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JACOBI-EISENSTEIN SERIES AND p-ADIC INTERPOLATION OF SYMMETRIC SQUARES OF CUSP FORMS

by Pavel I. GUERZHOY

1. INTRODUCTION AND STATEMENT OF THE MAIN THEOREM

Let

$$f=\sum_{\substack{n\in\mathbf{Z}\\ n\geq 0}} a(n)e(n\tau)\quad (e(x)=e^{2\pi i x})$$

be a cusp Hecke eigenform of even integral weight k on the full modular group $SL_2(\mathbf{Z})$. We denote the space of all such forms of weight k by S_k and the space of all modular forms of weight k by M_k. Let M be an integer, $3\leq M\leq k-1$, and χ be a Dirichlet character modulo r, $\chi(-1)=(-1)^{M+1}$. The special values of symmetric squares of the cusp form f are defined by the following:

$$D_f(M,\chi)=\sum_{n\geq 1}\frac{a(n^2)\chi(n)}{n^{k+M-1}}. \tag{1}$$

The values (1) are known to become algebraic numbers after multiplication by an appropriate constant. Below we extend in a natural way this definition for f being an Eisenstein series.

Key words: Jacobi forms – Eisenstein series – Symmetric square – Modular forms – p-adic interpolation – Rankin's method.
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Let $\{f_j \mid j = 1, \ldots, \dim M_k\}$ be a basis of the linear space M_k of modular forms of weight k. This basis consists of the normalized (*i.e.* $a(1) = 1$) Hecke eigenforms. The modular form

$$F(k, M, \chi) = \sum_{j=1}^{\dim M_k} \frac{f_j}{\langle f_j, f_j \rangle} D_{f_j}(M, \chi) \tag{2}$$

is the kernel function for the special values of the symmetric square with respect to the Petersson scalar product $\langle \cdot, \cdot \rangle$ in the following sense:

$$\langle F(k, M, \chi), f_j \rangle = D_{f_j}(M, \chi).$$

We construct a generating function from the modular forms $F(k, M, \chi)$ and their derivatives.

MAIN THEOREM. — *Let $M \geq 1$ be a fixed natural number, χ be a fixed Dirichlet character modulo r, and $t = (1 - \chi(-1))/2$.*

Then

$$\sum_{\nu \geq 0} z^{2\nu + t} \sum_{0 \leq \mu \leq \nu} \Lambda(\mu, \nu) F^{(\mu)}(2\nu - 2\mu + M + t + 1, M, \chi) = E^{\chi}_{M+1}(\tau, z), \tag{3}$$

with the Jacobi-Eisenstein series E^{χ}_{M+1} (of weight $M + 1$ and index r on $SL_2(\mathbf{Z})$) as explained below, where $\Lambda(\mu, \nu)$ is given by the following:

$$\Lambda(\mu, \nu) = 2^{1-4M-3t-2\nu} \pi^{1/2-M-t} r^{2\nu+t} \Gamma(M + 1/2)^{-1}$$
$$\times \sum_{0 \leq \mu \leq \nu} (-1)^{\nu-\mu} (2\pi i)^{\mu} \frac{\Gamma(M + 2\nu - 2\mu + t + 1)\Gamma(2M + 2\nu - 2\mu + t)}{\Gamma(M + 2\nu - \mu + t + 1)\Gamma(\mu + 1)\Gamma(2\nu - 2\mu + t + 1)}.$$

We must say a few words about the Jacobi-Eisenstein series $E^{\chi}_{M+1}(\tau, z)$ occuring in (3). Almost all necessary facts concerning Jacobi forms one can find in [2]. Taking into account that our notations are slightly different from those given in this work, we shall briefly recall some definitions and propositions of this theory. Hereafter the letter $\mathbf{H}$ denotes the complex upper-half plane, $\mathbf{C}$ denotes the whole complex plane, the letter $\mathbf{Z}$ denotes the set of integers. For $\tau \in \mathbf{H}$ and $\Gamma \in SL_2(\mathbf{Z})$ we assume $\gamma(\tau) = \dfrac{a\tau + b}{c\tau + d}$. The formulas

$$\left(\phi \mid_{k,r} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(\tau, z) = (c\tau + d)^{-k} e\left(\frac{-crz^2}{c\tau + d}\right) \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right),$$
$$(\phi \mid_r (\lambda, \mu))(\tau, z) = e(r(\lambda^2 \tau + 2\lambda z))\phi(\tau, z + \lambda\tau + \mu)$$

define the action of Jacobi group Γ^J (*i.e.* the semi direct product of $SL_2(\mathbf{Z})$ and $(\mathbf{Z} \times \mathbf{Z})$) in the space of holomorphic functions $\phi(\tau, z)$ of two variables

($\tau \in \mathbf{H}, z \in \mathbf{C}$). Let k and r be positive integers. A function ϕ is referred to as Jacobi form of weight k and index r if it satisfies the following conditions:

$$\phi \mid_{k,r} \xi(\tau, z) = \phi(\tau, z) \text{ for every element } \xi \text{ of } \Gamma^J,$$
$$\phi(\tau, z) = \sum_{n \geq 0} \sum_{\substack{m \in \mathbf{Z} \\ m^2 \leq 4rn}} c(n, m) e(n\tau + mz).$$

Denote as $J_{k,r}$ the finite-dimensional linear space of Jacobi forms of weight k and index r. For an integer $k > 2$ and any integer s the Eisenstein series $E_{k,r,s}$ in the space $J_{k,r}$ is defined, as in [2], p. 25, by the following:

$$E_{k,r,s}(\tau, z) = \sum_{\gamma \in \Gamma^J_\infty \backslash \Gamma^J} e(as^2\tau + 2absz) \mid_{k,r} \gamma,$$

where

$$\Gamma^J_\infty = \{\gamma \in \Gamma^J : 1 \mid_{k,r} \gamma = 1\} = \left\{ \left(\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, (0, \mu) \right) \mid (n, \mu) \in \mathbf{Z} \right\},$$

and where we use a, b for the unique natural numbers such that $r = ab^2$ and a is square-free. This series depends only on the residue of s modulo b. A Jacobi form is referred to as a cusp form if

$$\phi(\tau, z) = \sum_{n \geq 0} \sum_{\substack{m \in \mathbf{Z} \\ m^2 < 4rn}} c(n, m) e(n\tau + mz).$$

The Eisenstein series in (3) is now given by

$$E_k^\chi(\tau, z) = (4\pi i)^{-t} 1/2 \sum_{s \bmod r} \chi(s) E_{k,r,s}(\tau, z).$$

The idea to construct generating functions connected with special values of L-functions associated with modular forms appeared in [9]. In this paper a generating function associated with the period polynomials of modular forms was constructed and this generating function was calculated in terms of Jacobi theta function.

Section 2 is devoted to the proof of the Main Theorem. In section 3 we shall derive explicit formulas for the Fourier coefficients of the series $E_k^\chi(\tau, z)$ (*cf.* Theorem 2). In section 4 we shall use our Main Theorem and these formulas to prove the existence of a p-adic analytic function such that its special values coincide with those of the symmetric square of a p-ordinary cusp form (*cf.* Theorem 3). The main idea for this is to use the well-known p-adic interpolation properties of the special values of Dirichlet L-function. These special values appear in the formulas for

Fourier coefficients of our Jacobi-Eisenstein series. We construct the p-adic analytic function in question as the non-Archimedean Mellin transform of a bounded C_p-valued measure. The existence of this measure is proved using the abstract Kummer congruences.

A similar result on p-adic interpolation of symmetric squares one can find in [5]. Our method to prove it differs from those of [5]: we use Jacobi forms instead of non-holomorphic modular forms.

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2. PROOF OF THE MAIN THEOREM

To prove this theorem we must recall some facts concerning Jacobi forms and the Rankin's method of calculating the symmetric square special values.

2.1. Differential operators acting in the space of modular forms.

For two smooth functions f and g, a natural number ν and real positive k_1 and k_2 Cohen [1] defined smooth functions $F_\nu^{k_1,k_2}$ by the formulas

$$\begin{aligned} F_\nu^{k_1,k_2}(f,g) &= \sum_{0\le\mu\le\nu} (-1)^{\nu-\mu} \binom{\nu}{\mu} \frac{\Gamma(k_1+\nu)\Gamma(k_2+\nu)}{\Gamma(k_1+\mu)\Gamma(k_2+\nu-\mu)} f^{(\mu)} g^{(\nu-\mu)} \\ &= \sum_{0\le\mu\le\nu} (-1)^{\nu} \binom{\nu}{\mu} \frac{\Gamma(k_1+\nu)\Gamma(k_1+k_2+2\nu-\mu-1)}{\Gamma(k_1+\nu-\mu)\Gamma(k_1+k_2+\nu-1)} (f g^{(\nu-\mu)})^{(\mu)}. \end{aligned} \tag{4}$$

It is known that if f and g are modular forms on some group $H \subseteq SL_2(\mathbf{Z})$, with weights k_1 and k_2, then $F_\nu^{k_1,k_2}(f,g)$ is a modular form on H of weight $k_1+k_2+2\nu$.

A function $\tilde{f}^\ell(\tau,z)$ of two variables was constructed in [1] for a real positive number ℓ and a smooth function f:

$$\tilde{f}^\ell(\tau,z) = \sum_{\nu\ge 0} \frac{(2\pi i)^\nu \Gamma(\ell)}{\Gamma(\nu+1)\Gamma(\ell+\nu)} f^\nu(\tau) z^{2\nu}.$$

Both these operators are tightly connected:

$$\tilde{f_1}^{k_1}(\tau,z)\tilde{f_2}^{k_2}(\tau,iz) = \sum_{\nu\ge 0} z^{2\nu} \frac{(2\pi i)^{2\nu}}{\Gamma(\nu+1)\Gamma(k_1+\nu)\Gamma(k_2+\nu)} F_\nu^{k_1,k_2}(f_1,f_2).$$

2.2. Rankin's method for symmetric squares.

These operators can be used for the calculation of the symmetric square special values.

Let $\mathbf{Tr}_1^{4r^2} : M_k(\Gamma_0(4r^2)) \to M_k$ be the trace operator as in [4].

PROPOSITION 1 ([10]). — *Let ν and M be natural numbers. Let χ be a Dirichlet character modulo r, $t = (1-\chi(-1))/2$, and $\chi(-1) = (-1)^{M+1}$. Then there exists an Eisenstein series $S = S(2\nu + M + t + 1)$ in the space of modular forms of weight $2\nu + M + t + 1$ on $SL_2(\mathbf{Z})$ such that*

$$(5) \quad S+F^c(2\nu+M+t+1, M, \chi) = (2\pi i)^{-\nu} \frac{(4\pi)^{2\nu+M+t}\Gamma(M+1/2)}{\Gamma(M+2\nu+t)\Gamma(M+\nu+1/2)} \times \mathbf{Tr}_1^{4r^2} F_\nu^{t+1/2, M+1/2}(h_\chi, E^\chi_{M+1/2}),$$

where

$$F^c(k, m, \chi) = \sum_{j=1}^{\dim S_k} \frac{f_j}{\langle f_j, f_j \rangle} D_{f_j}(M, \chi),$$

and the sum is carried out through all normalized cusp Hecke eigenforms of weight k. Functions h_χ and $E^\chi_{M+1/2}$ are the modular forms of half integral weight introduced in [8]*:*

$$h_\chi(\tau) = 1/2 \sum_{n \in \mathbf{Z}} \chi(n) n^t e(n^2 \tau),$$

$$E^\chi_{M+1/2} = \sum_{\substack{(c,d)=1 \\ c \equiv 0 \bmod 4r}} \frac{\chi(d)\left(\frac{-1}{d}\right)\left(\frac{c}{d}\right)\varepsilon_d^{-2t-1}}{(c\tau+d)^{M+1/2}} |c\tau + d|^{-2s} \Big|_{s=0}.$$

Now we define the special value of symmetric square $D_f(M, \chi)$ when f is not necessary a cusp form by deleting the "addition member" S in (5). Assuming this definition one can rewrite (5):

$$(6) \qquad F(2\nu + M + t + 1, M, \chi) = (2\pi i)^{-\nu} \frac{(4\pi)^{2\nu+M+t}\Gamma(M+1/2)}{\Gamma(M+2\nu+t)\Gamma(M+\nu+1/2)} \times \mathbf{Tr}_1^{4r^2} F_\nu^{t+1/2, M+1/2}(h_\chi, E^\chi_{M+1/2}).$$

2.3. Taylor expansions of Jacobi forms.

PROPOSITION 2. — *Let $\phi \in J_{k,r}$ be a Jacobi form. We denote by $X_\nu(\phi)(\tau)$ the Taylor expansion coefficients of the function ϕ on z:*

$$\phi(\tau,z)=\sum_{\nu\geq 0} X_\nu(\phi)(\tau)z^\nu.$$

a) *The function*

$$\xi_\nu^r(\phi)(\tau)=\sum_{0\leq\mu\leq\nu/2}\frac{(-2\pi ir)^\mu\Gamma(k+\nu-\mu-1)}{\Gamma(k+\nu-1)\Gamma(\mu+1)}X_{\nu-2\mu}^{(\mu)}(\phi)(\tau)$$

is a modular form of weight $k+\nu$ on $SL_2(\mathbf{Z})$. In other words, one can define operators $\xi_\nu^r : J_{k,r}\to M_{k+\nu}$.

b) *The following identities take place:*

$$X_\nu(\phi)(\tau)=\sum_{0\leq\mu\leq\nu/2}\frac{(2\pi ir)^\mu\Gamma(k+\nu-2\mu)}{\Gamma(k+\nu-\mu)\Gamma(\mu+1)}(\xi_{\nu-2\mu}^r)^{(\mu)}(\phi)(\tau).$$

It means that the set of modular forms $\xi_\nu^r(\phi)(\tau)$ defines the Jacobi form ϕ uniquely.

c) *Let $SL_2(\mathbf{Z})=\bigcup_j H\sigma_j$ be a finite coset decomposition. Then*

$$\sum_j \xi_\nu^r(\phi)\mid_{k,r}\sigma_j=\xi_\nu^r\Big(\sum_j\phi\mid\sigma_j\Big).$$

In other words, the operators ξ_ν^r commute with the trace operator.

d) *One can construct the operators ξ_ν^r using the Fourier coefficients $c(n,m)$ of Jacobi form ϕ. If*

$$\phi(\tau,z)=\sum_{n\geq 0}\sum_{\substack{m\in\mathbf{Z}\\ m^2\leq 4r^2n}} c(n,m)e(n\tau+mz)\in J_{k,r^2}$$

then

$$(2\pi)^{-2\nu-t}\xi_{2\nu+t}^{r^2}(\phi)=\sum_{n\geq 0}\sum_{m\in\mathbf{Z}}\sum_{0\leq\mu\leq\nu}(-1)^\mu$$
$$\times\frac{\Gamma(2\nu+t+k-\mu-1)}{\Gamma(\mu+1)\Gamma(2\nu+t-2\mu+1)\Gamma(2\nu+k+t-1)}m^{2\nu+t-2\mu}r^{2\mu}n^\mu c(n,m)e(n\tau).$$

Here one must take t equal to 0 or 1 to make the number $k+t$ even.

Parts a), b) and d) of this proposition are contained in [2], Theorem 3.2. To prove c) we consider firstly the case when k is even. Consider the space $M_{k,r}$ of holomorphic functions ϕ of two variables with the property

$$\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e\left(\frac{rcz^2}{c\tau+d}\right)\phi(\tau,z)$$

for every element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $SL_2(\mathbf{Z})$. The differential operator

$$L_k = 8\pi i r\frac{\partial}{\partial\tau} - \frac{\partial^2}{\partial z^2} - \frac{2k-1}{z}\frac{\partial}{\partial z}$$

maps an element ϕ of $M_{k,r}$ to an element $L_k\phi$ of $M_{k+2,r}$. It is easy to see that

$$L_k(\phi|_{k,r}\sigma) = (L_k)|_{k+2,r}\sigma \quad \text{for all } \sigma \in SL_2(\mathbf{Z}), \phi \in M_{k,r},$$

$$\xi^r_{2\nu}(\phi)(\tau) = (L_{k+2\nu-2} \circ L_{k+2\nu-4} \circ \cdots \circ L_k\phi)(\tau, 0).$$

Part c) of the Proposition 2 in the case of even k follows immediately from these formulas. To prove it in the case when k is odd one must consider the function

$$\phi_1(\tau,z) = z\phi(\tau,z) \in M_{k+1,r}.$$

It has the same Fourier coefficients as ϕ, the number $k+1$ is even and it is enough to apply part a) to finish the proof.

2.4.

Let $\chi : \mathbf{Z}/r\mathbf{Z} \to \mathbf{C}$ be a Dirichlet character; $t = 0$ or 1, $\chi(-1) = (-1)^t$. We denote by θ_χ the theta-function associated with character χ:

$$\theta_\chi(\tau,z) = 1/2(4\pi i)^{-t}\sum_{m\in\mathbf{Z}}\chi(m)e(m^2\tau + 2mrz).$$

LEMMA 1. — *Let $SL_2(\mathbf{Z}) = \bigcup_j \Gamma_0(4r^2)\sigma_j$ be a right coset decomposition. Then for a natural number $k \geq 2$*

$$E^\chi_{M+1}(\tau,z) = \sum_j (\theta_\chi(\tau,z)E^\chi_{k-1/2}(\tau))\Big|_{k,r^2}\sigma_j.$$

To prove this lemma we use the following assertion connected with the action of elements of $\Gamma_0(4r^2)$ on the function θ_χ.

LEMMA 2. — *Let* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4r^2)$, *and let* χ *be a primitive Dirichlet character modulo* r. *Then*

$$\theta_\chi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = \chi(d)\left(\frac{-1}{d}\right)^t \left(\frac{c}{d}\right) \varepsilon_d^{-1}(c\tau+d)^{1/2} e\left(\frac{cr^2z^2}{c\tau+d}\right)\theta_\chi(\tau,z),$$

where $\varepsilon_d = 1$ *or* i *according as* $d \equiv 1$ *or* $3 \bmod 4$.

This lemma follows immediately from the modular properties of the function h_χ (cf. [8]) and the following three propositions.

PROPOSITION 3 ([8], Proposition 2.2, p. 457). — *If* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4r^2)$, *then*

$$h_\chi(\gamma(\tau)) = \chi(d)\left(\frac{-1}{d}\right)^t (c\tau+d)^{t+1/2} h_\chi(\tau).$$

PROPOSITION 4. — *The following identity holds true:*

$$\theta_\chi(\tau,z) = (rz)^t \tilde{h}_\chi^{1/2+t}(\tau, rz).$$

PROPOSITION 5. — *If* f *is a smooth function,* ℓ *a natural number, and* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$ *then*

$$\tilde{f}^\ell\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^\ell e\left(\frac{cr^2z^2}{c\tau+d}\right)\left[\widetilde{(c\tau+d)^{-\ell} f\left(\frac{a\tau+b}{c\tau+d}\right)}\right]^\ell.$$

The Cohen's operator in the right-hand side of this equation acts on the function in the square parentheses. Now we turn to the proof of the propositions.

Proof of Proposition 4. — After differentiation and changing the order of summation one has:

$$\tilde{h}_\chi^{1/2+t}(\tau,z) = \frac{\Gamma(t+1/2)}{2}\sum_{n\in\mathbf{Z}}\chi(n)e(n^2\tau)\sum_{\nu\geq 0}\frac{(2\pi i z)^{2\nu} n^{t+2\nu}}{\Gamma(\nu+1)\Gamma(t+1/2+\nu)}.$$

To finish the proof of Proposition 4, it is sufficient to use the Legendre formulas for Γ-function and to observe that

$$\sum_{n\in\mathbf{Z}}\chi(n)e(n^2\tau)\sum_{\nu\geq 0}\frac{(4\pi i n z)^{2\nu+(1-t)}}{\Gamma(2\nu+(1-t)+1)} = 0.$$

Proof of Proposition 5. — We consider a function $E(\tau) = 1/(\tau - \bar{\tau})$ on the upper-half plane. The bar denotes the complex conjugation. This function satisfies the following functional equation:

$$E\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 E(\tau) - c(c\tau+d)$$

for every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$. We construct a two-variable function $G_{f,\ell}(\tau, z)$ associated with the function $f(\tau)$:

$$G_{f,\ell}(\tau, z) = \exp(z^2 E(\tau)) \sum_{\nu \geq 0} \frac{z^{2\nu}}{\Gamma(\nu+1)\Gamma(t+\ell)} f^{(\nu)}(\tau).$$

From the following identities proved by Cohen ([1], p. 281) one gets the statement of Proposition 3:

$$G_{f,\ell}(\tau, z\sqrt{2\pi i}) = e(z^2 E(\tau)) \frac{1}{\Gamma(\ell)} \tilde{f}^{\ell}(\tau, z),$$

$$G_{f,\ell}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^{-\ell} G_{f|_{\ell}\gamma,\ell}(\tau, z).$$

Now we turn to the proof of Lemma 1. We claim that for every integer $k \geq 2$

$$E_k^{\chi}(\tau, z) = \sum_{(c,d)} \theta_\chi|_{k,r^2} \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau, z),$$

where for each pair of integers (c, d) the numbers a and b are such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$ and the sum is carried out through all such pairs (c, d) for which an appropriate pair (a, b) exists. One has:

$$\begin{aligned}
&\sum_{(c,d)} \theta_\chi|_{k,r^2} \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau, z) \\
&\quad = \sum_{(c,d)} (c\tau+d)^{-k} e\left(-\frac{cr^2z^2}{c\tau+d}\right) \theta_\chi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) \\
&\quad = \frac{1}{2}(4\pi i)^{-t} \sum_{(c,d)} (c\tau+d)^{-k} e\left(-\frac{cr^2z^2}{c\tau+d}\right) \\
&\qquad \times \sum_{m \in \mathbf{Z}} \chi(m) e\left(m^2 \frac{a\tau+b}{c\tau+d}\right) e\left(2mr\frac{z}{c\tau+d}\right) \\
&\quad = \frac{1}{2}(4\pi i)^{-t} \sum_{(c,d)} \sum_{\lambda \in \mathbf{Z}} \chi(\lambda)(c\tau+d)^{-k} e\left(\lambda^2 \frac{a\tau+b}{c\tau+d} + 2r\lambda\frac{z}{c\tau+d} - r^2\frac{cz^2}{c\tau+d}\right) \\
&\quad = E_k^{\chi}(\tau, z).
\end{aligned}$$

On the other side one can apply Lemma 2:

$$
\begin{aligned}
E_k^\chi(\tau,z) &= \sum_{(c,d)} \theta_\chi|_{k,r^2}\begin{pmatrix} a & b \\ c & d\end{pmatrix}(\tau,z) \\
&= \sum_j \left(\sum_{\substack{(c,d)\\ c\equiv 0 \bmod 4r^2}} \theta_\chi|_{k,r^2}\begin{pmatrix} a & b \\ c & d\end{pmatrix}(\tau,z)\right)\Bigg|_{k,r^2} \sigma_j \\
&= \sum_j \left(\theta_\chi(\tau,z) \sum_{\substack{(c,d)\\ c\equiv 0 \bmod 4r^2}} \frac{\chi(d)\left(\frac{-1}{d}\right)\left(\frac{c}{d}\right)\varepsilon_d^{-2t-1}}{(c\tau+d)^{k-1/2}}\right)\Bigg|_{k,r^2} \sigma_j \\
&= \sum_j \left(\theta_\chi(\tau,z)E_{k-1/2}(\tau)\right)\big|_{k,r^2}\,\sigma_j.
\end{aligned}
$$

Lemma 1 is proved.

2.5. Proof of the Main Theorem.

It is enough to calculate the Taylor expansion coefficients of the Jacobi-Eisenstein series E_{M+1}^χ. Taking into account Proposition 4 we have:

(7) $$\theta_\chi(\tau,z)E_{M+1/2}(\tau) = \Gamma(t+1/2)\sum_{\nu\geq 0}(2\pi i)^\nu \frac{(rz)^{2\nu+t}}{\Gamma(\nu+1)\Gamma(t+1/2+\nu)} h_\chi^{(\nu)} E_{M+1/2}.$$

We see from (7) that

$$X_{2\nu+t}(\theta_\chi(\tau,z)E_{M+1/2}(\tau)) = \frac{\sqrt{\pi}}{2^t} r^{2\nu+t}\frac{(2\pi i)^\nu}{\Gamma(\nu+1)\Gamma(t+1/2+\nu)} h_\chi^{(\nu)}E_{M+1/2}.$$

Using assertion a) of Proposition 2 and (4) we get

(8) $$\xi_{2\nu+t}(\theta_\chi(\tau,z)E_{M+1/2}(\tau)) = \frac{\sqrt{\pi}}{2^t}r^{2\nu+t}(2\pi i)^\nu \times\frac{\Gamma(M+t+\nu)}{\Gamma(\nu+1)\Gamma(M+2\nu+t)\Gamma(t+1/2+\nu)}\mathbf{F}_\nu^{t+1/2,M+1/2}(h_\chi, E_{M+1/2}).$$

Applying to (8) Lemma 1 and assertion c) of Proposition 2:

(9) $$\xi_{2\nu+t}(E_{M+1}^\chi) = \frac{\sqrt{\pi}}{2^t}r^{2\nu+t}(2\pi i)^\nu \times\frac{\Gamma(M+t+\nu)}{\Gamma(\nu+1)\Gamma(M+2\nu+t)\Gamma(t+1/2+\nu)}\mathbf{Tr}_1^{4r^2}\mathbf{F}_\nu^{t+1/2,M+1/2}(h_\chi, E_{M+1/2}).$$

Now one can rewrite the right-hand side of (9) in accordance with (6):

$$\xi_{2\nu+t}(E^{\chi}_{M+1}) = r^{2\nu+t}2^{1-4M-3t+2\nu}\pi^{1/2-M-t}(-1)^{\nu} \times \frac{\Gamma(2M+t+2\nu)}{\Gamma(1+2\nu+t)\Gamma(M+1/2)}F(2\nu+M+t+1,M,\chi).$$

Application of assertion b) of the Proposition 2 finishes the proof of the Main Theorem.

3. CALCULATION OF THE JACOBI-EISENSTEIN SERIES FOURIER

In the simplest case of the Jacobi-Eisenstein series $E_{k,1}$ of index one, these coefficients were calculated in [2]. To prove the p-adic interpolation theorem for symmetric squares of cusp forms we need some information about the Fourier coefficients $e_k^{\psi}(n,m)$ of the Jacobi-Eisenstein series

$$E = (4\pi i)^t E_k^{\psi}(\tau,z) = \sum_{n,m} e_k^{\psi}(n,m)e(n\tau+mz),$$

where ψ is a primitive Dirichlet character modulo $r = p^L$, $L \geq 0$, p is a fixed odd prime number. We denote by $G(\psi)$ the Gauss sum associated with the character ψ and by $L(s,\chi)$ the Dirichlet L-function associated with the character χ:

$$G(\psi) = \sum_{m \bmod r} \psi(m)e(m/r),$$

$$L(s,\chi) = \sum_{m\geq 0} \chi(m)m^{-s} \quad (\Re s > 1).$$

We set $G(\psi) = 1$ if the character ψ is trivial.

THEOREM 2. — *In the above notations one has:*

a) $e_k^{\psi}(n,m) = 0$ *if* $m^2 > 4r^2n$.

b) $e_k^{\psi}(n,m) = \psi(m/2r) = \psi(\sqrt{n})$ *if* $m^2 = 4r^2n$.

c) *If* $m^2 < 4r^2n$ *then*

$$e_k^{\psi}(n,m) = i^k \frac{\pi^{k-1/2}}{2^{k-2}\Gamma(k-1/2)} r^{2-2k} G(\psi) \frac{L(k-1,\xi_D\psi)}{L(2k-2,\psi^2)} g_1 Y. \tag{10}$$

In (10) $D = m^2 - 4r^2n < 0$, ξ_D *is the Dirichlet character associated with the imaginary quadratic field* $\mathbf{Q}(\sqrt{D})$,

$$Y = Y(\psi,m,n) = \prod_{q|D} Y_q,$$

where the product is taken over all prime numbers q dividing D and Y_q is a polynomial in the variable $\{\psi(q)q^{1-k}\}$, $g_1 = 1$ if ψ is the trivial character (i.e. $L = 0$). If $L > 0$ then

$$g_1 = g_1(\psi, m, n) = \sum_{\ell \geq 0} p^{\ell(1-k)-L}(p/(p-1))^{\delta_{\ell,0}} \tag{11}$$

$$\times \sum_{\substack{\lambda \bmod p^{L+\ell} \\ \lambda^2 r - \lambda m + rn \equiv 0 \bmod p^\ell}} \psi(L)\bar{\psi}((\lambda^2 r - \lambda m + rn)/p^\ell),$$

where $\delta_{\ell,0} = 1$ or 0 according as $\ell = 0$ or $\ell > 0$.

d) *Let*

$$H(\ell) = \sum_{\substack{\lambda \bmod p^{\ell+L} \\ Q(\lambda) \equiv 0 \bmod p^\ell}} \psi(\lambda)\bar{\psi}(Q(\lambda)/p^\ell),$$

be the internal sum in (11), $Q(\lambda) = \lambda^2 r - \lambda m + rn$; $r = p^L$, $m = p^a \hat{m}$, $a \geq 0$, $p \nmid \hat{m}$; $n = p^b \hat{n}$, $b > 2L > 0$, $p \nmid \hat{n}$.

Then $H(\ell) \neq 0$ implies $\ell = a$, and ℓ is equal to 0 or 1.

Remark. — One can prove that the summation over ℓ in (11) is finite also in the case when $b \leq 2L$, but we do not need this fact for our purposes.

The assertions a) and b) of Theorem 2 are contained in [2] and are almost evident. The proof of part c) is similar to the Fourier coefficients calculation in the case when the character ψ is trivial. This calculation is contained in [2], Theorem 2.1. We will prove now the assertion d). One can rewrite the condition $Q(\lambda) \equiv 0 \bmod p^\ell$ as:

$$p^L\lambda^2 - p^a\hat{m}\lambda + p^{b+L}\hat{n} \equiv 0 \bmod p^\ell. \tag{12}$$

PROPOSITION 6.

a) *If $a > L$ then $H(\ell) = 0$.*

b) *If $a \leq L$ and $\ell \neq a$ then $H(\ell) = 0$.*

To prove part a) of the proposition, we consider three cases: $0 \leq \ell < L$; $\ell = L$; $\ell > L$.

If $0 \leq \ell < L$ then (12) is true for any λ but $Q(\lambda)/p^\ell \equiv 0 \bmod p$ yields $\psi(Q(\lambda)/p) = 0$.

If $\ell = L$ then

$$\begin{aligned} H(\ell) &= \sum_{\lambda \bmod p^{2L}} \psi(\lambda)\bar{\psi}(\lambda^2 - \hat{m}\lambda p^{a-L} + \hat{n}p^b) \\ &= \sum_{\substack{\lambda \bmod p^{2L} \\ p \nmid \lambda}} \bar{\psi}(\lambda - \hat{m}p^{a-L}) \\ &= \sum_{\lambda \bmod p^{2L}} \bar{\psi}(\lambda - \hat{m}p^{a-L}) = 0. \end{aligned}$$

If $\ell > L$ then (12) implies $\lambda^2 \equiv 0 \bmod p$ and $\psi(\lambda) = 0$.

To prove part b) we assume that $a < L$ and consider the cases $0 \le \ell < a$ and $\ell > a$. If $0 \le \ell < a$, then $Q(\lambda)/p^\ell \equiv 0 \bmod p$ yields $\bar{\psi}(Q(\lambda)/p^\ell) = 0$. If $\ell > a$ then (12) implies $\lambda^2 \equiv 0 \bmod p$ and $\psi(\lambda) = 0$.

Now we assume that $a = L$ and consider three cases: $0 \le \ell < L$, $L < \ell \le 2L$ and $\ell > 2L$.

If $0 \le \ell < L$ then $Q(\lambda)/p^\ell \equiv 0 \bmod p$ and $\bar{\psi}(Q(\lambda)/p^\ell) = 0$.

If $L < \ell \le 2L$, then

$$\begin{aligned} H(\ell) &= \sum_{\substack{\lambda \bmod p^{\ell+L} \\ Q(\lambda) \equiv 0 \bmod p^\ell}} \psi(\lambda)\bar{\psi}((\lambda^2 - \hat{m}\lambda + \hat{n}p^b)/p^{\ell-L}) \\ &= \sum_{\substack{\lambda \bmod p^{\ell+L} \\ \lambda \equiv \hat{m} \bmod p^{\ell-L}}} \psi(\lambda)\bar{\psi}((\lambda^2 - \hat{m}\lambda)/p^{\ell-L}) \\ &= \sum_{\substack{\lambda \bmod p^{\ell+L} \\ \lambda = \hat{m} \bmod p^{\ell-L}}} \bar{\psi}((\lambda - \hat{m})/p^{\ell-L}) \\ &= \sum_{\alpha \bmod p^{2L}} \bar{\psi}(\alpha) = 0. \end{aligned}$$

Here we have done the variable change $\lambda = \hat{m} + \alpha p^{\ell-L}$.

If $\ell > 2L$, then $Q(\lambda + p^\ell)/p^\ell \equiv Q(\lambda)/p^\ell \bmod p^L$ implies

$$H(\ell) = p^L \sum_{\substack{\lambda \bmod p^\ell \\ Q(\lambda) \equiv 0 \bmod p^\ell}} \psi(\lambda)\bar{\psi}((\lambda^2 - \hat{m}\lambda + \hat{n}p^b)/p^{\ell-L}).$$

Let $\lambda = \alpha + \beta p^{\ell-L}$; $\alpha \bmod p^{\ell-L}$; $\beta \bmod p^L$. After this variable change we have

$$H(\ell) = p^L \sum_{\substack{\alpha \bmod p^{\ell-L} \\ Q(\alpha)\equiv 0 \bmod p^\ell}} \sum_{\beta \bmod p^L} \psi(\alpha+\beta p^{\ell-L})\bar{\psi}((\alpha^2-\hat{m}\alpha+\hat{n}p^b)/p^{\ell-L} + \beta(2\alpha-\hat{m})+\beta^2 p^{\ell-L})$$

$$= p^L \sum_{\substack{\alpha \bmod p^{\ell-L} \\ Q(\alpha)\equiv 0 \bmod p^\ell}} \sum_{\beta \bmod p^L} \bar{\psi}((\alpha^2-\hat{m}\alpha+\hat{n}p^b)/p^{\ell-L}+\beta(2\alpha-\hat{m})).$$

The condition $Q(\alpha)\equiv 0 \bmod p^\ell$ yields $\alpha^2-\alpha\hat{m}+\hat{n}p^b \equiv 0 \bmod p^{\ell-L}$. It takes place only if $\alpha\equiv 0$ or $\hat{m} \bmod p$. In both cases $H(\ell)=0$.

Now Proposition 6 is proved and we are able to finish the proof of part d) of Theorem 2. One can deduce from Proposition 6 that the Fourier expansion coefficient (10) may become non-zero only if $a\le L$ and $\ell=a$. If $1<a<L$, then

$$\begin{aligned}
H(\ell) &= \sum_{\substack{\lambda \bmod p^{a+L} \\ Q(\lambda)\equiv 0 \bmod p^a}} \psi(\lambda)\bar{\psi}(Q(\lambda)/p^a) \\
&= \sum_{\lambda \bmod p^{a+L}} \psi(\lambda)\bar{\psi}(p^{L-a}\lambda^2-\hat{m}\lambda+\hat{n}p^{b+L-a}) \\
&= \sum_{\substack{\lambda \bmod p^{a+L} \\ p\nmid\lambda}} \psi(p^{L-a}\lambda-\hat{m}) \\
&= p^L\bar{\psi}(-\hat{m}) \sum_{\substack{\lambda \bmod p^a \\ p\nmid\lambda}} \psi(p^{L-a}\lambda+1) \\
&= p^L\bar{\psi}(-\hat{m}) \left(\sum_{\lambda \bmod p^a} \psi(p^{L-a}\lambda+1) - \sum_{\lambda \bmod p^{a-1}} \psi(p^{L-a}\lambda+1) \right) = 0,
\end{aligned}$$

because for a primitive Dirichlet character ψ, both sums in the parentheses are equal to zero if $a>1$. It remains to consider only one case: $a=L=\ell$. Then

$$\begin{aligned}
H(\ell) &= \sum_{\lambda \bmod p^{2L}} \psi(\lambda)\bar{\psi}(\lambda^2-\hat{m}\lambda+\hat{n}p^b) \\
&= \sum_{\substack{\lambda \bmod p^{2L} \\ p\nmid\lambda}} \psi(\lambda-\hat{m}) = p^L\bar{\psi}(-\hat{m}) \sum_{\lambda \bmod p^{L-1}} \psi(p\lambda+1) = 0.
\end{aligned}$$

4. p-ADIC INTERPOLATION OF SYMMETRIC SQUARE SPECIAL VALUES

In this section we use the results of the two previous sections to construct the p-adic interpolation of the symmetric squares special values of cusp forms. We fix an odd prime number p and an embedding $i_p : \overline{\mathbf{Q}} \to \mathbf{C}_p$ of the algebraic closure of the field of rational numbers $\mathbf{Q}$ into Tate's field. We shall make no difference between $\mathbf{Q}$ and it's image under i_p and omit symbol i_p in formulas. One can construct a $\mathbf{C}_p$-analytical function on $X_p = \mathrm{Hom}_{\mathrm{contin}}(\mathbf{Z}_p^*, \mathbf{C}_p^*)$ as a non-archimedean Mellin transform of some bounded p-adic measure μ on $\mathbf{Z}_p^*$:

$$L_\mu(x) = \mu(x) = \int_{\mathbf{Z}_p^*} x d\mu.$$

We identify the elements of the torsion subgroup of $X_p^{\mathrm{tors}} \subset X_p$ with primitive Dirichlet characters modulo powers of p.

The symbol x_p will denote the natural embedding $\mathbf{Z}_p^* \to \mathbf{C}_p^*$, so that $x_p \in X_p$ and all integers k can be considered as the characters $x_p^k : y \mapsto y^k$.

The existence of a special values p-adic interpolation of some zeta function is equivalent to the existence of a p-adic measure with given special values ([5]). We shall use the following important fact to prove the existence of these measures.

4.1. The abstract Kummer congruences.

PROPOSITION 7 ([5], [7]). — *Let $\{f_j\}$ be a family of continuous functions from $\mathbf{Z}_p^*$ to the ring of integers $\mathcal{O}_p$ in $\mathbf{C}_p$. Assume that the set of finite $\mathbf{C}_p$-linear combinations of f_j is dense in the space of all such functions. Let $\{a_j\}$ be a family of elements in $\mathcal{O}_p$. Then the existence of a measure with the property*

$$\int_Y f\ d\mu = a_j$$

is equivalent to the fact that the following statement is true: for every finite set of elements $b_j \in \mathbf{C}_p$ it follows from $\left\{\sum_j b_j f_j(y) \in p^n\mathcal{O}_p \text{ for every } y \in Y\right\}$ that $\left\{\sum_j b_j a_j \in p^n\mathcal{O}_p\right\}$.

To formulate the theorem we need the definition of p-ordinary form.

A cusp form
$$f(\tau) = \sum_{n\geq 1} a(n)e(n\tau) \in S_k,$$
normalized by the condition $a(1) = 1$ which is an eigenform of Hecke algebra is called p-ordinary if $|a(p)|_p = 1$.

We denote the subspace of the p-ordinary forms of weight k by $S_k^0 \subseteq S_k$.

For a prime number q we denote by $\alpha = \alpha(q)$ and $\beta = \beta(q)$ the roots of the Hecke polynomial $X^2 - a(q)X + q^{k-1}$. We define by multiplicativity the numbers $\alpha(n)$ and $\beta(n)$ for every natural number n.

THEOREM 3. — *Let $c > 1$ be a natural number, $p \nmid c$. Let f be a p-ordinary form of even weight k. Then there exist a $\mathbf{C}_p$-analytic function $D^c : X_p \to \mathbf{C}_p$ such that its value $D^c(x_p^M \chi)$ for $3 \leq M \leq k-1$ equals*
$$2^{-4M-2t+1}\pi^{-2M-k+1}r^{k+2M-1}i^{k-M-2}\Gamma(k+M-1)\Gamma(M)G(\chi)^{-2}L(2M,\chi^2)$$
$$\times(1-\chi(c)^2c^{-2M})\frac{1}{\alpha(r^2)}TD_f(M,\chi),$$
where $\chi \in X_p^{\mathrm{tors}}$ is a Dirichlet character, $1 \leq M < r-1$, M is an integer,
$$T = \begin{cases} 1, & \text{if } \chi \text{ is a non-trivial character } (r>1) \\ (1-p^{M-1})(1-\alpha(p)^{-2}p^{M+k-2})(1-\alpha(p)^{-2}p^{k-2}), & \text{otherwise.}\end{cases}$$
Here r is the conductor of χ, $G(\chi)$ is the Gauss sum, associated with χ.

Proof. — One can assume that χ is primitive. Using the notation
$$\Lambda(k,M,t) = 2^{k-5M-4t}\pi^{1/2-M-t}i^{k-M-t-1}\frac{\Gamma(2M+2\nu+t)}{\Gamma(2\nu+t+1)\Gamma(M+1/2)},$$
$$\Lambda_1(k,M,t) = \frac{(2\pi)^{k-M-1}}{\Gamma(k-M)}\Lambda(k,M,t)^{-1}(4\pi i)^{-t},$$
definition (2), and statement (3) one gets
$$\sum_{j=1}^{\dim M_k}\frac{f_j}{\langle f_j, f_j\rangle}D_{f_j}(M,\chi) = \Lambda_1(k,M,t)\sum_{n\geq 1}\sum_{m}\sum_{0\leq\mu\leq(k-M-1)/2}(-1)^{\mu}$$
$$\times\frac{\Gamma(k-\mu-1)\Gamma(k-M)}{\Gamma(\mu+1)\Gamma(k-M-2\mu)\Gamma(k-1)} \tag{13}$$
$$\times m^{k-M-2\mu-1}r^{-(k-M-2\mu-1)}n^{\mu}e^{\chi}_{M+1}(n,m)e(n\tau).$$

The idea is to apply some operators to both sides of (13) to get some "good" p-adic properties in the right-hand side of the obtained identity keeping under control what happens in the left-hand side. We denote by V and U the operators

$$f|U(d) = \sum_{n\geq 0} a(dn)e(n\tau) = d^{k/2-1} \sum_{u \bmod d} f|_k \begin{pmatrix} 1 & u \\ 0 & d \end{pmatrix} \in M_k(Nd, \chi),$$

$$f|V(d) = f(d\tau) = d^{-k/2} f|_k \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \in M_k(Nd, \chi),$$

$$V(d) \circ U(d) = \mathrm{id}\,.$$

It is known [3] that for a natural number s there exists a limit

$$\mathcal{E}_s = \lim U(p)^{sp^v}.$$

We apply the operator $\mathcal{E}_s \circ V(r^2)$ to both sides of (13). To calculate the limit in the left-hand side we consider the modular forms on $\Gamma_0(p^2)$

$$f_{j,0}(\tau) = f(\tau) - \alpha_j f(p\tau), \quad f_{j,1}(\tau) = f(\tau) - \beta_j f(p\tau).$$

We denote

$$A_j = \lim_{v\to\infty} \alpha_j^{sp^v}, \quad B_j = \lim_{v\to\infty} \beta_j^{sp^v}.$$

It is clear that one of the numbers A_j,B_j is zero because $\alpha_j\beta_j = p^{k-1}$. For a p-ordinary form f_j one of them is non-zero. Without loss of generality one can assume that $A_j \neq 0$ (i.e. $|\alpha_j|_p = 1$). After noticing that

$$f_j = \frac{\alpha_j}{\alpha_j - \beta_j} f_{j,1} + \frac{\beta_j}{\beta_j - \alpha_j} f_{j,0},$$

we can write

$$\text{(14)} \qquad \Lambda_1(k, M, \chi)^{-1} F(k, M, \chi)|V(r^2)|\mathcal{E}_s = \sum_{j=1}^{\dim S_k^0} \frac{1}{\langle f_j, f_j\rangle} D_{f_j}(M, \chi) \frac{A_j f_{j,1}}{\alpha_j(r^2)} (\alpha_j(p) - \beta_j(p))^{-1}.$$

We denote by $c_s(k, M, \chi, n)$ the Fourier coefficients of the modular form in the left side of (14):

$$\text{(15)} \qquad \mathcal{F}(k, M, \chi) = F(k, M, \chi)|V(r^2)|\mathcal{E}_s = \sum_{n\geq 0} c_s(k, M, \chi, n)e(n\tau)$$

and consider the limit in the right side of (15). Now assume that χ is non-trivial. Using part d) of Theorem 2 and the notation

$$\Lambda_2(k, \mu, \chi) = i^{M+1} \frac{\pi^{M+1/2}}{2^{M-1}\Gamma(M + 1/2)} G(\chi) L(2M, \chi^2)^{-1} r^{-(k+M-1)},$$

we can rewrite (15):

$$\begin{aligned}&\Lambda_2(k,\mu,\chi)^{-1}\Lambda_1(k,M,\chi)^{-1}c_s(k,M,\chi,n)\\ &= \lim_{v\to\infty} \sum_{\substack{m\not\equiv 0 \bmod p\\ D_1=4np^{sp^v}-m^2<0}} {D_1}^{M-1/2}\bar{\chi}(-m)L(M,\xi_{D_1}\chi)Y(n,m,\chi,M)m^{k-M-1}\\ &\quad - p^{k-2} \sum_{\substack{m\not\equiv 0 \bmod p\\ D_2=4np^{sp^v-2}-m^2<0}} {D_2}^{M-1/2}\bar{\chi}(-m)L(M,\xi_{D_2}\chi)Y(n,m,\chi,M)m^{k-M-1},\end{aligned}$$

and apply the following assertion.

PROPOSITION 8 ([6]). — *Let ω be a primitive Dirichlet character modulo A, $(p,A)=1$. For an arbitrary integer $c>1$ such that $(c,pA)=1$, there exists a $\mathbf{C}_p$-valued measure $\mu^+(c,\omega)$ on $\mathbf{Z}_p^*$. This measure is uniquely defined by the following condition:*

$$\begin{aligned}&\int_{\mathbf{Z}_p^*} \bar{\chi}x_p^M\,d\mu^+(c,\omega)\\ &\qquad= (1-\chi\omega(c)c^{-M})\frac{A^M r^M}{G(\omega\chi)}L_{pA}(M,\chi\omega)\frac{2i^\delta\Gamma(M)\cos(\pi(M-\delta)/2)}{(2\pi)^M}X,\end{aligned}$$

where

$$X=\begin{cases}1, & \text{if } \chi \text{ is non-trivial}\\ (1-\overline{\omega(q)}q^{M-1})(1-\omega(q)q^{-M}), & \text{otherwise,}\end{cases}$$

$\delta=0$ or 1; $(-1)^\delta=\chi\omega(-1)$; M a positive integer.

Introducing the notation

$$\Lambda_3(M,\chi)=\frac{1}{2\Gamma(M)}r^{-M}G(\chi)(-2\pi i)^M,$$

one has

$$\begin{aligned}&(1-\chi^2(c)c^{-2M})\Lambda_3(M,\chi)^{-1}\Lambda_2(k,\mu,\chi)^{-1}\Lambda_1(k,M,\chi)^{-1}c_s(k,M,\chi,n)\\ &= \lim_{v\to\infty} \sum_{\substack{m\not\equiv 0 \bmod p\\ D_1=4np^{sp^v}-m^2<0}} {D_1}^{-1/2}m^{k-1}G(\xi_{D_1})(1+\chi\xi_{D_1}(c)c^{-M})\\ &\qquad\times(-D_1/Am)^M\bar{\chi}(-D_1/Am)Y(n,m,\chi,M)\int_{\mathbf{Z}_p^*}\bar{\chi}x_p^M\,d\mu^+(c,\xi_{D_1})\\ &\quad - p^{k-2} \sum_{\substack{m\not\equiv 0 \bmod p\\ D_2=4np^{sp^v-2}-m^2<0}} {D_2}^{-1/2}m^{k-1}G(\xi_{D_2})(1+\chi\xi_{D_2}(c)c^{-M})\\ &\qquad\times(-D_2/Am)^M\bar{\chi}(-D_2/Am)Y(n,m,\chi,M)\int_{\mathbf{Z}_p^*}\bar{\chi}x_p^M\,d\mu^+(c,\xi_{D_2}).\end{aligned}$$

Here we used the fact that $\xi_D(r)=1$ and $\chi(D)=\chi(m^2)$ for $D=m^2-4np^{sp^v}$ if v is sufficiently large.

It remains to consider the case when χ is trivial. This case is slightly different from previous, but the calculations contain no essentially new ideas. One must apply additionally the operator $(1 - p^{M+k-2}V(p^2))(1 - p^{k-2}V(p^2))$ to keep the "good" form of the formulas.

Now we notice that in the right-hand side of the obtained equation there are Fourier coefficients of modular forms of weight k on $\Gamma_0(p^2)$. They can be expressed as follows:

$$\sum_j \lambda_j \chi(y_j) y_j^{-M} \int_{\mathbf{Z}_p^*} \bar{\chi} x_p^M d\mu^+(c, \xi_D). \tag{16}$$

The numbers λ_j in (16) are p-integers, do not depend on M and χ and the sum is finite. Proposition 7 yields this Fourier coefficients to be $\mathbf{C}_p$-analytic functions on X_p. On the other hand the modular forms with these coefficients belong to the finite-dimensional linear space of the modular forms of weight k on $\Gamma_0(p^2)$. It implies the values of each functional on this modular form to be values of some $\mathbf{C}_p$-analytic function. To complete the proof of Theorem 3 it remains to consider the linear functional $\langle \cdot, f_j \rangle$ of the Petersson scalar product with a p-ordinary form f_j.

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