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FINITE MONODROMY OF POCHHAMMER EQUATION

by Yoshishige HARAOKA

Introduction.

Grothendieck's zero p-curvature conjecture was first intensively studied by T. Honda [5] and N. Katz [8], [9]. There followed several results, and it is known that the conjecture is true for equations of the first order [5], Picard-Fuchs equations [9], the Gauss hypergeometric equation [9] and the generalized hypergeometric equation [1]; however, in general the conjecture is still open. We note that, in the above examples, we can calculate their monodromy groups.

K. Okubo [12] developed a global theory of Fuchsian differential equations on the complex projective line. He reduced every Fuchsian equation to a normal form, defined a class of equations which is free from accessory parameters and gave an algorithm to calculate monodromy groups for equations free from accessory parameters (cf. [3], [4], [13], [14], [17]). Then we expect that, for such equations, the Grothendieck conjecture is true.

In this paper we study the Pochhammer equation. It is generically a Picard-Fuchs equation for which the conjecture holds, but is also an equation free from accessory parameters. Bearing an application to every equation free from accessory parameters in mind, we show that the Grothendieck conjecture holds for the Pochhammer equation by using its Okubo normal form (not using the integral representation of solutions).

Key words : Grothendieck's zero p-curvature conjecture – Okubo system – Equations free from accessory parameters – Pochhammer equation – Monodromy – Apparent singular point.
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Here we explain Grothendieck's conjecture in a form suitable for our purpose. Let $t_1, \dots, t_m$ be elements in $\mathbf{C}$, and set $K = \mathbf{Q}(t_1, \dots, t_m)$. To make the statement simple, we assume that $t_1, \dots, t_m$ are algebraically independent over $\mathbf{Q}$. Consider a linear ordinary differential equation over $K[x]$

$$\text{(E)} \qquad a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = 0,$$

where $a_i(x) \in K[x]$ for every i. For almost all primes $p \in \mathbf{Z}$ (i.e. except for a finite number of primes), we can reduce the coefficients of every $a_i(x)$ modulo p to obtain the equation $(\mathrm{E})_p$ over $K_p[x]$, where $K_p = \mathbf{F}_p(t_1, \dots, t_m)$. Then the following holds :

Proposition 0.1 ([5], [9]). — *If* (E) *has n algebraic function solutions which are linearly independent over* $\mathbf{C}$, *then, for almost all primes p,* $(\mathrm{E})_p$ *has n polynomial solutions in $K_p[x]$ which are linearly independent over $K_p(x^p)$.*

This is essentially Eisenstein's theorem (cf. [10]).

Note that the following three conditions are equivalent :

(i) (E) has n linearly independent algebraic function solutions,

(ii) every (analytic) solution of (E) is an algebraic function,

(iii) the monodromy group of (E) is of finite order.

Note also that $(\mathrm{E})_p$ has n polynomial solutions in $K_p[x]$ which are linearly independent over $K_p(x^p)$ if and only if (E) has zero p-curvature.

The converse of Proposition 0.1 is Grothendieck's zero p-curvature conjecture.

Conjecture. — *If, for almost all primes p,* $(\mathrm{E})_p$ *has n polynomial solutions in $K_p[x]$ which are linearly independent over $K_p(x^p)$, then every solution of* (E) *is an algebraic function.*

N. Katz gave an explicit proof of the conjecture for the Gauss hypergeometric equation in [9, §6]. We apply his manner to the Okubo normal form, and prove the conjecture for the Pochhammer equation. The Pochhammer equation is an n-th order Fuchsian differential equation with regular singular points at $x = t_1, \dots, t_n, \infty$, and is determined by fixing the characteristic exponents $(\lambda, \rho) \in \mathbf{C}^n \times \mathbf{C}$ at the singular points. In §1 we give the Okubo normal form of the Pochhammer equation, and obtain a condition on the exponents (λ, ρ) for the monodromy group to be finite

(Theorems 1.2 and 1.3). In §2 we consider a reduced Pochhammer equation modulo prime p, and obtain a condition for it to have n polynomial solutions (Theorem 2.1). Comparing these conditions, in §3 we prove the conjecture for the Pochhammer equation (Theorem 3.1).

The Pochhammer equations are divided into generic ones and non-generic ones (Definition 1.1; generic Pochhammer equations are irreducible and have no logarithmic solution at every finite singular point). Reduced equations modulo prime can be regarded as non-generic ones (§2.0); they may have logarithmic solutions. What we have obtained in Theorem 2.1 is essentially the condition that, for a non-generic Pochhammer equation, there is no logarithmic solution (i.e. the regular singular point is apparent; cf. Proposition 1.6 and §2.6). Hence in this paper we consider non-generic equations as well as generic ones. As a by-product we have obtained a necessary and sufficient condition for the monodromy group of a *reducible* Pochhammer equation to be finite (Theorem 1.3), which is the complementary result to Takano-Bannai [15] (where they give a list of the Pochhammer equations which are *irreducible* and have finite monodromy groups). The reader who is interested only in the generic case can skip §1.2, §2.6 and the latter half of the proof of Theorem 3.1.

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Notation.

$\mathbf{N}$: the set of positive integers.

$\mathbf{N}_0$: the set of non-negative integers.

1. Monodromy of Pochhammer system.

1.0. Let $t_1, \dots, t_n$ be n distinct points in $\mathbf{P}^1 \setminus \{\infty\}$, and let $\lambda_1, \dots, \lambda_n, \rho$ be complex numbers satisfying

$$\sum_{j=1}^{n} \lambda_j \neq n\rho. \tag{1.1}$$

Denote $(\lambda_1, \dots, \lambda_n)$ by λ. We call the system of differential equations in Okubo normal form

$$\mathcal{P}(\lambda, \rho): \qquad (x - T)\frac{dY}{dx} = A(\lambda, \rho)Y,$$

(1.2)

$$T = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}, \quad A(\lambda, \rho) = \begin{pmatrix} \lambda_1 & \lambda_1 - \rho & \cdots & \lambda_1 - \rho \\ \lambda_2 - \rho & \lambda_2 & \cdots & \lambda_2 - \rho \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_n - \rho & \lambda_n - \rho & \cdots & \lambda_n \end{pmatrix},$$

the *Pochhammer system* of rank n. First we note that $A(\lambda, \rho)$ is diagonalizable as follows : Set

$$P = \begin{pmatrix} 1 & 0 & \cdots & 0 & \lambda_1 - \rho \\ 0 & 1 & \cdots & 0 & \lambda_2 - \rho \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \lambda_{n-1} - \rho \\ -1 & -1 & \cdots & -1 & \lambda_n - \rho \end{pmatrix}, \tag{1.3}$$

then by (1.1) $\det P = \sum_{j=1}^{n} \lambda_j - n\rho \neq 0$, and we have

$$P^{-1} A(\lambda, \rho) P = \begin{pmatrix} \rho & & & \\ & \ddots & & \\ & & \rho & \\ & & & \rho' \end{pmatrix}, \tag{1.4}$$

where

$$\rho' = \sum_{j=1}^{n} \lambda_j - (n-1)\rho. \tag{1.5}$$

Rewriting $\mathcal{P}(\lambda, \rho)$ as

$$dY = \left(\sum_{i=1}^{n} A_i \frac{dx}{x - t_i} \right) Y, \tag{1.6}$$

$$A_i = i) \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} A(\lambda, \rho) \quad (i = 1, \ldots, n),$$

(where the above matrix is diagonal with the only non zero element 1 at the (i, i)-th position) we see that $\mathcal{P}(\lambda, \rho)$ is Fuchsian with regular singular points at $x = t_1, \ldots, t_n$ and ∞, that the characteristic exponents at $x = t_i$, which are the eigen values of A_i, are 0 of multiplicity $n-1$ and λ_i, and that the characteristic exponents at $x = \infty$, which are the eigen values of $\sum_{i=1}^{n}(-A_i) = -A(\lambda, \rho)$, are $-\rho$ of multiplicity $n-1$ and $-\rho'$ by (1.4). We sum up these facts into the scheme

$$\begin{bmatrix} x = t_1 & \cdots & x = t_n & x = \infty \\ 0 & \cdots & 0 & \rho \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & \rho \\ \lambda_1 & \cdots & \lambda_n & \rho' \end{bmatrix}. \tag{1.7}$$

(Moreover we see that $\mathcal{P}(\lambda, \rho)$ is free from accessory parameters (cf. [4])).

The classical *Pochhammer equation* is an n-th order Fuchsian differential equation

$$\mathcal{E}(\lambda, \rho): \qquad p_0(x)z^{(n)} + p_1(x)z^{(n-1)} + \cdots + p_n(x)z = 0$$

with

$$\begin{aligned} p_0(x) &= (x-t_1)\cdots(x-t_n), \\ p_k(x) &= \binom{-\rho+n-1}{k} q_0{}^{(k)}(x) + \binom{-\rho+n-1}{k-1} q_1{}^{(k-1)}(x) \\ &\qquad (k = 1, \ldots, n), \end{aligned}$$

where

$$q_0(x) = p_0(x), \quad q_1(x) = q_0(x) \sum_{j=1}^{n} \frac{\rho - \lambda_j}{x - t_j}.$$

The Riemann scheme of $\mathcal{E}(\lambda, \rho)$ is

$$\begin{Bmatrix} x = t_1 & \cdots & x = t_n & x = \infty \\ 0 & \cdots & 0 & 1-\rho \\ 1 & \cdots & 1 & 2-\rho \\ \vdots & & \vdots & \vdots \\ n-2 & \cdots & n-2 & n-1-\rho \\ \lambda_1 & \cdots & \lambda_n & -\rho' \end{Bmatrix}. \tag{1.8}$$

The Pochhammer equation $\mathcal{E}(\lambda, \rho)$ is introduced as an extension of the Gauss hypergeometric equation having a similar integral representation of Euler type of solutions ([6], [7], [16]) :

$$z_j(x) = \int_{\Gamma_j} (x-s)^{\rho-1}(s-t_1)^{\lambda_1-\rho} \cdots (s-t_n)^{\lambda_n-\rho} ds, \tag{1.9}$$

where the path Γ_j starts from a point x_0, encircles the point t_j in the positive direction to x_0, encircles x in the positive direction to x_0, encircles t_j in the negative direction to x_0 and then encircles x in the negative direction to x_0. $(z_1(x), \ldots, z_n(x))$ makes a fundamental system of solutions of $\mathcal{E}(\lambda, \rho)$ in the *generic* case (which we shall define later).

PROPOSITION 1.1. — *Let $Y = {}^t(y_1, \ldots, y_n)$ be a vector of differential indeterminates. Then*

$$z = y_1 + \cdots + y_n$$

induces a transformation of the system $\mathcal{P}(\lambda, \rho)$ into the equation $\mathcal{E}(\lambda, \rho)$.

This proposition is shown by a differential algebraic calculation. In particular the transformation of the Pochhammer system $\mathcal{P}(\lambda, \rho)$ into the Pochhammer equation $\mathcal{E}(\lambda, \rho)$

$$\begin{pmatrix} z \\ z' \\ \vdots \\ z^{(n-1)} \end{pmatrix} = F(x) \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \tag{1.10}$$

is a linear transformation with rational coefficients : $F(x) \in \mathrm{GL}(n; \mathbf{C}(x))$.

In this section we shall give a condition of the monodromy group of $\mathcal{P}(\lambda, \rho)$ to be finite. Before proceeding, we note two propositions.

From the scheme (1.7) it follows

PROPOSITION 1.2. — *Assume that $\lambda_j \notin \mathbf{Z}$ for some $j \in \{1, \ldots, n\}$. Then no solution of the system $\mathcal{P}(\lambda, \rho)$ around $x = t_j$ has a logarithmic term.*

The following proposition is obtained by N. Misaki [11].

PROPOSITION 1.3. — *The system $\mathcal{P}(\lambda, \rho)$ is irreducible if and only if*

$$\lambda_1 - \rho, \ldots, \lambda_n - \rho,\ \rho,\ \rho' \notin \mathbf{Z}. \tag{1.11}$$

DEFINITION 1.1. — *We call the Pochhammer system $\mathcal{P}(\lambda, \rho)$ generic if*

$$\lambda_1, \ldots, \lambda_n,\ \lambda_1 - \rho, \ldots, \lambda_n - \rho,\ \rho,\ \rho' \notin \mathbf{Z}. \tag{1.12}$$

Thus, by Propositions 1.2 and 1.3, a generic Pochhammer system is irreducible and has no logarithmic solution at every finite singular point.

1.1. Throughtout this subsection we consider a generic Pochhammer system $\mathcal{P}(\lambda,\rho)$; namely we assume (1.12). First we give generators of the monodromy group of $\mathcal{P}(\lambda,\rho)$.

Denote $\mathbf{P}^1 \setminus \{t_1,\dots,t_n,\infty\}$ by X. Let C be a simple closed curve in X passing through $t_1,\dots,t_n$ in this order in the positive direction, and take a base point t_0 inside C. We define a generator γ_j $(j=1,\dots,n)$ of $\pi_1(X,t_0)$ to be a homotopy class of a simple closed curve which starts from t_0, encircles t_j in the positive direction crossing C twice on the arc $t_{j-1}t_jt_{j+1}$ and ends at t_0.

Set

$$e(\alpha)=\exp(2\pi\sqrt{-1}\alpha)$$

for any $\alpha\in\mathbf{C}$, and set

$$\begin{aligned} e_0 &= e(\rho),\\ e_j &= e(\lambda_j) \quad (j=1,\dots,n). \end{aligned}$$

Then, applying Okubo's method to the system $\mathcal{P}(\lambda,\rho)$, or using the integral representation (1.9), we have

THEOREM 1.1. — *Suppose that the Pochhammer system $\mathcal{P}(\lambda,\rho)$ is generic. Then $\mathcal{P}(\lambda,\rho)$ has a fundamental system of solutions $(Y_1,\dots,Y_n)$ whose analytic continuation along γ_k becomes $(Y_1,\dots,Y_n)g_k$, where*

(1.13)

$$g_k=\begin{pmatrix} 1 &&&&&&& \\ & \ddots &&&&&& \\ && 1 &&&&& \\ (e_1-e_0)e_0^{-1} & \cdots & (e_{k-1}-e_0)e_0^{-1} & e_k & e_{k+1}-e_0 & \cdots & e_n-e_0 \\ &&&& 1 &&& \\ &&&&& \ddots && \\ &&&&&& 1 \end{pmatrix},$$

for every $k=1,\dots,n$. In particular $g_1,\dots,g_n$ generate the monodromy group of $\mathcal{P}(\lambda,\rho)$ with respect to $(Y_1,\dots,Y_n)$.

We denote by $G(\lambda,\rho)$ the monodromy group of $\mathcal{P}(\lambda,\rho)$ which is generated by $g_1,\dots,g_n$ in Theorem 1.1. It follows from the scheme (1.7) that, if $G(\lambda,\rho)$ is finite, then

$$\lambda_1,\dots,\lambda_n,\ \rho\in\mathbf{Q}. \tag{1.14}$$

From now on we assume (1.14), and call the Pochhammer system $\mathcal{P}(\lambda,\rho)$ with (1.14) *rational.*

By a simple calculation we obtain

PROPOSITION 1.4. — *Let h be a Hermitian form invariant under the monodromy group $G(\lambda,\rho)$ of a rational generic Pochhammer system $\mathcal{P}(\lambda,\rho)$. Then the Hermitian matrix associated with h is given by*

$$H=\alpha\cdot(h_{st})_{1\leqslant s,t\leqslant n},$$

$$\begin{aligned} h_{ss}&=4\sin\pi\lambda_s\cdot\sin\pi(\rho-\lambda_s)\quad(s=1,\dots,n),\\ h_{st}&=\frac{(e_s-e_0)(e_t-e_0)}{e_s{e_0}^{1/2}}\quad(s,t=1,\dots,n,\ s<t),\end{aligned} \tag{1.15}$$

where $\alpha\in\mathbf{R}$.

For any $\alpha\in\mathbf{R}$, we define its fractional part $\langle\alpha\rangle$ by

$$0\leqslant\langle\alpha\rangle<1,\quad \alpha-\langle\alpha\rangle\in\mathbf{Z}.$$

By calculating the principal minors of the Hermitian matrix H in Proposition 1.4, we obtain

PROPOSITION 1.5. — *The monodromy group $G(\lambda,\rho)$ of a rational generic Pochhammer system $\mathcal{P}(\lambda,\rho)$ has a positive definite invariant Hermitian form if and only if one of the following two conditions holds :*

$$\text{(1.16 : i)}\quad \langle\rho\rangle<\langle\lambda_j\rangle\quad(j=1,\dots,n),\quad \sum_{j=1}^{n}\langle\lambda_j\rangle<(n-1)\langle\rho\rangle+1;$$

$$\text{(1.16 : ii)}\quad \langle\lambda_j\rangle<\langle\rho\rangle\quad(j=1,\dots,n),\quad (n-1)\langle\rho\rangle<\sum_{j=1}^{n}\langle\lambda_j\rangle.$$

THEOREM 1.2. — *Let $G(\lambda,\rho)$ be the monodromy group of a rational generic Pochhammer system $\mathcal{P}(\lambda,\rho)$, and let D be the common denominator of the rational numbers $\lambda_1,\dots,\lambda_n,\ \rho$. Then $G(\lambda,\rho)$ is finite if and only* *if, for any $\Delta \in \mathbf{Z}$ prime to D, one of the following two conditions holds :*

$$(1.17:\text{i})\ \langle\Delta\rho\rangle < \langle\Delta\lambda_j\rangle \quad (j=1,\ldots,n), \quad \sum_{j=1}^{n}\langle\Delta\lambda_j\rangle < (n-1)\langle\Delta\rho\rangle + 1;$$

$$(1.17:\text{ii})\ \langle\Delta\lambda_j\rangle < \langle\Delta\rho\rangle \quad (j=1,\ldots,n), \quad (n-1)\langle\Delta\rho\rangle < \sum_{j=1}^{n}\langle\Delta\lambda_j\rangle.$$

Proof. — Let ζ_D be a primitive D-th root of 1. Then by Theorem 1.1, we see that

$$G(\lambda,\rho) \subset \mathrm{GL}(n;\mathbf{Z}[\zeta_D]).$$

As is explained in [3] (cf. [1]), $G(\lambda,\rho)$ is finite if and only if, for any $\sigma \in \mathrm{Gal}(\mathbf{Q}(\zeta_D)/\mathbf{Q})$, the transformed group $G(\lambda,\rho)^\sigma$ has a positive definite invariant Hermitian form.

To any $\sigma \in \mathrm{Gal}(\mathbf{Q}(\zeta_D)/\mathbf{Q})$, there corresponds a $\Delta \in \mathbf{Z}$ prime to D such that

$$\zeta_D \mapsto {\zeta_D}^{\Delta}$$

induces σ. Thus $G(\lambda,\rho)^\sigma$ is obtained from $G(\lambda,\rho)$ by replacing every λ_j $(j=1,\ldots,n)$ and ρ by $\Delta\lambda_j$ and $\Delta\rho$, respectively; namely,

$$G(\lambda,\rho)^\sigma = G(\Delta\lambda,\Delta\rho),$$

where $\Delta\lambda = (\Delta\lambda_1,\ldots,\Delta\lambda_n)$. Then Proposition 1.5 shows that (1.17 :i or ii) is a necessary and sufficient condition for $G(\lambda,\rho)^\sigma$ to be finite. Hence the theorem follows. Q.e.d.

1.2. Now we consider a non-generic Pochhammer system $\mathcal{P}(\lambda,\rho)$. In this case the g_j's in Theorem 1.1 do not necessarily generate a monodromy group of $\mathcal{P}(\lambda,\rho)$, so that we need a close study of solutions.

Assume that $\lambda_j \in \mathbf{Z}$ for some $j \in \{1,\ldots,n\}$. Proposition 1.2 asserts that there may be a logarithmic solution of $\mathcal{P}(\lambda,\rho)$ around $x=t_j$. When there is no logarithmic solution, the singular point $x=t_j$ is said to be *apparent.*

PROPOSITION 1.6. — *Consider a Pochhammer system $\mathcal{P}(\lambda,\rho)$ and assume that $\lambda_j \in \mathbf{Z}$ for some $j \in \{1,\ldots,n\}$. Then $x=t_j$ is apparent if and* *only if one of the following four conditions holds :*

$$(1.18:\text{i})\qquad \lambda_k-\rho\in\mathbf{N}_0\quad (k\neq j),\quad \lambda_j+\sum_{k\neq j}(\lambda_k-\rho)<0,$$

$$(1.18:\text{ii})\qquad \rho-\lambda_k\in\mathbf{N}\quad (k\neq j),\quad \lambda_j+\sum_{k\neq j}(\lambda_k-\rho)\geqslant 0,$$

$$(1.18:\text{iii})\qquad \lambda_j\geqslant 0,\quad \rho\in\mathbf{Z},\quad 0\leqslant\rho\leqslant\lambda_j,$$

$$(1.18:\text{iv})\qquad \lambda_j\leqslant -2,\quad \rho\in\mathbf{Z},\quad \lambda_j+1\leqslant\rho\leqslant -1.$$

This will be shown after the calculus of p-curvature (§2.6).

Consider a Pochhammer equation $\mathcal{E}(\lambda,\rho)$ which corresponds to a Pochhammer system $\mathcal{P}(\lambda,\rho)$ by Proposition 1.1.

PROPOSITION 1.7. — *Suppose that $\rho\notin\mathbf{Z}$ and that $\lambda_j-\rho\in\mathbf{Z}$ for every $j=1,\dots,n$.*

(i) *If $\lambda_j-\rho<0$ for every j, $\mathcal{E}(\lambda,\rho)$ has a fundamental system of solutions $(z_1,\dots,z_n)$ such that*

$$z_j=(x-t_j)^{\lambda_j}f_j(x-t_j)\quad (j=1,\dots,n),$$

where f_j is a polynomial of degree at most $(\rho-\lambda_j-1)$.

(ii) *If $\lambda_j-\rho\geqslant 0$ for every j, $\mathcal{E}(\lambda,\rho)$ has a fundamental system of solutions $(z_1,\dots,z_n)$ such that*

$$z_j=(x-t_j)^{\lambda_j}f_j(x-t_j)\quad (j=1,\dots,n),$$

where f_j is a polynomial of degree at most $(N-(\lambda_j-\rho))$, N denoting $\sum_{j=1}^{n}(\lambda_j-\rho)$.

Proof. — (i) Let $L(z)=0$ be the Pochhammer equation $\mathcal{E}(\lambda,\rho)$. Expanding every coefficient of L at $x=t_1$, we have

$$L(z)=\sum_{\ell=0}^{n}\Big(\sum_{k=0}^{\ell}p_{\ell k}(x-t_1)^k\Big)z^{(\ell)}.$$

To find a solution of $\mathcal{E}(\lambda,\rho)$ of exponent $1-\rho$ at $x=\infty$, we put

$$z=(x-t_1)^{\rho-1}\sum_{i=0}^{\infty}c_i(x-t_1)^{-i}$$

into $L(z) = 0$. Then we have

$$\begin{aligned} 0 &= L(z) \\ &= (x-t_1)^{\rho-1} \sum_{j=0}^{n-1} \Big\{ \sum_{i=0}^{\infty} g_j(i) c_i (x-t_1)^{-j-i} \Big\}, \end{aligned}$$

where

$$g_j(i) = \sum_{\ell=j}^{n} p_{\ell,\ell-j} (\rho - i - 1) \cdots (\rho - i - \ell) \quad (j = 0, \ldots, n).$$

(Note that $g_n(i) = 0$ for every i). Thus we have obtained an infinite system of linear equations

$$\begin{aligned} & g_0(0)c_0 = 0, \\ & g_0(1)c_1 + g_1(0)c_0 = 0, \\ & \qquad \vdots \\ & \sum_{j=0}^{n-1} g_j(s-j) c_{s-j} = 0. \\ & \qquad \vdots \end{aligned} \tag{1.19}$$

Set $n_1 = \rho - \lambda_1$, so that $n_1 \in \mathbf{N}$. Since $\mathcal{E}(\lambda, \rho)$ has a solution of exponent $\lambda_1 = \rho - n_1$ at $x = t_1$, from the indicial equation we obtain

$$(\rho - n_1)(\rho - n_1 - 1) \cdots (\rho - n_1 - n + 2)\{p_{n,1}(\rho - n_1 - n + 1) + p_{n-1,0}\} = 0.$$

By the assumption $\rho \notin \mathbf{Z}$, it follows that

$$p_{n,1}(\rho - n_1 - n + 1) + p_{n-1,0} = 0,$$

which gives

$$g_{n-1}(n_1 - 1) = 0.$$

Then the infinite system (1.19) has a system of solutions $(c_i)_{i=0}^{\infty}$ such that

$$c_{n_1 - 1} \neq 0, \quad c_i = 0 \quad \text{for any } i \geqslant n_1.$$

Hence $L(z) = 0$ has a special solution

$$\begin{aligned} z &= (x-t_1)^{\rho-1}(c_0 + c_1(x-t_1)^{-1} + \cdots + c_{n_1-1}(x-t_1)^{-n_1+1}) \\ &= (x-t_1)^{\rho-n_1}(c_{n_1-1} + c_{n_1-2}(x-t_1) + \cdots + c_0(x-t_1)^{n_1-1}). \end{aligned}$$

The similar holds for every j, and hence we obtain n solutions $z_1, \dots, z_n$ in the proposition. Every z_j is an algebraic function with branch points at t_j and ∞, so that $(z_1, \dots, z_n)$ is linearly independent.

(ii) If we consider a solution of exponent $-\rho' = -(\rho + N)$ at $x = \infty$:

$$z = (x - t_1)^{\rho+N} \sum_{i=0}^{\infty} c_i (x - t_1)^{-i},$$

the assertion (ii) is shown in a similar manner as (i). Q.e.d.

THEOREM 1.3. — *The monodromy group* $G(\lambda, \rho)$ *of a non-generic Pochhammer system* $\mathcal{P}(\lambda, \rho)$ *of rank* n *is finite if and only if one of the following eight conditions holds :*

(i) $\rho \notin \mathbf{Z},\ \rho - \lambda_j \in \mathbf{Z},\ \rho - \lambda_j > 0 \quad (j = 1, \dots, n)$,

(ii) $\rho \notin \mathbf{Z},\ \rho - \lambda_j \in \mathbf{Z},\ \rho - \lambda_j \leqslant 0 \quad (j = 1, \dots, n)$,

(iii) $\rho, \lambda_j \in \mathbf{Z},\ \lambda_j - \rho \geqslant 0 \quad (j = 1, \dots, n),\ \sum_{k=1}^{n} \lambda_k < (n-1)\rho$,

(iv) $\rho, \lambda_j \in \mathbf{Z},\ \lambda_j - \rho \geqslant 0 \quad (j = 1, \dots, n),\ \rho \geqslant 0$,

(v) $\rho, \lambda_j \in \mathbf{Z} \quad (j = 1, \dots, n),\ \lambda_k - \rho \geqslant 0$ for all but one $k = 1, \dots, n$, $\sum_{k=1}^{n} \lambda_k < (n-1)\rho$,

(vi) $\rho, \lambda_j \in \mathbf{Z},\ \lambda_j - \rho < 0 \quad (j = 1, \dots, n),\ \sum_{k=1}^{n} \lambda_k \geqslant (n-1)\rho$,

(vii) $\rho, \lambda_j \in \mathbf{Z},\ \lambda_j - \rho < 0 \quad (j = 1, \dots, n),\ \rho < 0$,

(viii) $\rho, \lambda_j \in \mathbf{Z} \quad (j = 1, \dots, n),\ \lambda_k - \rho < 0$ for all but one $k = 1, \dots, n$, $\sum_{k=1}^{n} \lambda_k \geqslant (n-1)\rho$.

Moreover, in the cases (i) *and* (ii), $G(\lambda, \rho)$ *is isomorphic to* n*-direct product of the cyclic group generated by* $e(\rho)$, *and in the cases* (iii) *to* (viii), $G(\lambda, \rho)$ *is the identity group.*

The proof will be completed in §3. Here we show that the conditions (i) - (viii) are sufficient conditions.

Suppose (i) or (ii). Then by Proposition 1.7, the corresponding Pochhammer equation has a finite monodromy group with generators

$$\begin{pmatrix} e(\rho) & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & e(\rho) & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & e(\rho) \end{pmatrix}.$$

Suppose one of (iii) to (viii). Then by Proposition 1.6 every singular point $x = t_j$ is apparent, so that any solution is meromorphic over $\mathbf{P}^1 \setminus \{\infty\} = \mathbf{C}$, and hence the monodromy group is the identity group.

2. Reduction of Pochhammer system modulo prime.

2.0. We consider a rational Pochhammer system $\mathcal{P}(\lambda, \rho)$; namely we assume (1.14). By definition $\mathcal{P}(\lambda, \rho)$ is a system over the differential field $\mathbf{Q}(t_1, \dots, t_n)(x)$. Throughout this paper we suppose

$$t_i \neq 0, \quad t_i \neq t_j \quad (1 \leqslant i, j \leqslant n,\ i \neq j).$$

Let m be the transcendence degree of $\mathbf{Q}(t_1, \dots, t_n)/\mathbf{Q}$. Take a transcendence basis $(\tau_1, \dots, \tau_m)$ contained in $\{t_1, \dots, t_n\}$.

Let D be the common denominator of the rational numbers $\lambda_1, \dots, \lambda_n,\ \rho$, and take a prime p satisfying

$$(D, p) = 1, \quad \rho \not\equiv \rho' \mod p, \tag{2.1}$$

where ρ' is defined by (1.5). Note that there are only finitely many primes which do not satisfy (2.1). The reduction modulo p

$$r_p : \mathbf{Q} \cap \mathbf{Z}_p \to \mathbf{F}_p$$

is extended to a homomorphism

$$r_p : (\mathbf{Q} \cap \mathbf{Z}_p)[\tau_1, \dots, \tau_m] \to \mathbf{F}_p[\tau_1, \dots, \tau_m]$$

by setting

$$r_p(\tau_i) = \tau_i \quad (1 \leqslant i \leqslant m).$$

Since $\mathbf{Q}(t_1, \dots, t_n)/\mathbf{Q}(\tau_1, \dots, \tau_m)$ is a finite algebraic extension, for almost all primes p the following holds :

(∗) r_p can be extended to homomorphisms of $(\mathbf{Q}\cap\mathbf{Z})[t_1,\dots,t_n]$, and for a extension (which is still denoted by r_p),

$$r_p(t_i)\neq 0,\quad r_p(t_i)\neq r_p(t_j)\quad (1\leqslant i,j\leqslant n,\ i\neq j) \tag{2.2}$$

hold.

In the following, for a prime p satisfying (2.1) and (∗), we fix an r_p satisfying (2.2). Now r_p is naturally extended to the homomorphism

$$r_p:(\mathbf{Q}\cap\mathbf{Z})[t_1,\dots,t_n,x]\{Y\}\to K_p(x)\{Y\},$$

where we have set

$$K_p=\mathbf{F}_p(r_p(t_1),\dots,r_p(t_n)).$$

Thus we obtain a system

$$\mathcal{P}(\lambda,\rho)_p=r_p(\mathcal{P}(\lambda,\rho))$$

over the differential field $K_p(x)$ of positive characteristic.

Define

$$R_p:\mathbf{Q}\cap\mathbf{Z}_p\to\mathbf{Z}$$

by

$$\begin{cases} r_p\circ R_p=r_p, \\ 0\leqslant R_p(\alpha)<p \quad\text{for}\quad \alpha\in\mathbf{Q}\cap\mathbf{Z}_p. \end{cases} \tag{2.3}$$

The main result in this section is the following

THEOREM 2.1. — *For a rational Pochhammer system $\mathcal{P}(\lambda,\rho)$ of rank n, take a prime p satisfying* (2.1) *and* (∗). *Then the reduced system $\mathcal{P}(\lambda,\rho)_p$ modulo p has n polynomial solutions in $K_p[x]$ of degrees at most $p-1$ which are linearly independent over the field of constants $K_p(x^p)$ if and only if one of the following two conditions holds :*

$$\text{(2.4 : i)}\quad R_p(\rho)\leqslant R_p(\lambda_j)\quad (j=1,\dots,n),\quad \sum_{j=1}^{n}R_p(\lambda_j)<(n-1)R_p(\rho)+p;$$

$$\text{(2.4 : ii)}\quad R_p(\lambda_j)<R_p(\rho)\quad (j=1,\dots,n),\quad (n-1)R_p(\rho)\leqslant\sum_{j=1}^{n}R_p(\lambda_j).$$

This section is devoted to the proof of Theorem 2.1. Here we explain the story.

Define $\tilde{R}_p(\lambda_1), \dots, \tilde{R}_p(\lambda_n),\ \tilde{R}_p(\rho) \in \mathbf{Z}$ by

$$\begin{aligned} \tilde{R}_p(\lambda_j) &= R_p(\lambda_j) + m_j p \quad (j = 1, \dots, n), \\ \tilde{R}_p(\rho) &= R_p(\rho), \end{aligned} \tag{2.5}$$

where $m_1, \dots, m_n \in \mathbf{Z}$ are so taken that

$$0 \leqslant \sum_{j=1}^{n} \tilde{R}_p(\lambda_j) - (n-1)\tilde{R}_p(\rho) < p \tag{2.6}$$

and fixed. It follows from (2.3) and (2.5) that

$$r_p(\mathcal{P}(\lambda, \rho)) = r_p(\mathcal{P}(\tilde{R}_p(\lambda), \tilde{R}_p(\rho))),$$

where $\tilde{R}_p(\lambda) = (\tilde{R}_p(\lambda_1), \dots, \tilde{R}_p(\lambda_n))$. Consider the intermediate system $\mathcal{P}(\tilde{R}_p(\lambda), \tilde{R}_p(\rho)))$ in stead of the reduced system $r_p(\mathcal{P}(\lambda, \rho)) = \mathcal{P}(\lambda, \rho)_p$. For simplicity we use $\lambda_1, \dots, \lambda_n, \rho$ for $\tilde{R}_p(\lambda_1), \dots, \tilde{R}_p(\lambda_n), \tilde{R}_p(\rho)$, respectively. Thus we consider the system

$$\tilde{\mathcal{P}}(\lambda, \rho) \qquad (x - T)Y' = A(\lambda, \rho)Y,$$

(2.7)

$$T = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}, \quad A(\lambda, \rho) = \begin{pmatrix} \lambda_1 & \lambda_1 - \rho & \cdots & \lambda_1 - \rho \\ \lambda_2 - \rho & \lambda_2 & \cdots & \lambda_2 - \rho \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_n - \rho & \lambda_n - \rho & \cdots & \lambda_n \end{pmatrix},$$

where we have assumed that

$$\lambda_1, \dots, \lambda_n, \ \rho \in \mathbf{Z}, \quad 0 \leqslant \rho, \rho' < p, \tag{2.8}$$

noticing that ρ' defined by (1.5) satisfies the above inequality because of (2.6).

In §2.1 we reduce the condition that $\tilde{\mathcal{P}}(\lambda, \rho)$ has n independent polynomial solutions, to a block triangularizability of some linear transformation L. §2.2 and 2.3 are devoted to the calculation of the elements which would become zero by the block triangularizaton of L. From the condition that these elements are zero, we extract conditions on $\lambda_1, \dots, \lambda_n, \rho$ in §2.4. Then finally in §2.5 we prove Theorem 2.1.

2.1. We denote by $V^n(K)$ the vector space of n-column vectors with entries in a field K. Set

$$Y = v_0 + v_1 x + \cdots + v_{p-1} x^{p-1}, \quad v_j \in V^n(\mathbf{Q}(t)) \ (j = 1, \ldots, n), \tag{2.9}$$

and put it into the system $\tilde{\mathcal{P}}(\lambda, \rho)$. Comparing the coefficients of the same power of x in both sides, we obtain

$$\begin{cases} [A-(p-1)]v_{p-1} = 0, \\ [A-(p-2)]v_{p-2} = (p-1)Tv_{p-1}, \\ [A-(p-3)]v_{p-3} = (p-2)Tv_{p-2}, \\ \quad \vdots \\ [A-1]v_1 = 2Tv_2, \\ Av_0 = Tv_1, \end{cases} \tag{2.10}$$

where we have denoted $A(\lambda, \rho)$ simply by A.

As we have seen in §1, (1.4), the eigen values of A are ρ of multiplicity $n-1$ and ρ'. First suppose that $\rho > \rho'$. Then from (2.10) it follows that

$$\begin{aligned} &v_{p-1} = \cdots = v_{\rho+1} = 0, \\ &v_\rho \in V_\rho, \\ &v_{\rho-1}, \ldots, v_{\rho'+1} : \text{ uniquely determined by } v_\rho, \end{aligned}$$

and

$$[A-\rho']v_{\rho'} = (\rho'+1)Tv_{\rho'+1}, \tag{2.11}$$

where V_ρ denotes the ρ-eigen space of A. Since ρ' is an eigen value of A, (2.11) requires that $Tv_{\rho'+1}$, which is uniquely determined by $v_\rho \in V_\rho$, is contained in the space spanned by the column vectors of the matrix $[A-\rho']$. Now we note a lemma from linear algebra.

LEMMA 2.1. — *Let A be an $n \times n$ matrix with entries in a field K. Suppose that A has just two distinct eigen values ρ_1 and ρ_2, of some multiplicities, in K, and that A is diagonalizable. Let V_1 be the ρ_1-eigen space of A. Then the space spanned by the column vectors of the matrix $[A-\rho_2]$ coincides with V_1.*

Hence we obtain from (2.11) that

$$Tv_{\rho'+1} \in V_\rho.$$

Now by (2.10) we have

$$(\rho' + 1)Tv_{\rho'+1} = (\rho' + 1)(\rho' + 2)\cdots \rho \cdot L_1 v_\rho, \tag{2.12}$$

where

$$L_1 = T[A - (\rho' + 1)]^{-1}T[A - (\rho' + 2)]^{-1}T \cdots T[A - (\rho - 1)]^{-1}T. \tag{2.13}$$

If $L_1 v_\rho \in V_\rho$ for any $v_\rho \in V_\rho$, then the system $\tilde{\mathcal{P}}(\lambda, \rho)$ has n linearly independent solutions of the form in (2.9). Thus we have proved : *The system $\tilde{\mathcal{P}}(\lambda, \rho)$ has n linearly independent solutions of the form in* (2.9) *if and only if V_ρ is an invariant subspace of L_1.*

Next suppose that $\rho' > \rho$. In a similar manner we obtain : *The system $\tilde{\mathcal{P}}(\lambda, \rho)$ has n linearly independent solutions of the form in* (2.9) *if and only if $V_{\rho'}$ is an invariant subspace of L_2*, where $V_{\rho'}$ is the ρ'-eigen space of A, and

$$L_2 = T[A - (\rho + 1)]^{-1}T[A - (\rho + 2)]^{-1}T \cdots T[A - (\rho' - 1)]^{-1}T. \tag{2.14}$$

Noticing that L_i $(i = 1, 2)$ is invertible, we see that the above statements hold if we replace L_i by ${L_i}^{-1}$ $(i = 1, 2)$. For later convenience we use ${L_i}^{-1}$. Since the matrix P defined in (1.3) diagonalizes A as (1.4), we can restate the above result in the following proposition.

PROPOSITION 2.1. — *The necessary and sufficient condition that the system $\tilde{\mathcal{P}}(\lambda, \rho)$ has n linearly independent polynomial solutions of degree at most $p - 1$ is,*

(i) *if $\rho > \rho'$,*

$$P^{-1}{L_1}^{-1}P = \begin{pmatrix} & & * \\ & * & \vdots \\ & & * \\ 0 & \cdots\ 0 & * \end{pmatrix}; \tag{2.15}$$

(ii) *if $\rho' > \rho$,*

$$P^{-1}{L_2}^{-1}P = \begin{pmatrix} & & 0 \\ & * & \vdots \\ & & 0 \\ * & \cdots\ * & * \end{pmatrix}, \tag{2.16}$$

where P, L_1 and L_2 are defined in (1.3), (2.13) *and* (2.14), *respectively.*

2.2.

Notation.

(i) Let $\ell \in \mathbf{N}$ and $m \in \mathbf{N}_0$. For $i_1, \dots, i_\ell \in \mathbf{N}_0$ satisfying $i_1 + \cdots + i_\ell \leqslant m$,

$$\begin{pmatrix} & m & \\ i_1 & \cdots & i_\ell \end{pmatrix} = \frac{m!}{i_1! \cdots i_\ell! (m - i_1 - \cdots - i_\ell)!}.$$

(ii) For $\alpha \in \mathbf{C}$ and $i \in \mathbf{N}$,

$$(\alpha, 0) = 1,$$
$$(\alpha, i) = \alpha(\alpha + 1) \cdots (\alpha + i - 1).$$

We introduce a polynomial which plays a central role in this section.

DEFINITION 2.1. — *Using the above notations, we define a polynomial* $\xi_j^{(\ell,m)}(\mu; u)$ *of* $u = (u_1, \dots, u_\ell)$ *with parameters* $\mu = (\mu_1, \dots, \mu_\ell)$ *by*

$$\xi_j^{(\ell,m)}(\mu; u) = \sum_{\substack{i_1 + \cdots + i_\ell = m \\ i_1, \dots, i_\ell \in \mathbf{N}_0}} \begin{pmatrix} & m & \\ i_1 & \cdots & i_\ell \end{pmatrix} \prod_{k=1}^{\ell} (\mu_k + \delta_{kj}, i_k) \cdot {u_1}^{i_1} \cdots {u_\ell}^{i_\ell}, \tag{2.17}$$

for $j = 1, \dots, \ell$, *where* δ_{kj} *denotes Kronecker's delta.*

Let $\rho > \rho'$. Recalling Proposition 2.1, (i), we proceed to obtain the (n, j)-entry of the matrix

$$M = P^{-1} {L_1}^{-1} P \tag{2.18}$$

for $j = 1, \dots, n-1$, where P and L_1 are as in (1.3) and (2.13), respectively.

PROPOSITION 2.2. — *Let* $\rho > \rho'$. *Then, for* $j = 1, \dots, n-1$, *the* (n, j)*-entry of the matrix* M *defined by* (2.18) *is*

$$-\frac{1}{m} u_j \xi_j^{(n-1,m-1)}(\mu; u), \tag{2.19}$$

where

$$\begin{aligned} & m = \rho - \rho' > 0, \\ & \mu = (\mu_1, \dots, \mu_{n-1}), \quad \mu_k = \lambda_k - \rho \quad (k = 1, \dots, n-1), \\ & u = (u_1, \dots, u_{n-1}), \quad u_k = {t_k}^{-1} - {t_n}^{-1} \quad (k = 1, \dots, n-1). \end{aligned} \tag{2.20}$$

Proof. — Using (1.4), (2.13), (2.15) and (2.18), we obtain

$$\begin{aligned} M &= P^{-1}{L_1}^{-1}P \\ &= P^{-1}T^{-1}[A-(\rho-1)]T^{-1}\cdots T^{-1}[A-(\rho'+1)]T^{-1}P \\ &= Q[1-mN]Q[2-mN]Q\cdots Q[(m-1)-mN]Q, \end{aligned} \tag{2.21}$$

where $m=\rho-\rho'$,

$$Q = P^{-1}T^{-1}P \tag{2.22}$$

and

$$N = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & 1 \end{pmatrix}. \tag{2.23}$$

Define Q_1 by

$$Q = Q_1 + {t_n}^{-1}I, \tag{2.24}$$

then by (2.7) and (2.20) we obtain

$$Q_1 = P^{-1}\begin{pmatrix} u_1 & & & \\ & \ddots & & \\ & & u_{n-1} & \\ & & & 0 \end{pmatrix}P. \tag{2.25}$$

Put (2.24) into (2.21) :
(2.26)
$$M = (Q_1+{t_n}^{-1})[1-mN](Q_1+{t_n}^{-1})\cdots(Q_1+{t_n}^{-1})[(m-1)-mN](Q_1+{t_n}^{-1}).$$

In general, let $\mathcal{U}$ be a K-module, K being a field, of matrices in $\mathrm{M}(n;K)$ whose (n,j)-entries are 0 for $j=1,\dots,n-1$. For $B_1,B_2\in\mathrm{M}(n;K)$, we denote $B_1\equiv B_2 \bmod \mathcal{U}$ if $B_1-B_2\in\mathcal{U}$. The following holds.

LEMMA 2.2. — *N being as in (2.23), for any $B\in\mathrm{M}(n;K)$ and any $s\in K$,*
(2.27)
$$\begin{aligned} &(B+s)[1-mN](B+s)[2-mN](B+s)\cdots(B+s)[(m-1)-mN](B+s) \\ &\equiv B[1-mN]B[2-mN]B\cdots B[(m-1)-mN]B \mod \mathcal{U} \end{aligned}$$

holds.

Define M_1 by

$$M_1 = Q_1[1 - mN]Q_1[2 - mN]Q_1 \cdots Q_1[(m-1) - mN]Q_1, \tag{2.28}$$

then by (2.26) and Lemma 2.2 we see that the (n, j)-entry of M is equal to the (n, j)-entry of M_1 for $j = 1, \dots, n-1$.

By (2.25) and (1.3), we can rewrite Q_1 as

$$Q_1 = \frac{1}{m} Q_2 + Q_3 \tag{2.29}$$

with

$$Q_2 = \begin{pmatrix} \mu_1 u_1 & \cdots & \mu_1 u_{n-1} & \mu_1 \theta \\ \vdots & \cdots & \vdots & \vdots \\ \mu_{n-1} u_1 & \cdots & \mu_{n-1} u_{n-1} & \mu_{n-1}\theta \\ -u_1 & \cdots & -u_{n-1} & -\theta \end{pmatrix}, \tag{2.30}$$

$$Q_3 = \begin{pmatrix} u_1 & & & \mu_1 u_1 \\ & \ddots & & \vdots \\ & & u_{n-1} & \mu_{n-1} u_{n-1} \\ 0 & \cdots & 0 & 0 \end{pmatrix},$$

where μ_k and u_k are defined in (2.20), and

$$\theta = \sum_{k=1}^{n-1} \mu_k u_k. \tag{2.31}$$

For later use we introduce more notations :

$$\theta^{(i)} = \sum_{k=1}^{n-1} \mu_k {u_k}^i, \tag{2.32}$$

$$Q_2^{(i)} = \begin{pmatrix} \mu_1 {u_1}^i & \cdots & \mu_1 {u_{n-1}}^i & \mu_1 \theta^{(i)} \\ \vdots & \cdots & \vdots & \vdots \\ \mu_{n-1} {u_1}^i & \cdots & \mu_{n-1} {u_{n-1}}^i & \mu_{n-1}\theta^{(i)} \\ -{u_1}^i & \cdots & -{u_{n-1}}^i & -\theta^{(i)} \end{pmatrix}, \tag{2.33}$$

$$Q_3^{(i)} = \begin{pmatrix} {u_1}^i & & & \mu_1 {u_1}^i \\ & \ddots & & \vdots \\ & & {u_{n-1}}^i & \mu_{n-1} {u_{n-1}}^i \\ 0 & \cdots & 0 & 0 \end{pmatrix} \tag{2.34}$$

for $i = 1, 2, \dots$. In particular $Q_2 = Q_2^{(1)}$ and $Q_3 = Q_3^{(1)}$. Between N, the $Q_2^{(i)}$'s and the $Q_3^{(i)}$'s, there are several relations.

LEMMA 2.3. — *For any $n \times n$ matrix B and any $i, j \in \mathbf{N}$,*

$$\begin{aligned} Q_3^{(i)} \cdot B &\equiv O \mod \mathcal{U}, \\ N \cdot Q_3^{(i)} &= O, \\ Q_2^{(i)} \cdot Q_2^{(j)} &= O, \\ Q_3^{(i)} \cdot Q_2^{(j)} &= O, \\ Q_2^{(i)} \cdot Q_3^{(j)} &= Q_2^{(i+j)}, \\ Q_2^{(i)} \cdot N \cdot Q_2^{(j)} &= -\theta^{(i)} Q_2^{(j)}. \end{aligned}$$

Using this lemma, we obtain

LEMMA 2.4.

$$\begin{aligned} M_1 \equiv \frac{1}{m} Q_2(1-N)(Q_2+Q_3)(2-N) \\ \times (Q_2+Q_3) \cdots (Q_2+Q_3)((m-1)-N)(Q_2+Q_3) \mod \mathcal{U}. \end{aligned}$$

Then set

$$M^{(m)} = Q_2(1-N)(Q_2+Q_3)(2-N)(Q_2+Q_3)\cdots (Q_2+Q_3)((m-1)-N)(Q_2+Q_3). \tag{2.35}$$

For $I = (i_1, i_2, \ldots) \in \mathbf{N}_0{}^\infty$ with $i_k = 0$ for any sufficiently large k, we define

$$\begin{aligned} ||I|| &= \sum_{k=1}^{\infty} k i_k, \\ \theta^I &= \prod_{k=1}^{\infty} (\theta^{(k)})^{i_k}. \end{aligned} \tag{2.36}$$

By Lemma 2.3, we see that $M^{(m)}$ has the following expansion :

$$M^{(m)} = \sum_{||I||+k=m} c_{Ik}^{(m)} \theta^I Q_2^{(k)}. \tag{2.37}$$

The following recurrence formulas of the $c_{Ik}^{(m)}$'s are shown by using (2.35) and Lemma 2.3.

LEMMA 2.5.

$$\begin{aligned} c_{I1}^{(m+1)} &= \sum_{J+1_k=I} c_{Jk}^{(m)}, \quad \text{if } ||I|| = m, \\ c_{I,m+1-||I||}^{(m+1)} &= m \cdot c_{I,m-||I||}^{(m)}, \quad \text{if } ||I|| < m, \end{aligned}$$

where 1_k denotes the element of $\mathbf{N}_0{}^\infty$ with the only non-zero entry 1 in the k-th position.

In the above we have shown that

$$M \equiv M_1 \equiv \frac{1}{m} M^{(m)} \mod \mathcal{U}.$$

Therefore it suffices to show that the (n, j)-entry of $M^{(m)}$ is $-u_j \xi_j^{(n-1,m-1)}$ $(\mu; u)$ for $j = 1, \dots, n-1$. Taking account of (2.37) and (2.33), we prove that

$$\sum_{||I||+k=m} c_{Ik}^{(m)} \theta^I (-u_j{}^k) = -u_j \xi_j^{(n-1,m-1)}(\mu; u) \tag{2.38}$$

for $j = 1, \dots, n-1$ by induction on m. When $m = 1$, $M^{(1)} = Q_2$, and hence (2.38) follows from (2.17), (2.30) and (2.37). Suppose that (2.38) for every $j \in \{1, \dots, n-1\}$ holds for m. Then, by Lemma 2.5, for every $j \in \{1, \dots, n-1\}$ we have

$$\begin{aligned}
&\sum_{||I||+k=m+1} c_{Ik}^{(m+1)} \theta^I (-u_j{}^k) \\
&= \sum_{||I||=m} c_{I1}^{(m+1)} \theta^I (-u_j) + \sum_{||I||<m} c_{I,m+1-||I||}^{(m+1)} \theta^I (-u_j{}^{m+1-||I||}) \\
&= \sum_{||J||+k=m} c_{Jk}^{(m)} \theta^{J+1_k} (-u_j) + m \sum_{||J||+k=m} c_{Jk}^{(m)} \theta^J (-u_j{}^{k+1}) \\
&= -u_j \Big[\sum_{||J||+k=m} c_{Jk}^{(m)} \theta^J \theta^{(k)} + m \sum_{||J||+k=m} c_{Jk}^{(m)} \theta^J u_j{}^k \Big] \\
&= -u_j \Big[\sum_{||J||+k=m} c_{Jk}^{(m)} \theta^J (\mu_1 u_1{}^k + \cdots \mu_{n-1} u_{N-1}{}^k) + m \sum_{||J||+k=m} c_{Jk}^{(m)} \theta^J u_j{}^k \Big] \\
&= -u_j \Big[\mu_1 u_1 \xi_1^{(n-1,m-1)}(\mu; u) + \cdots \\
&\qquad + \mu_{n-1} u_{n-1} \xi_{n-1}^{(n-1,m-1)}(\mu; u) + m u_j \xi_j^{(n-1,m-1)}(\mu; u) \Big] \\
&= -u_j \xi_j^{(n-1,m)}(\mu; u),
\end{aligned}$$

where the last equality follows from Lemma 2.9 which will be given later. Thus (2.38) holds for $m+1$. This completes the proof of Proposition 2.2.

Q.e.d.

2.3. Here we give a similar proposition to Proposition 2.2 when $\rho' > \rho$. Notice that, in the following, we use several letters common to §2.2 in different senses.

Let $\rho' > \rho$, and set

$$M = P^{-1}{L_2}^{-1}P, \tag{2.39}$$

where P and L_2 are defined in (1.3) and (2.14), respectively.

PROPOSITION 2.3. — *Let $\rho' > \rho$. For $j = 1, \dots, n-1$ the (j, n)-entry of the matrix M defined by* (2.39) *is*

$$\frac{(-1)^m}{m}\mu_j \xi_j^{(n-1,m)}(\mu; u), \tag{2.40}$$

where

$$\begin{aligned} &m = \rho' - \rho > 0, \\ &\mu = (\mu_1, \dots, \mu_{n-1}), \qquad \mu_k = \rho - \lambda_k \quad (k = 1, \dots, n-1), \\ &u = (u_1, \dots, u_{n-1}), \qquad u_k = {t_k}^{-1} - {t_n}^{-1} \quad (k = 1, \dots, n-1). \end{aligned} \tag{2.41}$$

The proof is similar to (and somewhat simpler than) that of Proposition 2.2, and is omitted.

2.4. Now we study common zeros of the polynomials $\xi_j^{(\ell,m)}$'s.

Let K be a field of characteristic 0, and let $\ell \in \mathbf{N}$, $m \in \mathbf{N}_0$. Define $(K^\ell)^* \subset K^\ell$ by

$$(K^\ell)^* = \{u = (u_1, \dots, u_\ell) \in K^\ell \mid u_1, \dots, u_\ell, 0 \text{ are mutually distinct}\}.$$

We consider $\xi_j^{(\ell,m)}(\mu; u)$ as a polynomial in $(\mu; u) \in K^\ell \times (K^\ell)^*$.

PROPOSITION 2.4. — *Consider the system*

$$\xi_1^{(\ell,m)}(\mu; u) = \cdots = \xi_\ell^{(\ell,m)}(\mu; u) = 0. \tag{2.42}$$

(i) *When $m < \ell$,* (2.42) *has no solution in $K^\ell \times (K^\ell)^*$.*

(ii) *When $m \geqslant \ell$,* (2.42) *holds if and only if $\mu = (\mu_1, \dots, \mu_\ell)$ satisfies the following :*

$$\begin{aligned} &-\mu_1, \dots, -\mu_\ell \in \mathbf{N}, \\ &0 \leqslant m + \sum_{j=1}^{\ell} \mu_j \leqslant m - \ell. \end{aligned} \tag{2.43}$$

To prove this, we need two lemmas. Let $m \geqslant 1$.

LEMMA 2.9.

$$\xi_j^{(\ell,m)}(\mu;u) = (\mu_j + m)u_j\xi_j^{(\ell,m-1)}(\mu;u) + \sum_{k\neq j}\mu_k u_k \xi_k^{(\ell,m-1)}(\mu;u)$$

for $j = 1, \dots, \ell$.

This is shown by comparing coefficients of monomials of the u_j's in both sides.

LEMMA 2.10. — *For* $s = 1, \dots, \ell - 1$,

$$\begin{aligned}(2.44)\qquad &[u_s\xi_s^{(\ell,m-1)}(\mu;u) - u_\ell\xi_\ell^{(\ell,m-1)}(\mu;u)]|_{\mu_\ell=-(\mu_1+\cdots+\mu_{\ell-1}+m)} \\ &= (u_s - u_\ell)\xi_s^{(\ell-1,m-1)}(\mu';u'-u_\ell)\end{aligned}$$

where $\mu' = (\mu_1, \dots, \mu_{\ell-1})$ *and* $u' - u_\ell = (u_1 - u_\ell, \dots, u_{\ell-1} - u_\ell)$.

Proof. — We show the lemma for $s = 1$. The assertion for any s is obtained similarly.

Expand the right hand side of (2.44) as

$$\begin{aligned}&(u_1 - u_\ell)\xi_1^{(\ell-1,m-1)}(\mu';u'-u_\ell) \\ &= \sum_{i_1+\cdots+i_{\ell-1}=m-1}\binom{m-1}{i_1\ \cdots\ i_{\ell-1}}(\mu_1+1,i_1)(\mu_2,i_2)\cdots(\mu_{\ell-1},i_{\ell-1}) \\ &\qquad\times(u_1-u_\ell)^{i_1+1}(u_2-u_\ell)^{i_2}\cdots(u_{\ell-1}-u_\ell) \\ &= \sum_{j_1+\cdots+j_\ell=m} c_{j_1\dots j_\ell}{u_1}^{j_1}\cdots{u_\ell}^{j_\ell},\end{aligned}$$

then we obtain

$$\begin{aligned}c_{j_1\dots j_\ell} = &\sum_{i_1\geqslant j_1-1, i_2\geqslant j_2,\dots,i_{\ell-1}\geqslant j_{\ell-1}}\binom{m-1}{i_1\ \cdots\ i_{\ell-1}}\binom{i_1+1}{j_1}\binom{i_2}{j_2}\cdots\binom{i_{\ell-1}}{j_{\ell-1}} \\ &\times(\mu_1+1,i_1)(\mu_2,i_2)\cdots(\mu_{\ell-1},i_{\ell-1})(-1)^{j_\ell}.\end{aligned}$$

On the other hand, μ_ℓ being $-(\mu_1+\cdots+\mu_{\ell-1}+m)$, expand the left hand side of (2.44) as

$$\begin{aligned}
&u_1\xi_1^{(\ell,m-1)}(\mu;u) - u_\ell\xi_\ell^{(\ell,m-1)}(\mu;u)\\
&= \sum_{i_1+\cdots i_\ell=m-1} \binom{m-1}{i_1\ \cdots\ i_{\ell-1}}(\mu_1+1,i_1)(\mu_2,i_2)\cdots(\mu_{\ell-1},i_{\ell-1})\\
&\quad\times(-(\mu_1+\cdots+\mu_{\ell-1}+m),i_\ell)u_1{}^{i_1+1}u_2{}^{i_2}\cdots u_\ell{}^{i_\ell}\\
&\quad-\sum_{i_1+\cdots i_\ell=m-1} \binom{m-1}{i_1\ \cdots\ i_{\ell-1}}(\mu_1,i_1)(\mu_2,i_2)\cdots(\mu_{\ell-1},i_{\ell-1})\\
&\quad\times(-(\mu_1+\cdots+\mu_{\ell-1}+m)+1,i_\ell)u_1{}^{i_1}\cdots u_{\ell-1}{}^{i_{\ell-1}}u_\ell{}^{i_\ell+1}\\
&= \sum_{j_1+\cdots+j_\ell=m} b_{j_1\dots j_\ell}u_1{}^{j_1}\cdots u_\ell{}^{j_\ell}.
\end{aligned}$$

Note that $b_{0j_2\dots j_{\ell-1}0}=0$, so that we assume $j_1\geqslant 1$ when $j_\ell=0$ and $j_\ell\geqslant 1$ when $j_1=0$. Then the coefficients $b_{j_1\dots j_\ell}$'s are obtained as follows :

$$\begin{aligned}
b_{j_1\dots j_{\ell-1}0} &= \binom{m-1}{j_1-1\ j_2\ \cdots\ j_{\ell-1}}(\mu_1+1,j_1-1)(\mu_2,j_2)\cdots(\mu_{\ell-1},j_{\ell-1}),\\
b_{0j_2\dots j_\ell} &= \binom{m-1}{j_2\ \cdots\ j_{\ell-1}\ j_\ell-1}(\mu_2,j_2)\cdots(\mu_{\ell-1},j_{\ell-1})\\
&\quad\times(-(\mu_1+\cdots+\mu_{\ell-1}+m)+1,j_\ell-1),
\end{aligned}$$

and when $j_1\geqslant 1$, $j_\ell\geqslant 1$,

$$\begin{aligned}
b_{j_1\dots j_\ell} &= \binom{m-1}{j_1-1\ j_2\ \cdots\ j_\ell}(\mu_1+1,j_1-1)(\mu_2,j_2)\cdots\\
&\qquad(\mu_{\ell-1},j_{\ell-1})(-(\mu_1+\cdots+\mu_{\ell-1}+m),j_\ell)\\
&\quad-\binom{m-1}{j_1\ \cdots\ j_{\ell-1}\ j_\ell-1}(\mu_1,j_1)\cdots\\
&\qquad(\mu_{\ell-1},j_{\ell-1})(-(\mu_1+\cdots+\mu_{\ell-1}+m)+1,j_\ell-1).
\end{aligned}$$

SUBLEMMA.

$$\begin{aligned}
&(\mu_1,j_1)\cdots(\mu_{\ell-1},j_{\ell-1})(\mu_1+\cdots+\mu_{\ell-1}+j_1+\cdots+j_{\ell-1},j_\ell)\\
&=\sum_{k_1+\cdots+k_{\ell-1}=j_\ell}\binom{j_\ell}{k_1\ \cdots\ k_{\ell-1}}(\mu_1,j_1+k_1)\cdots(\mu_{\ell-1},j_{\ell-1}+k_{\ell-1}).
\end{aligned}$$

This sublemma is shown by induction on j_ℓ.

Use Sublemma to reduce every $b_{j_1\dots j_\ell}$ to $c_{j_1\dots j_\ell}$, then the lemma follows. Q.e.d.

Proof of Proposition 2.4. — Suppose that $m > 0$. Then from Lemma 2.9 we obtain

$$\begin{pmatrix} \xi_1^{(\ell,m)}(\mu;u) \\ \xi_2^{(\ell,m)}(\mu;u) \\ \vdots \\ \xi_\ell^{(\ell,m)}(\mu;u) \end{pmatrix} = \begin{pmatrix} \mu_1+m & \mu_2 & \cdots & \mu_\ell \\ \mu_1 & \mu_2+m & \cdots & \mu_\ell \\ \vdots & \vdots & \ddots & \vdots \\ \mu_1 & \mu_2 & \cdots & \mu_\ell+m \end{pmatrix} \begin{pmatrix} u_1\xi_1^{(\ell,m-1)}(\mu;u) \\ u_2\xi_2^{(\ell,m-1)}(\mu;u) \\ \vdots \\ u_\ell\xi_\ell^{(\ell,m-1)}(\mu;u) \end{pmatrix}. \tag{2.45}$$

Using the notation in (1.2), we see that the coefficient matrix in the right hand side of (2.45) is ${}^tA(\mu+m,m)$, where $\mu+m=(\mu_1+m,\dots,\mu_\ell+m)$. As we have seen in §1.0, (1.4), the eigen values of ${}^tA(\mu+m,m)$ are m and $\sum_{j=1}^{\ell}\mu_j+m$. Since $m\neq 0$, the left hand side of (2.45) is zero if and only if one of the followings holds :

$$u_1\xi_1^{(\ell,m-1)}(\mu;u)=\cdots=u_\ell\xi_\ell^{(\ell,m-1)}(\mu;u)=0,$$

or

$$\begin{cases} \mu_1+\cdots\mu_\ell+m=0, \\ u_1\xi_1^{(\ell,m-1)}(\mu;u)=\cdots=u_\ell\xi_\ell^{(\ell,m-1)}(\mu;u). \end{cases}$$

By Lemma 2.10, the latter condition is equivalent to

$$(u_1-u_\ell)\xi_1^{(\ell-1,m-1)}(\mu';u'-u_\ell)=\cdots=(u_{\ell-1}-u_\ell)\xi_{\ell-1}^{(\ell-1,m-1)}(\mu';u'-u_\ell)=0$$

with $\mu_1+\cdots\mu_\ell+m=0$. Noting that $u=(u_1,\dots,u_\ell)\in(K^\ell)^*$, we obtain

$$\xi_1^{(\ell,m-1)}(\mu;u)=\cdots=\xi_\ell^{(\ell,m-1)}(\mu;u)=0, \tag{2.46}$$

or

$$\begin{cases} \mu_1+\cdots\mu_\ell+m=0, \\ \xi_1^{(\ell-1,m-1)}(\mu';u'-u_\ell)=\cdots=\xi_{\ell-1}^{(\ell-1,m-1)}(\mu';u'-u_\ell). \end{cases} \tag{2.47}$$

First we show the assertion (i) of the proposition by induction on ℓ and m. Assume that (i) holds for $\ell-1$ and for every m less than $\ell-1$. We shall prove (i) for ℓ and for every m less than ℓ. For $m=0$, every $\xi_j^{(\ell,0)}(\mu;u)$ is equal to 1, so that (i) holds. Assume that (i) holds for $m-1$, and consider the system (2.42) for (ℓ, m). As we have observed in the above, from (2.42) we obtain (2.46) or (2.47). By the induction assumption on m, we see that (2.46) has no solution in $K^\ell \times (K^\ell)^*$. Noting that $\ell - 1 > m - 1$, we see that (2.47) has no solution in $K^{\ell-1} \times (K^{\ell-1})^*$ by the induction assumption on ℓ. Hence (2.42) has no solution in $K^\ell \times (K^\ell)^*$, and this completes the induction.

Next we prove the assertion (ii) by induction on ℓ. For $\ell = 1$ and $m \geqslant 1$, (2.42) is reduced to

$$\xi_1^{(1,m)}(\mu; u) = (\mu_1 + 1, m){u_1}^m = 0,$$

so that we have $(\mu_1 + 1, m) = 0$ since $u_1 \neq 0$. Thus (ii) holds in this case. Assume that (ii) holds for $\ell - 1$, and consider the system (2.42) for (ℓ, m) with $m \geqslant \ell$. Denote $\sum_{j=1}^{\ell} \mu_j$ by M. Again by the above argument, we see that (2.42) holds if and only if (2.46) or (2.47) holds. From (2.47) and the induction assumption, we obtain

$$\begin{aligned} &M + m = 0, \\ &-\mu_1, \dots, -\mu_{\ell-1} \in \mathbf{N}, \\ &\sum_{j=1}^{\ell-1} \mu_j + m \geqslant 0, \end{aligned}$$

which is (2.43) with $M + m = 0$. From (2.46) we obtain (2.46) for $(\ell, m-2)$ or (2.47) for $(\ell-1, m-2)$. The latter gives (2.43) with $M + (m-1) = 0$, and the former gives (2.46) for $(\ell, m-3)$ or (2.47) for $(\ell-1, m-3)$. Proceeding recursively, we have (2.43) with $M + m = 0, 1, \dots, m - \ell$ and (2.47) for $(\ell, \ell - 1)$. Using the assertion (i), we see that (2.46) for $(\ell, \ell - 1)$ has no solution. Hence (2.43) with $M + m = 0, 1, \dots, m - \ell$ exhaust all possibilities, which establishes the assertion (ii). Q.e.d.

Now we return to the system $\tilde{\mathcal{P}}(\lambda, \rho)$. If we define $u = (u_1, \dots, u_{n-1})$ by $u_j = {t_j}^{-1} - {t_n}^{-1}$ for $j = 1, \dots, n-1$ as in (2.20) or (2.41), we see that

$$u = (u_1, \dots, u_{n-1}) \in (\mathbf{C}^{n-1})^*,$$

since $t_1, \dots, t_n$ are distinct. Then from Propositions 2.1, 2.2, 2.3 and 2.4, we obtain the following

PROPOSITION 2.5. — *The necessary and sufficient condition that the system $\tilde{\mathcal{P}}(\lambda, \rho)$ has n linearly independent polynomial solutions of degree at most $p - 1$ is,*

(i) *if* $\rho > \rho'$,

$$\lambda_j < \rho \quad \text{for } j = 1, \ldots, n, \tag{2.48}$$

(ii) *if* $\rho' > \rho$,

$$\rho \leqslant \lambda_j \quad \text{for } j = 1, \ldots, n, \tag{2.49}$$

where $\rho' = \sum\limits_{j=1}^{n} \lambda_j - (n-1)\rho$.

Proof. — Suppose that $\rho > \rho'$, and set $m = \rho - \rho'$. Then by Propositions 2.1 and 2.2, we see that the condition is

$$\xi_1^{(n-1,m-1)}(\mu, u) = \cdots = \xi_{n-1}^{(n-1,m-1)}(\mu, u) = 0,$$

where $\mu = (\lambda_1 - \rho, \ldots, \lambda_{n-1} - \rho)$. Applying Proposition 2.4 to this system, we obtain

$$\lambda_j < \rho \quad \text{for } j = 1, \ldots, n-1,$$
$$0 \leqslant \sum_{j=1}^{n-1} (\lambda_j - \rho) + (\rho - \rho' - 1).$$

(Note that every λ_j and ρ are integers.) From the last inequality it follows $\lambda_n < \rho$. Thus we obtain (2.48).

Next suppose that $\rho' > \rho$, and set $m = \rho' - \rho$. By Propositions 2.1 and 2.3, the condition is reduced to

$$\mu_1 \xi_1^{(n-1,m)}(\mu; u) = \cdots = \mu_{n-1} \xi_{n-1}^{(n-1,m)}(\mu; u) = 0,$$

where $\mu = (\rho - \lambda_1, \ldots, \rho - \lambda_{n-1})$. Similarly to the proof of Proposition 2.4, from this system we obtain

$$\rho \leqslant \lambda_j \quad \text{for } j = 1, \ldots, n-1,$$
$$0 \leqslant \sum_{j=1}^{n-1} (\rho - \lambda_j) + (\rho' - \rho).$$

Then $\rho \leqslant \lambda_n$ follows from the last inequality, so that we obtain (2.49).

Q.e.d.

2.5. *Proof of Theorem 2.1.* — Recalling §2.0 and 2.1, we see that, $\tilde{\mathcal{P}}(\lambda, \rho)$ has n polynomial solutions of degree at most $p-1$ which are linearly independent over the field of constants, if and only if so does $\mathcal{P}(\lambda, \rho)_p$. The former assertion is reduced to conditions on $\lambda_1, \dots, \lambda_n, \rho$ by Proposition 2.5. Noting that there we have used $\lambda_1, \dots, \lambda_n, \rho$ for $\tilde{R}_p(\lambda_1), \dots, \tilde{R}_p(\lambda_n), \tilde{R}_p(\rho)$, respectively, and using (2.5), we rewrite the conditions in the proposition as

(i)

$$R_p(\rho) > \sum_{j=1}^{n}(R_p(\lambda_j) + m_j p) - (n-1)R_p(\rho), \tag{2.50}$$

$$R_p(\lambda_j) + m_j p < R_p(\rho) \quad (j = 1, \dots, n); \text{ or} \tag{2.51}$$

(ii)

$$\sum_{j=1}^{n}(R_p(\lambda_j) + m_j p) - (n-1)R_p(\rho) > R_p(\rho), \tag{2.52}$$

$$R_p(\rho) \leqslant R_p(\lambda_j) + m_j p \quad (j = 1, \dots, n). \tag{2.53}$$

First we study the case (i). We show that $m_j = 0$ for every j. From (2.3) and (2.51) we obtain

$$p > R_p(\rho) > R_p(\lambda_j) + m_j p \geqslant m_j p,$$

so that $1 > m_j$. Set

$$m = R_p(\rho) - \left(\sum_{j=1}^{n}(R_p(\lambda_j) + m_j p) - (n-1)R_p(\rho)\right),$$

then we obtain

$$0 < m < p \tag{2.54}$$

from (2.3), (2.6) and (2.50). Now (2.51) and (2.54) yields

$$0 < R_p(\rho) - (R_p(\lambda_j) + m_j p) < p.$$

Hence we have

$$-p < R_p(\rho) - R_p(\lambda_j) < (m_j + 1)p,$$

so that $m_j > -2$. Thus $m_j = 0$ or -1 for every j. Set

$$J = \{j \in \{1, \dots, n\} \mid m_j = -1\},$$

and denote the cardinal number of J by ℓ. If $j \notin J$, $m_j = 0$ and then we have $R_p(\rho) - R_p(\lambda_j) > 0$ by (2.51). Since $R_p(\rho) \geqslant m$ by (2.50) and $R_p(\lambda_j) < p$ by (2.3), in general we have $R_p(\rho) - R_p(\lambda_j) > m - p$. Using the above, we obtain

$$\begin{aligned} m &= \sum_{j=1}^{n}(R_p(\rho) - (R_p(\lambda_j) + m_j p)) \\ &= \sum_{j=1}^{n}(R_p(\rho) - R_p(\lambda_j)) - \left(\sum_{j=1}^{n} m_j\right) p \\ &= \sum_{j\in J}(R_p(\rho) - R_p(\lambda_j)) + \sum_{j\notin J}(R_p(\rho) - R_p(\lambda_j)) + \ell p \\ &> \ell(m-p) + \ell p \\ &= \ell m, \end{aligned}$$

and hence $1 > \ell$, for $m > 0$. Since $0 \leqslant \ell \leqslant n$ by the definition, thus we have $\ell = 0$; namely $m_j = 0$ for every j. Put $m_j = 0$ into (2.51) and (2.6), then we obtain (2.4 : ii).

For the case (ii), in a similar manner we obtain $m_j = 0$ for every j. Then (2.4 : i) follows from (2.53) and (2.6) with $m_j = 0$.

Conversely, if (2.4 : i or ii) holds, we can take $m_j = 0$ for every j in order that (2.6) holds. Then the condition in Proposition 2.5 follows from (2.4 : i or ii), so that the system $\tilde{\mathcal{P}}(\lambda, \rho)$, and hence the system $\mathcal{P}(\lambda, \rho)_p$, has n linearly independent polynomial solutions. This completes the proof of Theorem 2.1. Q.e.d.

2.6. Here we prove Proposition 1.6 in §1.2 by using the results which we have obtained in this section. We consider the Pochhammer system $\mathcal{P}(\lambda, \rho)$ with parameters $(\lambda, \rho) = (\lambda_1, \dots, \lambda_n, \rho) \in \mathbf{C}^{n+1}$ satisfying (1.1). Direct computation shows the following

LEMMA 2.11. — *Suppose that*

$$\lambda_1 - \rho \neq 0, \quad \rho' = \sum_{i=1}^{n} \lambda_i - (n-1)\rho \neq 0.$$

Then by the change of the independent variable

$$x \to u: \quad u = \frac{1}{x - t_1} + s_1$$

and by the gauge transformation

$$Y \to Z: \quad Z = (u - s_1)^{\rho} \begin{pmatrix} -\rho' & & & \\ \lambda_2 - \rho & \rho - \lambda_1 & & \\ \vdots & & \ddots & \\ \lambda_n - \rho & & & \rho - \lambda_1 \end{pmatrix} Y,$$

$\mathcal{P}(\lambda, \rho)$ *is transformed into the system*

$$\mathcal{P}(\lambda', \rho): \qquad (u - S)\frac{dZ}{du} = A(\lambda', \rho)Z,$$

where

$$S = \begin{pmatrix} s_1 & & \\ & \ddots & \\ & & s_n \end{pmatrix}, \quad s_i = \frac{1}{t_i - t_1} + s_1 \quad (i = 2, \dots, n),$$

$$\lambda' = (\rho - \rho', \lambda_2, \dots, \lambda_n).$$

Proof of Proposition 1.6. — We may assume that $\lambda_1 \in \mathbf{Z}$ (i.e. $j = 1$).

First we suppose that $\lambda_1 - \rho \neq 0$ and $\rho' \neq 0$. Then by Lemma 2.11 it suffices to show that $u = \infty$ is an apparent singular point of

$$\mathcal{P}(\lambda', \rho): \qquad (u - S)\frac{dZ}{du} = A(\lambda', \rho)Z,$$

where

$$\text{(2.55)} \qquad \lambda' = (\lambda'_1, \dots, \lambda'_n), \quad \lambda'_1 = \rho - \rho', \quad \lambda'_j = \lambda_j \quad (j \geqslant 2).$$

We set $A_1 = A(\lambda', \rho)$. The eigen values of A_1 are ρ of multiplicity $n - 1$ and $\rho - \lambda_1$. Then $u = \infty$ is apparent if and only if there are $n - 1$ solutions of the form $u^{\rho} \sum_{i=0}^{\infty} v_i u^{-i}$ and a solution of the form $u^{\rho - \lambda_1} \sum_{i=0}^{\infty} v_i u^{-i}$ which are linearly independent over $\mathbf{C}$.

When $\lambda_1 = 0$, the eigen values of A_1 are ρ of multiplicity n, and $A_1 \neq \rho I_n$ by $\rho' \neq 0$. Thus A_1 is not diagonalizable, and hence $u = \infty$ is not apparent.

Suppose that $\lambda_1 > 0$. Set

$$Z = u^{\rho} \sum_{i=0}^{\infty} v_i u^{-i}, \quad v_i \in V^n(\mathbf{C}),$$

and put it into $\mathcal{P}(\lambda', \rho)$ to obtain

$$[A_1 - \rho]v_0 = 0,$$
$$[A_1 - (\rho - i)]v_i = ((i-1) - \rho)Sv_{i-1} \quad (i \geqslant 1).$$

Recalling §2.1, we see that $u = \infty$ is apparent if and only if, for any ρ-eigen vector v_0 of A_1, $((\lambda_1 - 1) - \rho)Sv_{\lambda_1 - 1}$ which is uniquely determined by v_0 becomes again a ρ-eigen vector. Note that

$$\begin{aligned} &((\lambda_1 - 1) - \rho)Sv_{\lambda_1 - 1} \\ (2.56) \quad &= (-\rho)(1-\rho)\cdots((\lambda_1 - 1) - \rho)S[A_1 - (\rho - (\lambda_1 - 1))]^{-1} \\ &\quad \times S[A_1 - (\rho - (\lambda_1 - 2))]^{-1}S \cdots S[A_1 - (\rho - 1)]^{-1}Sv_0. \end{aligned}$$

If $\rho(\rho - 1)\cdots(\rho - (\lambda_1 - 1)) = 0$, then clearly the left hand side of (2.56) lies in the ρ-eigen space V_ρ of A_1. Otherwise we can follow the arguments in §2.2 and 2.4 to obtain

$$\rho - \lambda'_j \in \mathbf{N}, \quad j = 1, \ldots, n.$$

By virtue of (2.55) it follows that

$$\rho' \in \mathbf{N}, \quad \rho - \lambda_j \in \mathbf{N} \quad (j \geqslant 2).$$

Hence we obtain the conditions

(a) $\rho \in \mathbf{Z}, \quad 0 \leqslant \rho \leqslant \lambda_1 - 1$, or

(b) $\rho - \lambda_j \in \mathbf{N} \quad (j \geqslant 2), \quad \lambda_1 + \sum_{j=2}^{n}(\lambda_j - \rho) > 0.$

When $\lambda_1 < 0$, similar argument and the assumption $\rho' \neq 0$ yield the conditions

(c) $\rho \in \mathbf{Z}, \quad \lambda_1 < \rho \leqslant -1$, or

(d) $\lambda_j - \rho \in \mathbf{N}_0 \quad (j \geqslant 2), \quad \lambda_1 + \sum_{j=2}^{n}(\lambda_j - \rho) < 0.$

Secondly we suppose that $\lambda_1 - \rho \neq 0$ and $\rho' = 0$. As Lemma 2.11, by the transformations

$$x \to u: \quad u = \frac{1}{x - t_1},$$

$$Y \to Z: \quad Z = u^{\rho} \begin{pmatrix} 1 & & & \\ \lambda_2 - \rho & \rho - \lambda_1 & & \\ \vdots & & \ddots & \\ \lambda_n - \rho & & & \rho - \lambda_1 \end{pmatrix} Y,$$

$\mathcal{P}(\lambda, \rho)$ is transformed into

$$(u - S)\frac{dZ}{du} = BZ, \tag{2.57}$$

where

$$S = \begin{pmatrix} 0 & & & \\ & s_2 & & \\ & & \ddots & \\ & & & s_n \end{pmatrix}, \quad s_i = \frac{1}{t_i - t_1} \quad (i \geqslant 2),$$

$$B = \begin{pmatrix} \rho & 1 & \cdots & 1 \\ 0 & \lambda_2 & \cdots & \lambda_2 - \rho \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \lambda_n - \rho & \cdots & \lambda_n \end{pmatrix}.$$

The system (2.57) is reducible. By setting

$$Z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}, \quad Z_1 = \begin{pmatrix} z_2 \\ \vdots \\ z_n \end{pmatrix}, \quad S_1 = \begin{pmatrix} s_2 & & \\ & \ddots & \\ & & s_n \end{pmatrix},$$

$$B_1 = \begin{pmatrix} \lambda_2 & \cdots & \lambda_2 - \rho \\ \vdots & \ddots & \vdots \\ \lambda_n - \rho & \cdots & \lambda_n \end{pmatrix},$$

it is decomposed into

$$(u - S_1)\frac{dZ_1}{du} = B_1 Z_1 \tag{2.58}$$

and

$$\begin{cases} z_1 = c(u) u^{\rho}, \\ u^{\rho+1} \dfrac{dc}{du} = z_2 + \cdots + z_n \end{cases} \tag{2.59}$$

(apply the method of variation of constants). Then $u = \infty$ is an apparent singular point of (2.57) if and only if it is an apparent singular point of (2.58) and, for any solution Z_1 of (2.58), the solution z_1 of (2.59) has no logarithmic term at $u = \infty$.

When $\lambda_1 = 0$, B is not diagonalizable, and hence $u = \infty$ is not apparent.

Suppose that $\lambda_1 > 0$. Set

$$Z_1 = u^{\rho} \sum_{i=0}^{\infty} v_i u^{-i}, \quad v_i = \begin{pmatrix} v_{i2} \\ \vdots \\ v_{in} \end{pmatrix} \in V^{n-1}(\mathbf{C}),$$

and put it into (2.58) to obtain

$$\begin{aligned} [B_1 - \rho]v_0 &= 0, \\ [B_1 - (\rho - i)]v_i &= ((i-1) - \rho)S_1 v_{i-1} \quad (i \geqslant 1). \end{aligned}$$

In particular v_0 is ρ-eigen vector of B_1, and, noting that $\lambda_1 \neq 0$, from it we obtain

$$v_{02} + \cdots + v_{0n} = 0. \tag{2.60}$$

Now by a similar argument as in the first case, we see that $u = \infty$ is an apparent singular point of (2.58) if and only if

(e) $\qquad \rho \in \mathbf{Z}, \quad 0 \leqslant \rho \leqslant \lambda_1 - 1, \quad$ or

(f) $\qquad \rho - \lambda_j \in \mathbf{N} \quad (j \geqslant 2)$

holds. We assume (e) or (f). Then, by virtue of (2.60), we have

$$\begin{aligned} \frac{dc}{du} &= u^{-\rho-1}(z_2 + \cdots + z_n) \\ &= u^{-1}\{(v_{02} + \cdots + v_{on}) + (v_{12} + \cdots + v_{in})u^{-1} + \cdots\} \\ &= O(u^{-2}), \end{aligned}$$

which shows that the solution z_1 of (2.59) has no logarithmic term at $u = \infty$. Thus the conditions (e) and (f) are sufficient.

Suppose that $\lambda_1 < 0$. Set

$$Z_1 = u^{\rho-\lambda_1} \sum_{i=0}^{\infty} v_i u^{-i}, \quad v_i = \begin{pmatrix} v_{i2} \\ \vdots \\ v_{in} \end{pmatrix} \in V^{n-1}(\mathbf{C}),$$

and put it into (2.58) to obtain

$$\begin{aligned}[B_1 - (\rho - \lambda_1)]v_0 &= 0, \\ [B_1 - (\rho - \lambda_1 - i)]v_i &= ((i-1) - (\rho - \lambda_1))S_1 v_{i-1} \quad (i \geqslant 1).\end{aligned} \tag{2.61}$$

We see that $u = \infty$ is an apparent singular point of (2.58) if and only if

(g) $\rho \in \mathbf{Z}, \quad \lambda_1 < \rho \leqslant 1, \quad$ or

(h) $\lambda_j - \rho \in \mathbf{N}_0 \quad (j \geqslant 2)$

holds (where we have used $\lambda_1 \neq \rho$). Assume (g) or (h). Now the integral of the equation

$$\frac{dc}{du} = u^{-\lambda_1 - 1}(z_2 + \cdots + z_n)$$

has no logarithmic term at $u = \infty$ if and only if

$$v_{-\lambda_1 2} + \cdots + v_{-\lambda_1 n} = 0.$$

Then we have

$$[B_1 - \rho]v_{-\lambda_1} = 0,$$

and it follows from (2.61) that

$$\begin{aligned} 0 &= -(\rho + 1)S_1 v_{-\lambda_1 - 1} \\ &= (-1)^{-\lambda_1}(\rho+1)(\rho+2)\cdots(\rho - \lambda_1)S_1 \\ &\quad \times [B_1 - (\rho+1)]^{-1}S_1 \cdots S_1[B_1 - (\rho - \lambda_1 - 1)]^{-1}S_1 v_0, \end{aligned}$$

which yields

$$(\rho+1)\cdots(\rho - \lambda_1) = 0.$$

Thus the condition (g) is sufficient. To sum up the above, in the second case we obtain three conditions (e), (f) and (g).

Thirdly we suppose that $\lambda_1 - \rho = 0$. In this case the system $\mathcal{P}(\lambda, \rho)$ is reducible, and is decomposed into

$$(x - t_1)\frac{dy_1}{dx} = \rho y_1$$

and

$$(x - T_1)\frac{dY_1}{dx} = B_1 Y_1 + \begin{pmatrix} \lambda_2 - \rho \\ \vdots \\ \lambda_n - \rho \end{pmatrix} y_1, \tag{2.62}$$

where we have set

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \; Y_1 = \begin{pmatrix} y_2 \\ \vdots \\ y_n \end{pmatrix}, \; T_1 = \begin{pmatrix} t_2 & & \\ & \ddots & \\ & & t_n \end{pmatrix},$$

$$B_1 = \begin{pmatrix} \lambda_2 & \cdots & \lambda_2 - \rho \\ \vdots & \ddots & \vdots \\ \lambda_n - \rho & \cdots & \lambda_n \end{pmatrix}.$$

Then we have

$$y_1 = c_1(x - t_1)^\rho.$$

Applying the method of variation of constants to (2.62), we see that Y_1 is holomorphic at $x = t_1$ if $\rho \geqslant 0$, and that Y_1 has logarithmic term at $x = t_1$ if $\rho = -1$. Thus we have the condition

(i) $$\rho = \lambda_1 \geqslant 0.$$

Suppose that $\rho = \lambda_1 < -1$. Set

$$Y = (x - t_1)^{\lambda_1} \sum_{i=0}^{\infty} v_i (x - t_1)^i, \quad v_i = \begin{pmatrix} v_{i1} \\ v_{i2} \\ \vdots \\ v_{in} \end{pmatrix} \in V^n(\mathbf{C}),$$

and put it into $\mathcal{P}(\lambda, \rho)$ to obtain

$$\begin{aligned} &\lambda_1(t_1 - T)v_0 = 0, \\ &[A - (\lambda_1 + i)]v_i = (\lambda_1 + i + 1)(t_1 - T)v_{i+1} \quad (i \geqslant 0). \end{aligned}$$

Then we have

$$v_{0j} = 0 \quad (j = 2, \ldots, n), \quad v_{i1} = 0 \quad (i > 1).$$

Now we set

$$v_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad v_i' = \begin{pmatrix} v_{i2} \\ \vdots \\ v_{in} \end{pmatrix} \; (i \geqslant 1), \quad T_2 = \begin{pmatrix} t_1 - t_2 & & \\ & \ddots & \\ & & t_1 - t_n \end{pmatrix}.$$

Then it follows that

$$\begin{pmatrix} \lambda_2 - \rho \\ \vdots \\ \lambda_n - \rho \end{pmatrix} = (\lambda_1 + 1)T_2 v_1',$$
$$[A_1 - (\lambda_1 + i)]v_i' = (\lambda_1 + i + 1)T_2 v_{i+1}' \quad (i \geqslant 1).$$

This system has a solution if and only if

$$[A_1 + 1]v_{-\lambda_1 - 1}' = 0.$$

Thus one of the eigen values of A_1 belongs to $\{-1, -2, \dots, \lambda_1 + 1\}$, so that it must be ρ', and $v_{\rho' - \lambda_1}'$ is a ρ'-eigen vector of A_1. This yields, by a similar argument as above, the condition

$$\text{(j)} \qquad \lambda_j - \rho \in \mathbf{N}_0 \quad (j \geqslant 2), \quad \lambda_1 + 1 \leqslant \lambda_1 + \sum_{j=2}^{n} (\lambda_j - \rho) \leqslant -1.$$

Thus in the above we have obtained nine conditions : (a), (b), (c) and (d) with $\lambda_1 - \rho \neq 0$ and $\rho' \neq 0$, (e), (f) and (g) with $\lambda_1 - \rho \neq 0$ and $\rho' = 0$, and (i) and (j) with $\lambda_1 - \rho = 0$. It is easy to see that

$$\begin{aligned} \text{(d) or (j)} &\iff (1.18 : \text{i}), \\ \text{(b) or (f)} &\iff (1.18 : \text{ii}), \\ \text{(a), (e) or (i)} &\iff (1.18 : \text{iii}), \\ \text{(c) or (g)} &\iff (1.18 : \text{iv}), \end{aligned}$$

and this completes the proof. Q.e.d.

3. From "local" to "global".

We consider the Pochhammer system $\mathcal{P}(\lambda, \rho)$ of rank n.

First we remark that, for our Pochhammer system $\mathcal{P}(\lambda, \rho)$, the following three conditions are equivalent :

(i) for almost all primes p, $\mathcal{P}(\lambda, \rho)$ has zero p-curvature,

(ii) for almost all primes p, the reduced system $\mathcal{P}(\lambda, \rho)_p$ modulo p has n polynomial solutions in $K_p[x]$ which are linearly independent over $K_p(x^p)$,

(iii) for almost all primes p, the reduced system $\mathcal{P}(\lambda,\rho)_p$ modulo p has n polynomial solutions in $K_p[x]$ of degree at most $p-1$ which are linearly independent over $K_p(x^p)$.

This will be shown similarly as in Katz [9, §6].

The following is the main result of this paper, and is an affirmative answer to the Grothendieck conjecture for the Pochhammer system.

THEOREM 3.1. — *The following conditions are equivalent :*

(i) *For almost all primes p, the Pochhammer system $\mathcal{P}(\lambda,\rho)$ with parameters $(\lambda_1,\dots,\lambda_n,\rho)$ has zero p-curvature.*

(ii) *Any solution of the Pochhammer system $\mathcal{P}(\lambda,\rho)$ with parameters $(\lambda_1,\dots,\lambda_n,\rho)$ is an algebraic function.*

Proof. — Owing to the above remark and Proposition 0.1, it suffices to show that, if the reduced system $\mathcal{P}(\lambda,\rho)_p$ has n linearly independent polynomial solutions of degree at most $p-1$ for almost all primes p, then the monodromy group of $\mathcal{P}(\lambda,\rho)$ is finite. By Theorem 2.1 the former condition is reduced to that, for almost all primes p,

$$(I)_p \quad R_p(\rho)\leqslant R_p(\lambda_j)\quad (j=1,\dots,n),\quad \sum_{j=1}^{n}R_p(\lambda_j)<(n-1)R_p(\rho)+p$$

or

$$(II)_p \qquad R_p(\lambda_j)<R_p(\rho)\quad (j=1,\dots,n),\quad (n-1)R_p(\rho)\leqslant\sum_{j=1}^{n}R_p(\lambda_j)$$

holds. Then we suppose that $(I)_p$ or $(II)_p$ holds for almost all primes p.

We quote a lemma from Katz [9, (6.5.2), (6.5.3)].

LEMMA 3.1. — *Let $\alpha\in\mathbf{Z}$, $D\in\mathbf{Z}$ with $D\neq 0$.*

(i) *If $\alpha/D\notin\mathbf{Z}$, $(p,D)=1$, $p\Delta\equiv 1$ (D) and $p>|\alpha|$, we have*

$$\frac{1}{p}R_p\left(\frac{-\alpha}{D}\right)=\left\langle\frac{\alpha\Delta}{D}\right\rangle-\frac{\alpha}{pD}.$$

(ii) *For each invertible element Δ in $\mathbf{Z}/D\mathbf{Z}$, we have the limit formula*

$$\lim_{\substack{p\to\infty \\ p\Delta\equiv 1(D)}} \frac{1}{p} R_p\left(\frac{-\alpha}{D}\right) = \begin{cases} \left\langle \dfrac{\alpha\Delta}{D} \right\rangle & \text{if } \dfrac{\alpha}{D} \notin \mathbf{Z}, \\ 0 & \text{if } \dfrac{\alpha}{D} \in \mathbf{Z},\ \dfrac{\alpha}{D} \leqslant 0, \\ 1 & \text{if } \dfrac{\alpha}{D} \in \mathbf{Z},\ \dfrac{\alpha}{D} > 0. \end{cases} \tag{3.1}$$

Let D denote the common denominator of $\lambda_1, \dots, \lambda_n, \rho$.

First assume that $\mathcal{P}(\lambda, \rho)$ is generic; i.e.

$$\lambda_j,\ \rho,\ \lambda_j - \rho \notin \mathbf{Z} \quad \text{for } j = 1, \dots, n.$$

Take $\Delta \in \mathbf{Z}$ invertible in $\mathbf{Z}/D\mathbf{Z}$. Then $(I)_p$ or $(II)_p$ holds for infinitely many primes p with $p\Delta \equiv 1\ (D)$. Suppose that $(I)_p$ holds for infinitely many such primes p. By the limit formula (3.1) we obtain

$$\langle -\Delta\rho \rangle \leqslant \langle -\Delta\lambda_j \rangle \quad (j = 1, \dots, n), \quad \sum_{j=1}^{n} \langle -\Delta\lambda_j \rangle \leqslant (n-1)\langle -\Delta\rho \rangle + 1,$$

and hence

$$\langle \Delta\lambda_j \rangle \leqslant \langle \Delta\rho \rangle \quad (j = 1, \dots, n), \quad (n-1)\langle \Delta\rho \rangle \leqslant \sum_{j=1}^{n} \langle \Delta\lambda_j \rangle.$$

From $\lambda_j - \rho \notin \mathbf{Z}$ it follows that

$$\langle \Delta\lambda_j \rangle < \langle \Delta\rho \rangle \quad (j = 1, \dots, n). \tag{3.2}$$

We shall show that $(n-1)\langle \Delta\rho \rangle < \sum_{j=1}^{n} \langle \Delta\lambda_j \rangle$. Suppose

$$(n-1)\langle \Delta\rho \rangle = \sum_{j=1}^{n} \langle \Delta\lambda_j \rangle. \tag{3.3}$$

By $(I)_p$ and Lemma 3.1 (i), we have

$$(n-1)\left(\langle \Delta\rho \rangle - \frac{\rho}{p}\right) < \sum_{j=1}^{n} \left(\langle \Delta\lambda_j \rangle - \frac{\lambda_j}{p}\right)$$

for sufficiently large p with $p\Delta \equiv 1\ (D)$. Using (3.3), we have

$$-(n-1)\frac{\rho}{p} < -\sum_{j=1}^{n} \frac{\lambda_j}{p},$$

and hence

$$\sum_{j=1}^{n} \lambda_j < (n-1)\rho. \tag{3.4}$$

Use $-\Delta$ in stead of Δ, then for infinitely many p with $-\Delta p \equiv 1$ (D) $(II)_p$ holds; in fact, if $(I)_p$ would hold, by the limit formula (3.1) we should obtain

$$\langle \Delta\rho \rangle \leqslant \langle \Delta\lambda_j \rangle \quad (j = 1, \ldots, n),$$

which contradicts to (3.2). Then we have

$$(n-1)\left(\langle \Delta\rho \rangle + \frac{\rho}{p}\right) \leqslant \sum_{j=1}^{n} \left(\langle \Delta\lambda_j \rangle + \frac{\lambda_j}{p}\right).$$

By using (3.3), we obtain

$$(n-1)\rho \leqslant \sum_{j=1}^{n} \lambda_j,$$

which contradicts to (3.4). Thus we have proved that

$$\langle \Delta\lambda_j \rangle < \langle \Delta\rho \rangle \quad (j = 1, \ldots, n), \quad (n-1)\langle \Delta\rho \rangle < \sum_{j=1}^{n} \langle \Delta\lambda_j \rangle. \tag{3.5}$$

If $(II)_p$ holds for infinitely many p with $p\Delta \equiv 1$ (D), in a similar manner we obtain

$$\langle \Delta\rho \rangle < \langle \Delta\lambda_j \rangle \quad (j = 1, \ldots, n), \quad \sum_{j=1}^{n} \langle \Delta\lambda_j \rangle < (n-1)\langle \Delta\rho \rangle + 1. \tag{3.6}$$

Hence for every $\Delta \in \mathbf{Z}$ invertible in $\mathbf{Z}/D\mathbf{Z}$, (3.5) or (3.6) holds, which implies that the monodromy group of $\mathcal{P}(\lambda, \rho)$ is finite (Theorem 1.2).

Secondly we assume that

$$\lambda_j, \ \rho \notin \mathbf{Z} \quad (j = 1, \ldots, n),$$

and

$$\lambda_k - \rho \in \mathbf{Z}$$

for some k.

LEMMA 3.2. — *Let $r \in \mathbf{Q}$, $n \in \mathbf{N}_0$. For any sufficiently large prime p, we have*

$$R_p(n+r) = n + R_p(r).$$

Suppose $\lambda_k - \rho = n_k \geqslant 0$. Then by Lemma 3.2, for sufficiently large p, we have

$$R_p(\lambda_k) = R_p(\lambda_k - \rho + \rho) = n_k + R_p(\rho),$$

and hence

$$R_p(\rho) \leqslant R_p(\lambda_k).$$

Then, since $(II)_p$ does not hold, $(I)_p$ holds for any sufficiently large p. The limit formula (3.1) for $\Delta = 1$ gives

$$\langle \lambda_j \rangle \leqslant \langle \rho \rangle \quad (j = 1, \dots, n),$$

and that for $\Delta = -1$ gives

$$\langle \rho \rangle \leqslant \langle \lambda_j \rangle \quad (j = 1, \dots, n).$$

Then $\langle \rho \rangle = \langle \lambda_j \rangle$, and hence $\lambda_j - \rho \in \mathbf{Z}$ for every $j = 1, \dots, n$. Moreover we obtain

$$\lambda_j - \rho \geqslant 0 \quad (j = 1, \dots, n)$$

from $(I)_p$ and Lemma 3.2. This is the case (ii) of Theorem 1.3 in §1.2, then by the argument there we know that the monodromy group of $\mathcal{P}(\lambda, \rho)$ is finite. If $\lambda_k - \rho < 0$, then in a similar manner we obtain

$$\lambda_j - \rho < 0 \quad (j = 1, \dots, n).$$

This is the case (i) of Theorem 1.3, and again the monodromy group is finite.

Finally we assume that one of

$$\lambda_1, \dots, \lambda_n, \rho$$

belongs to $\mathbf{Z}$. Then in a similar argument we know that all belongs to $\mathbf{Z}$:

$$\lambda_j, \ \rho \in \mathbf{Z} \quad (j = 1, \dots, n).$$

Suppose that $(I)_p$ holds for infinitely many p. If $\rho < 0$, for any sufficiently large p we have

$$R_p(\rho) = p + \rho.$$

Then $(I)_p$ gives

$$R_p(\lambda_j) \geqslant p + \rho \quad (j = 1, \ldots, n). \tag{3.7}$$

If $\lambda_j \geqslant 0$, $R_p(\lambda_j) = \lambda_j$, which contradicts to (3.7) when p is large enough. Hence $\lambda_j < 0$, $R_p(\lambda_j) = p + \lambda_j$, and we have

$$\lambda_j \geqslant \rho$$

for every j. Now by $(I)_p$ we have

$$\sum_{j=1}^{n}(p + \lambda_j) < (n-1)(p+\rho) + p,$$

and hence

$$\sum_{j=1}^{n} \lambda_j < (n-1)\rho.$$

Thus we have obtained the case (iii) of Theorem 1.3, which implies that the monodromy group of $\mathcal{P}(\lambda, \rho)$ is finite. If $\rho \geqslant 0$, for sufficiently large p we have

$$R_p(\rho) = \rho.$$

By $(I)_p$ we have

$$R_p(\lambda_j) \geqslant \rho \quad (j = 1, \ldots, n), \quad \sum_{j=1}^{n} R_p(\lambda_j) < (n-1)\rho + p.$$

Let ℓ be the number of negative λ_j's. Then

$$\sum_{j=1}^{n} R_p(\lambda_j) = \ell p + \sum_{j=1}^{n} \lambda_j,$$

and hence

$$\ell p + \sum_{j=1}^{n} \lambda_j < (n-1)\rho + p.$$

Since this inequality holds for sufficiently large p, we obtain $\ell \leqslant 1$. When $\ell = 0$, every λ_j is non-negative, and from $(I)_p$ we obtain

$$\lambda_j \geqslant \rho \quad (j = 1, \dots, n).$$

This is the case (iv) of Theorem 1.3, and then the monodromy group is finite. When $\ell = 1$, let $\lambda_i < 0$. Then $\lambda_k \geqslant 0$ for any $k \neq i$, and by $(I)_p$ we have

$$\lambda_k \geqslant \rho \quad (k \neq i).$$

Again by $(I)_p$ we obtain

$$\sum_{j=1}^{n} \lambda_j < (n-1)\rho,$$

so that we have obtained the case (v) of Theorem 1.3. Hence the monodromy group is finite also in this case. Supposing that $(II)_p$ holds for infinitely many p, similarly we obtain the cases (vi), (vii) and (viii) of Theorem 1.3, and hence also in this case the monodromy group of $\mathcal{P}(\lambda, \rho)$ is finite.

On the exponents $\lambda_1, \dots, \lambda_n, \rho$, we have examined all cases, and in any case the monodromy group of the Pochhammer system $\mathcal{P}(\lambda, \rho)$ is finite. This completes the proof. Q.e.d.

In the above proof we have shown that, if the Pochhammer system $\mathcal{P}(\lambda, \rho)$ is non-generic and has finite monodromy group, one of the eight conditions (i) to (viii) in Theorem 1.3 holds. Thus, together with the proof of sufficiency given after Theorem 1.3, we have completed the proof of Theorem 1.3.

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