ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 44, nº 2 (1994), p. 601-628

<http://www.numdam.org/item?id=AIF_1994__44_2_601_0>

Ann. Inst. Fourier, Grenoble
44, 2 (1994), 601-628

FERROMAGNETIC INTEGRALS, CORRELATIONS AND MAXIMUM PRINCIPLES

by Johannes SJÖSTRAND

0. Introduction.

In [S] the author developed an idea of Singer-Wong-Yau-Yau [SiWYY] to use the maximum principle in the study of the logarithm of the first eigenfunction of a Schrödinger operator on $\mathbf{R}^m$ with a strictly convex potential, and for suitable sequences of potentials, he was able to establish the exponential convergence of the first eigenvalue divided by m, when m tends to infinity. More recently, Helffer and the author [HS] employed a similar method to study expectation values of the form :

$$\langle f\rangle = \frac{\int_{\mathbf{R}^m} e^{-\phi(x)/h} f(x)dx}{\int_{\mathbf{R}^m} e^{-\phi(x)/h}dx}, \tag{0.1}$$

and in particular the correlations,

$$\langle (x_j - \langle x_j\rangle)(x_k - \langle x_k\rangle)\rangle, \tag{0.2}$$

for large m and $|j-k|$. Under suitable assumptions, implying uniform strict convexity for ϕ, we proved that (0.2) can be estimated by $\mathcal{O}(1)\exp(-|j-k|/C)$. Such bounds have previously been obtained for models in statistical mechanics by many people, see Ellis [E], Sokal [So]. In the second part of [HS] we further improved the estimates and were able to avoid a certain "ϵ-loss" in the exponential decay rate.

In this paper we shall further improve the use of the maximum principle and as an application, we consider in section 2 correlations of the form (0.2) where now j, k vary in some finite subset Γ of $\mathbf{Z}^d$, so that $m = \sharp\Gamma$.

Key words : Correlation – Statistical mechanics – Maximum principle.
A.M.S. Classification : 81C05 – 82A99 – 35C20 – 35P15 – 35P20.

Assuming that $\phi = \phi_\Gamma(x)$ satisfies $\partial_{x_j}^2 \phi \geq 1$, $-v(j-k) \leq \partial_{x_j}\partial_{x_k}\phi \leq 0$, $j \neq k$, where $v : \mathbf{Z}^d \to [0, \infty[$ is even, independent of Γ, with $v(0) = 0$, $\sum v(j) = 1$, we prove under some additional assumptions that for $d \geq 3$ the correlation (0.2) (which is positive in view of a general result of Cartier [C]) is $\leq F_0(j-k) + o(\langle j-k\rangle^{2-d})$, $|j-k| \to \infty$ uniformly with respect to Γ, where $F_0(x)$ is the fundamental solution positively homogeneous of degree $2-d$ of a certain 2 :nd order elliptic operator. Estimates of this kind have been obtained by Bricmont, Fontaine, Lebowitz, Spencer [BrFLeSp] for certain polynomial ϕ :s by using the Brascamp-Lieb estimates [BL]. See also [BrFLeLSp] for a different and more special model. For $d = 2$ we get a corresponding logarithmic bound and for $d = 1$ we get a linear bound. (See Theorem 2.4 for a more complete statement.) According to Glimm-Jaffe [GlJ] and Ellis [E] it is expected by physicists that the correlation (0.2) should behave like $|j-k|^{-(d-2+\eta)}$, where $\eta \geq 0$ is a so called anomalous dimension, depending on the 4 :th order derivatives of ϕ. Similar conjectures and results exist in percolation theory, see Grimmett [Gr]. Since we only make assumptions about the second order derivatives it is quite natural that we only get estimates with $\eta = 0$. It cannot be excluded however, that our methods could be further refined to give estimates with $\eta > 0$ under more specific assumptions about the behaviour of the 4 :th order derivatives of ϕ.

In section 3, we assume that $\partial_{x_j}^2 \phi \geq 1 + \epsilon$, $-v(j-k) \leq \partial_{x_j}\partial_{x_k}\phi \leq 0$, $j \neq k$, with v roughly as above and with $\epsilon \geq 0$ small, and we get bounds which are uniform in j, k, ϵ, Γ. In particular, when $d = 3$, we get the bound

$$\mathcal{O}(h)e^{-|j-k|g(\sqrt{2\epsilon})}\frac{1}{1+|j-k|},$$

where g also depends on $\dfrac{j-k}{|j-k|}$ and we have

$$g(\sqrt{2\epsilon})|j-k| = (1+\mathcal{O}(\epsilon))\sqrt{2\epsilon\langle q''(0)^{-1}(j-k),(j-k)\rangle},$$

and $q''(0)$ is a positive definite matrix appearing as the Hessian of $1 - \hat{v}(\theta)$ at $\theta = 0$. Here $\hat{v}(\theta) = \sum_{j\in\mathbf{Z}^d} v(j)e^{ij\theta}$. The elliptic operator mentioned earlier is equal to $\frac{1}{2}\langle q''(0)D_x, D_x\rangle$. More detailed estimates, can be given when $|j-k|\sqrt{\epsilon}$ tend to infinity or to zero. We may here think of ϵ as the difference of the temperature and the "critical" temperature in statistical mechanics.

Section 4 can be read independently of the others. We formulate a general maximum principle and show how this maximum principle permits to generalize some results of [S], [HS].

It is a pleasure to acknowledge the hospitality and the peaceful working conditions of the Mittag-Leffler Institute. We thank C. Tracy for an interesting discussion, which stimulated us to add section 3, and we also thank B. Helffer and T. Spencer for useful comments and references.

1. Some simple results on matrices.

In this section, we shall consider real matrices of the form

$$A = A_\Gamma = (a_{j,k})_{j,k \in \Gamma}, \tag{1.1}$$

where Γ is some finite set. We shall first discuss a weak maximum principle.

PROPOSITION 1.1. — *The following two properties are equivalent :*

$$\begin{aligned} &\textit{If } \Gamma \ni j \mapsto u(j) \in \mathbf{R} \textit{ is maximal } = m \geq 0 \textit{ at} \\ &j = j_0, \textit{ then } Au(j_0) \geq 0, \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} &a_{j,j} \geq 0, \quad a_{j,k} \leq 0, \textit{ for all } j,k \in \Gamma \textit{ with } j \neq k, \\ &\textit{and } \sum_{k \in \Gamma} a_{j,k} \geq 0 \textit{ for all } j \in \Gamma. \end{aligned} \tag{1.3}$$

Proof. — We first assume (1.2). Taking $u = \delta_j$ (the function that is equal to 1 at j and zero elsewhere) we see that $a_{j,j} \geq 0$. If $k \neq j$, we can take $u = -\delta_k$ to see that $a_{j,k} \leq 0$. If we then take $u = 1$, we obtain $\sum_{k \in \Gamma} a_{j,k} \geq 0$ for all $j \in \Gamma$. Thus (1.3) holds.

Conversely, assume (1.3). Let $\Gamma \ni j \mapsto u(j)$ be maximal $= m \geq 0$ at $j = j_0$. Then

$$Au(j_0) = \sum_k a_{j_0,k} u(k) = \sum_{k \neq j_0} a_{j_0,k}(u(k) - m) + m \sum_k a_{j_0,k},$$

and since $u(k) - m \leq 0$, it follows from (1.3) that the last member is ≥ 0. Hence we have (1.2). $\square$

We observe that if A satisfies (1.2) (equivalent to (1.3)), then $\epsilon I + A$ is bijective for every $\epsilon > 0$. Indeed, if $u \neq 0$ and $(\epsilon + A)u = 0$, we may assume that u has at least one component $u(j)$ which is > 0. Assuming then that $u(j_0) = \max_j u(j) > 0$, we get from (1.2) : $(\epsilon + A)u(j_0) \geq \epsilon u(j_0)$ in contradiction with the assumption that $(\epsilon + A)u = 0$.

Proposition 1.2. — *Let A satisfy (1.2) and be bijective. If $v \in \mathbf{R}^\Gamma$ is ≥ 0 (component wise), and u is the solution of $Au = v$, then $u \geq 0$ (component wise).*

Proof. — Let u_ϵ be the solution of $(\epsilon I + A)u_\epsilon = v$, so that $u_\epsilon \to u$ when $\epsilon \to 0$. If $u_\epsilon(j) < 0$ for some $\epsilon > 0$ and some $j \in \Gamma$, let j_0 be a point where $u_\epsilon(j_0)$ is minimal < 0. Then we get $v(j_0) < 0$ from (1.2), which contradicts the assumption on v. □

For a similar result, see Appendix A in Guerra, Rosen, Simon [GRSi].

Definition 1.3. — *Let $A = (a_{j,k})_{j,k\in\Gamma}$, $B = (b_{j,k})_{j,k\in\Gamma}$ be real matrices. We write $A \leq' B$ if $a_{j,k} \leq b_{j,k}$ for all $(j,k) \in \Gamma \times \Gamma$.*

We notice that if $A \leq' B$, A satisfies (1.3) and $b_{j,k} \leq 0$, for $j \neq k$, then B also satisfies (1.3). Assume in addition that both A and B are bijective, and let u_A, u_B be the solutions of

$$Au_A = v, \quad Bu_B = v, \tag{1.4}$$

where $v \geq 0$. Then we have

$$0 \leq u_B \leq u_A. \tag{1.5}$$

In fact, we have $Bu_A = v + (B - A)u_A \geq v$, so $B(u_A - u_B) \geq 0$. Hence $u_B \leq u_A$. By a limiting procedure we also see that we can suppress the assumption that B is bijective, and we then get the bijectivity of B as a consequence of (1.5) (with $v = 0$). Summing up, we have

Proposition 1.4. — *Assume that A, B satisfy (1.2), that $A \leq' B$, and that A is bijective. Then B is bijective and if u_A, u_B are the solutions in (1.4), with $v \geq 0$, then we have (1.5).*

When A is symmetric, we have some positivity for the lowest eigenfunctions. First notice that in this case, if A satisfies (1.2), then the lowest eigenvalue is ≥ 0.

PROPOSITION 1.5. — *Assume that A is symmetric and satisfies (1.2). If $\lambda_0 \geq 0$ is the lowest eigenvalue, then the dimension of the corresponding eigenspace is at most equal to the number of connected components of Γ, where the connectedness is defined by means of paths $j_1, j_2, .., j_k$, with $a_{j_\nu, j_{\nu+1}} \neq 0$. A corresponding orthonormal basis is given by eigenfunctions which are strictly positive on precisely one of the components and zero elsewhere.*

Proof. — We may assume that Γ is connected. Let u be an eigenfunction associated to λ_0 and write $u = u_+ - u_-$, with $u_\pm = \max(\pm u, 0)$. Then

$$\lambda_0 \|u\|^2 = (Au|u) = (Au_+|u_+) + (Au_-|u_-) + 2((-A)u_+|u_-) \geq$$
$$\lambda_0(\|u_+\|^2 + \|u_- \|^2) + 2((-A)u_+|u_-) = \lambda_0\|u\|^2 + 2((-A)u_+|u_-) \geq \lambda_0\|u\|^2,$$

so we have equality and even $(Au_\pm|u_\pm) = \lambda_0\|u_\pm\|^2$. If for instance $u_+ \neq 0$, then u_+ is an eigenvector : $Au_+ = \lambda_0 u_+$. It is easy to see that the set of $j \in \Gamma$ with $u_+(j) = 0$ is connected, and hence reduces to $\varnothing$. $\square$

Let $A = A_\Gamma = (a_{j,k})$ satisfy (1.2). If $\tilde{\Gamma} \subset \Gamma$, let $A_{\tilde{\Gamma}}$ denote the matrix $(a_{j,k})_{j,k\in\tilde{\Gamma}}$. Then $A_{\tilde{\Gamma}}$ also satisfies (1.2). Assume that A and $A_{\tilde{\Gamma}}$ are both bijective. Let $0 \leq v \in \ell^2(\tilde{\Gamma})$ and identify v with its 0-extension to Γ. Let $u_\Gamma \in \ell^2(\Gamma)$, $u_{\tilde{\Gamma}} \in \ell^2(\tilde{\Gamma})$ be the solutions of $A_\Gamma u_\Gamma = v$, $A_{\tilde{\Gamma}} u_{\tilde{\Gamma}} = v$. We observe that $A_{\tilde{\Gamma}}(u_{\Gamma|\tilde{\Gamma}}) \geq A_\Gamma u_\Gamma = v$ on $\tilde{\Gamma}$ and consequently $u_{\tilde{\Gamma}} \leq u_{\Gamma|\tilde{\Gamma}}$. By a limiting procedure, we also see that the bijectivity of $A_{\tilde{\Gamma}}$ does not have to be assumed but follows from that of A. Summing up, we have the following result, which can be viewed as a special case of Proposition 1.4.

PROPOSITION 1.6. — *Let $A = A_\Gamma$ satisfy (1.2) and be bijective. If $\tilde{\Gamma} \subset \Gamma$, define $A_{\tilde{\Gamma}}$ as above, as the restriction of the matrix of A_Γ to $\tilde{\Gamma} \times \tilde{\Gamma}$. Then $A_{\tilde{\Gamma}}$ is bijective and satisfies (1.2). Moreover, if $v \in \ell^2(\tilde{\Gamma})$ is non-negative and we identify v with its 0-extension to Γ, then for the solutions of $A_\Gamma u_\Gamma = v$, $A_{\tilde{\Gamma}} u_{\tilde{\Gamma}} = v$, we have $0 \leq u_{\tilde{\Gamma}} \leq u_\Gamma|_{\tilde{\Gamma}}$.*

We are particularly interested in examples produced by convolution matrices. Let $v \in \ell^1(\mathbf{Z}^d)$ with

$$v(j) \geq 0, \ v(j) \neq 0 \text{ for some } j \neq 0, \ \|v\|_{\ell^1} \leq 1 \tag{1.6}$$

and let $A = \delta_{j,k} - v(j-k)$, be the corresponding $\mathbf{Z}^d \times \mathbf{Z}^d$-matrix.

PROPOSITION 1.7. — *Under the above assumptions, A_Γ satisfies (1.2) and is bijective for any finite $\Gamma \subset \mathbf{Z}^d$.*

Proof. — It is immediate to check (1.3) (equivalent to (1.2)) for A_Γ. The bijectivity is also obvious in the case when $\|v\|_{\ell^1} < 1$. In the case $\|v\|_{\ell^1} = 1$, assume that A_Γ is not bijective, so that $A_\Gamma u = 0$ for some u which is not identically zero. We may assume that $\|u\|_{\ell^\infty} = u(j_0) = m > 0$. Writing $u(j_0) = \sum_k v(j_0 - k)u(k)$, we see that $u(k) = m$ for every k with $v(j_0 - k) \neq 0$. In other words, if $K = \{j \in \Gamma;\ u(j) = m\}$, then $K - \operatorname{supp} v \subset K$ where $\operatorname{supp} v$ denotes the set of j for which $v \neq 0$. This contradicts the boundedness of Γ, since $\operatorname{supp} v$ contains at least one point different from 0. □

Keeping the assumption (1.6), we assume now that $d \geq 3$, that v is even, and that

$$\|v\|_{\ell^1} = 1. \tag{1.7}$$

For $\epsilon \in]0,1]$, the convolution equation

$$(\delta_0 - (1-\epsilon)v) * E_\epsilon = \delta_0 \tag{1.8}$$

has a unique solution in $\ell^1(\mathbf{Z}^d)$ given by

$$E_\epsilon = \delta_0 + (1-\epsilon)v + (1-\epsilon)^2 v * v + \ldots$$

which is ≥ 0. Taking Fourier transforms gives :

$$(1 - (1-\epsilon)\hat{v}(\theta))\hat{E}_\epsilon(\theta) = 1, \quad \theta \in \mathbf{T}^d, \quad \mathbf{T} = \mathbf{Z}/2\pi\mathbf{Z}, \tag{1.9}$$

where $\hat{v}(\theta) = \sum_{j \in \mathbf{Z}^d} v(j)e^{ij\theta}$ and similarly for $\hat{E}_\epsilon(\theta)$.

The assumptions on v imply that $\hat{v}(\theta) \leq 1$ with equality for $\theta = 0$ (in $\mathbf{T}^d$). We add the assumption,

$$\sum \langle j \rangle^{\max(2,d-2)} v(j) < \infty, \tag{1.10}$$

where we use the standard notation $\langle j \rangle = \sqrt{1 + j^2}$. Then $\hat{v}(\theta)$ belongs to $C^{\max(2,d-2)}$ and we add the assumption that

(1.11) a) $1 - \hat{v}(\theta) > 0$ for $\theta \neq 0$ in $\mathbf{T}^d$

b) $1 - \hat{v}(\theta) = \sum_{j,k} a_{j,k}\theta_j\theta_k + \mathcal{O}(|\theta|^3)$, where $\sum_{j,k} a_{j,k}\theta_j\theta_k > 0$, $|\theta| \neq 0$.

Of course the only assumption in (1.11) is that the inequalities are strict. Then if we recall that $d \geq 3$, we get

(1.12)

$$E_\epsilon(j) = \frac{1}{(2\pi)^d}\int \frac{e^{-ij\theta}}{(1-(1-\epsilon)\hat{v}(\theta))}\,d\theta \to \frac{1}{(2\pi)^d}\int \frac{e^{-ij\theta}}{(1-\hat{v}(\theta))}\,d\theta \underset{\text{def}}{=} E_0(j),$$

when $\epsilon \to 0$. We conclude that $E_0(j) \geq 0$ belongs to $\ell^\infty(\mathbf{Z}^d)$ and satisfies

$$(\delta_0 - v) * E_0 = \delta_0, \quad E_0(j) \to 0, \ |j| \to \infty.$$

We can be more precise about the asymptotics of E_0 near infinity. Let $q(\theta) = \sum a_{j,k}\theta_j\theta_k$ and let $\chi \in C^\infty(\mathbf{T}^d)$ be equal to 1 near $\theta = 0$ and have its support in a small neighborhood of 0. Then we can write,

$$\frac{1}{1-\hat{v}(\theta)} = \frac{\chi}{q(\theta)} + \frac{\chi(\theta)(\hat{v}(\theta) - (1-q(\theta)))}{(1-\hat{v}(\theta))q(\theta)} + \frac{1-\chi(\theta)}{1-\hat{v}(\theta)}. \tag{1.14}$$

Here,

$$\frac{1}{(2\pi)^d}\int e^{-ij\theta}\frac{\chi(\theta)}{q(\theta)}d\theta = F_0(j) + \mathcal{O}(\langle j\rangle^{-\infty}), \tag{1.15}$$

where F_0 is the fundamental solution of $q(D)$ which is homogeneous of degree $2-d$.

Since $\dfrac{1-\chi(\theta)}{1-\hat{v}(\theta)} \in C^{d-2}$, we have

$$\int e^{-ij\theta}\frac{1-\chi(\theta)}{1-\hat{v}(\theta)}d\theta = o(\langle j\rangle^{2-d}), \ |j| \to \infty. \tag{1.16}$$

We have

$$\partial_\theta^\alpha\Big(\frac{\chi(\theta)(\hat{v}(\theta) - (1-q(\theta)))}{(1-\hat{v}(\theta))q(\theta)}\Big) = \mathcal{O}(|\theta|^{-1-|\alpha|}) \in \ell^1, \text{ for } |\alpha| \leq d-2, \tag{1.17}$$

so

$$\int e^{-ij\theta}\frac{\chi(\theta)(\hat{v}(\theta) - (1-q(\theta)))}{(1-\hat{v}(\theta))q(\theta)}d\theta = o(\langle j\rangle^{2-d}), \ |j| \to \infty. \tag{1.18}$$

From (1.14), (1.15), (1.18), we conclude that,

$$E_0(j) = F_0(j) + o(\langle j\rangle^{2-d}), \quad |j| \to \infty, \tag{1.19}$$

where we recall that F_0 is the fundamental solution of $q(D)$ which is positively homogeneous of degree $2-d$.

Let $\Gamma \subset \mathbf{Z}^d$, $d \geq 3$, be a finite set containing 0. Let A denote the $\mathbf{Z}^d \times \mathbf{Z}^d$-matrix $\delta_{j,k} - v(j-k)$ and let A_Γ denote the restriction of this matrix to $\Gamma \times \Gamma$ as above. Similarly, we put $E_{0,\Gamma} = E_{0|\Gamma}$. Then $A_\Gamma E_{0,\Gamma} = \delta_0 + r_\Gamma$, where $r_\Gamma \geq 0$, so if $E_\Gamma(j,0)$ denotes the solution of $A_\Gamma E_\Gamma = \delta_0$, we have

$0 \le E_\Gamma(j,0) \le E_0(j)$. More generally, if $E_\Gamma(j,k)$, $j,k \in \Gamma$, denotes the solution of

$$A_\Gamma E_\Gamma(\cdot,k) = \delta_k$$

for $k \in \Gamma$, so that $E_\Gamma(j,k)$ is the matrix of A_Γ^{-1}, we get

$$0 \le E_\Gamma(j,k) \le E_0(j-k). \tag{1.20}$$

It may be of some interest, to estimate the boundary influence on $E_\Gamma(j,k)$, when $\mathrm{dist}\,(j,k) << \mathrm{dist}\,(j,\mathbf{Z}^d \setminus \Gamma)$. For that, we shall consider $\Gamma = \Gamma_R = B(0,R) \cap \mathbf{Z}^d$, where $B(0,R)$ denotes the open unit ball in $\mathbf{R}^d$, and we sharpen the assumption (1.10) to

$$\sum \langle j\rangle^N v(j) < \infty, \text{ for every } N \in \mathbf{N}. \tag{1.10$'$}$$

We also assume that $R \ge 2\max\,(|j|,|k|)$, and we shall estimate $E_\Gamma(j,k) - E_0(j-k)$. We have

$$A_\Gamma(E_0(\cdot - k)_{|\Gamma})(\ell) = \delta_{\ell,k} + \sum_{\nu\in\mathbf{Z}^d\setminus\Gamma} v(\ell-\nu)E_0(\nu-k) = \delta_{\ell,k} + r(\ell),$$

where (as already noted) $r(\ell) \ge 0$. We have

$$\begin{aligned} r(\ell) &\le C \sum_{\nu\in\mathbf{Z}^d,\ |\nu|\ge R} v(\ell-\nu)|\nu|^{2-d} \\ &= C \sum_{\nu\in\mathbf{Z}^d,\ |\nu|\ge R} \langle \ell-\nu\rangle^N v(\ell-\nu)\langle\ell-\nu\rangle^{-N}|\nu|^{2-d} \\ &\le C_N \sup_{\nu\in\mathbf{Z}^d,\ |\nu|\ge R} \langle\ell-\nu\rangle^{-N}|\nu|^{2-d} \le \tilde{C}_N R^{2-d}(1+R-|\ell|)^{-N}. \end{aligned}$$

Then,

$$E_0(j-k) - E_\Gamma(j,k) = \sum_{\ell\in\Gamma} E_\Gamma(j,\ell)r(\ell) \le \mathcal{O}(1)R^{2-d}\sum_{|\ell|\le R}\langle j-\ell\rangle^{2-d}(1+R-|\ell|)^{-N}$$

$$\le \mathcal{O}(1)R^{2-d-N}\sum_{|\ell|\le 3R/4}\langle j-\ell\rangle^{2-d} + \mathcal{O}(1)R^{2-d}\sum_{3R/4<|\ell|\le R} R^{2-d}(1+R-|\ell|)^{-N}$$

$$= \mathcal{O}(1)R^{4-d-N} + \mathcal{O}(1)R^{3-d} = \mathcal{O}(1)R^{3-d},$$

if we choose N large enough. So under the more special assumptions above, we have obtained

(1.21)
$$E_\Gamma(j,k) = E_0(j-k) + \mathcal{O}(1)R^{3-d} \text{for } \Gamma = B(0,R)\cap\mathbf{Z}^d, \quad R \ge 2\max\,(|j|,|k|).$$

The result is empty for $d = 3$, but of interest for $d \geq 4$.

We next assume that $d = 2$. Under the assumptions (1.6), (1.7), (1.10) and (1.11), we still have a real-valued function $E_0(j)$ on $\mathbf{Z}^2$, such that

$$(\delta_0 - v) * E_0 = \delta_0, \tag{1.22}$$

but it is no longer true that E_0 is non-negative, since

$$E_0(j) = -C_0(1 + o(1)) \log |j|, \quad |j| \to \infty, \tag{1.23}$$

where $C_0 > 0$ is determined by the fact that E_0 is asymptotic to a temperate fundamental solution of $\sum a_{j,k} D_{x_j} D_{x_k}$ which is of sublinear growth at infinity.

Let $\Gamma \subset B(0, R_1) \cap \mathbf{Z}^2$ for some $R_1 \geq 2$ and let A_Γ denote the same matrix as before. Let $R_2 \geq 3R_1$. After adding a positive constant (depending on R_2) to E_0, we may assume that $E_0 \geq 0$ pointwise on $B(0, R_2)$. The price we have to pay for this is that we only have the estimate $E_0 \leq (1 + o(1)) \log R_2$ on $B(0, R_2)$, assuming of course that we added the smallest possible constant.

The aim, as in the case $d \geq 3$, is to estimate the matrix elements of the positive matrix A_Γ^{-1} under some additional assumption on v. Let $j, k \in \Gamma$, and consider

$$(A_\Gamma \circ E_{0,\Gamma}\delta_k)(j) = \sum_{\nu \in \Gamma} a_{j,\nu} E_0(\nu - k) = \delta_{j,k} + \sum_{\nu \in \mathbf{Z}^2 \setminus \Gamma} (-a_{j,\nu}) E_0(\nu - k)$$

$$= \delta_{j,k} + \sum_{\nu \in \mathbf{Z}^2 \setminus \Gamma,\ |\nu - k| \leq R_2} (-a_{j,\nu}) E_0(\nu - k) + \sum_{|\nu - k| > R_2} (-a_{j,\nu}) E_0(\nu - k)$$

$$\geq \delta_{j,k} + \sum_{|\nu - k| > R_2} (-a_{j,\nu}) E_0(\nu - k).$$

Assume that

$$v(j) \leq \mathrm{Const.} \langle j \rangle^{-k_0} \tag{1.24}$$

for some $k_0 > 2$. Then by a simple estimate, comparing with an integral, we get for $j, k \in \Gamma$:

$$\Big| \sum_{|\nu - k| > R_2} (-a_{j,\nu}) E_0(\nu - k) \Big| = \mathcal{O}(1) R_2^{2 - k_0} \log R_2.$$

It then follows from the earlier computation, that

$$(A_\Gamma \circ E_{0,\Gamma}) \geq' I - R,$$

where $R = (r_{j,k})$, $0 \leq r_{j,k} \leq \mathcal{O}(R_2^{2-k_0} \log R_2)$ (and we write $A \geq' B$, if $B \leq' A$). With $F = E_{0,\Gamma} - A_\Gamma^{-1} B$, where $B \underset{\text{def}}{=} A_\Gamma \circ E_{0,\Gamma} - (I - R) \geq' 0$, we have $F \leq' E_{0,\Gamma}$ and $A_\Gamma \circ F = I - R$. Assume that R_2 is sufficiently large depending on R_1, so that

$$R_2^{2-k_0} (\log R_2) R_1^2 << 1. \tag{1.25}$$

Then $(I - R)^{-1} = 1 + R + R^2 + .. \underset{\text{def}}{=} I + S$, where

$$S = (s_{j,k}),\ 0 \leq s_{j,k} \leq \mathcal{O}(R_2^{2-k_0} \log R_2) \tag{1.26}$$

and for $A_\Gamma^{-1} = E_{0,\Gamma}(I - R)^{-1}$, we then conclude the following 2-dimensional analogue of (1.19), (1.20) :

PROPOSITION 1.8. — *Let $d = 2$ and assume (1.6), (1.7), (1.10), (1.11), (1.24) (1.25) for some fixed $k_0 > 2$. Then uniformly for $\Gamma \subset B(0, R_1) \cap \mathbf{Z}^2$, $R_2 \geq 3R_1$, we have $A_\Gamma^{-1} = (b_{j,k})$, with*

$$0 \leq b_{j,k} \leq \mathcal{O}(1) \log R_2. \tag{1.27}$$

One could certainly replace the last member by some more precise expression in order to make the "$\mathcal{O}(1)$" more explicit, say in terms of the asymptotics of E_0. If we choose to ignore such fine questions, we also observe that for given large R_1, we can achieve (1.25) with R_2 equal to some power of R_1, and then we may forget about R_2 all together and get from (1.27) :

$$0 \leq b_{j,k} \leq \mathcal{O}(1) \log R_1. \tag{1.28}$$

We finally treat the case $d = 1$, as the case $d = 2$. We still assume (1.6), (1.7), (1.10) and (1.11). Then we have the fundamental solution $E_0(j)$ satisfying (1.22) and

$$E_0(j) = -C_0(1 + o(1))|j|, \quad |j| \to \infty. \tag{1.30}$$

Take again $\Gamma \subset B(0, R_1) \cap \mathbf{Z}$, $R_1 \geq 2$, $3R_1 \leq R_2$, and make E_0 positive on $B(0, R_2)$ by adding a constant $= \mathcal{O}(R_2)$. We make the same computation as prior to (1.24). Assuming (1.24) for some $k_0 > 2$, we get this time

$$\Big| \sum_{|\nu-k|>R_2} (-a_{j,\nu}) E_0(\nu - k) \Big| = \mathcal{O}(1) R_2^{2-k_0}.$$

Then we get $A_\Gamma \circ E_{0,\Gamma} \geq' I - (r_{j,k})$, $0 \leq r_{j,k} \leq \mathcal{O}(R_2^{2-k_0})$. Instead of (1.25), we now assume

$$R_2^{2-k_0} R_1 << 1 \tag{1.31}$$

and we get

PROPOSITION 1.9. — *Let $d = 1$ and assume (1.6), (1.7), (1.10), (1.11), (1.24), (1.31) for some fixed $k_0 > 2$. Then uniformly for $\Gamma \subset B(0, R_1) \subset \mathbf{Z}$, $R_2 \geq 3R_1$, we have $A_\Gamma^{-1} = (b_{j,k})$, with*

$$0 \leq b_{j,k} \leq \mathcal{O}(1) R_2. \tag{1.32}$$

If we choose $R_2 = \max(3R_1, \delta^{-1} R_1^{1/(k_0-2)})$ for some sufficiently small $\delta > 0$, (1.32) applies and gives :

$$0 \leq b_{j,k} \leq \mathcal{O}(1) R_1^{\max(1, 1/(k_0-2))}. \tag{1.33}$$

2. Correlation estimates in "critical" cases.

Let Γ be some finite set, and write $x = (x_j)_{j \in \Gamma} \in \mathbf{R}^\Gamma$. We shall apply a suitable version of the maximum principle, to estimate the gradient ∇u, of solutions of

$$\nabla\phi \cdot \partial_x u - h\Delta u = v + \text{Const.}, \quad x \in \mathbf{R}^\Gamma. \tag{2.1}$$

Assuming enough regularity, we get :

$$\phi''(x) \nabla u - h\Delta \nabla u + \nabla\phi \cdot \partial_x \nabla u = \nabla v. \tag{2.2}$$

We assume that $\phi \in C^\infty(\mathbf{R}^\Gamma; \mathbf{R})$ and that

(2.3) $\phi''(x)$ satisfies (1.2) and $\phi''(x) \geq' A > 0$, where $A = A_\Gamma$ is symmetric and satisfies (1.2). Moreover, $|\partial^\alpha \phi| \leq C_\Gamma$ for $|\alpha| = 2$.

Let u be a C^2-solution of (2.1) with $\nabla u \longrightarrow 0$, when $|x| \longrightarrow \infty$. We also assume that $\nabla v(x)$ is bounded. Let $U_\pm \in \mathbf{R}^\Gamma$ be independent of x with

$$U_\pm \geq 0, \quad AU_\pm \mp \nabla v(x) \geq 0 \text{ component wise.} \tag{2.4}$$

One possibility is to let $V_+ \in \mathbf{R}^\Gamma$ be the vector ≥ 0, given by

$$V_+(j) = \sup_x \max(0, \partial_{x_j} v(x))$$

and put $U_+ = A^{-1}V_+$. Similarly, we can take $U_- = A^{-1}V_-$, where

$$V_-(j) = \sup_x \max(0, -\partial_{x_j} v(x)).$$

With these choices of $U_\pm$ we also notice that $U_\pm = \lim\limits_{\epsilon \to 0} U_{\pm,\epsilon}$, where $U_{\pm,\epsilon} > 0$ and $AU_{\pm,\epsilon} \mp \nabla v > 0$ component wise. For instance, we could take $U_{\pm,\epsilon} = A^{-1}(V_\pm + \epsilon\mathbf{1})$, where $\mathbf{1}$ is the vector given by $\mathbf{1}(j) = 1$.

From (2.2), (2.3), (2.4), we get $\phi''(x)U_\pm \mp \nabla v(x) \geq 0$ component wise, and replacing $U_\pm$ by $U_{\pm,\epsilon}$, we even get $\phi''(x)U_{\pm,\epsilon} \mp \nabla v(x) > 0$.

Proposition 2.1. — *Let u be a C^2-solution of (2.1) and assume that $\nabla u(x) \longrightarrow 0$, $|x| \longrightarrow \infty$ and that $\nabla v(x)$ is bounded. Choose $U_\pm = A^{-1}V_\pm$ as above, so that (2.4) holds. Then*

$$-U_- \leq \nabla u(x) \leq U_+ \text{component wise.} \tag{2.5}$$

Proof. — Since $U_\pm$ are independent of x, we get from (2.2) :

$$\begin{aligned} &\phi''(x)(\pm\nabla u - U_\pm) - h\Delta(\pm\nabla u - U_\pm) + \nabla\phi \cdot \partial_x(\pm\nabla u - U_\pm) \\ &= -(\phi''(x)U_\pm \mp \nabla v(x)). \end{aligned} \tag{2.6}$$

Let (2.6ϵ) denote the correponding identity with $U_\pm$ replaced by $U_{\pm,\epsilon}$. Assume that there exists an $\epsilon > 0$ such we do not have $\nabla u \leq U_{+,\epsilon}$ everywhere in all components. Let $x_0 \in \mathbf{R}^\Gamma$, $j_0 \in \Gamma$ have the property that

$$0 < \partial_{x_{j_0}} u(x_0) - U_{+,\epsilon}(j_0) = \sup_x \max_j \partial_{x_j} u(x) - U_{+,\epsilon}(j).$$

Considering then (2.6ϵ) in the (+)-case, for $x = x_0$, $j = j_0$, we see that the left hand side is ≥ 0, using (1.2) and a corresponding maximum principle for the Laplacian, while the right hand side is

$$-(\phi''(x_0)U_{+,\epsilon} - \nabla v(x_0))_{j_0} \leq -(AU_{+,\epsilon} - \nabla v(x_0))_{j_0} < 0.$$

Here we get a contradiction, so $\nabla u(x) \leq U_{+,\epsilon}$ for every $\epsilon > 0$. Letting ϵ tend to zero, we obtain $\nabla u(x) \leq U_+$. The proof of the first inequality in (2.5) is identical. $\square$

In order to get estimates on the correlations, we also need to discuss the existence of suitable solutions of (2.1), and this will be obtained by applying the arguments of [HS] (using also some arguments of [S]). For the arguments to follow, we do not need any uniformity with respect to $\sharp\Gamma$. Let us first recall the Propositions 2.1, 2.3, 3.3 of [HS], which imply that if $\phi(x) \in C^\infty(\mathbf{R}^\Gamma)$ is strictly convex and if $\phi(x) - \frac{x^2}{2}$ has compact support, then for every $v \in C_0^\infty(\mathbf{R}^\Gamma)$, we have a C^∞-solution of (2.1), which satisfies $|\partial_x^\alpha u(x)| \leq C_\alpha \langle x\rangle^{\frac{1}{2}-|\alpha|}$.

We now keep the assumption that $v \in C_0^\infty(\mathbf{R}^\Gamma)$, but take $\phi \in C^\infty(\mathbf{R}^\Gamma)$ satisfying $\phi''(x) \geq \epsilon_0 I > 0$, for some fixed $\epsilon_0 > 0$ and $|\partial^\alpha \phi| \leq$ Const., for $|\alpha| = 2$. (These assumptions are weaker than (2.3).) As in [S], [HS], we let $\chi = \chi_\epsilon \in C_0^\infty(\mathbf{R}^\Gamma)$ be equal to 1 on $B(0, e^{1/\epsilon})$ (the open Euclidean ball of center 0 and radius $e^{1/\epsilon}$) and satisfy $|\partial^\alpha \chi| \leq \epsilon |x|^{-|\alpha|}$, $|\alpha| \geq 1$. Put $\phi_\epsilon(x) = \chi_\epsilon(x)\phi(x) + (1-\chi_\epsilon(x))\frac{x^2}{2}$. Then $\phi_\epsilon - \frac{x^2}{2}$ has compact support and $\phi_\epsilon''(x) \geq (\epsilon_0 - \mathcal{O}(\epsilon))I > 0$, when $\epsilon > 0$ is small enough. From the results of [HS] already cited, we conclude that there exist $u_\epsilon \in C^\infty(\mathbf{R}^\Gamma)$, solving

$$\nabla\phi_\epsilon \cdot \partial_x u_\epsilon - h\Delta u_\epsilon = v + \text{Const.}$$

with

$$|\partial_x^\alpha u_\epsilon(x)| \leq C_{\alpha,\epsilon}\langle x\rangle^{\frac{1}{2}-|\alpha|}.$$

Writing the differential equation

$$\phi_\epsilon''(x)\nabla u_\epsilon - h\Delta\nabla u_\epsilon + \nabla\phi_\epsilon(x)\cdot\partial_x\nabla u_\epsilon = \nabla v,$$

we get from the maximum principle, that

$$|\nabla u_\epsilon(x)| \leq C_0 \mathrm{sup}_x |\nabla v(x)|,$$

where C_0 is independent of ϵ. If we introduce suitable weights of the form $e^{\Phi(x)}$, with Φ constant near infinity and with $\langle x\rangle\nabla\Phi(x)$ and $|\partial^\alpha\Phi(x)|$, $|\alpha| = 2$ sufficiently small independently of $\epsilon > 0$, we even get,

$$|\nabla u_\epsilon(x)| \leq C(v)\langle x\rangle^{-\rho}$$

for some $\rho > 0$ and $C(v)$ which are independent of ϵ. Taking a weak limit in the sense of distributions, we get a C^∞-solution $u(x)$ of (2.1) with the property that $\nabla u(x) \to 0$, $|x| \to \infty$. If ϕ satisfies (2.3), we can therefore apply the conclusion of Proposition 2.1.

We finally eliminate the assumption that v has compact support. Assume (2.3) and let $v \in C^\infty(\mathbf{R}^\Gamma; \mathbf{R})$ with $\nabla v(x)$ bounded. With χ_ϵ as above, we put $v_\epsilon = \chi_\epsilon v \in C_0^\infty$ and notice that $\nabla v_\epsilon = \chi_\epsilon \nabla v + \mathcal{O}(\epsilon)$. Applying Proposition 2.1 in the case of v_ϵ and passing to the limit, we obtain a solution $u \in C^\infty(\mathbf{R}^\Gamma; \mathbf{R})$ of (2.1), which satisfies (2.5) (although we cannot state that $\nabla u(x) \to 0$, $x \to \infty$).

Summing up, we have :

THEOREM 2.2. — *Let $\phi \in C^\infty(\mathbf{R}^\Gamma; \mathbf{R})$ satisfy (2.3) and let $v \in C^\infty(\mathbf{R}^\Gamma; \mathbf{R})$ with $\nabla v(x)$ uniformly bounded. Choose $U_\pm = A^{-1} V_\pm$ as prior to Proposition 2.1. Then (2.2) has a C^∞-solution u which satisfies (2.5).*

COROLLARY 2.3. — *Under the same assumptions on ϕ as in the theorem, we have*

$$0 \leq \langle (x_j - \langle x_j \rangle)(x_k - \langle x_k \rangle) \rangle \leq h(A^{-1} e_k)(j), \tag{2.7}$$

where we have put

$$\langle f \rangle = \frac{\int f(x) e^{-\phi(x)/h} dx}{\int e^{-\phi(x)/h} dx} \tag{2.8}$$

and e_k is the unit vector in $\mathbf{R}^\Gamma$ corresponding to the index k.

Proof. — The first inequality is a special case of a more general result of Cartier [C]. Let $v = x_k$ and let $u(x)$ be given by Theorem 2.2. In this case, we have $U_- = 0$, so (2.5) implies that

$$0 \leq \frac{\partial u}{\partial x_j} \leq U_+(j) = (A^{-1} e_k)(j)$$

(where the first inequality was proved in [HS] by means of the maximum principle). As in [HS], we get $\langle (x_j - \langle x_j \rangle)(x_k - \langle x_k \rangle) \rangle = h \left\langle \dfrac{\partial u}{\partial x_j} \right\rangle$ and (2.7) follows. $\square$

It is now straight forward to give detailed estimates on the correlations in the special situations with $\Gamma \subset \mathbf{Z}^d$.

THEOREM 2.4. — *Let* $A = (\delta_{j,k} - v(j-k))_{j,k\in \mathbf{Z}^d}$ *satisfy (1.6), (1.7), (1.10), (1.11). Let* $\Gamma \subset \mathbf{Z}^d$ *be finite and assume that* $\phi \in C^\infty(\mathbf{R}^\Gamma;\mathbf{R})$ *satisfies (2.3) with* $A = A_\Gamma$ *(the restriction of the matrix* A *to* $(j,k) \in \Gamma\times\Gamma$*). Depending on* d *we have the following results, valid uniformly in* (j,k,Γ) *with* $j,k \in \Gamma$ *:*

1) *Assume* $d \geq 3$*. Then we have*

(2.9)
$$\frac{1}{h}|\langle(x_j - \langle x_j\rangle)(x_k - \langle x_k\rangle)\rangle| \leq F_0(j-k) + o(\langle j-k\rangle^{-1}), \quad |j-k| \to \infty,$$

where $F_0(x)$ *is the fundamental solution of* $\sum a_{j,k}D_{x_j}D_{x_k}$ *with* $a_{j,k}$ *appearing in (1.11).*

2) *Assume d=2 and add the assumption (1.24) for some* $k_0 > 2$*. Then for* $\Gamma \subset B(0,R_1)\cap \mathbf{Z}^d$*, we have*

$$\frac{1}{h}|\langle(x_j - \langle x_j\rangle)(x_k - \langle x_k\rangle)\rangle| = \mathcal{O}(1)\log R_1. \tag{2.10}$$

3) *Assume d=1 and add the assumption (1.24) for some* $k_0 > 2$*. Then for* $\Gamma \subset B(0,R_1)$*,* $R_1 \geq 2$*, we have*

$$\frac{1}{h}|\langle(x_j - \langle x_j\rangle)(x_k - \langle x_k\rangle)\rangle| = \mathcal{O}(1)R_1^{\max(1,1/(k_0-2))}. \tag{2.11}$$

3. Uniform correlation estimates near a "critical" case.

Let $v \in \ell^1(\mathbf{Z}^d)$ be even, real-valued, ≥ 0, with $v(0) = 0$, $\sum_{\mathbf{Z}^d} v(j) = 1$. We strengthen the power decay assumptions of the preceding section to :

$$\sum e^{|j|/C_0}v(j) \leq \infty \tag{3.1}$$

for some $C_0 > 0$. In this section, we are interested in uniform estimates of the inverse of

$$((1+\epsilon)\delta_{j,k} - v(j-k))_{j,k\in\mathbf{Z}^d} = A_\epsilon, \tag{3.2}$$

where $\epsilon \geq 0$ is a small parameter, and (as an application) in the corresponding estimates on correlations. Our estimates could easily be extended to the case when v depends on ϵ in a such a way that all the assumptions on v are uniformly satisfied.

Changing slightly the definition of E_ϵ in section 1, we have

$$((1+\epsilon)\delta_0 - v) * E_\epsilon = \delta_0 \tag{3.3}$$

for $\epsilon > 0$, where $E_\epsilon \geq 0$ is given by

$$E_\epsilon(j) = \frac{1}{(2\pi)^d} \int_{\mathbf{T}^d} \frac{e^{ij\theta}}{\epsilon + 1 - \hat{v}(\theta)} d\theta. \tag{3.4}$$

We make the assumption (1.11) so that the analytic function $q(\theta) = 1 - \hat{v}(\theta)$ is ≥ 0 with equality precisely at 0 (on $\mathbf{T}^d$) and has a non-degenerate maximum at that point. Writing x instead of θ and replacing j by $\lambda\omega$, $\lambda > 0$, $\omega \in S^{d-1}$ we are then interested in the asymptotics when $\lambda \to \infty$ of

$$\int_{\mathbf{T}^d} \frac{e^{i\lambda x\cdot\omega}}{\epsilon + q(x)} dx. \tag{3.5}$$

A slight contour deformation shows that we can concentrate on a neighborhood of $x = 0$. After an orthogonal change of coordinates, we can assume that $\omega = (0, 0, .., 1)$, $x \cdot \omega = x_d$. We can make an analytic change of coordinates of the form

$$x' = x'(y', y_d),\ x_d = y_d,\ y = (y', y_d),\ x = (x', x_d), \tag{3.6}$$

such that

$$q(x) = (y')^2/2 + f(y_d), \tag{3.7}$$

where $f(y_d) \geq 0$, $f(0) = 0$, $f''(0) > 0$. In order to compute $f''(0)$, we may assume that q is quadratic : $q(x) = \frac{1}{2}\langle Qx, x\rangle$. Then

$$\frac{1}{2}\langle Qx, x\rangle = \frac{f''(0)}{2}\langle x, \omega\rangle^2,\ x \in \gamma,$$

where γ is the line characterized by $t(\partial_x)\langle Qx, x\rangle = 0$ on γ for all $t \in (\omega)^\perp$. In other words,

$$x \in \gamma \Leftrightarrow \langle Qx, t\rangle = 0, \forall t \in (\omega)^\perp$$

so $\gamma = Q^{-1}((\omega))$, and we get

$$f''(0) = \langle Q^{-1}\omega, \omega\rangle^{-1}. \tag{3.8}$$

A further change of coordinates of the form

$$z' = y',\ z_d = \mathrm{sgn}(y_d)\sqrt{2f(y_d)}$$

gives

$$q(x) = \frac{1}{2}z^2 \tag{3.9}$$

and we may assume that the map $x \mapsto z$ is odd. Writing x instead of z, we get the integral

$$\int \frac{e^{i\lambda\tilde{g}(x_d)}}{\epsilon + x^2/2} J(x)dx, \tag{3.10}$$

where $J > 0$ is even with

$$J(0) = (\det q''(0))^{-1/2} \tag{3.11}$$

and $\tilde{g}$ is odd with

$$\tilde{g}'(0) = \sqrt{\langle q''(0)^{-1}\omega, \omega\rangle} > 0. \tag{3.12}$$

Here we view (3.10) as an integral over a neighborhood of 0 after a suitable contour deformation in x_d. Eliminating the x_d-integration by residue calculus, we get

$$2\pi \int e^{i\lambda\tilde{g}(i\sqrt{2\epsilon+x'^2})} \frac{J(x', i\sqrt{2\epsilon + x'^2})}{\sqrt{2\epsilon + x'^2}} dx'. \tag{3.13}$$

Here, we assume that $d \geq 2$, and we postpone the easier case $d = 1$.

In (3.13), we may replace $J(x', i\sqrt{2\epsilon + x'^2})$ by the real valued function

$$J_0(x', 2\epsilon + x'^2) = \frac{1}{2}(J(x', i\sqrt{2\epsilon + x'^2}) + J(x', -i\sqrt{2\epsilon + x'^2})), \tag{3.14}$$

since $J(-x', i\sqrt{2\epsilon + x'^2}) = J(x', -i\sqrt{2\epsilon + x'^2})$. Notice that $J_0(x', 2\epsilon + x'^2)$ is an even function of x'. Since $\tilde{g}$ is odd, we can also write

$$i\tilde{g}(i\sqrt{2\epsilon + x'^2}) = -g(\sqrt{2\epsilon + x'^2}),$$

where g is real valued, odd and satisfies (3.12). Using these remarks, we rewrite (3.13) using polar coordinates, as

$$2\pi \int_{S^{d-2}} d\omega \int_0^\infty e^{-\lambda g(\sqrt{2\epsilon+r^2})} \frac{J_0(r\omega, 2\epsilon + r^2)}{\sqrt{2\epsilon + r^2}} r^{d-2} dr, \tag{3.15}$$

where in reality, we integrate in r over an interval of the form $[0, \delta_0[$ for some $\delta_0 > 0$ and we shall in the following neglect contributions which are $\mathcal{O}(e^{-\lambda/C_0})$.

Here we replace r by $s = \sqrt{2\epsilon + r^2}$, so that $r = \sqrt{s^2 - 2\epsilon}$, $ds = \frac{rdr}{\sqrt{2\epsilon + r^2}}$, and (3.15) becomes

$$2\pi \int_{S^{d-2}} d\omega \int_{\sqrt{2\epsilon}}^{\infty} e^{-\lambda g(s)} J_0(\sqrt{s^2 - 2\epsilon}\,\omega, s^2)(s^2 - 2\epsilon)^{(d-3)/2} ds, \tag{3.16}$$

where we recall that J_0 is even in its first argument, so $J_0(\sqrt{s^2 - 2\epsilon}\,\omega, s^2)$ is smooth in $s^2 - 2\epsilon$.

Before continuing, we make some remarks about the function

$$F_{k,\ell,m}(\mu) = \int_0^{\infty} e^{-t} t^k (t + \mu)^\ell (t + 2\mu)^m dt, \ \mu > 0,$$

where $k, \ell, m \in \mathbf{R}$, $k > -1$. When μ tends to ∞, we have

$$F_{k,\ell,m}(\mu) = \mu^{\ell+m} \int_0^{\infty} e^{-t} t^k (1+t/\mu)^\ell (2+t/\mu)^m dt = (1+o(1)) 2^m \Gamma(k+1) \mu^{\ell+m}.$$

For the limit $\mu \to 0$, we assume that $k + \ell + m \geq -1$. When $k + \ell + m = -1$, we get

$$F_{k,\ell,m}(\mu) = (1 + o(1)) \log \frac{1}{\mu}, \ \mu \to 0.$$

When $k + \ell + m > -1$, $F_{k,\ell,m}$ extends to a continuous function on $[0, \infty[$, which satisfies

$$F_{k,\ell,m}(0) = \Gamma(k + \ell + m + 1).$$

From this we see that for $k > -1$, $k + \ell + m \geq -1$, the function

$$f_{k,\ell,m}(\lambda, \mu) = \int_0^{\infty} e^{-t\lambda} t^k (t + \mu)^\ell (t + 2\mu)^m dt = \lambda^{-(1+k+\ell+m)} F_{k,\ell,m}(\lambda\mu)$$

defined for $\lambda \geq 1$, $0 < \mu \leq 1$, has the following bounds and asymptotic behaviours :

$$\begin{aligned} &= (1 + o(1)) \Gamma(k + 1) 2^m \lambda^{-(1+k)} \mu^{\ell+m}, \ \lambda\mu \to \infty, \\ &= \mathcal{O}(1) \lambda^{-(1+k+\ell+m)}, \ C^{-1} \leq \lambda\mu \leq C, \\ &= (1+o(1)) \Gamma(k+\ell+m+1) \lambda^{-(1+k+\ell+m)}, \ \lambda\mu \to 0, \text{ if } k+\ell+m > -1, \\ &= (1 + o(1)) \log \frac{1}{\lambda\mu}, \ \lambda\mu \to 0 \text{ in the case } k + \ell + m = -1. \end{aligned} \tag{3.17}$$

We observe in all the cases that an increase in k gives an additional decay by some power of $1/\lambda$. We now return to (3.16) which is equal to

$$(3.18) \qquad 2\pi \operatorname{vol}(S^{d-2}) \int_{\sqrt{2\epsilon}}^{\infty} e^{-\lambda g(s)} (s^2 - 2\epsilon)^{(d-3)/2} (J(0) + \mathcal{O}(\epsilon + s^2)) ds.$$

Put $t = g(s) - g(\sqrt{2\epsilon})$, $s = g^{-1}(t + g(\sqrt{2\epsilon}))$. Write

$$s^2 - 2\epsilon = (s + \sqrt{2\epsilon})(s - \sqrt{2\epsilon}), \ s - \sqrt{2\epsilon} = f(t)t,$$

$$s = \tilde{f}(t)(t + g(\sqrt{2\epsilon})), \ s + \sqrt{2\epsilon} = \hat{f}(t)(t + 2g(\sqrt{2\epsilon})),$$

where

$$f(0), \ \tilde{f}(0), \ \hat{f}(0) = \frac{1}{g'(0)} + \mathcal{O}(\sqrt{\epsilon}).$$

The expression (3.18) then becomes

$$\begin{aligned}
(3.19) \quad & 2\pi \operatorname{vol}(S^{d-2}) J(0) g'(0)^{2-d} e^{-\lambda g(\sqrt{2\epsilon})} \\
& \times \int_0^{\infty} e^{-\lambda t} t^{(d-3)/2} (t + 2g(\sqrt{2\epsilon}))^{(d-3)/2} (1 + \mathcal{O}(\sqrt{\epsilon}) + \mathcal{O}(t)) dt \\
& = (1 + \mathcal{O}(\sqrt{\epsilon})) 2\pi \operatorname{vol}(S^{d-2}) J(0) g'(0)^{2-d} e^{-\lambda g(\sqrt{2\epsilon})} \\
& \times (f_{\frac{d-3}{2},0,\frac{d-3}{2}}(\lambda, g(\sqrt{2\epsilon})) + \mathcal{O}(1) f_{\frac{d-1}{2},0,\frac{d-3}{2}}(\lambda, g(\sqrt{2\epsilon})) + \mathcal{O}(e^{-\lambda/C})).
\end{aligned}$$

Using that g is odd and satisfies (3.12), we obtain

$$(3.20) \qquad g(\sqrt{2\epsilon}) = \sqrt{2\epsilon \langle q''(0)^{-1}\omega, \omega \rangle} + \mathcal{O}(\epsilon^{3/2}),$$

and in particular,

$$(3.21) \qquad e^{-\lambda g(\sqrt{2\epsilon})} = e^{\sqrt{2\epsilon \langle q''(0)^{-1} j, j \rangle} + \mathcal{O}(\epsilon^{3/2})|j|},$$

where we recall that $j = \lambda \omega$.

Combining (3.17), (3.19), (3.20), (3.11), (3.12), we finally get the following uniform asymptotics for $E_\epsilon(j)$ when $\epsilon > 0$ is small and $|j|$ is large $(j = \lambda\omega)$:

When $|j|\sqrt{\epsilon} \to \infty$:
(3.22)

$$(1 + \mathcal{O}(\sqrt{\epsilon}) + o(1)) \frac{\operatorname{vol}(S^{d-2})}{(2\pi)^{d-1}} \Gamma(\frac{d-1}{2}) 2^{\frac{d-3}{2}} \frac{(2\epsilon)^{\frac{d-3}{4}} e^{-|j| g(\sqrt{2\epsilon})}}{\sqrt{\det q''(0)} \langle q''(0)^{-1} j, j \rangle^{\frac{d-1}{4}}}.$$

When $\frac{1}{C} \leq |j|\sqrt{\epsilon} \leq C$:

$$\mathcal{O}(1)e^{-|j|g(\sqrt{2\epsilon})}\langle j\rangle^{-(d-2)}. \tag{3.23}$$

When $|j|\sqrt{\epsilon} \to 0$, $|j| \to \infty$ we get in the case $d \geq 3$:

$$\frac{(1+o(1))\mathrm{vol}(S^{d-2})\Gamma(d-2)}{(2\pi)^{d-1}\sqrt{\det q''(0)}\langle q''(0)j,j\rangle^{\frac{d-2}{2}}} \tag{3.24}$$

and in the case $d = 2$:

$$(1+o(1))\frac{\log\frac{1}{|j|\sqrt{\epsilon}}}{\pi\sqrt{\det q''(0)}}. \tag{3.25}$$

In the case $d \geq 2$ it remains to discuss the region where $|j| = \mathcal{O}(1)$. For $d \geq 3$ we get (as in section 1) :

$$\mathcal{O}(1). \tag{3.26}$$

For $d = 2$ a direct estimate of (3.5) shows that $E_\epsilon(j)$ is

$$\leq \frac{1+o(1)}{\pi\sqrt{\det q''(0)}}\log\frac{1}{\sqrt{\epsilon}}. \tag{3.27}$$

In the case $d = 1$ we get instead of (3.13) :

$$E_\epsilon(j) = \frac{(1+\mathcal{O}(\epsilon))}{\sqrt{2\epsilon q''(0)}}e^{-|j|g(\sqrt{2\epsilon})} + \mathcal{O}(e^{-|j|/C}) \tag{3.28}$$

and $E_\epsilon = \mathcal{O}\left(\frac{1}{\sqrt{\epsilon}}\right)$, for $|j| = \mathcal{O}(1)$.

We can then formulate an application to correlations, analogous to Theorem 2.4 :

THEOREM 3.1. — *Let $0 \leq v \in \ell^1(\mathbf{Z}^d)$ be even with $v(0) = 0$, $\sum v(j) = 1$ and assume (1.11), (3.1). For $0 < \epsilon \leq 1$, we put*

$$A = A_\epsilon = ((1+\epsilon)\delta_{j,k} - v(j-k))_{j,k\in\mathbf{Z}^d}.$$

Let $\Gamma \subset \mathbf{Z}^d$ be finite and assume that $\phi \in C^\infty(\mathbf{R}^\Gamma;\mathbf{R})$ satisfies (2.3) with $A = A_{\epsilon,\Gamma}$, the restriction of A_ϵ to $\Gamma\times\Gamma$. Then for $j,k \in \Gamma$, we have

$$0 \leq \frac{1}{h}\langle(x_j - \langle x_j\rangle)(x_k - \langle x_k\rangle)\rangle \leq E_\epsilon(j-k),$$

where $E_\epsilon(j) \geq 0$ is defined by (3.4) and has the properties (3.22-28) when $\epsilon > 0$ is sufficiently small. Here $g(\sqrt{2\epsilon})$ satisfies (3.20) (and (3.21)).

We may think of ϵ as the difference between the temperature and the critical temperature in statistical mechanics, but of course, we have not proved (and this may very well be wrong) that $\epsilon = 0$ corresponds to some kind of phase transition.

4. A general maximum principle and applications.

Starting in the second half of [HS] and continued in the present paper, we have seen a more precise form of the maximum principle than what was used in [S] and in the first half of [HS]. In this section, we formulate a general version of the maximum principle and then show how it permits to sharpen some of the results of [HS] and [S]. It is the author's impression that this generalized version permits an important gain in flexibility.

Let $B \cong \mathbf{R}^m$ be an m-dimensional ($m < \infty$) real Banach space and let B^* be the dual space. Since we are in the finite dimensional case, we know that B is reflexive. Let $A : B \to B$ be a linear map, and let $\delta \geq 0$. We say that A satisfies (mp δ) (with respect to the space B), if we have

$$\text{If } t \in B,\ \ s \in B^* \text{ and } \langle t, s\rangle = \|t\|_B \|s\|_{B^*} \text{ then } \langle At, s\rangle \geq \delta \|t\|_B \|s\|_{B^*}.$$

Notice that if A satisfies (mp δ) with respect to the space B then the adjoint A^* satisfies (mp δ) with respect to B^*.

Examples.

1) Let $B = B^* = \ell^2$ and let A be symmetric $\geq \delta \geq 0$. Then A satisfies (mp δ).

2) Let $B = \ell^p_\rho$ for some $p \in [1, \infty]$, $\rho : \{1, .., m\} \to]0, \infty[$, where ℓ^p_ρ is equipped with the norm $\|x\|_{\ell^p_\rho} = \|\rho x\|_{\ell^p}$, $(\rho x)_j = \rho(j) x_j$. Assume $A = D + W$, where D is diagonal $\geq r_0 \geq r_1 \geq 0$ and where $\|W\|_{\mathcal{L}(\ell^p_\rho, \ell^p_\rho)} \leq r_1$. Then A satisfies (mp$(r_0 - r_1)$). In fact, we have $B^* = \ell^q_{1/\rho}$, where $1 = \dfrac{1}{p} + \dfrac{1}{q}$, so if $\langle t, s\rangle = \|t\|_B \|s\|_{B^*}$, then $t_j s_j \geq 0$, for every j. Consequently, $\langle Dt, s\rangle \geq r_0 \|t\| \|s\|_{B^*}$, while $|\langle Wt, s\rangle| \leq r_1 \|t\|_B \|s\|_{B^*}$.

We may also notice in general, that if A satisfies (mp δ) for some Banach space B, then for $\epsilon \geq 0$, $\epsilon I + A$ satisfies (mp$(\delta + \epsilon)$). We indicate briefly how to work with the principles (mp δ) in the situations encountered in [HS] and in [S], and in particular, we shall see how to eliminate (in both cases) the assumption, that certain Hessians should be close to the identity. For simplicity, we shall not review the whole regularization processes, and mainly concentrate on the use of the maximum principle.

We start with the situation of [HS]. Assume $\Gamma = \{1, 2, .., m\}$ and consider the equations (2.1),(2.2), where ϕ, u, v are assumed to be sufficiently regular. We identify $\mathbf{R}^m$ with some Banach space B and assume that $\phi''(x)$ satisfies (mpδ) for all $x \in \mathbf{R}^m$ for some fixed $\delta > 0$. We also assume that $\nabla u(x) \to 0$, $|x| \to \infty$. (To be more precise about the regularity, we assume to start with that $\phi \in C^2$, $u \in C^1$.) Let $x_0 \in \mathbf{R}^m$ be a point where $\|\nabla u(x)\|_B$ is maximal $= m_1$. Let $s \in B^*$ be a unit vector such that $\langle \nabla u(x_0), s\rangle = m_1$. Then $\langle -h\Delta\nabla u(x_0), s\rangle \geq 0$, $\langle \nabla\phi \cdot \nabla u(x_0), s\rangle = 0$, $\langle \phi''(x_0)\nabla u(x_0), s\rangle \geq \delta m_1$ (by (mp δ)), so if we put $x = x_0$ and take the scalar product with s, we get from (2.2) :

$$\delta m_1 \leq \langle \nabla v(x_0), s\rangle \leq \|\nabla v(x_0)\|_B,$$

so we obtain for every $x \in \mathbf{R}^m$:

$$\|\nabla u(x)\|_B \leq \delta^{-1} \sup_{x\in\mathbf{R}^m} \|\nabla v(x)\|_B. \tag{4.1}$$

We keep the preceding assumptions and assume in addition, that $\phi \in C^3$, $u \in C^2$, $v \in C^2$ and that $\nabla^2 u(x) \to 0$, $|x| \to \infty$. Differentiating (2.2), we get (as in [HS]) :

(4.2)
$$(\nabla\phi \cdot \partial_x - h\Delta)\nabla^2 u + \nabla^2\phi \circ \nabla^2 u + \nabla^2 u \circ \nabla^2\phi = \nabla^2 v - \langle \nabla^3\phi, \nabla u\rangle,$$

where the last term is the contraction of $\nabla^3\phi$ by ∇u (defined in [S]). We define norms of the higher order Hessians as in [S]. Let $x_0 \in \mathbf{R}^m$ be a point where

$$\sup_{x\in\mathbf{R}^m} \|\nabla^2 u(x)\|_{(B\otimes B^*)^*} \underset{\mathrm{def}}{=} m_2$$

is attained and let $s \in B$, $t \in B^*$ be normalized vectors with

$$\langle \nabla^2 u(x_0), s \otimes t\rangle (= \langle \nabla^2 u(x_0)s, t\rangle) = m_2.$$

(Notice that $\|\nabla^2 u(x)\|_{(B\otimes B^*)^*} = \|\nabla^2 u(x)\|_{\mathcal{L}(B,B)}$ where $\nabla^2 u$ to the left is viewed as a bilinear form and to the right as a linear map.) Applying (4.2) to $s \otimes t$ and putting $x = x_0$, we get

$$\begin{aligned}\langle \nabla^2\phi(x_0)\nabla^2 u(x_0)s, t\rangle &+ \langle s, \nabla^2\phi(x_0)\nabla^2 u(x_0)t\rangle \\ &\leq \langle \nabla^2 v(x_0), t \otimes s\rangle - \langle \nabla^3\phi(x_0), \nabla u(x_0) \otimes s \otimes t\rangle.\end{aligned}$$

Since

$$\langle \nabla^2 u(x_0)s, t\rangle = \langle s, \nabla^2 u(x_0)t\rangle = m_2$$
$$= \|\nabla^2 u(x_0)s\|_B \|t\|_{B^*} = \|s\|_B \|\nabla^2 u(x_0)t\|_{B^*},$$

we can use (mp δ) for $\phi''(x_0)$ to bound the left hand side from below by $2\delta m_2$. We conclude that for every $x \in \mathbf{R}^m$:

$$2\delta\|\nabla^2 u(x)\|_{(B\otimes B^*)^*} \le \sup_x\|\nabla^2 v(x)\|_{(B\otimes B^*)^*}$$
$$+\sup_x|\nabla u(x)|\sup_x\|\nabla^3\phi\|_{(\ell^\infty\otimes B\otimes B^*)^*},$$

where $|\cdot|$ denotes the ℓ^∞-norm. We now add the assumption that $\phi''(x)$ satisfies (mpδ) for all x with respect to ℓ^∞. (B. Helffer has indicated to us that other spaces than ℓ^∞ can be useful in this part for the applications.) Then (4.1) holds with B replaced by ℓ^∞, and we get

$$(4.3)\quad \|\nabla^2 u(x)\|_{(B\otimes B^*)^*}$$
$$\le \frac{1}{2\delta}\sup_x\|\nabla^2 v(x)\|_{(B\otimes B^*)^*} + \frac{1}{2\delta^2}\sup_x|\nabla v(x)|\sup_x\|\nabla^3\phi\|_{(\ell^\infty\otimes B\otimes B^*)^*},$$

where $|\cdot|$ denotes the ℓ^∞-norm. Notice that (4.3) remains valid with B replaced by ℓ^∞.

Keeping the earlier assumptions we strengthen the regularity assumptions to $\phi \in C^4$, $u, v \in C^3$ and assume that $\nabla^3 u \to 0$, $x \to \infty$. First we rewrite (4.2) in a more systematic form :

$$\langle \nabla^2 v, t\otimes s\rangle = -h\Delta\langle \nabla^2 u, t\otimes s\rangle + \langle \nabla^3 u, \nabla\phi\otimes t\otimes s\rangle + \langle \nabla^3\phi, \nabla u\otimes t\otimes s\rangle$$
$$+\langle \nabla^2 u, \langle \nabla^2\phi, t\rangle\otimes s\rangle + \langle \nabla^2 u, t\otimes\langle \nabla^2\phi, s\rangle\rangle$$

and differentiate it in the constant direction r :

$$\begin{aligned}(4.4)\ \langle \nabla^3 v, t\otimes s\otimes r\rangle &= -h\Delta\langle \nabla^3 u, t\otimes s\otimes r\rangle + \nabla\phi\cdot\partial_x\langle \nabla^3 u, t\otimes s\otimes r\rangle\\ &+\langle \nabla^3 u, \phi'' t\otimes s\otimes r\rangle + \langle \nabla^3 u, t\otimes\phi'' s\otimes r\rangle + \langle \nabla^3 u, t\otimes s\otimes\phi'' r\rangle\\ &+\langle \nabla^3\phi, u'' t\otimes s\otimes r\rangle + \langle \nabla^3\phi, t\otimes u'' s\otimes r\rangle + \langle \nabla^3\phi, t\otimes s\otimes u'' r\rangle\\ &+\langle \nabla^4\phi, \nabla u\otimes t\otimes s\otimes r\rangle.\end{aligned}$$

Let $x_0 \in \mathbf{R}^m$ be a point, where $\sup_x\|\nabla^3 u\|_{(\ell^\infty\otimes B\otimes B^*)^*} = m_3$ is attained, and let $t \in \ell^\infty$, $s \in B$, $r \in B^*$ be normalized vectors such that $\langle \nabla^3 u(x_0), t\otimes s\otimes r\rangle = m_3$. We rewrite this as $\langle\langle \nabla^3 u(x_0), s\otimes r\rangle, t\rangle = m_3$, noting that $|\langle \nabla^3 u(x_0), s\otimes r\rangle|_1 = m_3$. It follows from (mp δ) with respect to ℓ^∞, that

$$\langle \nabla^3 u(x_0), \phi'' t\otimes s\otimes r\rangle = \langle\langle \nabla^3 u(x_0), s\otimes r\rangle, \phi''(x_0)t\rangle \geq \delta m_3.$$

The terms $\langle \nabla^3 u(x_0), t\otimes \phi'' s\otimes r\rangle$, $\langle \nabla^3 u(x_0), t\otimes s\otimes \phi'' t\rangle$ have the same lower bound, so if we use this in (4.4), we get :

$$\begin{aligned}3\delta \mathrm{sup}_x \|\nabla^3 u\|_{(\ell^\infty\otimes B\otimes B^*)^*} \leq& \mathrm{sup}\|\nabla^3 v\|_{(\ell^\infty\otimes B\otimes B^*)^*}\\ &+\mathrm{sup}\|\nabla^3\phi\|_{(\ell^\infty\otimes B\otimes B^*)^*}(2\mathrm{sup}\|\nabla^2 u\|_{(B\otimes B^*)^*}+\mathrm{sup}\|\nabla^2 u\|_{(\ell^\infty\otimes l^1)^*})\\ &+\mathrm{sup}\|\nabla^4\phi\|_{(\ell^\infty\otimes l^\infty\otimes B\otimes B^*)^*}\mathrm{sup}|\nabla u|\end{aligned}$$

and applying (4.3), (4.1), this gives :

$$\begin{aligned}\mathrm{sup}_x\|\nabla^3 u\|_{(\ell^\infty\otimes B\otimes B^*)^*} \qquad (4.5)\\ \leq \frac{1}{3\delta}\mathrm{sup}\|\nabla^3 v\|_{(\ell^\infty\otimes B\otimes B^*)^*} + \frac{1}{3\delta^2}\mathrm{sup}\|\nabla^3\phi\|_{(\ell^\infty\otimes B\otimes B^*)^*}\\ \times\left(\mathrm{sup}\left\|\nabla^2 v\right\|_{(B\otimes B^*)^*} + \frac{1}{2}\mathrm{sup}\left\|\nabla^2 v\right\|_{(l^\infty\otimes\ell^1)^*}\right)\\ +\left(\frac{1}{3\delta^3}(\mathrm{sup}\|\nabla^3\phi\|_{(\ell^\infty\otimes B\otimes B^*)*}\right)^2 + \frac{1}{6\delta^3}\mathrm{sup}\|\nabla^3\phi\|_{(\ell^\infty\otimes B\otimes B^*)^*}\\ \mathrm{sup}\|\nabla^3\phi\|_{(\ell^\infty\otimes\ell^\infty\otimes\ell^1)^*})\mathrm{sup}|\nabla v|\\ +\frac{1}{3\delta^2}\mathrm{sup}\|\nabla^4\phi\|_{(\ell^\infty\otimes\ell^\infty\otimes B\otimes B^*)^*}\mathrm{sup}\|\nabla v\|_B.\end{aligned}$$

It seems clear that the derivatives to all orders can be estimates in the same way, for instance in the framework of 0-standard functions.

We end this section by showing how to generalize some estimates of [S] of the Hessian of the logarithm of the first eigenfunction for a Schrödinger operator with a convex potential. Let $V\in C^\infty(\mathbf{R}^m;\mathbf{R})$ be convex and satisfy

$$\begin{aligned}V''(x) = D + W(x),\ \text{where } D \text{ is diagonal } \geq r_0 > r_1 \geq 0\\ \text{and } \|W(x)\|_{\mathcal{L}(B,B)} \leq r_1.\end{aligned} \qquad (4.6)$$

Here $B=\ell^p_\rho$ for some $p\in[1,\infty]$, $\rho:\{1,..,m\}\to]0,\infty[$.

THEOREM 4.1. — *Let $e^{-\phi(x)/h}$ be the first eigenfunction of $-\frac{h^2}{2}\Delta + V(x)$. Then $\phi''(x) = \sqrt{D} + \psi''(x)$, where $\|\psi''(x)\|_{\mathcal{L}(B,B)} \leq \sqrt{r_0} - \sqrt{r_0 - r_1}$.*

Proof. — If μ denotes the corresponding lowest eigenvalue, we have the Riccati equation :

$$V - \mu = \frac{1}{2}(\partial_x \phi)^2 - \frac{h}{2}\Delta\phi. \tag{4.7}$$

Following Singer-Wong-Yau-Yau [SiWYY] as in [S], we get

$$V''(x) = \nabla\phi \cdot \partial_x \phi'' - \frac{h}{2}\Delta\phi'' + \phi'' \circ \phi'', \tag{4.8}$$

which we rewrite as
(4.9)

$$W(x) = \nabla\phi \cdot \partial_x \psi'' - \frac{h}{2}\Delta\psi'' + \left(\sqrt{D} + \frac{1}{2}\psi''\right)\psi'' + \psi''\left(\sqrt{D} + \frac{1}{2}\psi''\right),$$

where we have written

$$\phi''(x) = \sqrt{D} + \psi''(x). \tag{4.10}$$

We assume that $\psi''(x) \to 0$, $|x| \to \infty$, and that

$$\sup_x \|\psi''(x)\|_{\mathcal{L}(B,B)} \underset{\text{def}}{=} m \leq \sqrt{r_0}. \tag{4.11}$$

Then $\sqrt{D} + \frac{1}{2}\psi''(x)$ satisfies (mp δ) with $\delta = \sqrt{r_0} - m/2$. Let x_0 be a point, where the supremum in (4.11) is attained and let $t \in B$, $s \in B^*$ be normalized vectors with $\langle \psi''(x_0)t, s\rangle = m$. We apply (4.9) to t, take the scalar product with s and put $x = x_0$, and obtain

$$\begin{aligned}(4.12)\ r_1 &\geq \sup_x \|W(x)\|_{\mathcal{L}(B,B)} \\ &\geq \left\langle \left(\sqrt{D} + \frac{1}{2}\psi''(x_0)\right)\psi''(x_0)t, s\right\rangle + \left\langle \left(\sqrt{D} + \frac{1}{2}\psi''(x_0)\right)t, \psi''(x_0)s\right\rangle \\ &\geq 2(\sqrt{r_0} - m/2)m,\end{aligned}$$

so $m^2 - 2\sqrt{r_0}m + r_1 \geq 0$ and hence $m \geq \sqrt{r_0} + \sqrt{r_0 - r_1}$, or $m \leq \sqrt{r_0} - \sqrt{r_0 - r_1}$. The first possibility can be excluded because of the assumption (4.11) and we get the conclusion of the theorem.

In order to treat the general case, we shall first extend the argument above to the case, when the diagonal matrix D may depend on x. Thus instead of (4.6), we assume that

$$V''(x) = D(x) + W(x), \text{ where } D \text{ is diagonal } \geq r_0 > r_1 \geq 0 \tag{4.13}$$
$$\text{is of class } C^\infty \text{ and } \left\| W(x) - \nabla\phi \cdot \partial_x D - \frac{h}{2}\Delta D \right\|_{\mathcal{L}(B,B)} \leq r_1.$$

Then write

$$\phi''(x) = \sqrt{D(x)} + \Psi(x), \tag{4.14}$$

and assume that $\Psi(x) \to 0$, $|x| \to \infty$ and that

$$\sup_x \|\Psi(x)\|_{\mathcal{L}(B,B)} \underset{\text{def}}{=} m \leq \sqrt{r_0}. \tag{4.15}$$

Instead of (4.9) we have

$$W(x) - \nabla\phi \cdot \partial_x D - \frac{h}{2}\Delta D = \nabla\phi \cdot \partial_x \Psi - \frac{h}{2}\Delta\Psi$$
$$+ \left(\sqrt{D} + \frac{1}{2}\Psi\right)\Psi + \Psi\left(\sqrt{D} + \frac{1}{2}\Psi\right) \tag{4.16}$$

and the earlier argument now gives

$$\|\Psi(x)\|_{\mathcal{L}(B,B)} \leq \sqrt{r_0} - \sqrt{r_0 - r_1}. \tag{4.17}$$

Let V satisfy (4.6). Choose χ_ϵ as in section 2 and put for $\epsilon << 1$:

$$V_\epsilon(x) = \chi_\epsilon(x)V(x) + r_0(1 - \chi_\epsilon(x))\frac{x^2}{2}$$

and for $0 \leq t \leq 1$:

$$V_{\epsilon,t}(x) = r_0(1-t)\frac{x^2}{2} + tV_\epsilon.$$

Let $e^{-\phi_\epsilon(x)/h}$, $e^{-\phi_{\epsilon,t}(x)/h}$ be the corresponding lowest eigenfunctions. It is proved in [S], that $\phi''_\epsilon - \sqrt{r_0}I$, $\phi''_{\epsilon,t}(x) - \sqrt{r_0}I \to 0$, $|x| \to \infty$. On the other hand,

$$V''_\epsilon(x) = D_\epsilon(x) + W_\epsilon(x),$$

where

$$D_\epsilon(x) = \chi_\epsilon(x)D + r_0(1 - \chi_\epsilon(x))I \geq r_0 I, \;\; \|W_\epsilon(x)\|_{\mathcal{L}(B,B)} \leq r_1 + \mathcal{O}(\epsilon).$$

Notice that $\nabla D_\epsilon(x) = \mathcal{O}(\epsilon/|x|)$, $\Delta D_\epsilon(x) = \mathcal{O}(\epsilon/|x|^2)$, so if we make the a priori assumption (4.15) for Ψ_ϵ given by (4.14) with $\phi = \phi_\epsilon$, $D = D_\epsilon$, it follows that

$$\|W_\epsilon(x) - \nabla\phi_\epsilon \cdot \partial_x D - \frac{h}{2}\Delta D\|_{\mathcal{L}(B,B)} \leq r_1 + \mathcal{O}(\epsilon).$$

The argument leading to (4.17) now gives

$$\|\Psi_\epsilon(x)\|_{\mathcal{L}(B,B)} \leq \sqrt{r_0} - \sqrt{r_0 - r_1} + \mathcal{O}(\epsilon). \tag{4.18}$$

These arguments apply uniformly to $V_{\epsilon,t}$, $0 \leq t \leq 1$, when $\epsilon > 0$ is small enough, and for $t > 0$ sufficiently small the a priori assumption (4.15) will be satisfied by $\Psi_{\epsilon,t}$ when $D = D_{\epsilon,t} = tD_\epsilon(x) + (1-t)r_0 I$. It is then clear that we can deform from $t = 0$ to $t = 1$ and get (4.18) in general. Letting ϵ tend to 0, we get the theorem. □

Remark 4.2. — In the case $B = \ell^2$, Theorem 4.1 still holds if we replace the word "diagonal" by "symmetric" in the assumption (4.6).

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Manuscrit reçu le 7 juillet 1993.

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