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ON THE K-THEORY AND HATTORI–STALLINGS TRACES OF MINIMAL PRIMITIVE FACTORS OF ENVELOPING ALGEBRAS OF SEMISIMPLE LIE ALGEBRAS : THE SINGULAR CASE

by Patrick POLO

Introduction.

Let G be a semisimple complex algebraic group, let $\mathfrak{g} = \mathrm{Lie}(G)$, and let $U = U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. Let X be the flag variety of G. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and let W be the Weyl group. For $\mu \in \mathfrak{h}^*$, let J_μ be the corresponding minimal primitive ideal of U, let $U_\mu = U/J_\mu$, and let $\mathcal{T}_{U_\mu} : K_0(U_\mu) \to \mathbb{C}$ be the Hattori–Stallings trace. One says that a weight $\mu \in \mathfrak{h}^*$ is regular if its Weyl group stabilizer W_μ is trivial, and singular otherwise. For a regular weight μ, T.J. Hodges has shown [11] that $K_0(U_\mu)$ is isomorphic to $K_0(X)$ and is therefore generated by the classes corresponding to G-linearized line bundles on X.

Moreover, in [12], Hodges used the Hattori–Stallings trace to classify, in the case where $\mathfrak{g} = \mathfrak{sl}_2$, the $\mathbb{C}$-algebras U_μ up to Morita equivalence and to obtain a short proof of Dixmier's earlier description of the isomorphism classes. For an arbitrary semisimple $\mathfrak{g}$ and a regular weight μ, it was shown in [13], in a special case, and then in [21], in general, that the value of $\mathcal{T}_{U_\mu}$ on the generators corresponding to G-linearized line bundles on X was given by Weyl's dimension formula.

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In this paper, we obtain a similar description of the Hattori–Stallings trace for singular weights. This is done in three steps. Let $G_0(U_\mu)$ be the Grothendieck group of the category of finitely generated left U_μ-modules. Firstly, we show that $G_0(U_\mu) \cong K_0(X)/\Sigma_s \operatorname{Im}(1-s)$, where the sum runs over a set of simple reflections generating W_μ. This fact, which results from [3], Quillen's localization theorem, [19], and [11], was certainly known to some specialists, but we are not aware if it was known more widely. One deduces, in particular, that $G_0(U_\mu)$ has rank $|W/W_\mu|$. Secondly, we prove that the Cartan map $K_0(U_\mu) \to G_0(U_\mu)$ is an isomorphism up to torsion. It follows that $K_0(U_\mu)_{\mathbb{Q}}$ is generated by the images of the G-linearized line bundles on X. Finally, as in [21], we show, using the Bernstein trace, that the value of $\mathcal{T}_{U_\mu}$ on these generators is given by a certain polynomial formula. It is hoped that this will bring some information about the isomorphism and Morita equivalence classes of primitive factors.

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1. Preliminaries.

1.1. Throughout the paper, the base field k is algebraically closed and of characteristic zero. Let $\mathfrak{g}$ be a semisimple Lie algebra over k and let $U = U(\mathfrak{g})$ be its enveloping algebra. Let $\mathfrak{h} \subset \mathfrak{b}$ be a Cartan subalgebra inside a Borel subalgebra of $\mathfrak{g}$, let W be the Weyl group of $(\mathfrak{g}, \mathfrak{h})$, let R^+ be the set of roots of $\mathfrak{h}$ in $\mathfrak{b}$, let Δ be the corresponding set of simple roots, and let ρ be the half-sum of the elements of R^+. For $\alpha \in R^+$, let $H_\alpha \in \mathfrak{h}$ be the corresponding coroot and let s_α be the corresponding reflection in W.

For a weight $\lambda \in \mathfrak{h}^*$, let $M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} k_{\lambda-\rho}$ be the Verma module with highest weight $\lambda - \rho$ and let $J_\lambda = \operatorname{Ann} M(\lambda)$. Let Z denote the centre of U. Via the Harish-Chandra isomorphism $Z \xrightarrow{\sim} S(\mathfrak{h})^W$, every $\lambda \in \mathfrak{h}^*$ defines a central character χ_λ. One has $U(\operatorname{Ker} \chi_\lambda) = J_\lambda$ and $J_\lambda = J_{\lambda'}$ if and only if λ and λ' are W-conjugate. Further, J_λ is a minimal primitive ideal of U. For all this, see [9], §§7.4, 8.4. Finally, let $U_\lambda = U/J_\lambda$ and let U_λ-Modf denote the category of finitely generated left U_λ-modules.

Let $\mu \in \mathfrak{h}^*$. One says that μ is antidominant if $\mu(H_\alpha) \notin \mathbb{N}^+$, for $\alpha \in R^+$. Let W_μ denote the stabilizer of μ in W and let $\Delta_\mu = \{\alpha \in \Delta \mid s_\alpha \mu = \mu\}$. Recall that μ is said to be regular if W_μ is trivial, and singular otherwise. Moreover, we shall say that μ is tamely singular if W_μ is generated by the simple reflections s_α, for $\alpha \in \Delta_\mu$. By [7], Chap. V, n° 3.3, every weight $\mu \in \mathfrak{h}^*$ is W-conjugate to at least one tamely singular, antidominant weight and hence, for the study of U_μ, there is no loss of generality in assuming that μ is antidominant and tamely singular.

1.2. Let G be a semisimple, connected and simply-connected, algebraic group over k such that $\mathrm{Lie}(G) = \mathfrak{g}$ and let $T \subset B$ be the connected subgroups corresponding to $\mathfrak{h} \subset \mathfrak{b}$. Since G is simply-connected, the character group of T identifies with the lattice of integral weights $\mathcal{P} := \{\nu \in \mathfrak{h}^* \mid \nu(H_\beta) \in \mathbb{Z},\ \forall \beta \in R^+\}$. For every rational B-module V, let $\mathcal{L}(V)$ denote the associated G-equivariant vector bundle on G/B. For $\nu \in \mathcal{P}$, the line bundle associated with $V = k_\nu$ will be denoted simply by $\mathcal{L}(\nu)$.

Let $\mathbb{Z}\mathcal{P}$ be the group algebra of $\mathcal{P}$, with its natural basis $e^\nu : \nu \in \mathcal{P}$. By [20], §6, the map $e^\nu \mapsto [\mathcal{L}(\nu)]$ induces an isomorphism $\mathbb{Z}\mathcal{P}/I \xrightarrow{\sim} K_0(G/B)$, where I denotes the ideal generated by the W-invariants in the augmentation ideal of $\mathbb{Z}\mathcal{P}$. Thus, via this isomorphism, the natural action of W on $\mathbb{Z}\mathcal{P}$ induces a W-module structure on $K_0(G/B)$.

1.3. For $\lambda \in \mathfrak{h}^*$, let $\mathcal{D}_\lambda$ denote the sheaf of twisted differential operators on G/B associated with $\lambda + \rho$. By [3], one has $\Gamma(G/B, \mathcal{D}_\lambda) \cong U_\lambda$ (see also [19] 6.1–6.2). Let $\mathcal{D}_\lambda$-Coh denote the category of coherent left $\mathcal{D}_\lambda$-modules and let Γ_λ denote the restriction to $\mathcal{D}_\lambda$-Coh of the global sections functor $\Gamma(G/B, -)$. Suppose that λ is antidominant and regular. Then, by [3], Γ_λ induces an equivalence of categories $\mathcal{D}_\lambda$-Coh $\xrightarrow{\sim}$ U_λ-Modf. By a spectral sequence argument as in [6], VI.1.10, this implies that U_λ has finite global dimension (see also [14] Theorem 3.9). Let us then recall the following theorem.

THEOREM ([11], Theorem 2). — *Let λ be a regular antidominant weight. Then the exact functor $\mathcal{E} \mapsto \Gamma(G/B, \mathcal{D}_\lambda \otimes_{\mathcal{O}_{G/B}} \mathcal{E})$ induces isomorphisms $K_n(G/B) \simeq K_n(U_\lambda)$, for $n \geq 0$. In particular, $K_0(U_\lambda)$ is a free $\mathbb{Z}$-module of rank $|W|$.*

1.4. For U-modules M and N, let $L(M, N)$ denote the set of $\mathfrak{g}$-finite vectors in $\mathrm{Hom}_k(M, N)$; this is a U-sub-bimodule of $\mathrm{Hom}_k(M, N)$, see [18], 1.2. By [17] Corollary 7.25, one has

$$U_\lambda \cong L(M(\lambda), M(\lambda)), \qquad \forall\, \lambda \in \mathfrak{h}^*. \tag{1.4.1}$$

Further, one deduces from the proof of [23] Theorem 6, the following

PROPOSITION. — *For $\lambda \in \mathfrak{h}^*$ and $\nu \in \mathcal{P}$, there is an isomorphism of U-bimodules $\Gamma(G/B, \mathcal{D}_\lambda \otimes_{\mathcal{O}_{G/B}} \mathcal{L}(\nu)) \cong L(M(\lambda-\nu), M(\lambda))$.*

Thus, one obtains the

COROLLARY. — *Let λ be a regular antidominant weight. Then, under the isomorphism $K_0(G/B) \cong K_0(U_\lambda)$ of Theorem 1.3, the class of $\mathcal{L}(\nu)$ corresponds to the class of $L(M(\lambda-\nu), M(\lambda))$, for $\nu \in \mathcal{P}$.*

2. On $G_0(U_\mu)$ and $K_0(U_\mu)$ for singular μ.

2.1. For the remainder of this paper, we fix a tamely singular, antidominant weight μ. Let $G_0(\mathcal{D}_\mu)$ denote the Grothendieck group of the category $\mathcal{D}_\mu$-Coh and let U_μ-Proj denote the category of finitely generated projective left U_μ-modules. By [3], the functor Γ_μ is exact and takes $\mathcal{D}_\mu$-Coh to U_μ-Modf. Thus, it induces a map $\gamma_\mu : G_0(\mathcal{D}_\mu) \to G_0(U_\mu)$. Also, let $\phi_\mu : K_0(U_\mu) \to G_0(\mathcal{D}_\mu)$ be the map induced by the localization functor $\Phi_\mu := \mathcal{D}_\mu \otimes_{U_\mu} -$. Clearly, $\Gamma_\mu \circ \Phi_\mu(U_\mu) \cong U_\mu$ and hence, by additivity, $\Gamma_\mu \circ \Phi_\mu(P) \cong P$, for every $P \in U_\mu$-Proj. Thus, $\gamma_\mu \circ \phi_\mu$ equals the Cartan map $c_\mu : K_0(U_\mu) \to G_0(U_\mu)$. By [11] Theorem 1, the functor $\mathcal{D}_\mu \otimes_{\mathcal{O}_{G/B}} -$ induces an isomorphism $\psi_\mu : K_0(G/B) \xrightarrow{\sim} G_0(\mathcal{D}_\mu)$. Thus, in particular, $G_0(\mathcal{D}_\mu)$ is free and hence ϕ_μ factors through a map $\overline{\phi}_\mu : K_0(U_\mu)/K_0(U_\mu)_{\mathrm{tor}} \to G_0(\mathcal{D}_\mu)$, where $K_0(U_\mu)_{\mathrm{tor}}$ denotes the torsion part of $K_0(U_\mu)$. Note that, by Proposition 1.4,

$$\gamma_\mu \circ \psi_\mu\left([\mathcal{L}(\nu)]\right) = [L(M(\mu-\nu), M(\mu))]\,, \qquad \forall\, \nu \in \mathcal{P}. \tag{2.1.1}$$

Then, one has the

THEOREM. — *There is a commuting diagram*

$$\begin{array}{ccccc}
K_0(U_\mu)/K_0(U_\mu)_{\mathrm{tor}} & \xrightarrow{\overline{\phi}_\mu} & G_0(\mathcal{D}_\mu) & \xrightarrow{\gamma_\mu} & G_0(U_\mu) \\
\cong\big\downarrow & & \cong\big\downarrow\psi_\mu^{-1} & & \cong\big\downarrow \\
K_0(G/B)^{W_\mu} & \xrightarrow{i} & K_0(G/B) & \xrightarrow{p} & \dfrac{K_0(G/B)}{\sum_{\alpha\in\Delta_\mu} \mathrm{Im}(1-s_\alpha)},
\end{array}$$

where i and p are the natural injection and projection. Both $K_0(U_\mu)$ and $G_0(U_\mu)$ have rank $|W/W_\mu|$ and the kernel and cokernel of c_μ are annihilated by $|W_\mu|$.

Remark. — The theorem gives a partial solution to the conjecture made in [11] that $K_0(U_\mu)$ is a free $\mathbb{Z}$-module of rank $|W/W_\mu|$[1].

2.2. We shall prove the theorem in several steps. First, by [3], Γ_μ is exact and induces an equivalence from the quotient category $\mathcal{D}_\mu$- Coh / $\operatorname{Ker}\Gamma_\mu$ to U_μ-Modf and hence, by Quillen's localization theorem [22] §5, Theorem 5, $G_0(U_\mu)$ is isomorphic to $G_0(\mathcal{D}_\mu)/\operatorname{Ker}\gamma_\mu$. Thus, the commutativity of the right-hand square follows from the

PROPOSITION. — *One has $\psi_\mu^{-1}(\operatorname{Ker}\gamma_\mu) = \sum_{\alpha\in\Delta_\mu} \operatorname{Im}(1-s_\alpha)$ and hence ψ_μ^{-1} induces an isomorphism $G_0(U_\mu) \cong K_0(G/B)/\sum_{\alpha\in\Delta_\mu} \operatorname{Im}(1-s_\alpha)$.*

Proof. — For $\alpha \in \Delta$, let $P_\alpha \supset B$ be the corresponding parabolic subgroup and let π_α denote the projection $G/B \to G/P_\alpha$. For $\alpha \in \Delta_\mu$, let $\mathcal{A}_\mu^\alpha$ be the sheaf of twisted differential operators on G/P_α associated with μ (see, for instance, [19] 4.9.2). If $\mathcal{N}$ is an $\mathcal{A}_\mu^\alpha$-module then, by [19] 8.1.1, $\pi_\alpha^*(\mathcal{N})$ is a $\mathcal{D}_{\mu-\rho}$-module. Moreover, by [10] p. 328-329, there is an exact sequence of left $\mathcal{D}_{\mu-\rho}$-modules

$$0 \longrightarrow \mathcal{D}_{\mu-\rho} \otimes \mathcal{L}(-\alpha) \longrightarrow \mathcal{D}_{\mu-\rho} \longrightarrow \pi_\alpha^*(\mathcal{A}_\mu^\alpha) \longrightarrow 0.$$

Tensoring this exact sequence by $\mathcal{L}(\rho)$ on the left, one obtains, using [10] A.3.1, an exact sequence of left $\mathcal{D}_\mu$-modules

$$0 \longrightarrow \mathcal{D}_\mu \otimes \mathcal{L}(\rho-\alpha) \longrightarrow \mathcal{D}_\mu \otimes \mathcal{L}(\rho) \longrightarrow \mathcal{L}(\rho) \otimes \pi_\alpha^*(\mathcal{A}_\mu^\alpha) \longrightarrow 0. \tag{2.2.1}$$

In particular, $\mathcal{L}(\rho) \otimes \pi_\alpha^*(\mathcal{A}_\mu^\alpha)$ is a coherent $\mathcal{D}_\mu$-module. One deduces that the exact functor $\mathcal{N} \mapsto \mathcal{L}(\rho) \otimes \pi_\alpha^*(\mathcal{N})$ takes $\mathcal{A}_\mu^\alpha$-Coh to $\mathcal{D}_\mu$-Coh and hence induces a map $f_\alpha : G_0(\mathcal{A}_\mu^\alpha) \to G_0(\mathcal{D}_\mu)$.

Now, it follows from [19] Theorem 8.3.1, that

$$\operatorname{Ker}\gamma_\mu = \sum_{\alpha\in\Delta_\mu} \operatorname{Im} f_\alpha. \tag{2.2.2}$$

For $\alpha \in \Delta_\mu$, let us denote by $\mathcal{L}_\alpha(V)$ the G-equivariant vector bundle on G/P_α associated with a rational P_α-module V. Then, it follows from

(1) The freeness of $K_0(U_\mu)$ has now been obtained in collaboration with M. Holland, using different ideas and techniques [15].

[11] Theorem 1, applied to G/P_α, together with [20] Proposition 6, that $\operatorname{Im} f_\alpha$ is generated by the classes of the objects

$$\mathcal{L}(\rho) \otimes \pi_\alpha^*(\mathcal{A}_\mu^\alpha \otimes \mathcal{L}_\alpha(V)) \cong \mathcal{L}(\rho) \otimes \pi_\alpha^*(\mathcal{A}_\mu^\alpha) \otimes \mathcal{L}(V),$$

for V an irreducible rational P_α-module. But, tensoring (2.2.1) by $\mathcal{L}(V)$ on the right, one obtains an exact sequence of left $\mathcal{D}_\mu$-modules

(2.2.3)

$$0 \longrightarrow \mathcal{D}_\mu \otimes \mathcal{L}(k_{\rho-\alpha} \otimes V) \longrightarrow \mathcal{D}_\mu \otimes \mathcal{L}(k_\rho \otimes V) \longrightarrow \mathcal{L}(\rho) \otimes \pi_\alpha^*(\mathcal{A}_\mu^\alpha) \otimes \mathcal{L}(V) \longrightarrow 0.$$

Let $\mathcal{P}_\alpha^+ = \{\nu \in \mathcal{P} \mid \nu(H_\alpha) \geq 0\}$ and, for $\nu \in \mathcal{P}_\alpha^+$, let $V_\alpha(\nu)$ denote the irreducible rational P_α-module with highest weight ν. Since the formal character of $V_\alpha(\nu)$ equals $e^\nu + e^{\nu-\alpha} + \cdots + e^{s_\alpha \nu}$, one deduces from (2.2.3) that

$$\begin{aligned}
[\mathcal{L}(\rho) \otimes \pi_\alpha^*(\mathcal{A}_\mu^\alpha) \otimes \mathcal{L}(V_\alpha(\nu))] &= [\mathcal{D}_\mu \otimes \mathcal{L}(k_\rho \otimes V_\alpha(\nu))] \\
&\quad - [\mathcal{D}_\mu \otimes \mathcal{L}(k_{\rho-\alpha} \otimes V_\alpha(\nu))] \\
&= [\mathcal{D}_\mu \otimes \mathcal{L}(\rho + \nu)] - [\mathcal{D}_\mu \otimes \mathcal{L}(s_\alpha \rho + s_\alpha \nu)].
\end{aligned}$$

It follows that $\psi_\mu^{-1}(\operatorname{Ker} \gamma_\mu)$ is the $\mathbb{Z}$-submodule of $K_0(G/B)$ spanned by $[\mathcal{L}(\nu + \rho)] - [\mathcal{L}(s_\alpha(\nu + \rho))] : \alpha \in \Delta_\mu,\ \nu \in \mathcal{P}_\alpha^+$. But, by [20] Proposition 4, this submodule is exactly $\sum_{\alpha \in \Delta_\mu} (1 - s_\alpha) K_0(G/B)$. This completes the proof of the proposition.

2.3. Let us briefly recall the definition of the translation functors (see [16], [5]). Let $\mathcal{C}$ be the category of finitely generated left U-modules which are locally Z-finite. For $M \in \mathcal{C}$ and $\eta \in \mathfrak{h}^*$, let $\pi_\eta M = \{x \in M \mid (\operatorname{Ker} \chi_\eta)^n x = 0, \text{ for } n \gg 0\}$. Let $\eta, \xi \in \mathfrak{h}^*$ such that $\xi - \eta \in \mathcal{P}$ and let E denote the finite dimensional irreducible left U-module whose highest weight is W-conjugate to $\xi - \eta$. Then, the functor T_η^ξ is defined by $\mathrm{T}_\eta^\xi M = \pi_\xi(E \otimes \pi_\eta M)$, for $M \in \mathcal{C}$. By [5] Corollary 2.6, $\mathrm{T}_\eta^\xi M$ belongs to $\mathcal{C}$ and it follows from the definition that T_η^ξ and T_ξ^η are both left and right adjoint.

We shall also need the analogous functors for right modules. Let $\mathcal{C}'$ be the category of finitely generated, locally Z-finite, right U-modules. For $M' \in \mathcal{C}'$ and $\eta \in \mathfrak{h}^*$, $M'{}_\eta\pi$ is defined in the obvious way. If η, ξ and E are as above then E^* is in a natural way a right U-module and, for $M' \in \mathcal{C}'$, we set $M'\,{}_\eta^\xi T = (M'{}_\eta\pi \otimes E^*)\,{}_\xi\pi$. Then, for $M, N \in \mathcal{C}$, it is easily seen that $L(M, N)$, regarded as a left resp. right U-module, belongs to $\mathcal{C}$ resp. $\mathcal{C}'$ and that there are bimodule isomorphisms

$$\text{(2.3.1)} \quad \mathrm{T}_\eta^\xi L(M, N) \cong L(M, \mathrm{T}_\eta^\xi N) \quad \text{and} \quad L(M, N)\,{}_\eta^\xi T \cong L(\mathrm{T}_\eta^\xi M, N).$$

Recall also the definition of the category $\mathcal{O}$, see, for instance, [17] Chap. 4. Let $\mathcal{P}^+ = \{\nu \in \mathcal{P} \mid \nu(H_\beta) \geq 0,\ \forall\, \beta \in R^+\}$. Then, one has the

LEMMA.

(a) *Let $N \subset M$ be objects in $\mathcal{O}$ such that $M/N \cong M(\xi)$, for some $\xi \in \mathfrak{h}^*$, and let κ be an antidominant weight. Then the sequence $0 \to L(M(\xi), M(\kappa)) \to L(M, M(\kappa)) \to L(N, M(\kappa)) \to 0$ is exact.*

(b) *Let κ, ζ be antidominant weights and let $\nu \in \mathcal{P}^+$. Then $L(\mathrm{T}_\zeta^{\zeta-\nu} M(\zeta), M(\kappa))$ has a bimodule composition series with factors exactly the $L(M(\xi), M(\kappa))$, for $\xi \in W_\zeta(\zeta-\nu)$. Moreover, if $W_\zeta \subseteq W_\kappa$ then these composition factors are all isomorphic to $L(M(\zeta-\nu), M(\zeta))$.*

Proof. — Let δ denote the duality functor in $\mathcal{O}$. For an antidominant weight κ, $M(\kappa)$ is simple (see [9] 7.6.24) and hence $M(\kappa) \cong \delta M(\kappa)$. Thus, assertion (a) follows from the proof of 6.9.(9) and Lemmas 4.7, 4.11 in [17]. Moreover, by [16] 2.9.c), 2.17, $\mathrm{T}_\zeta^{\zeta-\nu} M(\zeta)$ has a composition series with factors exactly the $M(w(\zeta-\nu))$, for $w \in W_\zeta$ and hence the first part of assertion (b) follows from assertion (a). Finally, the last part follows from [17] Cor. 7.24 (see also the proof of [18] Prop. 4.19).

2.4. Let $\lambda = \mu - \rho$. Note that λ is antidominant and regular. Let U_λ-Proj denote the category of finitely generated projective left U_λ-modules. Let $L = U_\lambda\, {}^\mu_\lambda\mathrm{T}$. By the definition of ${}^\mu_\lambda\mathrm{T}$, L belongs to U_λ-Proj. By (1.4.1), (2.3.1), and [16], 2.10.a), one has $L \cong L(M(\mu), M(\lambda))$ and hence L is a right U_μ-module. Thus, the functor $L \otimes_{U_\mu} -$ takes U_μ-Proj to U_λ-Proj and hence induces a map $\theta^{\mathrm{out}} : K_0(U_\mu) \to K_0(U_\lambda)$.

Recall that the functor

$$\mathcal{L}^+ : \mathcal{M} \mapsto \mathcal{L}(\rho) \otimes_{\mathcal{O}_{G/B}} \mathcal{M}$$

induces an equivalence from $\mathcal{D}_\lambda$-Coh to $\mathcal{D}_\mu$-Coh, with inverse

$$\mathcal{L}^- : \mathcal{N} \mapsto \mathcal{L}(-\rho) \otimes_{\mathcal{O}_{G/B}} \mathcal{N},$$

see, for example, [10] A.3.1. Then one has the

LEMMA. — *One has a commuting diagram*

$$\begin{array}{ccc} K_0(U_\mu) & \xrightarrow{\phi_\mu} & G_0(\mathcal{D}_\mu) \\ {\scriptstyle\theta^{\mathrm{out}}}\big\downarrow & & \big\uparrow{\scriptstyle \cong\ \mathcal{L}^+\circ\phi_\lambda} \\ K_0(U_\lambda) & \xrightarrow{\cong} & G_0(U_\lambda). \end{array}$$

Proof. — Since λ is antidominant and regular then, by [3], the functors Γ_λ and Φ_λ are mutually inverse. Thus, it suffices to show that the functor $F := \Gamma_\lambda \circ \mathcal{L}^- \circ \Phi_\mu$ is isomorphic to $L \otimes_{U_\mu} -$. But F is exact and commutes with direct limits (since this is true for each of Φ_μ, $\mathcal{L}^-$ and Γ_λ) and hence it is isomorphic to the functor $F(U_\mu) \otimes_{U_\mu} -$, by [5], Prop. 1.3. Further, $\mathcal{L}^- \circ \Phi_\mu(U_\mu) \cong \mathcal{D}_\lambda \otimes_{\mathcal{O}_{G/B}} \mathcal{L}(-\rho)$ and hence $F(U_\mu) \cong L$, by Proposition 1.4. This proves the lemma.

2.5. Let $L^\natural$ denote the (U_μ, U)–bimodule $U_\mu\, {}^{\lambda}_{\mu}\mathrm{T}$. By the definition of ${}^{\lambda}_{\mu}\mathrm{T}$, $L^\natural$ is a finitely generated projective left U_μ-module and its right annihilator contains a power J_λ^n of J_λ, for some $n > 0$. Therefore, the functor $L^\natural \otimes_{(U/J_\lambda^n)} -$ induces a map $\theta^\natural : K_0(U/J_\lambda^n) \to K_0(U_\mu)$. Note also that $L^\natural \cong L(\mathrm{T}_\mu^\lambda M(\mu), M(\mu))$, by (1.4.1) and (2.3.1).

Consider now the $(U/J_\lambda^n, U)$–bimodule $\widehat{L} := (U/J_\lambda^n)\, {}^{\mu}_{\lambda}T$. By the definition of ${}^{\mu}_{\lambda}\mathrm{T}$, again, $\widehat{L}$ is a finitely generated projective left (U/J_λ^n)-module and its right annihilator contains some power J_μ^r of J_μ. Therefore the functor $\widehat{L} \otimes_{(U/J_\mu^r)} -$ induces a map $\widehat{\theta^{\mathrm{out}}} : K_0(U/J_\mu^r) \to K_0(U/J_\lambda^n)$. Moreover, by [1] IX.1.3, the natural maps $f : K_0(U/J_\lambda^n) \to K_0(U_\lambda)$ and $g : K_0(U/J_\mu^r) \to K_0(U_\mu)$ are isomorphisms. Then, one has the

PROPOSITION. — *There is a commuting diagram*

$$\begin{tikzcd}
K_0(U_\mu) \arrow[d, "\theta^{\mathrm{out}}"'] & K_0(U/J_\mu^r) \arrow[l, "g"', "\cong"] \arrow[r, "g", "\cong"'] \arrow[d, "\widehat{\theta^{\mathrm{out}}}"'] & K_0(U_\mu) \arrow[d, "|W_\mu|\ \mathrm{id}"] \\
K_0(U_\lambda) & K_0(U/J_\lambda^n) \arrow[l, "f"', "\cong"] \arrow[r, "\theta^\natural"] & K_0(U_\mu).
\end{tikzcd}$$

Thus, $\theta^\natural \circ f^{-1} \circ \theta^{\mathrm{out}} = |W_\mu|\ \mathrm{id}_{K_0(U_\mu)}$.

Proof. — By [5] 1.3, one has $U_\lambda \otimes_{U/J_\lambda^n} \widehat{L} \cong L$ and hence the left-hand square commutes. Further, by [5] 1.3, again, and (1.4.1), (2.3.1), one has

$$L^\natural \otimes_{U/J_\lambda^n} \widehat{L} \cong L^\natural\, {}^{\mu}_{\lambda}\mathrm{T} \cong L(\mathrm{T}_\lambda^\mu \mathrm{T}_\mu^\lambda M(\mu), M(\mu)).$$

Moreover, by [17] 4.7, 4.13(2), $\mathrm{T}_\lambda^\mu \mathrm{T}_\mu^\lambda M(\mu)$ is isomorphic to a direct sum of $|W_\mu|$ copies of $M(\mu)$ and hence, by (1.4.1), again, $L^\natural \otimes_{U/J_\lambda^n} \widehat{L}$ is isomorphic to a direct sum of $|W_\mu|$ copies of U_μ. This proves the commutativity of the right-hand square and the proposition follows.

2.6. For a subset J of Δ, let W_J denote the subgroup of W generated by $s_\alpha : \alpha \in J$ and let $\mathcal{P}_J^+ = \{\nu \in \mathcal{P} \mid \nu(H_\beta) \geq 0,\ \forall \beta \in J\}$. For $\nu \in \mathcal{P}_J^+$, denote by $\zeta_J(\nu)$ the element $\sum\limits_{\xi \in W_J \nu} e^\xi$ of $\mathbb{Z}\mathcal{P}$. We shall need the following

LEMMA. — *$K_0(G/B)^{W_J}$ is generated by the image of $\zeta_J(\nu) : \nu \in \mathcal{P}^+$.*

Proof. — By [20] Prop. 6(a), it suffices to prove that $(\mathbb{Z}\mathcal{P})^{W_J}$ is generated over $(\mathbb{Z}\mathcal{P})^W$ by $\zeta_J(\nu) : \nu \in \mathcal{P}^+$. Clearly, $\zeta_J(\nu) : \nu \in \mathcal{P}_J^+$ is a $\mathbb{Z}$-basis of $(\mathbb{Z}\mathcal{P})^{W_J}$ and hence, being finitely generated over $(\mathbb{Z}\mathcal{P})^W$, $(\mathbb{Z}\mathcal{P})^{W_J}$ is generated over $(\mathbb{Z}\mathcal{P})^W$ by a finite subset $\zeta_J(\nu_1), \dots, \zeta_J(\nu_r)$. Then one may pick $\nu \in \mathcal{P}^+$ so that $\nu(H_\alpha) = 0$, for $\alpha \in J$, and $\nu + \nu_i \in \mathcal{P}^+$, for $i = 1, \dots, r$. Then one has $\zeta_J(\nu + \nu_i) = e^\nu \zeta_J(\nu_i)$, for $i = 1, \dots, r$, and these form another system of generators, since e^ν is an invertible element of $(\mathbb{Z}\mathcal{P})^{W_J}$. This proves the lemma.

Then, one has the

PROPOSITION. — *One has* $\mathrm{Im}(\psi_\mu^{-1} \circ \phi_\mu) = K_0(G/B)^{W_\mu}$.

Proof. — For $\nu \in \mathcal{P}^+$, let $Q_\nu = L(\mathrm{T}_\mu^{\mu-\nu} M(\mu), M(\mu))$. By (1.4.1) and (2.3.1), Q_ν is isomorphic to $U_\mu\ {}^{\mu-\nu}_{\ \ \mu}T$ and belongs therefore to U_μ-Proj. By Lemma 2.4, one has

$$\Phi_\mu(Q_\nu) \cong \mathcal{L}^+ \circ \Phi_\lambda(L \otimes_{U_\mu} Q_\nu).$$

Since $L \cong L(M(\mu), M(\lambda))$ and $Q_\nu \cong U_\mu\ {}^{\mu-\nu}_{\ \ \mu}T$ then, by [5] 1.3 and (2.3.1),

$$L \otimes_{U_\mu} Q_\nu \cong L(\mathrm{T}_\mu^{\mu-\nu} M(\mu), M(\lambda)). \tag{2.6.1}$$

Then, using Lemma 2.3, Proposition 1.4 and the fact that $\mathcal{L}^+ \circ \Phi_\lambda$ and $\Gamma_\lambda \circ \mathcal{L}^-$ are mutually inverse, one deduces that

$$\phi_\mu([Q_\nu]) = \sum_{\xi \in W_\mu \nu} [\mathcal{D}_\mu \otimes_{\mathcal{O}_{G/B}} \mathcal{L}(\xi)].$$

Let $J = \Delta_\mu$. Then, one has

$$\psi_\mu^{-1} \circ \phi_\mu([Q_\nu]) = \zeta_J(\nu), \qquad \forall \nu \in \mathcal{P}^+. \tag{2.6.2}$$

By the previous lemma, this implies that

$$\psi_\mu^{-1} \circ \phi_\mu(K_0(U_\mu)) \supseteq K_0(G/B)^{W_\mu}. \tag{2.6.3}$$

Further, for $\nu \in \mathcal{P}^+$ let $P_\nu = (U/J_\lambda^n)\ {}^{\lambda-\nu}_{\ \ \lambda}T$. Note that $\lambda - \nu$ is antidominant and regular. By the definition of ${}^{\lambda-\nu}_{\ \ \lambda}T$, P_ν is a finitely generated projective left (U/J_λ^n)-module and, using [5] 1.3, (1.4.1), (2.3.1), and [16] 2.10a), one obtains

$$U_\lambda \otimes_{U/J_\lambda^n} P_\nu \cong L(M(\lambda-\nu), M(\lambda)) \tag{2.6.4}$$

$$L^\natural \otimes_{U/J_\lambda^n} P_\nu \cong L(\mathrm{T}_\lambda^{\lambda-\nu}\,\mathrm{T}_\mu^\lambda M(\mu), M(\lambda)). \tag{2.6.5}$$

Then, one deduces from (2.6.4), combined with [1] IX.1.3, Cor. 1.4, and the previous lemma, that $K_0(U/J_\lambda^n)$ is generated by $[P_\nu] : \nu \in \mathcal{P}^+$. Further, by [5] 3.5, coupled with [17] 2.10.c), the functors $\mathrm{T}_\lambda^\mu\,\mathrm{T}_{\lambda-\nu}^\lambda$ and $\mathrm{T}_{\lambda-\nu}^\mu$ are isomorphic and hence so are their adjoints $\mathrm{T}_\lambda^{\lambda-\nu}\,\mathrm{T}_\mu^\lambda$ and $\mathrm{T}_\mu^{\lambda-\nu}$. Coupled with (2.6.5), this gives

$$L^\natural \otimes_{U/J_\lambda^n} P_\nu \cong L(\mathrm{T}_\mu^{\lambda-\nu} M(\mu), M(\lambda)) \cong Q_{\nu-\rho}. \tag{2.6.6}$$

Combining the previous paragraph with (2.6.2), one obtains $\mathrm{Im}(\psi_\mu^{-1} \circ \phi_\mu \circ \theta^\natural) \subseteq K_0(G/B)^{W_\mu}$ and hence, by Proposition 2.5, it follows that

$$|W_\mu|\,\mathrm{Im}(\psi_\mu^{-1} \circ \phi_\mu) \subseteq K_0(G/B)^{W_\mu}. \tag{2.6.7}$$

Finally, by [20] Prop. 6, $K_0(G/B)^{W_\mu}$ is a direct summand of $K_0(G/B)$ and hence (2.6.3) and (2.6.7) together imply that $\mathrm{Im}(\psi_\mu^{-1} \circ \phi_\mu) = K_0(G/B)^{W_\mu}$. This completes the proof of the proposition.

2.7. Now, to complete the proof of Theorem 2.1 it suffices, by virtue of Lemma 2.4 and Propositions 2.5, 2.6, to prove the following easy lemma.

LEMMA.

(a) *The map $p \circ i$ is injective and its cokernel is annihilated by $|W_\mu|$.*

(b) *Both $K_0(G/B)^{W_\mu}$ and $K_0(G/B)/\sum_{\alpha\in\Delta_\mu} \mathrm{Im}(1-s_\alpha)$ have rank $|W/W_\mu|$.*

Proof. — It is well-known that $K_0(G/B)^{W_\mu}$ has rank $|W/W_\mu|$, see, for example, [20] Prop. 6b). Thus, the second assertion follows from the first, which we now prove. Let σ_μ denote the operator $\sum_{w\in W_\mu} w$. For $x \in K_0(G/B)$, one has $p(x) = p(s_\alpha x)$, for $\alpha \in \Delta_\mu$, and hence $p(x) = p(wx)$, for $w \in W_\mu$. Thus, $|W_\mu| p(x) = p(\sigma_\mu(x))$. This shows that $\mathrm{Coker}(p \circ i)$ is annihilated by $|W_\mu|$.

Next, let $x \in K_0(G/B)^{W_\mu} \bigcap \sum_{\alpha\in\Delta_\mu} \mathrm{Im}(1-s_\alpha)$. Then, on the one hand, $\sigma_\mu(x) = |W_\mu| x$ and, on the other hand, $\sigma_\mu(x) = 0$, since $\sigma_\mu \circ (1-s_\alpha) = 0$

for $\alpha \in \Delta_\mu$. This yields $x = 0$, since $K_0(G/B)$ is torsion free. Thus, the lemma is proved and the proof of Theorem 2.1 is complete.

2.8. Let us then derive the following corollary.

COROLLARY. — *$K_0(U_\mu)/K_0(U_\mu)_{\mathrm{tor}}$ is generated by the classes*
$$[L(\mathrm{T}_\mu^{\mu-\nu}\, M(\mu), M(\mu))] : \nu \in \mathcal{P}^+.$$

Proof. — Let $Q_\nu = L(\mathrm{T}_\mu^{\mu-\nu}\, M(\mu), M(\mu))$, for $\nu \in \mathcal{P}^+$. We saw in the proof of Proposition 2.6 that Q_ν belongs to U_μ-Proj and that $\psi_\mu^{-1} \circ \phi_\mu([Q_\nu]) = \zeta_J(\nu)$, where $J = \Delta_\mu$. But, by Theorem 2.1, $\psi_\mu^{-1} \circ \phi_\mu$ induces an isomorphism from $K_0(U_\mu)/K_0(U_\mu)_{\mathrm{tor}}$ to $K_0(G/B)^{W_\mu}$ and, by Lemma 2.6, the latter is generated by the image of $\zeta_J(\nu) : \nu \in \mathcal{P}^+$. The lemma follows.

3. Hattori-Stallings traces.

3.1. Let μ be as in 2.1 and let $\mathcal{T}_{U_\mu} : K_0(U_\mu) \to U_\mu/[U_\mu, U_\mu]$ denote the Hattori–Stallings trace, see, for example, [2] §2. It is well-known that $U_\kappa/[U_\kappa, U_\kappa] = k$, for $\kappa \in \mathfrak{h}^*$ (see, for example, [9] 7.8.4) and hence $\mathcal{T}_{U_\mu}$ takes values in k. Note also that $\mathcal{T}_{U_\mu}$ factors through $K_0(U_\mu)/K_0(U_\mu)_{\mathrm{tor}}$.

For $\xi \in \mathfrak{h}^*$, let τ_ξ denote the translation operator on $S(\mathfrak{h})$ defined by $\tau_\xi F(\eta) = F(\eta + \xi)$, for $F \in S(\mathfrak{h})$, $\eta \in \mathfrak{h}^*$. Let $R_\mu^+ = \{\alpha \in R^+ \mid \mu(H_\alpha) = 0\}$. Let $P = P_{R^+}$ denote the element $\prod_{\alpha \in R^+} H_\alpha$ of $S(\mathfrak{h})$ and let $P_{R_\mu^+}$ and $P_{R^+ \setminus R_\mu^+}$ be defined in the obvious way.

By Corollary 2.8, the classes of the projective modules $L(\mathrm{T}_\mu^{\mu-\nu}\, M(\mu), M(\mu)) : \nu \in \mathcal{P}^+$ generate $K_0(U_\mu)/K_0(U_\mu)_{\mathrm{tor}}$ and, similarly to [21] §2, one deduces from Bernstein's trace formula [4] §2 that the value of $\mathcal{T}_{U_\mu}$ on these generators is given by the following proposition.

PROPOSITION. — *For $\nu \in \mathcal{P}^+$ one has* $\mathcal{T}_{U_\mu} L(\mathrm{T}_\mu^{\mu-\nu}\, M(\mu), M(\mu)) = \left(\dfrac{\sum_{\xi \in W_\mu \nu} \tau_{-\xi} P}{P} \right)(\mu).$

Let us evaluate the right-hand side. Let a_μ denote the operator $\sum_{w \in W_\mu} \varepsilon(w) w$ and let w_μ be the unique element of W_μ such that $w_\mu(\Delta_\mu) =$

$-\Delta_\mu$. For $w \in W$, let D_w denote the corresponding Demazure operator on $S(\mathfrak{h})$, see [8] §4.

Let $\nu \in \mathcal{P}^+$. Since the stabilizer of ν in W_μ equals $W_{\mu-\nu}$ and since $wP = \varepsilon(w)P$, for $w \in W$, one has

$$(3.1.1) \qquad |W_{\mu-\nu}| \sum_{\xi \in W_\mu \nu} \tau_{-\xi} P = \sum_{w \in W_\mu} \tau_{-w\nu} P = a_\mu \tau_{-\nu} P.$$

By [8] Prop. 3(b), one has $D_{w_\mu}(F) = a_\mu(F)/P_{R^+_\mu}$, for $F \in S(\mathfrak{h})$. Thus, combining (3.1.1) with the previous proposition and noting that $P_{R^+ \setminus R^+_\mu}(\mu) \neq 0$, one obtains

$$(3.1.2) \qquad \mathcal{T}_{U_\mu} L(\mathrm{T}_\mu^{\mu-\nu} M(\mu), M(\mu)) = |W_{\mu-\nu}|^{-1} \frac{(D_{w_\mu} \tau_{-\nu} P)(\mu)}{P_{R^+ \setminus R^+_\mu}(\mu)}.$$

Then, for $\eta \in \mathfrak{h}^*$, let ∂_η denote the corresponding derivation of $S(\mathfrak{h})$. For $F \in S(\mathfrak{h})$, $\alpha \in \Delta$ and $\eta \in \mathfrak{h}^*$ such that $\eta(H_\alpha) = 0$, it is easily seen that $(D_{s_\alpha} F)(\eta) = (\partial_\alpha F)(\eta)$. Thus, denoting by $\partial_{R^+_\mu}$ the differential operator $\prod_{\alpha \in R^+_\mu} \partial_\alpha$, one deduces that

$$(3.1.3) \qquad (D_{w_\mu} \tau_{-\nu} P)(\mu) = (\partial_{R^+_\mu} \tau_{-\nu} P)(\mu) = (\partial_{R^+_\mu} P)(\mu-\nu).$$

Moreover, since $P_{R^+_\mu}$ vanishes with multiplicity $|R^+_\mu|$ at μ and since $(\partial_{R^+_\mu} P_{R^+_\mu})(\mu)$ equals $(D_{w_\mu} P_{R^+_\mu})(\mu) = |W_\mu|$, then one has $(\partial_{R^+_\mu} P)(\mu) = |W_\mu| \, P_{R^+ \setminus R^+_\mu}(\mu)$. Combined with (3.1.2) and (3.1.3), this gives the following

COROLLARY. — *For $\nu \in \mathcal{P}^+$ one has* $\mathcal{T}_{U_\mu} L(\mathrm{T}_\mu^{\mu-\nu} M(\mu), M(\mu)) = \dfrac{|W_\mu|}{|W_{\mu-\nu}|} \dfrac{(\partial_{R^+_\mu} P)(\mu-\nu)}{(\partial_{R^+_\mu} P)(\mu)}$.

3.2. Since the cokernel of the Cartan map $K_0(U_\mu) \to G_0(U_\mu)$ is torsion, by Theorem 2.1, we can extend $\mathcal{T}_{U_\mu}$ to a map $\mathcal{T}'_{U_\mu} : G_0(U_\mu) \to k$. Let $\nu \in \mathcal{P}^+$. By the last assertion of Lemma 2.3(b), one has in $G_0(U_\mu)$ the equality

$$(3.2.1) \qquad [L(\mathrm{T}_\mu^{\mu-\nu} M(\mu), M(\mu))] = |W_\mu / W_{\mu-\nu}| \, [L(M(\mu-\nu), M(\mu))].$$

Combined with Corollary 3.1, this yields

$$(3.2.2) \qquad \mathcal{T}'_{U_\mu}[L(M(\mu-\nu), M(\mu))] = \frac{(\partial_{R^+_\mu} P)(\mu-\nu)}{(\partial_{R^+_\mu} P)(\mu)}, \qquad \forall \nu \in \mathcal{P}^+.$$

Let F_μ denote the polynomial $\partial_{R_\mu^+}\tau_\mu P$. Note that F_μ is W_μ-invariant. Moreover, it is well-known that P satisfies the difference equation

$$\sum_{\eta\in W\xi} P(\kappa+\eta) = |W\xi|\, P(\kappa), \qquad \forall\, \kappa,\, \xi \in \mathfrak{h}^*,$$

and therefore so does F_μ. Combining these facts, one obtains that the map $\mathbb{Z}\mathcal{P} \to k$, $e^\nu \mapsto F_\mu(-\nu)$ factors through $K_0(G/B)/\sum_{\alpha\in\Delta_\mu}\mathrm{Im}(1-s_\alpha)$ and hence, by Proposition 2.2, induces a $\mathbb{Z}$-linear map $\varphi_\mu : G_0(U_\mu) \to k$. Moreover, by Lemma 2.6, coupled with (2.1.1), the classes $[L(M(\mu-\nu), M(\mu))]$: $\nu \in \mathcal{P}^+$ generate $G_0(U_\mu)$ and hence (3.2.2) shows that $\mathcal{T}'_{U_\mu}$ and $\varphi_\mu/((\partial_{R_\mu^+}P)(\mu))$ coincide on a generating set of $G_0(U_\mu)$. Therefore we obtain the following result, which generalizes [21] Th. 3.

THEOREM. — *For every $\nu \in \mathcal{P}$, one has* $\mathcal{T}'_{U_\mu}[L(M(\mu+\nu), M(\mu))] = \dfrac{(\partial_{R_\mu^+}P)(\mu+\nu)}{(\partial_{R_\mu^+}P)(\mu)}$.

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