pi a \,\frac{1}{h_1(q,0)} = \delta_2 > 0,\\
F_3(q) &= (b-a) \int_I h_3 \sqrt{\frac{1-x}{x}}\, \mathrm{d}x > \tfrac{1}{2}\pi (b-a) h_3(q,0) = \delta_3 > 0,\\
F_4(q) &= (b-a) \int_I \frac{1}{h_3} \sqrt{\frac{x}{1-x}}\, \mathrm{d}x > \tfrac{1}{2}\pi (b-a) \,\frac{1}{h_3(q,1)} = \delta_4 > 0,\\
F_5(q) &= (c-b) \int_I h_5 \sqrt{\frac{x}{1-x}}\, \mathrm{d}x > \tfrac{1}{2}\pi (c-b) h_5(q,1) = \delta_5 > 0\\
F_6(q) &= (c-b) \int_I \frac{1}{h_5} \sqrt{\frac{1-x}{x}}\, \mathrm{d}x > \tfrac{1}{2}\pi (c-b) \,\frac{1}{h_5(q,0)} = \delta_6 > 0.
\end{aligned}$$

We take $\delta = \min\{\delta_i,\ i = 1, 2, \ldots, 6\}$. Then the proposition is proved.

Let $\overline{M}_3$ be the compact Riemann surface of genus 3 which is the normalization (see [G]), of the algebraic curve

$$\omega^2 - z(z^2 - a^2)(z^2 - b^2)(z^2 - c^2) = 0, \qquad 0 < a < b < c < \infty.$$

We define in $\overline{M}_3$ a meromorphic function g and a meromorphic differential η as follows :

$$g(z) = \frac{\lambda \omega}{z(z^2 - b^2)}, \quad \eta = \frac{z(z^2 - b^2)}{\omega}\, \mathrm{d}z, \qquad 0 < \lambda < \infty. \tag{5}$$

We want to prove that there are real constants a, b, c, λ such that the pair (g, η) is the Weierstrass representation of an immersion as in theorem A. Indeed, we need to show that there are values a, b, c, λ such that the properties (c_1), (c_2), (c_3) (see (2)) are satisfied.

The table (Fig. 1) shows that the first condition (c_1) is satisfied for all a, b, c, λ with $0 < a < b < c < \infty$, $0 < \lambda < \infty$, where $0, 0$ and ∞, ∞ denote double zero and double pole, respectively.

z	$-c$	$-b$	$-a$	0	a	b	c	∞
g	0	∞	0	∞	0	∞	0	∞
η		$0,0$		$0,0$		$0,0$		∞,∞

Figure 1

For the proof of condition (c_2) first we observe that the differentials ϕ_j, $j = 1, 2, 3$, have no poles at infinity. This occurs because in the expression of ϕ_1 and ϕ_2 the function ω has odd degree and ϕ_3 is exact. Let γ_r, $r = 1, \ldots, 6$ be curves which generate the fundamental group of $\overline{M}_3$ (Fig. 2).

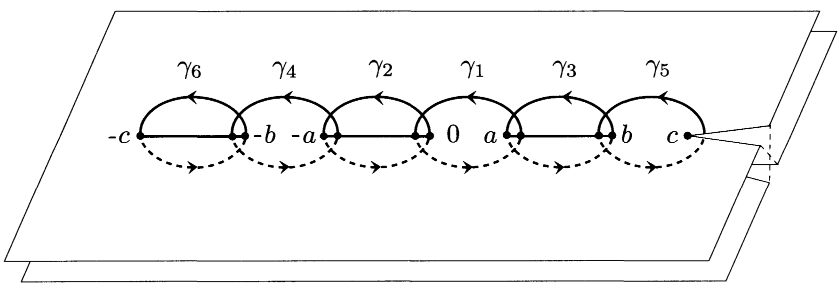

Figure 2

We have :

$$\phi_1 = \tfrac{1}{2} f(1 - g^2)\, \mathrm{d}z = \frac{z(z^2 - b^2) - \lambda^2(z^2 - a^2)(z^2 - c^2)}{2\,\omega}\, \mathrm{d}z,$$

$$\phi_2 = \tfrac{1}{2} i f(1 + g^2)\, \mathrm{d}z = \frac{z(z^2 - b^2) + \lambda^2(z^2 - a^2)(z^2 - c^2)}{2\,\omega}\, \mathrm{d}z,$$

$$\phi_3 = g f\, \mathrm{d}z = \lambda\, \mathrm{d}z.$$

For each $z \in \mathbb{C} \setminus \{0, \pm a, \pm b, \pm c\}$, we consider $\omega_+(z)$ and $\omega_-(z)$ the distinct values such that :

$$\omega^2 = z(z^2 - a^2)(z^2 - b^2)(z^2 - c^2). \tag{6}$$

The points (z,ω_+) and (z,ω_-) belong to the Riemann surface $\overline{M}_3$ and they are in distinct branches. We denote these branches $(+)$ and $(-)$ respectively. In Fig. 2, $(+)$ denote the lower branch, where the curves are dotted and $(-)$ denote the uper branch where the curves are full. If $z(z^2-a^2)(z^2-b^2)(z^2-c^2)$ is a real positive number, then ω_+ and ω_- represent the positive and negative root of (6) respectively.

We parametrize the path γ_1 in the following way :

$$\gamma_1(x) = \begin{cases} (x,\omega(x)), \quad \omega(x) = \sqrt{x(x^2-a^2)(x^2-b^2)(x^2-c^2)}, & \text{if } 0 \le x \le a, \\ (2a-x, \omega(2a-x)), \quad \omega(x) = -\sqrt{x(x^2-a^2)(x^2-b^2)(x^2-c^2)}, & \text{if } a \le x \le 2a. \end{cases}$$

Then,

$$\int_{\gamma_1} \phi_1 = \tfrac{1}{2}\int_{[0,a]} \frac{x(x^2-b^2) - \lambda^2(x^2-a^2)(x^2-c^2)}{\omega(x)}\,\mathrm{d}x$$
$$-\tfrac{1}{2}\int_{[a,2a]} \frac{(2a-x)[(2a-x)^2-b^2] - \lambda^2[(2a-x)^2-a^2][(2a-x)^2-c^2]}{\omega(2a-x)}\,\mathrm{d}x.$$

By a change of variable $y = 2a - x$, we have $\int_{\gamma_1}\phi_1 = iF_1 + \lambda^2 F_2$ where

$$F_1 = \int_{[0,a]} \frac{x(b^2-x^2)}{\sqrt{x(a^2-x^2)(b^2-x^2)(c^2-x^2)}}\,\mathrm{d}x,$$

$$F_2 = \int_{[0,a]} \frac{(a^2-x^2)(x^2-x^2)}{\sqrt{x(a^2-x^2)(b^2-x^2)(c^2-x^2)}}\,\mathrm{d}x.$$

By similar calculations we get $\int_{\gamma_i}\phi_j$, $(i,j) \neq (1,1)$, (see Fig. 3).

	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6
$\int\phi_1$	$iF_1+i\lambda^2F_2$	$F_1-\lambda^2F_2$	$-F_3+\lambda^2F_4$	$iF_3+i\lambda^2F_4$	$-iF_5-i\lambda^2F_6$	$-F_5+\lambda^2F_6$
$\int\phi_2$	$-F_1+\lambda^2F_2$	$iF_1+i\lambda^2F_2$	$-iF_3-i\lambda^2F_4$	$F_3-\lambda^2F_4$	$F_5-\lambda^2F_6$	$-iF_5-i\lambda^2F_6$

Figure 3

where

$$F_3 = \int_{[a,b]} \frac{x(b^2-x^2)}{\sqrt{x(x^2-a^2)(b^2-x^2)(c^2-x^2)}}\,\mathrm{d}x,$$

$$F_4 = \int_{[a,b]} \frac{(x^2-a^2)(c^2-x^2)}{\sqrt{x(x^2-a^2)(b^2-x^2)(c^2-x^2)}}\,\mathrm{d}x,$$

$$F_5 = \int_{[b,c]} \frac{x(x^2-b^2)}{\sqrt{x(x^2-a^2)(x^2-b^2)(c^2-x^2)}}\,\mathrm{d}x,$$

$$F_6 = \int_{[b,c]} \frac{(x^2-a^2)(c^2-x^2)}{\sqrt{x(x^2-a^2)(x^2-b^2)(c^2-x^2)}}\,\mathrm{d}x.$$

Observe that the functions that appear in the integrand are strictly positive in $(0,a)$, (a,b) and (b,c) respectively.

Note that $\operatorname{Re}\int_{\gamma_i}\phi_1 = \operatorname{Re}\int_{\gamma_k}\phi_2 = 0$ if $i = 2,3,6$ and $k = 1,4,5$. Then to conclude the proof of condition (c_2) (see Fig. 3) it is sufficient to show there are real numbers a, b, c, λ, such that

$$F_1 - \lambda^2 F_2 = 0, \quad -F_3 + \lambda^2 F_4 = 0, \quad -F_5 + \lambda^2 F_6 = 0. \tag{7}$$

Let D (see (3)) be the domain of the functions $F_i = F_i(a,b,c)$, $i = 1,2,\dots,6$. By a change of variables we express these integrals in the interval $I = [0,1]$. We obtain that F_i, $i = 1,2,\dots,6$ are the same functions of (4), with $(q,x) \in D \times I$.

Choose $a = a_0 = 1.1632$ and the compact set $K = J_1 \times J_2$, $J_1 = [2.55, 2.85]$ and $J_2 = [3.45, 3.8]$, then $\{a_0\} \times K \subset D$. Since a is fixed we denote the functions $f_i(q,x)$ and $h_i(q,x)$, (see (4)) $i = 1,\dots,6$ by $f_i(b,c,x)$ and $h_i(b,c,x)$, respectively. Define the functions $F, G : K \to \mathbb{R}$ by

$$\begin{cases} F(b,c) = \dfrac{F_2(b,c)}{F_1(b,c)} - \dfrac{F_4(b,c)}{F_3(b,c)} = \dfrac{\int_I f_1(b,c,x)\,\mathrm{d}x}{\int_I f_1(b,c,x)\,\mathrm{d}x} - \dfrac{\int_I \frac{1}{f_3(b,c,x)}\,\mathrm{d}x}{\int_I f_3(b,c,x)\,\mathrm{d}x}, \\[2ex] G(b,c) = \dfrac{F_2(b,c)}{F_1(b,c)} - \dfrac{F_6(b,c)}{F_5(b,c)} = \dfrac{\int_I \frac{1}{f_1(b,c,x)}\,\mathrm{d}x}{\int_I f_1(b,c,x)\,\mathrm{d}x} - \dfrac{\int_I \frac{1}{f_5(b,c,x)}\,\mathrm{d}x}{\int_I f_5(b,c,x)\,\mathrm{d}x}. \end{cases} \tag{8}$$

Obviously the functions F and G are continuous in K and we have the following :

LEMMA 1. — *Let $F, G : K \to \mathbb{R}$ be functions as above. Then there are sequences of functions $F_n, G_n : K \to \mathbb{R}$ such that :*

(a) *for every $n \geq 3$, F_n and G_n are differentiable,*

(b) *the sequences $\{F_n\}$ and $\{G_n\}$ converge uniformly to F and G respectively,*

(c) *there is $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$, F_n has a unique curve of zeroes α_n in K and G_n has a unique curve of zeros β_n in K and $\alpha_n \cap \beta_n \neq \emptyset$.*

Proof. — For each $n \geq 3$ we consider the interval $I_n = \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$ and we define the functions $F_{i,n} : K \to \mathbb{R}$, $i = 1, \dots, 6$

$$F_{i,n}(b,c) = \begin{cases} \displaystyle\int_{I_n} f_i(b,c,x)\,\mathrm{d}x, & \text{if } i = 1,3,5\,; \\ \displaystyle\int_{I_n} \frac{1}{f_{i-1}(b,c,x)}\,\mathrm{d}x, & \text{if } i = 2,4,6. \end{cases}$$

Consider the sequences of functions in K

$$\text{(9)} \quad \begin{cases} F_n(b,c) = \dfrac{F_{2,n}(b,c)}{F_{1,n}(b,c)} - \dfrac{F_{4,n}(b,c)}{F_{2,n}(b,c)} = \dfrac{\int_{I_n} \frac{1}{f_1}\,\mathrm{d}x}{\int_{I_n} f_1\,\mathrm{d}x} - \dfrac{\int_{I_n} \frac{1}{f_3}\,\mathrm{d}x}{\int_{I_n} f_3\,\mathrm{d}x}, \\ G_n(b,c) = \dfrac{F_{2,n}(b,c)}{F_{1,n}(b,c)} - \dfrac{F_{6,n}(b,c)}{F_{5,n}(b,c)} = \dfrac{\int_{I_n} \frac{1}{f_1}\,\mathrm{d}x}{\int_{I_n} f_1\,\mathrm{d}x} - \dfrac{\int_{I_n} \frac{1}{f_5}\,\mathrm{d}x}{\int_{I_n} f_5\,\mathrm{d}x}. \end{cases}$$

The property (a) is consequence of the Lebesgue dominated convergence theorem, since the functions $\partial f_i/\partial b$, $\partial f_i/\partial c$ and $\partial f_i/\partial x$ are bounded in $K \times I_n$ for all $n \geq 3$. The same happens with the functions $1/f_i$, $i = 1,3,5$.

From Proposition 1.2 we obtain that the integrals $\int_I f_i\,dx$ and $\int_I 1/f_i\,\mathrm{d}x$, $i = 1,3,5$, are continuous functions in K. Then

$$\begin{aligned} &\left| \int_I f_i(b,c,x)\,\mathrm{d}x - \int_{I_n} f_i(b,c,x)\,\mathrm{d}x \right| \\ &= \left| \int_{[0,1/n]} f_i(b,c,x)\,\mathrm{d}x + \int_{[1-1/n,1]} f_i(b,c,x)\,\mathrm{d}x \right| \\ &\leq \left| \int_{[0,1/n]} f_i(b,c,x)\,\mathrm{d}x \right| + \left| \int_{[1-1/n,1]} f_i(b,c,x)\,\mathrm{d}x \right| \underset{n\to\infty}{\longrightarrow} 0, \end{aligned}$$

that is, $\{F_{i,n}\}$ converges uniformly to F_i, for $i = 1,3,5$. Analogously we obtain $\{F_{i,n}\}$ converges uniformly to F_i, for $i = 2,4,6$, respectively.

Then F_n and G_n converge uniformly to F and G, respectively in K. This concludes the proof of (b).

To prove (c) first we show that there is $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$,

(i) $F_{n|J_1 \times \{c_1\}} > 0$, $\quad F_{n|J_1 \times \{c_2\}} < 0$ $\quad$ and $\quad \partial F_n/\partial c < 0$,

(ii) $G_{n|\{b_1\} \times J_2} > 0$, $\quad G_{n|\{b_2\} \times J_2} < 0$ $\quad$ and $\quad \partial G_n/\partial b < 0$,

where $c_1 = 3.45$, $c_2 = 3.8$, $b_1 = 2.55$ and $b_2 = 2.85$.

In order to prove (i) we denote

$$\widetilde{F}_j(b, I_n) = F_{n|J_1\times\{c_j\}} = \frac{1}{\left(\int_{I_n} f_1\,\mathrm{d}x\right)\left(\int_{I_n} f_3\,\mathrm{d}x\right)} m_j(b, I_n), \quad j = 1, 2,$$

where

$$\begin{aligned}(10)\qquad m_j(b, I_n) = \Big(\int_{I_n} \frac{1}{f_1(b_1 c_j, x)}\,\mathrm{d}x\Big)\Big(\int_{I_n} f_3(b, c_j, x)\,\mathrm{d}x\Big) \\ - \Big(\int_{I_n} \frac{1}{f_3(b, c_j, x)}\,\mathrm{d}x\Big)\Big(\int_{I_n} f_1(b, c_j, x)\,\mathrm{d}x\Big).\end{aligned}$$

Then, $\widetilde{F}_1(b, I_n) > 0$ and $\widetilde{F}_2(b, I_n) < 0$ for all b in J_1 if and only if $m_1(b, I_n) > 0$ and $m_2(b, I_n) < 0$, respectively. Since $\{F_n\}$ converges to F uniformly, it is sufficient to show that $m_1(b, I) > 0$ and $m_2(b, I) < 0$.

From now on $\{t_0, t_1, \ldots, t_k\}$, $0 = t_0 < t_1 < \cdots < t_k = 1$ is a partition of I, $E_i = [t_{i-1}, t_i]$, $i = 1, \ldots, k$ and χ_{E_i} is the characteristic function of E_i. Then, from Proposition 1.1 we have, for all $(b, c, x) \in K \times I$,

$$(11)\quad \begin{cases} h_1(b, c_1, x) < \sum_{i=1}^{k} h_1(b, c_1, t_{i-1})\chi_{E_i}(x) \\ \qquad = \sum_{i=1}^{k} \Big[\dfrac{p(q, t_{i-1})}{a(1 + t_{i-1})\tilde{p}(q, t_{i-1})}\Big]^{1/2} \chi_{E_i}(x), \\ h_3(b, c_1, x) > \sum_{i=1}^{k} h_3(b, c_1, t_{i-1})\chi_{E_i}(x) \\ \qquad = \sum_{i=1}^{k} \Big[\dfrac{u(q, t_{i-1})}{v(q, t_{i-1})\omega(q, t_{i-1})}\Big]^{1/2} \chi_{E_i}(x), \end{cases}$$

where $p, \tilde{p}, u, v, \omega$ are the functions in (4) and $q = (a, b, c_1)$. Then, from Proposition 1.1 and (11) we have

$$(12)\quad \begin{cases} \displaystyle\int_I f_1(b, c_1, x)\,\mathrm{d}x < \sum_{i=1}^{k} \Big[\frac{p(q, t_{i-1})}{a(1 + t_{i-1})(\tilde{p}(q, t_{i-1}))}\Big]^{1/2} \int_{E_i} \sqrt{\frac{x}{1 - x}}\,\mathrm{d}x, \\ \displaystyle\int_I \frac{1}{f_1(b, c_1, x)}\,\mathrm{d}x > \sum_{i=1}^{k} \Big[\frac{a(1 + t_{i-1})\tilde{p}(q, t_{i-1})}{p(q, t_{i-1})}\Big]^{1/2} \int_{E_i} \sqrt{\frac{1 - x}{x}}\,\mathrm{d}x, \\ \displaystyle\int_I f_3(b, c_1, x)\,\mathrm{d}x > \sum_{i=1}^{k} \Big[\frac{u(q, t_{i-1})}{v(q, t_{i-1})(\omega(q, t_{i-1})}\Big]^{1/2} \int_{E_i} \sqrt{\frac{1 - x}{x}}\,\mathrm{d}x, \\ \displaystyle\int_I \frac{1}{f_3(b, c_1, x)}\,\mathrm{d}x < \sum_{i=1}^{k} \Big[\frac{v(q, t_{i-1})\omega(q, t_{i-1})}{u(q, t_{i-1})}\Big]^{1/2} \int_{E_i} \sqrt{\frac{x}{1 - x}}\,\mathrm{d}x. \end{cases}$$

For each t_i, $i = 1, \dots, k$, we define the functions in K

$$(13) \quad \begin{cases} p_i = p_i(b, t_i) = \dfrac{1}{b^2} p(q, t_i), & \tilde{p}_i = \tilde{p}_i(c_1, t_i) = \dfrac{1}{c_1^2} \tilde{p}(q, t_i), \\ u_i = u_i(b, t_i) = \dfrac{1}{a(b+a)}\, u(q, t_i), & v_i = v_i(b, t_i) = \dfrac{1}{2a} v(q, t_i), \\ \omega_i = \omega_i(b, c_1, t_i) = \dfrac{1}{c_1^2 - a^2}\, \omega(q, t_i). \end{cases}$$

Observe that

$$\begin{aligned} &p(q,0) = b^2, && \tilde{p}(q,0) = c_1^2, \\ &u(0) = a(b+a), && v(q,0) = 2a, \\ &\omega(q,0) = c_1^2 - a^2. \end{aligned}$$

We write J_1 as the following union :

$$J_1 = \bigcup_{s=1}^{10} A_s, \quad A_s = \big[2.55 + (s-1)0.03, 2.55 + s0.03\big].$$

Let b_{1s} and b_{2s} be the minimal and maximum values of b in A_s, $s = 1, \dots, 10$. Then, for each A_s we get from (13)

$$\begin{aligned} &p(q, t_{i-1}) = b^2 p_i(b, t_{i-1}) \le b^2 p_{i-1}(b_{2s}, t_{i-1}), \qquad \tilde{p}(q, t_{i-1}) = c_1^2 \tilde{p}_i(c_1, t_{i-1}) \\ &u(q, t_{i-1}) = a(b+a) u_{i-1}(b, t_{i-1}) \ge a(b+a) u_{i-1}(b_{1s}, t_{i-1}), \\ &v(q, t_{i-1}) = 2a v_{i-1}(b, t_{i-1}) \le 2a v_{i-1}(b_{2s}, t_{i-1}), \\ &\omega(q, t_{i-1}) = (c_1^2 - a^2)\omega_{i-1}(b, c_1, t_{i-1}) \le (c_1^2 - a^2)\omega_{i-1}(b_{1s}, c_1, t_{i-1}). \end{aligned}$$

From the expressions above and (12) we get for all $b \in A_s$ and for each s, $s = 1, \dots, 10$

$$(14) \quad \begin{cases} \displaystyle\int_I f_1(b, c_1, x)\,\mathrm{d}x \; < \frac{b}{a^{1/2} c_1} K_1(s), \\ \displaystyle\int_I \frac{1}{f_1(b, c_1, x)} \; > \frac{a^{1/2} c_1}{b} K_2(s), \\ \displaystyle\int_I f_3(b, c_1, x)\,\mathrm{d}x \; > \Big[\frac{b+a}{2(c_1^2 - a^2)}\Big]^{1/2} K_3(s), \\ \displaystyle\int_I \frac{1}{f_3(b, c_1, x)}\,\mathrm{d}x < \Big[\frac{2(c_1^2 - a^2)}{b+a}\Big]^{1/2} K_4(s), \end{cases}$$

where

$$(15)\quad \begin{cases} K_1(s) = \sum_{i=1}^{k} \Big[\dfrac{p_{i-1}(b_{2s}, t_{i-1})}{(1+t_{i-1})\tilde{p}_{i-1}(c_1, t_{i-1})} \Big]^{1/2} \displaystyle\int_{E_i} \sqrt{\frac{x}{1-x}}\, dx, \\ K_2(s) = \sum_{i=1}^{k} \Big[\dfrac{(1+t_{i-1})\tilde{p}_{i-1}(c_1, t_{i-1})}{p_{i-1}(b_{2s}, t_{i-1})} \Big]^{1/2} \displaystyle\int_{E_i} \sqrt{\frac{1-x}{x}}\, dx, \\ K_3(s) = \sum_{i=1}^{k} \Big[\dfrac{u_{i-1}(b_{1s}, t_{i-1})}{v_{i-1}(b_{2s}, t_{i-1})\omega_{i-1}(b_{1s}, c_1, t_{i-1})} \Big]^{1/2} \displaystyle\int_{E_i} \sqrt{\frac{1-x}{x}}\, dx, \\ K_4(s) = \sum_{i=1}^{k} \Big[\dfrac{v_{i-1}(b_{2s}, t_{i-1})\omega_{i-1}(b_{1s}, c_1, t_{i-1})}{u_{i-1}(b_{1s}, t_{i-1})} \Big]^{1/2} \displaystyle\int_{E_i} \sqrt{\frac{x}{1-x}}\, dx. \end{cases}$$

Then, for each A_s, $s = 1, \dots, 10$, we obtain from (10) and (14),

$$\begin{aligned} m_1(b, I) &> \frac{a^{1/2}c_1}{b} K_2(s) \Big[\frac{b+a}{2(c_1^2 - a^2)} \Big]^{1/2} K_3(s) \\ &\qquad - \Big[\frac{2(c_1^2 - a^2)}{b+a} \Big]^{1/2} K_4(s)\, \frac{b}{a^{1/2}c_1} K_1(s) \\ &= \frac{1}{bc_1[2a(b+a)(c_1^2 - a^2)]^{1/2}} \Big[a(b+a)c_1^2 K_2(s)K_3(s) \\ &\qquad - 2(c_1^2 - a^2)b^2 K_1(s)K_4(s) \Big]. \end{aligned}$$

So, $m_1(b, I) > 0$ if, for all $b \in A_s$, for each s, $s = 1, \dots, 10$:

$$(16)\qquad a(b+a)c_1^2 K_2(s)K_3(s) - 2(c_1^2 - a^2)b^2 K_1(s)K_4(s) > 0.$$

Let $k = 50$ (i.e. $t_i = i/50$), $1 \le i \le 50$, we take an upper bound or a lower bound, as we want, of the values $p_{i-1}(b_{2s}, t_{i-1})$, $\tilde{p}_{i-1}(c_1, t_{i-1})$, $u_{i-1}(b_{1s}, t_{i-1})$, $v_{i-1}(b_{2s}, t_{i-1})$, $\omega_{i-1}(b_{1s}, c_1, t_{i-1})$. Then we get from (15)

$$K_2(s)K_3(s) < 3.4 \quad \text{and} \quad K_1(s)K_4(s) > 1,$$

for each $s = 1, \dots, 10$. This implies, for all $(b, c) \in K$, for each s :

$$\begin{aligned} &\frac{\partial}{\partial b} \Big[a(b+a)c_1^2 K_2(s)K_3(s) - 2(c_1^2 - a^2)b^2 K_1(s)K_4(s) \Big] \\ &\qquad\qquad < 3.4ac_1^2 - 4b(c_1^2 - a^2) < 0. \end{aligned}$$

Therefore the left hand side of (16) is a decreasing function of b, for all $b \in A_s$. Then, for all $s = 1, \dots, 10$,

$$\begin{aligned} (17)\quad & a(b+a)c_1^2 K_2(s)K_3(s) - 2(c_1^2 - a^2)b^2 K_1(s)K_4(s) \\ &\ge ac_1^2(b_{2s} + a)K_2(s)K_3(s) - 2(c_1^2 - a^2)b_{2s}^2 K_3(s)K_4(s) > \frac{3}{4}. \end{aligned}$$

For the last inequality, we take an upper bound and a lower bound of $K_1(s)K_4(s)$ and $K_2(s)K_4(s)$, respectively, in the left hand side of (17), for each $s = 1, \dots, 10$ (see table I in the end). This shows that $m_1(b, I) > 0$. Then, from the uniform convergence $F_n \to F$ as $n \to \infty$, there is $n_2 \in \mathbb{N}$ such that $m_1(b, I_n) > 0$ for all $n \geq n_2$. This implies $\widetilde{F}_1(b, I_n) > 0$ for all $n \geq n_2$.

To prove that $m_2(b, I) < 0$, we take $J_1 = \bigcup_{r=1}^{27} B_r$ with

$$B_r = \begin{cases} [2.55 + (r-1)0.004, 2.55 + r0.004], & \text{if } r = 1, 2, \dots, 15, \\ [2.61 + (r-16)0.02, 2.61 + (r-15)0.02], & \text{if } r = 16, 12, \dots, 27. \end{cases}$$

Then

$$(18) \quad \begin{cases} \displaystyle\int_I f_1(b, c_2, x)\, \mathrm{d}x > \frac{b}{a^{1/2} c_2} K_1'(r), \\ \displaystyle\int_I \frac{1}{f_1(b, c_2, x)}\, \mathrm{d}x < \frac{a^{1/2} c_2}{b} K_2'(r), \\ \displaystyle\int_I f_3(b, c_2, x)\, \mathrm{d}x < \left[\frac{b+a}{2(c_2^2 - a^2)}\right]^{1/2} K_3'(r), \\ \displaystyle\int_I \frac{1}{f_3(b, c_2, x)}\, \mathrm{d}x > \left[\frac{2(c_2^2 - a^2)}{b+a}\right]^{1/2} K_4'(r), \end{cases}$$

where we obtain $K_j(r)$ from $K_j(s)$, $j = 1, 2, 3, 4$, respectively (see (15)) by the following substitutions :

$$c_1 \mapsto c_2, \quad t_{i-1} \mapsto t_i, \quad b_{1s} \mapsto b_{2r}, \quad b_{2s} \mapsto b_{1r},$$

where b_{1r} and b_{2r} are the minimal and maximum values of $b \in B_r$ respectively, $r = 1, 2, \dots, 27$.

Then, from (10) and (18), $m_2(b, I) < 0$, if for each B_r, $r = 1, \dots, 27$, and for all $b \in B_r$:

$$a(b+a)c_2^2 K_2'(r)K_3'(r) - 2(c_2^2 - a^2)b^2 K_1'(r)K_4'(r) < 0.$$

Analogously for the proof of $m_1(I, b) > 0$, we consider $k = 60$, $t_i = i/60$ and we obtain $K_2'(r)K_3'(r) < 3.4$ and $K_1'(r)K_4(r) > 1$, for all $r = 1, \dots, 27$. Then,

$$(19) \quad \begin{aligned} &a(b+a)c_2^2 K_2'(r)K_3'(r) - 2(c_2^2 - a^2)b^2 K_1'(r)K_4'(r) \\ &\leq ac_2^2(b_{1r} + a)K_2'(r)K_3'(r) - 2(c_2^2 - a^2)b_{1r}^2 K_1'(r)K_4'(r) \\ &< -\frac{1}{100}. \end{aligned}$$

For the last inequality we take an upper bound $K_2'(r)K_3'(r)$ and a lower bound of $K_1'(r)K_4'(r)$, respectively, for each $r = 1, 2, \ldots, 23$ in the left hand side of (19) (see table II in the end). This shows that $m_2(I, b) < 0$. So, there is $n_3 \in \mathbb{N}$ such that $m_2(I_n, b) < 0$ for all $n \geq n_3$. This implies $\widetilde{F}_2(I_n, b) < 0$, for all $n \geq n_3$.

To prove that $\partial F_n/\partial c < 0$ observe that for all $n \geq 3$

$$\frac{\partial F_n}{\partial c}(b, c) = \frac{1}{\left[\left(\int_{I_n} f_1 \,\mathrm{d}x\right)\left(\int_{I_n} f_3 \,\mathrm{d}x\right)\right]^2} Q(b, c, I_n),$$

where

$$\begin{aligned}(20)\quad Q(b, c, I_n) = {} & \Big(\int_{I_n} f_3 \,\mathrm{d}x\Big)^2 \Big(\int_{I_n} f_1 \,\mathrm{d}x\Big)\Big(\int_{I_n} \frac{\partial}{\partial c}\Big(\frac{1}{f_1}\Big) \mathrm{d}x\Big) \\ & + \Big(\int_{I_n} f_1 \,\mathrm{d}x\Big)^2 \Big(\int_{I_n} f_3 \,\mathrm{d}x\Big)\Big(-\int_{I_n} \frac{\partial}{\partial c}\Big(\frac{1}{f_3}\Big) \mathrm{d}x\Big) \\ & + \Big(\int_{I_n} f_3 \,\mathrm{d}x\Big)^2 \Big(\int_{I_n} \frac{1}{f_1} \,\mathrm{d}x\Big)\Big(-\int_{I_n} \frac{\partial f_1}{\partial c} \,\mathrm{d}x\Big) \\ & + \Big(\int_{I_n} f_1 \,\mathrm{d}x\Big)^2 \Big(\int_{I_n} \frac{1}{f_3} \,\mathrm{d}x\Big)\Big(\int_{I_n} \frac{\partial f_3}{\partial c} \,\mathrm{d}x\Big)\end{aligned}$$

and

$$\frac{\partial h_1}{\partial c} = -c\Big[\frac{p(q, x)}{a(1 + x)(\tilde{p}(q, x))^3}\Big]^{1/2}, \quad \frac{\partial}{\partial c}\Big(\frac{1}{h_1}\Big) = c\Big[\frac{a(1 + x)}{p(q, x)\tilde{p}(q, x)}\Big]^{1/2},$$

$$\frac{\partial h_3}{\partial c} = -c\Big[\frac{u(q, x)}{v(q, x)(\omega(q, x))^3}\Big]^{1/2}, \quad \frac{\partial}{\partial c}\Big(\frac{1}{h_3}\Big) = c\Big[\frac{v(q, x)}{u(q, x)\omega(q, x)}\Big]^{1/2},$$

where $p, \tilde{p}, u, v, \omega$ are the functions in (4) and $q = (a, b, c)$.

The functions,

$$\int_{I_n} \frac{\partial f_i}{\partial c}(b, c, x) \,\mathrm{d}x, \quad \int_{I_n} \frac{\partial}{\partial c}\Big(\frac{1}{f_i}\Big)(b, c, x) \,\mathrm{d}x, \qquad i = 1, 3,$$

converge uniformly to

$$\int_{I} \frac{\partial f_i}{\partial c}(b, c, x) \,\mathrm{d}x, \quad \int_{I} \frac{\partial}{\partial c}\Big(\frac{1}{f_i}\Big)(b, c, x) \,\mathrm{d}x$$

in $K \times I$ respectively. This occurs because $\partial h_i/\partial c$ and $\partial(1/h_i)/\partial c$, $i = 1, 3$, are bounded in $K \times I$ and the integrals of the functions $\sqrt{(1 - x)/x}$

and $\sqrt{x/(1-x)}$ are bounded in I. Hence, the sequence of functions $\{Q(\cdot,\cdot,I_n)\}_{n\geq 3}$ converge uniformly to $Q(\cdot,\cdot,I)$ in K. It is easy to verify that $\partial h_1/\partial c$ and $\partial(1/h_1)/\partial c$ are strictly increasing functions in x and $\partial h_3/\partial c$ is a strictly decreasing function in x, for all $x \in I$ and for all $(b,c) \in K$. Then :

$$
(21)\quad \begin{cases}
-\int_I \frac{\partial f_1}{\partial c} < \frac{b}{a^{1/2}c^2}\int_{E_i}\sqrt{\frac{x}{1-x}}\,\mathrm{d}x \\
\qquad\qquad \sum_{i=1}^{k}\Big[\frac{p_{i-1}(2.85,t_{i-1})}{(1+t_{i-1})(\tilde{p}_{i-1}(3.45,t_{i-1}))^3}\Big]^{1/2}, \\
\int_I \frac{\partial}{\partial c}\Big(\frac{1}{f_1}\Big) < \frac{a^{1/2}}{b}\int_{E_i}\sqrt{\frac{1-x}{x}}\,\mathrm{d}x \\
\qquad\qquad \sum_{i=1}^{k}\Big[\frac{1+t_i}{p_i(2.55,t_i)\tilde{p}_i(3.45,t_i)}\Big]^{1/2}, \\
-\int_I \frac{\partial f_3}{\partial c} > c\Big[\frac{b+a}{2(c^2-a^2)^3}\Big]^{1/2}\int_{E_i}\sqrt{\frac{1-x}{x}}\,\mathrm{d}x \\
\qquad\qquad \sum_{i=1}^{k}\Big[\frac{u_{i-1}(2.55,t_{i-1})}{v_{i-1}(2.85,t_{i-1})(\omega_{i-1}(2.55,3.8,t_{i-1}))^3}\Big]^{1/2}, \\
\int_I \frac{\partial}{\partial c}\Big(\frac{1}{f_3}\Big) > c\Big[\frac{2}{(b+a)(c^2-a^2)}\Big]^{1/2}\int_{E_i}\sqrt{\frac{x}{1-x}}\,\mathrm{d}x \\
\qquad\qquad \sum_{i=1}^{k}\Big[\frac{v_{i-1}(2.5,t_{i-1})}{u_i(2.85,t_i)\omega_{i-1}(2.55,3.8,t_{i-1})}\Big]^{1/2}.
\end{cases}
$$

The last inequality occurs because $1/u$ is a decreasing function of x and v/ω is an increasing function of x, for all $x \in I$ and for all $(b,c) \in K$ (see (4)). And from Proposition 1.1 we have

$$
(22')\quad \begin{cases}
\int_I f_1(b,c,x)\,\mathrm{d}x > \frac{b}{a^{1/2}c}\int_{E_i}\sqrt{\frac{x}{1-x}}\,\mathrm{d}x \\
\qquad\qquad \sum_{i=1}^{k}\Big[\frac{p_i(2.55,t_i)}{(1+t_i)\tilde{p}_i(3.8,t_i)}\Big]^{1/2}, \\
\int_I \frac{1}{f_1(b,c,x)}\,\mathrm{d}x < \frac{a^{1/2}c}{b}\int_{E_i}\sqrt{\frac{1-x}{x}}\,\mathrm{d}x \\
\qquad\qquad \sum_{i=1}^{k}\Big[\frac{(1+t_i)\tilde{p}_i(3.8,t_i)}{p_i(2.55,t_i)}\Big]^{1/2},
\end{cases}
$$

$$(22'')\quad \begin{cases} \displaystyle\int_I f_3(b,c,x)\,\mathrm{d}x < \Big[\frac{2(c^2-a^2)}{b+a}\Big]^{-1/2} \int_{E_i}\sqrt{\frac{1-x}{x}}\,\mathrm{d}x \\ \qquad\qquad \displaystyle\sum_{i=1}^{k}\Big[\frac{u_i(2.85,t_i)}{v_i(2.55,t_i)\omega_i(2.85,3.45,t_i)}\Big]^{1/2}, \\ \displaystyle\int_I \frac{1}{f_3(b,c,x)}\,\mathrm{d}x > \Big[\frac{b+a}{2(c^2-a^2)}\Big]^{-1/2} \int_{E_i}\sqrt{\frac{x}{1-x}}\,\mathrm{d}x \\ \qquad\qquad \displaystyle\sum_{i=1}^{k}\Big[\frac{v_i(2.55,t_i)\omega_i(2.85,3.45,t_i)}{u_i(2.85,t_i)}\Big]^{1/2}. \end{cases}$$

Let $k = 50$ and $t_i = i/50$. We obtain the following bounds of (21) and (22′), (22″) :

$$\begin{aligned} &\int_I f_1\,\mathrm{d}x > 1.1482\,\frac{b}{a^{1/2}c}, \\ &\int_I \frac{1}{f_1}\,\mathrm{d}x < 1.7721\,\frac{a^{1/2}c}{b}, \\ &\int_I f_3\,\mathrm{d}x < 1.9732\Big[\frac{b+a}{2(c^2-a^2)}\Big]^{1/2}, \\ &\int_I \frac{1}{f_3}\,\mathrm{d}x > 0.8501\Big[\frac{2(c^2-a^2)}{b+a}\Big]^{1/2}, \\ -&\int_I \frac{\partial f_1}{\partial c}\,\mathrm{d}x < 1.2712\,\frac{b}{a^{1/2}c^2}, \\ &\int_I \frac{\partial}{\partial c}\Big(\frac{1}{f_1}\Big)\,\mathrm{d}x < 1.8014\,\frac{a^{1/2}}{b}, \\ -&\int_I \frac{\partial f_3}{\partial c}\,\mathrm{d}x > 1.9904\,c\Big[\frac{b+a}{2(c^2-a^2)^3}\Big]^{1/2}, \\ &\int_I \frac{\partial}{\partial c}\Big(\frac{1}{f_3}\Big)\,\mathrm{d}x > 1.3319\,c\Big[\frac{2}{(b+a)(c^2-a^2)}\Big]^{1/2}. \end{aligned}$$

Let us denote I, II, III, IV the terms of $Q(b,c,I)$ respectively (see 20). Then we get from above, for all $(b,c) \in K$,

$$\begin{aligned} \mathrm{I} + 0.78\,\mathrm{II} &= \Big(\int_I f_3 dx\Big)\Big(\int_I f_1\,\mathrm{d}x\Big)\Big[\Big(\int_I f_3\,\mathrm{d}x\Big)\Big(\int_I \frac{\partial}{\partial c}\Big(\frac{1}{f_1}\Big)\,\mathrm{d}x\Big) \\ &\qquad + 0.78\Big(\int_I f_1\,\mathrm{d}x\Big)\Big(-\int_I \frac{\partial}{\partial c}\Big(\frac{1}{f_3}\Big)\,\mathrm{d}x\Big)\Big] \\ &< \Big(\int_I f_3\,\mathrm{d}x\Big)\Big(\int_I f_1\,\mathrm{d}x\Big)\frac{3.5572\,a(b+a) - 2.3836\,b^2}{b[2a(b+a)(c^2-a^2)]^{1/2}} < 0, \end{aligned}$$

$$0.22\,\mathrm{II}+0.24\,\mathrm{III}=\Big(\int_I f_3\,\mathrm{d}x\Big)\Big[0.22\Big(\int_I f_1\,\mathrm{d}x\Big)^2\Big(-\int_I \frac{\partial}{\partial c}\frac{1}{f_3}\Big)$$
$$+0.24\Big(\int_I f_3\,\mathrm{d}x\Big)\Big(\int_I \frac{1}{f_1}\mathrm{d}x\Big)\Big(-\int_I \frac{\partial f_1}{\partial c}\mathrm{d}x\Big)\Big]$$
$$<\Big(\int_I f_3\,\mathrm{d}x\Big)\frac{-0.7718\,b^2+1.0685\,a(b+a)}{ac[2(b+a)(c^2-a^2)]^{1/2}}<0,$$

$$0.76\,\mathrm{III}+\mathrm{IV}<\frac{6.6658\,a(b+a)-4.4614\,b^2}{2ac(c^2-a^2)}<0.$$

Therefore $Q(b,c,I)<0$, for all $(b,c)\in K$. From uniform convergence, $Q(\cdot,\cdot,I_n)\to Q(\cdot,\cdot,I)$ as $n\to\infty$, there is $n_4\in\mathbb{N}$ such that $Q(b,c,I_n)<0$, for all $n\geq n_4$. This implies $\partial F_n/\partial c(b,c)<0$ for all $n\geq n_4$. We take $n_5=\max\{n_2,n_3,n_4\}$. Then, for all $n\geq n_5$ the condition (i) is satisfied.

To prove (ii) we denote

$$\widetilde{G}_j(c,I_n)=G_{n}{}_{|\{b_j\}\times J_2}=\frac{1}{\big(\int_{I_n} f_1\,\mathrm{d}x\big)\big(\int_{I_n} f_5\,\mathrm{d}x\big)}\widetilde{m}_j(c,I_n),$$

where $j=1,2$ and

$$\widetilde{m}_j(c,I_n)=\Big(\int_{I_n}\frac{1}{f_1(b_j,c,x)}\,\mathrm{d}x\Big)\Big(\int_{I_n} f_5(b_j,c,x)\,\mathrm{d}x\Big)-\Big(\int_{I_n}\frac{1}{f_5(b_j,c,x)}\,\mathrm{d}x\Big)\Big(\int_{I_n} f_1(b_j,c,x)\,\mathrm{d}x\Big). \tag{23}$$

Therefore, $\widetilde{G}_1(c,I_n)>0$ and $\widetilde{G}_2(c,I_n)<0$, for all $c\in J_2$, if and only if, $\widetilde{m}_1(c,I_n)>0$ and $\widetilde{m}_2(c,I_n)<0$, for all $c\in J_2$, respectively.

Observe that for all $n\geq 3$:

$$\frac{d}{dc}\widetilde{m}_j(c,I_n)=\Big(\int_{I_n}\frac{\partial}{\partial c}\Big(\frac{1}{f_1}\Big)\mathrm{d}x\Big)\Big(\int_{I_n} f_5\,\mathrm{d}x\Big)+\Big(\int_{I_n}\frac{1}{f_1}\mathrm{d}x\Big)\Big(\int_{I_n}\frac{\partial f_5}{\partial c}\mathrm{d}x\Big)+\Big(-\int_{I_n}\frac{\partial}{\partial c}\Big(\frac{1}{f_5}\Big)\mathrm{d}x\Big)\Big(\int_{I_n} f_1\,\mathrm{d}x\Big)+\Big(\int_{I_n}\frac{1}{f_5}\mathrm{d}x\Big)\Big(-\int_{I_n}\frac{\partial f_1}{\partial c}\mathrm{d}x\Big). \tag{24}$$

We have the following

$$\text{(25)}\qquad \begin{cases} c\Big(-\dfrac{\partial f_1}{\partial c}\Big) \geq f_1, & c\dfrac{\partial}{\partial c}\Big(\dfrac{1}{f_1}\Big) \geq \dfrac{1}{f_1}, \\ \dfrac{1}{c} f_5 + \dfrac{\partial f_5}{\partial c} > 0, & \dfrac{1}{c f_5} - \dfrac{\partial}{\partial c}\Big(\dfrac{1}{f_5}\Big) > 0, \end{cases}$$

for all $(b,c,x) \in K \times I_n$. The two first inequalities are easy to verify. To show the two others we observe that

$$\begin{aligned} \frac{1}{c} f_5 + \frac{\partial}{\partial c} f_5 = {} & \frac{1}{2c}\sqrt{\frac{x}{1-x}}\, \tilde{u}^{-1/2}(\tilde{v}\tilde{\omega})^{-3/2} \\ & \times \Big[2\tilde{u}\tilde{v}\tilde{\omega} + c\Big(\frac{\partial \tilde{u}}{\partial c}\Big)\tilde{v}\tilde{\omega} - c\tilde{u}\Big(\frac{\partial \tilde{v}}{\partial c}\Big)\tilde{\omega} - c\tilde{u}\tilde{v}\frac{\partial \tilde{\omega}}{\partial c}\Big], \end{aligned}$$

where

$$\begin{aligned} 2\tilde{u}\tilde{v}\tilde{\omega} & + c\Big(\frac{\partial \tilde{u}}{\partial c}\Big)\tilde{v}\tilde{\omega} - c\tilde{u}\Big(\frac{\partial \tilde{v}}{\partial c}\Big)\tilde{\omega} - c\tilde{u}\tilde{v}\frac{\partial \tilde{\omega}}{\partial c} \\ & = \tilde{u}\tilde{\omega}\Big(\tilde{v} - c\frac{\partial \tilde{v}}{\partial c}\Big) + \tilde{v}\Big[c\Big(\frac{\partial \tilde{u}}{\partial c}\Big)\tilde{\omega} - c\tilde{u}\frac{\partial \tilde{\omega}}{\partial c} + \tilde{u}\tilde{\omega}\Big] \\ & = \tilde{u}\tilde{w}(b - bx) + \tilde{v}[(c-b)^4x^4 + b(4c-5b)(c-b)^2x^3 \\ & \qquad + (7b^2c - 9b^3 - 3a^2c + a^2b)(c-b)x^2 \\ & \qquad\quad + (6b^3c - 7b^4 - 6a^2bc + 3a^2b)x + 2b^2(b^2-a^2)]. \end{aligned}$$

Since $(b - bx)\tilde{u}\tilde{w} \geq 0$, for all $(b,c) \in K$, we get :

$$\begin{gathered} 4c - 5b \geq -0.45, \quad 7b^2c - 9b^3 - 3a^2c + a^2b > -22.34, \\ 6b^3c - 7b^4 - 6a^2bc + 3a^2b > -29.50. \end{gathered}$$

Then,

$$\frac{1}{c} f_5 + \frac{\partial}{\partial c} f_5 > \frac{1}{2c}\sqrt{\frac{x}{1-x}}\, \tilde{u}^{-1/2}(\tilde{v}\tilde{\omega})^{-3/2}\tilde{v}\ell_3$$

where

$$\begin{aligned} \ell_3 & = \ell_3(b,c,x) \\ & = (c-b)^4x^4 - 0.45\, b(c-b)^2x^3 - 22.34(c-b)x^2 - 29.50\, x + 2b^2(b^2-a^2). \end{aligned}$$

Observe that $\partial \ell_3/\partial x < 0$, for all $x \in I_n$ and $(b,c) \in K$; and $\ell_3(b,c,1)$ is an increasing function of b and a decreasing function of c for all $(b,c) \in K$. Then $\ell_3(b,c,1) \geq \ell_3(2.55, 3.8, 1) > 5$. So, $\ell_3(b,c,x) > 0$ and therefore

$(1/c)f_5 + \partial f_5/\partial c > 0$, for all $(b, c, x) \in K \times I_n$. By the same calculations we obtain

$$\frac{1}{cf_5} - \frac{\partial}{\partial c}\Big(\frac{1}{f_5}\Big) = \frac{1}{2c}\sqrt{\frac{1-x}{x}}\,\tilde{u}^{-3/2}(\tilde{v}\tilde{\omega})^{-1/2} \times \Big[2\tilde{u}\tilde{v}\tilde{\omega} + c\Big(\frac{\partial \tilde{u}}{\partial c}\Big)\tilde{v}\tilde{\omega} - c\tilde{u}\Big(\frac{\partial \tilde{v}}{\partial c}\Big)\omega - c\tilde{u}\tilde{\omega}\frac{\partial \tilde{\omega}}{\partial c}\Big] > 0.$$

Therefore, from (24) and (25) we get, for all $n \geq 3$,

$$\frac{\mathrm{d}}{\mathrm{d}c}\widetilde{m}_j(c, I_n) \geq \Big(\int_{I_n}\frac{1}{f_1}\,\mathrm{d}x\Big)\Big[\frac{1}{c}\int_{I_n} f_5\,\mathrm{d}x + \int_{I_n}\frac{\partial f_5}{\partial c}\,\mathrm{d}x\Big] + \Big(\int_{I_n} f_1\,\mathrm{d}x\Big)\Big[-\int_{I_n}\frac{\partial}{\partial c}\Big(\frac{1}{f_5}\Big)\,\mathrm{d}x + \frac{1}{c}\int_{I_n}\frac{1}{f_5}\,\mathrm{d}x\Big] > 0.$$

That is, $\widetilde{m}_j(c, I_n)$ is an increasing function for all $c \in J_2$. Then

$$\widetilde{m}_1(c, I_n) \geq \widetilde{m}_1(3.45, I_n), \quad \widetilde{m}_2(c, I_n) \leq \widetilde{m}_2(3.8, I_n),$$

for all $n \geq 3$. From (23) we have

$$\begin{aligned}\widetilde{m}_1(3.45, I) > {} & \Big(\sum_{i=1}^{k}\frac{1}{h_1(2.55, 3.45, t_{i-1})}\int_{E_i}\sqrt{\frac{1-x}{x}}\,\mathrm{d}x\Big) \times \Big(\sum_{i=1}^{k} h_5(2.55, 3.45, t_i)\int_{E_i}\sqrt{\frac{x}{1-x}}\,\mathrm{d}x\Big)\\ & - \Big(\sum_{i=1}^{k}\frac{1}{h_5(2.55, 3.45, t_i)}\int_{E_i}\sqrt{\frac{1-x}{x}}\,\mathrm{d}x\Big) \times \Big(\sum_{i=1}^{k} h_1(2.55, 3.45, t_{i-1})\int_{E_i}\sqrt{\frac{x}{1-x}}\,\mathrm{d}x\Big)\end{aligned}$$

and

$$\begin{aligned}\widetilde{m}_2(3.8, I) < {} & \Big(\sum_{i=1}^{k}\frac{1}{h_1(2.85, 3.8, t_i)}\int_{E_i}\sqrt{\frac{1-x}{x}}\,\mathrm{d}x\Big) \times \Big(\sum_{i=1}^{k} h_5(2.85, 3.8, t_{i-1})\int_{E_i}\sqrt{\frac{x}{1-x}}\,\mathrm{d}x\Big)\\ & - \Big(\sum_{i=1}^{k}\frac{1}{h_5(2.85, 3.8, t_{i-1})}\int_{E_i}\sqrt{\frac{1-x}{x}}\,\mathrm{d}x\Big) \times \Big(\sum_{i=1}^{k} h_1(2.85, 3.8, t_i)\int_{E_i}\sqrt{\frac{x}{1-x}}\,\mathrm{d}x\Big).\end{aligned}$$

If we take $k = 20$ and $t_i = i/20$, we get

$$\widetilde{m}_1(3.45, I) > \frac{1}{10}, \quad \widetilde{m}_2(3.8, I) < -\frac{9}{100}.$$

Then, from the uniform convergence, $\widetilde{m}_j(c, I_n) \to \widetilde{m}_j(c, I)$ as $n \to \infty$, $j = 1, 2$, there is $n_6 \in \mathbb{N}$ such that $\widetilde{m}_1(3.45, I_n) > 0$ and $\widetilde{m}_2(3.8, I_n) < 0$, for all $n \geq n_6$. It follows $\widetilde{m}_1(c, I_n) > 0$ and $\widetilde{m}_2(c, I_n) < 0$, for all $n \geq n_6$ and for all $c \in J_2$.

Now we want to prove that there is $n_7 \in \mathbb{N}$ such that for all $n \geq n_7$ $\partial G_n(b, c)/\partial b < 0$, for all $(b, c) \in K$. From expression of G_n we get

$$\frac{\partial G_n}{\partial b}(b, c) = \frac{1}{\left[\left(\int_{I_n} f_1 \, dx\right)\left(\int_{I_n} f_5 \, dx\right)\right]^2} \widetilde{Q}(b, c, I_n),$$

where

$$\begin{aligned}
\widetilde{Q}(b, c, I_n) = {} & \left(\int_{I_n} f_5 \, dx\right)^2 \left(\int_{I_n} f_1 \, dx\right) \left(\int_{I_n} \frac{\partial}{\partial b}\left(\frac{1}{f_1}\right) dx\right) \\
& + \left(\int_{I_n} f_1 \, dx\right)^2 \left(\int_{I_n} f_5 \, dx\right) \left(-\int_{I_n} \frac{\partial}{\partial b}\left(\frac{1}{f_5}\right) dx\right) \\
& + \left(\int_{I_n} f_5 \, dx\right)^2 \left(\int_{I_n} \frac{1}{f_1} \, dx\right) \left(-\int_{I_n} \frac{\partial f_1}{\partial b} \, dx\right) \\
& + \left(\int_{I_n} f_1 \, dx\right)^2 \left(\int_{I_n} \frac{1}{f_5} \, dx\right) \left(\int_{I_n} \frac{\partial f_5}{\partial b} \, dx\right).
\end{aligned} \tag{26}$$

Then, $\partial G_n(b, c)/\partial b < 0$ if and only if, $\widetilde{Q}(b, c, I_n) < 0$. First we will show that $\widetilde{Q}(b, c, I) < 0$. From (3) we get

$$\begin{cases}
\dfrac{\partial h_1}{\partial b}(q, x) = b\left[\dfrac{1}{a(1+x)p(q, x)\tilde{p}(q, x)}\right]^{1/2}, \\
\dfrac{\partial}{\partial b}\left(\dfrac{1}{h_1}\right)(q, x) = -b\left[\dfrac{a(1+x)\tilde{p}(q, x)}{(p(q, x))^3}\right]^{1/2}, \\
\dfrac{\partial}{\partial b} h_5(q, x) = \frac{1}{2}\tilde{u}^{-1/2}(\tilde{v}\tilde{\omega})^{-3/2} H(b, c, x), \\
\dfrac{\partial}{\partial b}\left(\dfrac{1}{h_5}\right)(q, x) = \frac{1}{2}\tilde{u}^{-3/2}(\tilde{v}\tilde{\omega})^{-1/2}(-H(b, c, x)),
\end{cases} \tag{27}$$

where

$$H(b, c, x) = \tilde{v}\tilde{\omega}\frac{\partial \tilde{u}}{\partial b} - \tilde{u}\left(\frac{\partial \tilde{v}}{\partial b}\right)\tilde{\omega} - \tilde{u}\tilde{v}\frac{\partial \tilde{\omega}}{\partial b}. \tag{28}$$

Since $\partial^3\tilde{u}/\partial x^2\partial b = \partial^3\widetilde{\omega}/\partial x^2\partial b = -4(c-b)$, $\partial^3\tilde{v}/\partial x^2\partial b = 0$, we get :

$$
\begin{aligned}
(29)\quad & \frac{\partial^2}{\partial x^2} H(b,c,x) \\
&= 2(c-b)^2\tilde{v}\Big(\frac{\partial\tilde{u}}{\partial b} - \frac{\partial\widetilde{\omega}}{\partial b}\Big) + 2(c-b)\Big(\frac{\partial\tilde{u}}{\partial b}\frac{\partial\widetilde{\omega}}{\partial x} - \frac{\partial\tilde{u}}{\partial x}\frac{\partial\widetilde{\omega}}{\partial b}\Big) \\
&\quad + 2\tilde{v}\Big[\frac{\partial^2\tilde{u}}{\partial x\partial b}\frac{\partial\widetilde{\omega}}{\partial x} - \frac{\partial\tilde{u}}{\partial x}\frac{\partial^2\widetilde{\omega}}{\partial x\partial b}\Big] + 2(c-b)\Big[\widetilde{\omega}\frac{\partial^2\tilde{u}}{\partial x\partial b} - \tilde{u}\frac{\partial^2\widetilde{\omega}}{\partial x\partial b}\Big] \\
&\quad + 2(c-b)\Big[2\tilde{v}[b(c-b)x + b^2 + a^2)] - (c-b)(\tilde{u}+\widetilde{\omega})\frac{\partial\tilde{v}}{\partial b}\Big] \\
&\quad + 2\tilde{u}\Big(\frac{\partial\widetilde{\omega}}{\partial x}\Big) - 2(1-x)\frac{\partial}{\partial x}\tilde{u}\frac{\partial}{\partial x}\widetilde{\omega} + 2\Big(\frac{\partial\tilde{u}}{\partial x}\Big)\widetilde{\omega}.
\end{aligned}
$$

We want to show that, for all $(b,c,x) \in K \times I$,

$$
(30)\qquad \frac{\partial^2}{\partial x^2} H(b,c,x) > 0.
$$

It is easy to verify that the first five terms of (29) are positive for all $(b,c,x) \in K \times I$.

Observe that the three last terms of (29) satisfy

$$
\begin{aligned}
&\Big[2\tilde{u} - (1-x)\frac{\partial\tilde{u}}{\partial x}\Big]\frac{\partial\widetilde{\omega}}{\partial x} + \Big[2\widetilde{\omega} - (1-x)\frac{\partial\widetilde{\omega}}{\partial x}\Big]\frac{\partial\tilde{u}}{\partial x} \\
&\quad > \Big[2\tilde{u}(b,c,0) - \frac{\partial\tilde{u}}{\partial x}(b,c,0)\Big]\frac{\partial\widetilde{\omega}}{\partial x} + \Big[2\widetilde{\omega}(b,c,0) - \frac{\partial\widetilde{\omega}}{\partial x}(b,c,0)\Big]\frac{\partial\tilde{u}}{\partial x} \\
&\quad = b(7b-3c)\frac{\partial\widetilde{\omega}}{\partial x} + 2(2b^2 - bc - a^2)\frac{\partial\tilde{u}}{\partial x} > 0,
\end{aligned}
$$

where the inequality is from the facts : for all $x \in I$ and $(b,c) \in K$,

$$
2\tilde{u} - (1-x)\frac{\partial\tilde{u}}{\partial x}, \quad 2\widetilde{\omega} - (1-x)\frac{\partial\widetilde{\omega}}{\partial x}
$$

are increasing functions of x. This show (30).

Furthermore, for all $(b,c) \in K$,

$$
\begin{aligned}
\frac{\partial H}{\partial x}(b,c,0) &= (3c-6b)(c+b)(b^2-a^2) \\
&\qquad + 4b(c-b)(b^2-a^2) + 8b^2(c+b)(c-b) \\
&= (c+b)(2b-c)(b^2+3a^2) + b\big[4.6b^3 - 2.6b^2c - a^2(c+b)\big] \\
&\qquad + 2b^2(c-b)(c+b-0.2) > 0.
\end{aligned}
$$

This and (30) imply, for all $(b, c, x) \in K \times I$,

$$\frac{\partial H}{\partial x}(b, c, x) > 0. \tag{31}$$

Observe that, for all $(b, c) \in K$ (see (28)) :

$$H(b, c, 0) = -2b(b^3 + 2a^2c + a^2b) < 0, \tag{32}$$

$$H(b, c, \tfrac{1}{2}) = \tfrac{1}{32}\Big\{8a^2(b-c) + (c+b)^2\big[2.9c^2 - 2.7bc - 3b^2\big] + (3c+b)\big[0.7c(c+b)^2 - 4a^2(3c+5b)\big]\Big\} < 0, \tag{33}$$

$$H(b, c, 1) = 2c^2(c^2 - a^2) > 0. \tag{34}$$

For each $(b, c) \in K$ we consider the segments of straight lines

$$\begin{cases} H_1(b, c, x) = (1-2x)H(b, c, 0), & x \in [0, \tfrac{1}{2}]; \\ H_2(b, c, x) = (2x-1)H(b, c, 1), & x \in [\tfrac{1}{2}, 1]. \end{cases} \tag{35}$$

Then, from (30), (31), (32), (33), (34) we obtain

$$H(b, c, x) \le \begin{cases} H_1(b, c, x) & \text{if } x \in [0, \tfrac{1}{2}], \\ H_2(b, c, x) & \text{if } x \in [\tfrac{1}{2}, 1]. \end{cases} \tag{36}$$

(See Fig. 4.)

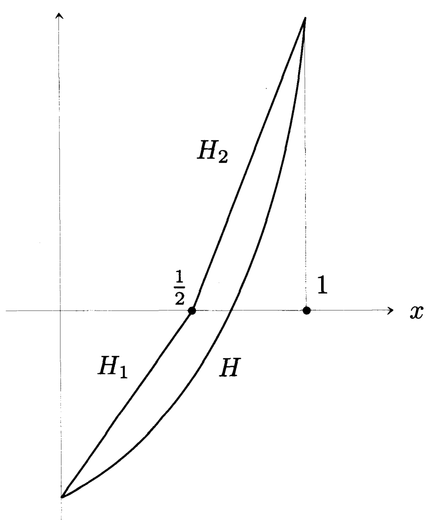

Figure 4

Then, from (27) and (36),

$$\int_I \frac{\partial f_5}{\partial b}\,\mathrm{d}x < \frac{1}{2}\Big[\int_{[0,\frac{1}{2}]} H_1(b,c,x)\tilde{u}^{-1/2}(\tilde{v}\widetilde{\omega})^{-3/2}\sqrt{\frac{x}{1-x}}\,\mathrm{d}x + \int_{[\frac{1}{2},1]} H_2(b,c,x)\tilde{u}^{-1/2}(\tilde{v}\widetilde{\omega})^{-3/2}\sqrt{\frac{x}{1-x}}\,\mathrm{d}x\Big]. \tag{37}$$

Furthermore, $H_1(b,c,x)\tilde{u}^{-1/2}(\tilde{v}\widetilde{\omega})^{-3/2}$ is a strictly increasing function of x, for all $x \in [0, \frac{1}{2}]$ and $(b,c) \in K$. This occurs because :

- $\frac{\partial}{\partial x}\left[H_1\tilde{u}^{-1/2}(\tilde{v}\widetilde{\omega})^{-3/2}\right] = \tilde{u}^{-1/2}(\tilde{v}\widetilde{\omega})^{-3/2}\frac{\partial H_1}{\partial x} + H_1\frac{\partial}{\partial x}\left[\tilde{u}^{-1/2}(\tilde{v}\widetilde{\omega})^{-3/2}\right]$;
- $\frac{\partial}{\partial x}\left[\tilde{u}^{-1/2}(\tilde{v}\widetilde{\omega})^{-3/2}\right] < 0$, for all $x \in I$ and $(b,c) \in K$ (remember that $\tilde{u}, \tilde{v}$ and $\widetilde{\omega}$ are strictly increasing functions of x, for all $(b,c,x) \in K \times I$);
- $\frac{\partial H_1}{\partial x} = -2H(b,c,0) > 0$, $H_1(b,c,x) \leq 0$, for all $x \in [0, \frac{1}{2}]$ and $(b,c) \in K$.

Observe (see (4)) that :

$$\tilde{u}(q,0) = 2b^2, \quad \tilde{v}(q,0) = (c+b), \quad \widetilde{\omega}(q,0) = b^2 - a^2.$$

As in (15), we define the functions in K

$$\tilde{u}_i = \tilde{u}_i(b,c,t_i) = \frac{1}{2b^2}\tilde{u}(b,c,t_i),$$
$$\tilde{v}_i = \tilde{v}_i(b,c,t_i) = \frac{1}{c+b}\tilde{v}(b,c,t_i),$$
$$\widetilde{\omega}_i = \widetilde{\omega}_i(b,c,t_i) = \frac{1}{b^2-a^2}\widetilde{\omega}(b,c,t_i),$$

which are strictly decreasing functions of b and strictly increasing functions of c for all $(b,c) \in K$. Then

$$\begin{cases} \tilde{u}(b,c,t_i) \leq 2b^2\tilde{u}_i(2.55, 3.8, t_i) \\ \qquad\qquad \geq 2b^2\tilde{u}_i(2.85, 3.45, t_i), \\ \tilde{v}(b,c,t_i) \leq (c+b)\tilde{v}_i(2.55, 3.8, t_i) \\ \qquad\qquad \geq (c+b)\tilde{v}_i(2.85, 3.45, t_i), \\ \widetilde{\omega}(b,c,t_i) \leq (b^2-a^2)\widetilde{\omega}_i(2.55, 3.8, t_i) \\ \qquad\qquad \geq (b^2-a^2)\widetilde{\omega}_i(2.85, 3.45, t_i). \end{cases} \tag{38}$$

Then, if $\{t_0, t_1, \ldots, t_k\}$, is a partition of I, $0 = t_0 < \cdots < t_k = 1$, where k is even, $t_{k/2} = \frac{1}{2}$, and $E_i = [t_{i-1}, t_i]$, $i = 1, 2, \ldots, k$, we get from (35), (37) and (38)

$$(39)\quad \int_I \frac{\partial f_5}{\partial b}\,\mathrm{d}x < \frac{1}{2}\Big\{H(b,c,0)\Big[\frac{1}{2b^2((c+b)(b^2-a^2))^3}\Big]^{1/2} \times \sum_{i=1}^{k/2}(1-2t_i)\Big(\frac{1}{D_i'}\Big)^{1/2}\int_{E_i}\sqrt{\frac{x}{1-x}}\,\mathrm{d}x + H(b,c,1)\Big[\frac{1}{2b^2((c+b)(b^2-a^2))^3}\Big]^{1/2} \times \sum_{i=(k/2)+1}^{k}(2t_i-1)\Big(\frac{1}{D_i''}\Big)^{1/2}\int_{E_i}\sqrt{\frac{x}{1-x}}\,\mathrm{d}x\Big\},$$

with

$$D_i' = \tilde{u}_i(2.55, 3.8, t_i)\big[\tilde{v}_i(2.55, 3.8, t_i)\widetilde{\omega}(2.55, 3.8, t_i)\big]^3,$$

$$D_i'' = \tilde{u}_{i-1}(2.85, 3.45, t_{i-1})\big[\tilde{v}_{i-1}(2.85, 3.45, t_{i-1})\widetilde{\omega}_{i-1}(2.85, 3.45, t_{i-1})\big]^3.$$

Analogously, from (27) and (36)

$$(40)\quad -\int_I \frac{\partial}{\partial b}\Big(\frac{1}{f_5}\Big)\,\mathrm{d}x < \frac{1}{2}\Big\{H(b,c,0)\Big[\frac{1}{(2b^2)^3(c+b)(b^2-a^2)}\Big]^{1/2} \times \sum_{i=1}^{k/2}(1-2t_i)\Big(\frac{1}{D_i'''}\Big)^{1/2}\int_{E_i}\sqrt{\frac{1-x}{x}}\,\mathrm{d}x + H(b,c,1)\Big[\frac{1}{(2b^2)^3(c+b)(b^2-a^2)}\Big]^{1/2} \times \sum_{i=k/2+1}^{k}(2t_i-1)\Big(\frac{1}{D_i''''}\Big)^{1/2}\int_{E_i}\sqrt{\frac{1-x}{x}}\,\mathrm{d}x\Big\},$$

with

$$D_i''' = \big[\tilde{u}_i(2.55, 3.8, t_i)\big]^3\tilde{v}_i(2.55, 3.8, t_i)\widetilde{\omega}(2.55, 3.8, t_i),$$

$$D_i'''' = \big[\tilde{u}_{i-1}(2.85, 3.45, t_{i-1})\big]^3\tilde{v}_{i-1}(2.85, 3.45, t_{i-1})\widetilde{\omega}_{i-1}(2.85, 3.45, t_{i-1}).$$

Consider $k = 50$ (i.e. $t_i = i/50$) in (39) and (40). We obtain the following bounds :

$$\int_I \frac{\partial f_5}{\partial b} < \frac{1}{2}\Big\{0.0617\,H(b,c,0)\Big[\frac{1}{2b^2((c+b)(b^2-a^2))^3}\Big]^{1/2} + 0.3941\,H(b,c,1)\Big[\frac{1}{2b^2((c+b)(b^2-a^2))^3}\Big]^{1/2}\Big\}$$

$$= \frac{1}{2}\Big[\frac{1}{2b^2((c+b)(b^2-a^2))^3}\Big]^{1/2}(0.0617H(b,c,0) + 0.3941H(b,c,1))$$

and

$$-\int_I \frac{\partial}{\partial b}\left(\frac{1}{f_5}\right) < \frac{1}{2}\left[\frac{1}{(2b^2)^3(c+b)(b^2-a^2)}\right]^{1/2} \times \big(0.7322H(b,c,0) + 0.0667H(b,c,1)\big).$$

From (32) and (34) we have, for all $(b,c) \in K$,

$$\begin{aligned} 0.7322\,H(b,c,0) &+ 0.0667\,H(b,c,1) \\ &= -1.4644\,b(b^3 + 2a^2c + a^2b) + 0.1334\,c^2(c^2 - a^2) \\ &< -1.46\,b(b^3 + 2a^2c + a^2b) + 0.14\,c^2(c^2 - a^2) < -80. \end{aligned}$$

For all $(b,c) \in K$, this implies

$$-\int_I \frac{\partial(1/f_5)}{\partial b}\,\mathrm{d}x < 0.$$

Since $-\int_I \partial f_1/\partial b\,\mathrm{d}x < 0$ and $\int_I \partial(1/f_1)/\partial b\,\mathrm{d}x < 0$ (see (27)), the only possibly positive term in $\widetilde{Q}(b,c,I)$ (see 26) is the last one. We will show that the sum of the first and fourth terms of $\widetilde{Q}(b,c,I)$ has a negative upper bound. From (4) and (27), we obtain that $\partial(1/h_1)/\partial b$ is a strictly decreasing function of x, for all $(b,c,x) \in K \times I$. Then, we have :

$$\int_I \frac{\partial}{\partial b}\left(\frac{1}{f_1}\right)\mathrm{d}x < -\frac{a^{1/2}c}{b^2}\sum_{i=1}^{k}\left[\frac{(1+t_{i-1})\tilde{p}_{i-1}(3.45,t_{i-1})}{(\tilde{p}_{i-1}(2.85,t_{i-1}))^3}\right]^{1/2}\int_{E_i}\sqrt{\frac{1-x}{x}}\,\mathrm{d}x.$$

Furthermore, from (4), (13), (38) and Proposition 1.1 we get :

$$\int_I f_1\,\mathrm{d}x < \frac{b}{a^{1/2}c}\sum_{i=1}^{k}\left[\frac{p_{i-1}(2.85,t_{i-1})}{(1+t_{i-1})\tilde{p}_{i-1}(3.45,t_{i-1})}\right]^{1/2}\int_{E_i}\sqrt{\frac{x}{1-x}}\,\mathrm{d}x,$$

$$\int_I f_5\,\mathrm{d}x > \left[\frac{2b^2}{(c+b)(b^2-a^2)}\right]^{1/2} \times \sum_{i=1}^{k}\left[\frac{\tilde{u}_i(2.85,3.45,t_i)}{\tilde{v}_i(2.55,3.8,t_i)\widetilde{\omega}_i(2.55,3.8,t_i)}\right]^{1/2}\int_{E_i}\sqrt{\frac{x}{1-x}}\,\mathrm{d}x,$$

$$\int_I \frac{1}{f_5}\,\mathrm{d}x < \left[\frac{(c+b)(b^2-a^2)}{2b^2}\right]^{1/2} \times \sum_{i=1}^{k}\left[\frac{\tilde{v}_i(2.55,3.8,t_i)\widetilde{\omega}_i(2.55,3.8,t_i)}{\tilde{u}_i(2.85,3.45,t_i)}\right]^{1/2}\int_{E_i}\sqrt{\frac{1-x}{x}}\,\mathrm{d}x.$$

We take $k = 50$ (i.e. $t_i = i/50$) and obtain the following bounds :

$$\int_I f_1 \,\mathrm{d}x < 1.1815 \,\frac{b}{a^{1/2}c},$$

$$\int_I \frac{\partial}{\partial c}\Big(\frac{1}{f_1}\Big)\,\mathrm{d}x < -1.7482\,\frac{a^{1/2}c}{b^2},$$

$$\int_I f_5 \,\mathrm{d}x > 1.1394\Big[\frac{2b^2}{(c+b)(b^2-a^2)}\Big]^{1/2},$$

$$\int_I \frac{1}{f_5}\,\mathrm{d}x < 1.7884\Big[\frac{(c+b)(b^2-a^2)}{2b^2}\Big]^{1/2}$$

for all $(b,c) \in K$. Then

$$\begin{aligned}
&\Big(\int_I f_1\,\mathrm{d}x\Big)\Big[\Big(\int_I f_5\,\mathrm{d}x\Big)^2\Big(\int_i \frac{\partial}{\partial b}\Big(\frac{1}{f_1}\Big)\,\mathrm{d}x\Big) \\
&\qquad + \Big(\int_I f_1\,\mathrm{d}x\Big)\Big(\int_I \frac{1}{f_5}\,\mathrm{d}x\Big)\Big(\int_I \frac{\partial f_5}{\partial b}\,\mathrm{d}x\Big)\Big] \\
&< \Big(\int_I f_1\,\mathrm{d}x\Big)\Big\{(1.13)^2(-1.74)\frac{2b^2}{(c+b)(b^2-a^2)}\,\frac{a^{1/2}c}{b^2} \\
&\qquad + (1.19)(1.79)\frac{b}{a^{1/2}c}\Big[\frac{(c+b)(b^2-a^2)}{2b^2}\Big]^{1/2} \\
&\qquad \times \frac{1}{2}\Big[\frac{1}{2b^2((c+b)(b^2-a^2))^3}\Big]^{1/2} \\
&\qquad \times (0.061H(b,c,0) + 0.395H(b,c,1))\Big\} \\
&= \Big(\int_I f_1\,\mathrm{d}x\Big)\Big[2(1.13)^2(-1.74)\frac{a^{1/2}c}{(c+b)(b^2-a^2)} \\
&\qquad + \frac{1}{4}(1.19)(1.79)\frac{(0.061H(b,c,0)+0.395H(b,c,1))}{a^{1/2}cb(c+b)(b^2-a^2)}\Big] \\
&= \Big(\int_I f_1\,\mathrm{d}x\Big)\frac{1}{4ba^{1/2}c(c+b)(b^2-a^2)} \\
&\qquad \times \big[-8(1.13)^2(1.74)abc^2 + (1.19)(1.79)(0.061H(b,c,0) \\
&\qquad\qquad + 0.395H(b,c,1))\big].
\end{aligned}$$

We replace $H(b,c,0)$ and $H(b,c,1)$ by their respective expressions (see (32) and (34)) and we get, for all $(b,c) \in K$:

$$\begin{aligned}
&-8(1.13)^2(1.74)abc^2 + (1.19)(1.79)(0.061H(b,c,0) + 0.395H(b,c,1)) \\
&\qquad < -20bc^2 - 0.25b(b^3 + 2a^2c + a^2b) + 1.7c^2(c^2 - a^2) \\
&\qquad = c^2(-20b + 1.7c^2) - 0.25b(b^3 + 2a^2c + a^2b) - 1.7a^2c^2 < 0.
\end{aligned}$$

This prove that the sum of the first and fourth term of $Q(b,c,I)$ has negative upper bound. Then $Q(b,c,I) < 0$, for all $(b,c) \in K$. Since $Q(\cdot,\cdot,I_n)$ converges uniformly to $Q(\cdot,\cdot,I)$ in K, there exists $n_7 \in \mathbb{N}$ such that, $Q(b,c,I_n) < 0$, for all $n \geq n_7$, for all $(b,c) \in K$. Then $\partial G_n/\partial b(b,c) < 0$, for all $n \geq n_7$ and for all $(b,c) \in K$. We take $n_1 = \max\{n_5, n_6, n_7\}$. Then (i) and (ii) are satisfied for all $n \geq n_1$. Now we can prove the condition (c) of Lemma 1.

Let n_1 as above. From (i) and the intermediate value theorem we have that for each $b \in J_1$, there exists an unique $c \in \{b\} \times \overset{\circ}{J}_2$ such that $F_n(b,c) = 0$, for all $n \geq n_1$, where $\overset{\circ}{J}_2$ denote the interior of J_2. Also the gradient of F_n is not null in K, for all $n \geq n_1$, then zero is a regular value of F_n, indeed $F_n^{-1}(\{0\})$ is a submanifold of K of dimension 1. Then $F_n^{-1}(\{0\}) = \alpha_n$, where α_n is a regular curve in K which join the segments $\{2.55\} \times J_2$ and $\{2.85\} \times J_2$, of the boundary of K, for all $n \geq n_1$.

By the same way, from (ii) we obtain $G_n^{-1}(\{0\}) = \beta_n$, where β_n is a regular curve in K which join the segments $J_1 \times \{3.45\}$ and $J_1 \times \{3.8\}$ of the boundary of K, for all $n \geq n_1$. Then, $\alpha_n \cap \beta_n \neq \emptyset$, for all $n \geq n_1$. This concludes the proof of Lemma 1.

Corollary 1. — *Let $n_1 \in \mathbb{N}$ be as Lemma 1. Then there exist real numbers b_0, c_0, with $(b_0, c_0) \in K$, such that $F(b_0, c_0) = G(b_0, c_0) = 0$.*

Proof. — For each $n \geq n_1$, let (b_n, c_n) be a point of $\alpha_n \cap \beta_n$. Since K is compact the sequence $\{(b_n, c_n)\}_{n \geq n_1}$ has a convergent subsequence in K which we denote $\{(b_{n_k}, c_{n_k})\}_{k \in \mathbb{N}}$.

Let $(b_0, c_0) = \lim_{k \to \infty} (b_{n_k}, c_{n_k})$. Since $\{F_n\}$ and $\{G_n\}$ converge uniformly to F and G, respectively, then the same occurs for the subsequences $\{F_{n_k}\}$ and $\{G_{n_k}\}$. Therefore, for each $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$

$$|F_{n_k}(b,c) - F(b,c)| < \tfrac{1}{2}\varepsilon, \qquad |G_{n_k}(b,c) - G(b,c)| < \tfrac{1}{2}\varepsilon,$$
$$|F(b_{n_k}, c_{n_k}) - F(b,c)| < \tfrac{1}{2}\varepsilon, \quad |G(b_{n_k}, c_{n_k}) - G(b,c)| < \tfrac{1}{2}\varepsilon,$$

where the two last inequalities follow from the continuity of F and G in K. Then, for all $k \geq k_0$

$$\big|F_{n_k}(b_{n_k}, c_{n_k}) - F(b_0, c_0)\big| < \big|F_{n_k}(b_{n_k}, c_{n_k}) - F(b_{n_k}, c_{n_k})\big| + \big|F(b_{n_k}, c_{n_k}) - F(b_0, c_0)\big| < \varepsilon.$$

Analogously, we obtain $|G_{n_k}(b_{n_k}, c_{n_k}) - G(b_0, c_0)| < \varepsilon$. Indeed

$$\lim_{k\to\infty} F_{n_k}(b_{n_k}, c_{n_k}) = F(b_0, c_0), \quad \lim_{k\to\infty} G_{n_k}(b_{n_k}, c_{n_k}) = G(b_0, c_0).$$

Since $F_{n_k}(b_{n_k}, c_{n_k}) = G_{n_k}(b_{n_k}, c_{n_k}) = 0$, for all $k \geq k_0$, then $F(b_0, c_0) = G(b_0, c_0) = 0$. This complete the proof of Corollary 1.

Now we can finish the proof of Theorem A.

Let $a_0 = 1.1632$ be the value that we fixed at the beginning and (b_0, c_0) which satisfy the Corollary 1. Then

$$\frac{F_2(a_0, b_0, c_0)}{F_1(a_0, b_0, c_0)} = \frac{F_4(a_0, b_0, c_0)}{F_3(a_0, b_0, c_0)} = \frac{F_6(a_0, b_0, c_0)}{F_5(a_0, b_0, c_0)}.$$

Thus, for $\lambda_0 = F_1(a_0, b_0, c_0)/F_2(a_0, b_0, c_0)$, $a = a_0$, $b = b_0$ and $c = c_0$, the equalities in (7) are satisfied. Observe that the differentials ϕ_j, $j = 1, 2, 3$, have no residue at ∞. This is because the product $g^2\eta$ in the expression of ϕ_1 and ϕ_2 ω has an odd exponent. Furthermore $\phi_3 = g\eta$ is exact. Then the pair (g, η) defined in (5) satisfies the condition c_2 of (2).

It remains to show that (g, η) satisfies the condition c_3. Let ℓ be a path in M. Then

$$\begin{aligned}\int_\ell (1 + |g|^2)|\eta| &= \int_\ell \Big(1 + \frac{\lambda_0^2|\omega|^2}{|z(z^2 - b_0^2)|^2}\Big) \frac{|z(z^2 - b_0^2)|}{|\omega|} |dz| \\ &= \int_\ell \Big(\frac{|z(z^2 - b_0^2)|}{|\omega|} + \lambda_0^2 \frac{|\omega|}{|z(z^2 - b_0^2)|}\Big)|dz|.\end{aligned}$$

If ℓ is a divergent path in M then $|z| \to \infty$ and $|\omega|/|z(z^2 - b_0^2)| \to \infty$, so

$$\int_\ell (1 + |g|^2)|\eta| = +\infty.$$

Then Theorem A is proved.

Remarks.

1) Let a_0, b_0, c_0 the real values that satisfy (7). We consider the conformal map $T : \mathbb{C}^2 \to \mathbb{C}^2$, such that $T(z, w) = ((1/a_0)z, (1/a_0^{7/2})\omega) = (\tilde{z}, \widetilde{\omega})$. Then $(z, w) \in \mathbb{C}^2$ satisfies $P_0(z, w) = 0$, where

$$P_0(z, w) = \omega^2 - z(z^2 - a_0^2)(z^2 - b_0^2)(z^2 - c_0^2),$$

if and only if $(\tilde{z}, \tilde{\omega})$ satisfies $\widetilde{P}(z,\omega) = 0$, where

$$\widetilde{P}(z,\omega) = \omega^2 - z(z^2-1)(z^2-b_1^2)(z^2-c_1^2)$$

and $b_1 = b_0/a_0$, $c_1 = c_0/a_0$. This implies (in local coordinates) that the map $\tilde{z} : \mathbb{C} \to \mathbb{C}$, $\tilde{z}(z) = (1/a_0)z$, is an conformal equivalence between M and $\widetilde{M}$ which are the Riemann surfaces obtained from $P_0(z,\omega) = 0$ and $\widetilde{P}(z,\omega) = 0$, respectively. Then, Theorem A is true for $\widetilde{M}$.

2) We began to solve this problem by search of a numerical solution of the equation

$$\big[F_n(a,b,c)\big]^2 + \big[G_n(a,b,c)\big]^2 = 0, \tag{41}$$

where F_n and G_n are the functions in (9) and n is large. We obtained an approximate solution a', b', c' with a' close to 1.1632. Then we fixed $a_0 = 1.1632$ and we considered a compact neighborhood K of (b', c'), $\{a_0\} \times K \subset D$, D as (3) and we proved the result.

3) We had knowledge that Hermann Karcher [K] constructed an example of a minimal immersion in $\mathbb{R}^3$ which satisfies Theorem A. The conformal structure of his example is distinct from the conformal structure of the Riemann surfaces which we considered.

4) For the genus four case, it would be natural to start with the Riemann surface $\overline{M}_4$ produced by an algebraic curve

$$P(z,\omega) = z(z^2-a^2)(z^2-b^2)(z^2-c^2)(z^2-d^2),$$

with $0 < a < b < c < d < \infty$, and the Weierstrass representation (g,η), where

$$g(z) = \frac{\lambda\omega}{(z^2-a^2)(z^2-c^2)}, \qquad \eta = \frac{\lambda}{g}\,\mathrm{d}z, \qquad 0 < \lambda < \infty.$$

We obtain four equations of type (7). It is possible to find a numerical solution by using a equation similar to (41). We think that the method used to solve (7) may possibly be extended to this case.

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Table I

s	$K_2(s) \cdot K_3(s)$		$K_1(s) \cdot K_4(s)$		$m_1(b, I) >$
	$>$	$<$	$>$	$<$	
1	3.2240	3.2241	1.1720	1.1721	2.46
2	3.2345	3.2346	1.1616	1.1617	2.00
3	3.2452	3.2453	1.1511	1.1512	1.59
4	3.2561	3.2562	1.1403	1.1404	1.27
5	3.2672	3.2673	1.1294	1.1295	1.01
6	3.2786	3.2787	1.1184	1.1185	0.83
7	3.2902	3.2903	1.1071	1.1072	0.75
8	3.3020	3.3021	1.0957	1.0958	0.75
9	3.3142	3.3143	10.084	1.0841	0.87
10	3.3266	3.3267	1.0722	1.0723	1.06

Table II (beginning)

r	$K_2'(r) \cdot K_3'(r)$		$K_1'(r) \cdot K_4'(r)$		$m_2(b, I) <$
	$>$	$<$	$>$	$<$	
1	3.2502	3.2504	1.1912	1.1913	−0.01
2	3.2515	3.2516	1.1901	1.1902	−0.16
3	3.2527	3.2528	1.1890	1.1891	−0.32
4	3.2539	3.2540	1.1879	1.1880	−0.47
5	3.2551	3.2553	1.1868	1.1869	−0.62
6	3.2564	3.2565	1.1857	1.1858	−0.78
7	3.2576	3.2577	1.1846	1.1847	−0.93
8	3.2588	3.2589	1.1834	1.1835	−1.06
9	3.2601	3.2602	1.1823	1.1824	−1.21
10	3.2613	3.2614	1.1812	1.1813	−1.36
11	3.2626	3.2627	1.1801	1.1802	−1.51

Table II (end)

r	$K_2'(r) \cdot K_3'(r)$		$K_1'(r) \cdot K_4'(r)$		$m_2(b, I) <$
	$>$	$<$	$>$	$<$	
12	3.2638	3.2639	1.1790	1.1791	−1.66
13	3.2650	3.2652	1.1779	1.1780	−1.81
14	3.2663	3.2664	1.1767	1.1768	−1.94
15	3.2675	3.2676	1.1756	1.1757	−2.09
16	3.2770	3.2771	1.1670	1.1671	−0.38
17	3.2834	3.2835	1.1614	1.1615	−1.06
18	3.2898	3.2899	1.1557	1.1558	−1.71
19	3.2962	3.2963	1.1499	1.4500	−2.32
20	3.3028	3.3029	1.1441	1.1442	−2.92
21	3.3094	3.3095	1.1382	1.1383	−3.48
22	3.3160	3.3161	1.1323	1.1324	−4.03
23	3.3228	3.3229	1.1263	1.1264	−4.53
24	3.3296	3.3297	1.1203	1.1204	−5.01
25	3.3364	3.3365	1.1142	1.1143	−5.46
26	3.3434	3.3435	1.1080	1.1081	−5.85
27	3.3504	3.3505	1.1018	1.1019	−6.23

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