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REDUCTIVE GROUP ACTIONS ON AFFINE VARIETIES AND THEIR DOUBLING

by Dmitri I. PANYUSHEV[(1)]

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0. Introduction.

Let G be a connected reductive algebraic group and X an algebraic variety (the ground field k is assumed to be algebraically closed and of characteristic 0). Given the action $(G : X)$, one may define another one, which will be denoted by $(G : X^*)$. Here X^* is a copy of X, but equipped with a twisted G-action (see sect. 1 for exact definitions). It may be regarded as a generalization of the notion of the dual G-module. The diagonal action $(G : X \times X^*)$ will be referred to as the *doubled* one. It is our main object in this paper. A part of our results is true for any X, but whenever we consider algebras of (regular) invariants, it is assumed that X is affine. Unless otherwise stated, all varieties are assumed to be irreducible. All actions under consideration are assumed to be left ones.

Denote by $k[X]$ (resp. $k[X]^G$) the algebra of regular functions (resp. the subalgebra of G-invariant functions) on X. By Hilbert's theorem, $k[X]^G$ is finitely generated and affine variety $X/\!\!/G := \operatorname{Spec} k[X]^G$ is the *quotient* (in the sense of Invariant Theory) of X by G. The inclusion $k[X]^G \hookrightarrow k[X]$ induces a surjective morphism $\pi_{G,X} : X \to X/\!\!/G$ (see *e.g.* [VP2] concerning the properties of $\pi_{G,X}$). Let us indicate several good properties of quotients.

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(FA) $k[X]^G$ is a polynomial algebra or, in other words, $X/\!\!/G$ is an affine space.

(EQ) $\pi_{G,X}$ is equidimensional, *i.e.* all its fibers have dimension $\dim X - \dim X/\!\!/G$.

(FM) $k[X]$ is a free $k[X]^G$-module.

These properties are not independent, for instance, the following equivalence is true in the case of G-modules [Sch2], §17: $(EQ)\&(FA) \Leftrightarrow (FM)$. The subalgebra $k[X]^U$, where U is a maximal unipotent subgroup of G, plays an important role in these considerations. By the Hadžiev-Grosshans theorem, $k[X]^U$ is also finitely generated. Therefore it makes sense to consider the affine variety $X/\!\!/U = \operatorname{Spec} k[X]^U$ and the morphism $\pi_{U,X} : X \to X/\!\!/U$. It is an empirical fact that if X has "good" properties as U-variety, then its properties as G-variety is better. For instance, if $\pi_{U,X}$ has (FA), then $\pi_{G,X}$ has (FA) and (FM). Denote by B the subgroup $N_G(U)$. Our general philosophy is that properties of X as U- and B-variety are closely related with the ones of G-variety $X\times X^*$. For stabilizers in general position of these actions it has been demonstrated in [P2], sect. 1.

One of our observations in this paper is that given X with a "very good" property, namely, X is a spherical G-variety, then the doubled variety $X\times X^*$ enjoys also a good one. (Recall that G-variety X is called *spherical*, if a Borel subgroup of G has a dense orbit in X. If X is affine, this condition is equivalent to the following one: any irreducible G-module occurs in $k[X]$ at most once.) Another exciting fact is that the doubled action is closely related with the action $(G^\theta : X)$, where G^θ is the subgroup of fixed points of an involution of maximal rank of G. Most of our results on spherical varieties gathered in the following theorems.

THEOREM 0.1. — *Suppose X is an affine normal spherical G-variety. Then* (i) *$k[X\times X^*]^G$ is a deformation of $k[X]^U$ (or of $k[X^*]^U$). Moreover, if $k[X]^U$ is a polynomial algebra, then $k[X\times X^*]^G$ is a polynomial one as well;* (ii) *there is a finite surjective morphism $X/\!\!/G^\theta \to X\times X^*/\!\!/G$.*

Spherical G-modules enjoy more good properties than arbitrary spherical G-varieties. (In this case V^* is nothing but the dual G-module.)

THEOREM 0.2. — *Suppose V is a spherical G-module. Then G-module $V\oplus V^*$ has properties (FA) and (EQ). If $d_1,\dots,d_r$ are degrees of free homogeneous generators of $k[V]^U$, then free generators of $k[V\oplus V^*]^G$ have bi-degrees $(d_1,d_1),\dots,(d_r,d_r)$ with respect to the natural bi-grading* *on $V \oplus V^*$. (An explicit relation between two sets of generators will be explained in sect. 2).*

These assertions are mostly proved in sect. 2. In sect. 1, a machinery is developed for dealing with doubled actions. We consider the diagonal Δ in $X \times X^*$. It is the usual diagonal of the Cartesian product, if one forgets about G-action. But, it is not a G-invariant subset relative to the doubled action and my goal is to obtain some general results on the subvariety $\overline{G\cdot\Delta} \subset X \times X^*$ (the bar over a subset denotes its Zariski-closure). It is also shown that there is a link between the actions $(G^\theta : X)$ and $(G : \overline{G\cdot\Delta})$. My interest in sect. 3 is concentrated on questions related with the Poincaré series of algebras of invariants under consideration, provided that they are graded. It is shown that a part of information about the Poincaré series of $k[X]^U$ fully determines that of $k[X \times X^*]^G$, when G-variety X is spherical or of complexity one. Examples are considered in sect. 4. For instance, we describe invariants of orthogonal group on the variety of complexes and on the product of two Grassmanian cones.

Example. — One can obtain 'almost all' irreducible spherical modules by using the following procedure. Let $\mathfrak{g}$ be a simple Lie algebra, endowed with a $\mathbb{Z}$-grading: $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(i)$. Suppose $d = \max\{i \mid \mathfrak{g}(i) \neq 0\}$ and L is a connected reductive group having Lie algebra $\mathfrak{g}(0)$. Then $\mathfrak{g}(i)$ is a spherical L-module for $i > \frac{d}{2}$ [P4]. By considering a suitable subalgebra of $\mathfrak{g}$, one immediately reduces to the case $i = d = 1$. For $d = 1$, *i.e.* $\mathfrak{g} = \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1)$, the series of representations related with Hermitian symmetric spaces is obtained. In this particular case properties (EQ) and (FA) for $(L : \mathfrak{g}(1) \oplus \mathfrak{g}(-1))$ are consequences of the general theory developed in [KR]. That is, theorem 0.2 is an extension of these results.

Our main reference in Invariant Theory is [VP2]. We use mainly notation and terminology from there.

1. Doubled G-variety and its diagonal.

Throughout the paper G denotes a connected reductive group with a fixed Borel subgroup B and a fixed maximal torus $T \subset B$. Denote by U the unipotent radical of B. Thereupon one also obtains the respective objects: root system, dominant weights $\mathcal{X}_+$, Weyl group, *etc.* . We let V_λ be an irreducible G-module with highest weight λ and denote by λ^* the highest weight of the dual G-module. Having passed to a finite covering of G, one may assume that there is an involution $\theta \in \operatorname{Aut} G$ such that $\theta(t) = t^{-1}$, $t \in T$ (a so-called involution of maximal rank). Then $\theta(B) \cap B = T$ and $\dim G^\theta = \dim U$, where G^θ is the subgroup of fixed points of θ. Since involutions of maximal rank form a single $\operatorname{Ad} G$-orbit, our next considerations do not depend essentially on the choice of θ. Without loss of generality, we may assume that G is a direct product of its connected center and a simply-connected semisimple group. Then $G^\theta = (G^\theta)^0 \times \mathbb{Z}_2^d$, where d is dimension of the center and $(-)^0$ denotes the identity component of a group.

In this section we begin with considering a G-action on arbitrary irreducible G-variety X. The integer $c_G(X) = \min_{x\in X} \operatorname{codim} Bx$ is called the *complexity* of G-variety X. (That is, spherical varieties are exactly those of complexity 0.) The integer $r_G(X) = \min_{x\in X} \operatorname{codim} Ux - \min_{x\in X} \operatorname{codim} Bx$ is called the *rank* of G-variety X. By Rosenlicht's theorem (see *e.g.*[VP2], 2.3), $c_G(X) = \operatorname{trdeg} k(X)^B$ and $c_G(X) + r_G(X) = \operatorname{trdeg} k(X)^U$.

Recall the definition of the dual G-variety X^* [P2]. We let X^* be isomorphic to X as an abstract variety, but endowed with a twisted G-action. Let $i : X \to X^*$ be a (fixed) isomorphism and $x^* := i(x)$. The twisted action is defined by

$$(g, x^*) \mapsto g \circ x^* := (\theta(g)x)^*, \quad x \in X, g \in G.$$

Example. — Suppose $X = V$ is a G-module. Then V^* is nothing but the dual G-module. Moreover, if $v \in V$ is a T-weight vector, then $v^* \in V^*$ has the opposite weight relative to T.

Remark. — It is easy to see that whenever θ is inner, then X^* is isomorphic X as G-variety.

We shall consider the doubled variety $X \times X^*$ with the diagonal ("doubled") action of G. The subvariety $\Delta = \{(x, x^*) \mid x \in X\}$ is said to be the *diagonal* of $X \times X^*$. Remark that it is not a G-stable subvariety of $X \times X^*$.

THEOREM 1. — *Codimension of the subvariety $\overline{G \cdot \Delta} \subset X \times X^*$ equals $c_G(X)$.*

Proof. — Consider the morphism $\tau : G \times \Delta \to \overline{G \cdot \Delta}$, $(g, z) \mapsto g \cdot z$ defined via the doubled action. It is immediate that $\tau^{-1}(z) \cong$

$\mathrm{Tran}_G(z,\Delta) := \{g \in G \mid g \cdot z \in \Delta\}$, $z \in \Delta$. By the theorem on dimension of fibers, there is an open subset $\Delta_0 \subset \Delta$ such that $\tau^{-1}(z)$ is unmixed and $\dim \tau^{-1}(z) = \dim G + \dim \Delta - \dim \overline{G \cdot \Delta}$ for any $z \in \Delta_0$. On the other hand, by the definition of twisted action, $\mathrm{Tran}_G(z,\Delta) = \{g \in G \mid gx = \theta(g)x\}$, where $z = (x, x^*)$. Since θ is an involution, it is easy to see this coincides with the subset $\{g \in G \mid g^{-1}\theta(g) \in G_x \cap \theta(G_x) = G_z\}$.

Let us consider the mapping $\psi : G \to G$, $\psi(g) = g^{-1}\theta(g)$. It is known that the image of ψ is a closed subvariety of G (the Cartan model of the symmetric variety G/G^θ) [Ri], 9.1. More precisely, $\psi(G)$ is a connected component of the smooth variety $\mathcal{Q}_G = \{s \in G \mid \theta(s) = s^{-1}\}$. Obviously, $\dim \psi(G) = \dim B$ and the mapping $G \to \psi(G)$ is smooth. One sees that

$$\mathrm{Tran}_G(z,\Delta) = \psi^{-1}(G_z) = \psi^{-1}(\psi(G) \cap G_z). \tag{1}$$

According to [P2], sect. 1, there exists $z \in \Delta_0$ such that $S := G_z$ is reductive, $B_x = S \cap B$, B_x^0 is a Borel subgroup of S, and $c_G(X) = \mathrm{codim}_X\, Bx$. Obviously, S is a θ-stable subgroup of G and $\theta\mid_S$ is again an involution of maximal rank. Now, $\psi(S) \subset \psi(G) \cap S \subset \mathcal{Q}_S$ and therefore $\psi(S^0)$ is a connected component of $\psi(G) \cap S$. Since $\psi(G) \cap S = \psi(\mathrm{Tran}_G(z,\Delta))$ is unmixed as well, we have $\dim \psi(G) \cap S = \dim \psi(S^0) = \dim B_x$. Hence, by Eq. (1) we get $\dim \mathrm{Tran}_G(z,\Delta) = \dim U + \dim B_x$. Thus $\dim \overline{G \cdot \Delta} = \dim \Delta + \dim B - \dim B_x = 2\dim X - c_G(X)$. □

Remark. — The subgroup S is the *stabilizer in general position* (s.g.p.) for the doubled action [P2]. (See [VP2], §7 about the s.g.p.) Further we always specify a choice of S as it suggested by the proof of Theorem 1. That is, (1) $S = G_z$, where $z = (x, x^*) \in \Delta$ and (2) $B_x = S \cap B$.

COROLLARY 1. — *$\overline{G \cdot \Delta} = X \times X^*$ if and only if X is a spherical G-variety.* □

Denote by $\tilde{G}$ the natural semi-direct product of G and $\langle\theta\rangle \cong \mathbb{Z}_2$. The diagonal action $(G : X \times X^*)$ is extended on $\tilde{G}$ by $\theta(x, y^*) = (y, x^*)$. Let us denote by $k(Y)^G$ the field of G-invariant rational functions on an irreducible G-variety Y.

PROPOSITION 1. — (i) *$k(X \times X^*)^G = k(X \times X^*)^{\tilde{G}}$ if and only if $c_G(X) = 0$.*

(ii) *Suppose X is affine. Then $k[X \times X^*]^G = k[X \times X^*]^{\tilde{G}}$ if and only if $c_G(X) = 0$.*

Proof. — "if" part in both cases is a direct consequence of Theorem 1.

"only if". We prove only affine part (ii) and remark that the non-affine case (i) is easily reduced to the affine one by a standard procedure (Sumihiro's theorem + affine cone).

Assume $c_G(X) > 0$. Then there exists a dominant weight λ such that V_λ appears at least twice in $k[X]$. Let us denote by ${V_\lambda}^{(1)}, {V_\lambda}^{(2)}$ two different copies of V_λ in $k[X]$, ${V_\lambda}^{(i)} \subset k[X] \otimes 1 \subset k[X] \times k[X^*]$. Then $\theta(V_\lambda^{(i)}) =: V_{\lambda^*}^{(i)} \subset 1 \otimes k[X^*] \subset k[X] \times k[X^*]$. According to the Schur lemma, there are G-invariants $f_{ij} \in ({V_\lambda}^{(i)} \otimes V_{\lambda^*}^{(j)})^G \subset (k[X] \otimes k[X^*])^G$. It is clear that $\theta(f_{ij}) = cf_{ji}$, $c \in k \setminus \{0\}$. Therefore $f_{12} \in k[X \times X^*]^G \setminus k[X \times X^*]^{\tilde{G}}$. □

From now on, we assume X is affine and let $D = \overline{G \cdot \Delta}$. Our next goal is to compare two actions: $(G : D)$ and $(G^\theta : \Delta)$. Further we shall identify the actions $(G^\theta : X)$ and $(G^\theta : \Delta)$ via G^θ-equivariant isomorphism $x \mapsto (x, x^*)$.

Recall that a G-action on an irreducible variety Z is called *stable*, if there is an open subset $Z_{\rm pr} \subset Z$ such that $G \cdot z$ is closed (in Z) for any $z \in Z_{\rm pr}$ and all orbits in $Z_{\rm pr}$ have maximal dimension. (To guarantee this, it suffices to find only one closed orbit of maximal dimension [Po1].)

In what follows, we keep the notation introduced in the proof of Theorem 1. Take $f \in k[D]^G$. Being restricted on $\Delta \subset D$, it gives a G^θ-invariant function. This defines an algebra homomorphism $k[D]^G \to k[\Delta]^{G^\theta}$ and the dual morphism of the respective affine varieties $\varrho : \Delta /\!\!/ G^\theta \to D /\!\!/ G$.

THEOREM 2. — (i) *The mapping* $\varrho : \Delta /\!\!/ G^\theta \to D /\!\!/ G$ *is finite and surjective;*

(ii) *each closed* G*-orbit in* D *has a non-empty intersection with* Δ*;*

(iii) *the action* $(G^\theta : X)$ *is stable and* S^θ *is the s.g.p. for it;*

(iv) $\dim X /\!\!/ G^\theta = \dim X /\!\!/ U = r_G(X) + c_G(X)$;

(v) *degree of* ϱ *equals n.c.c.*$(\psi(G) \cap S)$*/n.c.c.* $\psi(S)$*, where n.c.c.= the number of connected components.*

Proof. — (i) It follows from the construction of D that $k[D]^G = k[D]^{\tilde{G}}$, *i.e.* $D /\!\!/ G = D /\!\!/ \tilde{G}$. By [Lu] the natural mapping $D^\theta /\!\!/ N_{\tilde{G}}(\langle\theta\rangle) \to D /\!\!/ \tilde{G}$ (induced by the inclusion $D^\theta \hookrightarrow D$) is finite. In our case $D^\theta = \Delta$ and $N_{\tilde{G}}(\langle\theta\rangle) = G^\theta \times \langle\theta\rangle$, *i.e.* $\Delta /\!\!/ G^\theta \to D /\!\!/ G$ is finite. Since $\overline{G \cdot \Delta} = D$, it is also dominant. (ii) By [Lu], cor. 1 we know that

(2) $$G^{\theta}\cdot z \text{ is closed } \Leftrightarrow G\cdot z \text{ is closed},$$

for $z \in \Delta$. Therefore the assertion follows from (i) and the standard properties of quotients by reductive group actions.

(iii) By [P5], 1.6, one knows that the action $(G : X\times X^*)$ is stable and

(3) $$(X\times X^*)_{pr} \cap \Delta \neq \emptyset.$$

Hence the action $(G : D)$ is stable as well and by Eq. (2) it implies the stability for $(G^{\theta} : X)$. The assertion about S^{θ} is now evident.

(iv) According to part (i) $\dim X/\!\!/G^{\theta} = \dim D/\!\!/G$. It follows from Eq. (3) that codimension $D/\!\!/G$ in $X\times X^*/\!\!/G$ is equal codimension D in $X\times X^*$ and by Theorem 1 the latter equals $r_G(X)$. Finally, one knows that $\dim(X\times X^*/\!\!/G) = r_G(X) + 2c_G(X)$ and $\dim X/\!\!/U = r_G(X) + c_G(X)$ [P2], sect. 1.

(v) Take $z \in \Delta \cap (X\times X^*)_{pr}$. Then $G_z = S$ and by [Lu] $G\cdot z\cap\Delta$ is a finite union of (closed) G^{θ}-orbits. Hence degree of ϱ is the number of G^{θ}-orbits in $G\cdot z \cap \Delta$. Since almost all G^{θ}-orbits in Δ are isomorphic, we obtain

$$\deg \varrho = n.c.c.(G\cdot z\cap\Delta)/n.c.c.\, G^{\theta}\cdot z.$$

Consider (a part of) the orbit mapping

$$\varsigma : g \in \mathrm{Tran}_G(z,\Delta) \mapsto g\cdot z \in G\cdot z \cap \Delta.$$

Then $\varsigma^{-1}(G^{\theta}z) = G^{\theta}S$. According to (1) $\varsigma^{-1}(G\cdot z\cap\Delta) = \psi^{-1}(\psi(G)\cap S)$ and since the fibers of ψ are exactly the left cosets by G^{θ}, we have $\varsigma^{-1}(G^{\theta}z) = \psi^{-1}(\psi(S))$. Finally, both ψ and ς are fibrations, hence ς^{-1} and ψ^{-1} do not affect the ratio of n.c.c. □

By to the last claim of Theorem 2, it is quite useful to know when

(*) $$\psi(S) = \psi(G) \cap S$$

holds. We describe below several sufficient conditions.

PROPOSITION 2. — (i) *If Q_S is connected, then (*) is fulfilled.*

(ii) *Suppose S is a direct product of groups GL (or SL) and a torus. Then Q_S is connected.*

Proof. — (i) It follows from the facts, mentioned before, that $\psi(S) \subset \psi(G)\cap S \subset Q_S$ and $\psi(S^0)$ is a connected component of Q_S.

(ii) If $S = S_1 \times S_2$, then $\mathcal{Q}_{S_1 \times S_2} = \mathcal{Q}_{S_1} \times \mathcal{Q}_{S_2}$. Therefore it suffices to handle two cases.

1. S is a torus. Then obviously, $\mathcal{Q}_S = S$.

2. $S = GL_n$ or SL_n. Then $\theta(A) = (A^t)^{-1}$, $A \in S$ and $\mathcal{Q}_S$ is the set of nonsingular symmetric matrices (with determinant one). □

The most interesting applications of these results related with spherical varieties, see sect. 2 and 4.

2. Doubling of spherical varieties.

In this section, X is an affine spherical G-variety. In particular, X contains a dense G-orbit, that is, X is a prehomogeneous G-variety. Recall that we select the subgroups B, T, U and the automorphism θ. Take a point $x \in X$ whose $\theta(B)$-orbit is dense in X. For $H = G_x$, we then have

$$\overline{\theta(B) \cdot H} = \overline{H \cdot \theta(B)} = G. \tag{4}$$

Since X is spherical, any irreducible G-module occurs in $k[X]$ at most once. Denote by Γ the semigroup of dominant weights, which counts irreducible G-modules in $k[X]$, that is $k[X] = \bigoplus_{\lambda \in \Gamma} V_\lambda$. The set Γ is actually a semigroup, because $k[X]$ has not zero divisors. We call it the *weight semigroup*. Then $k[X^*] = \bigoplus_{\lambda \in \Gamma} V_{\lambda^*} = \bigoplus_{\mu \in \Gamma^*} V_\mu$. Later on denote by $\langle M \rangle$ the linear span of a subset M in a linear space. The next assertion is a slight sharpening of [P2], thm.7.

PROPOSITION 3. — (i) *$k[X^*]^H$ is a deformation of $k[X^*]^U$.*

(ii) *Suppose $k[X^*]^U$ is a polynomial algebra, then $k[X^*]^H$ is a polynomial one as well.*

Proof. — (i) Take an arbitrary irreducible G-module V_μ. A general property of prehomogeneous varieties is that $V_\mu \subset k[X]$ if and only if $(V_{\mu^*})^H \neq \{0\}$. So by the sphericality condition and the Frobenius reciprocity it must be $\dim(V_{\mu^*})^H = 1$, if $\mu \in \Gamma$. Define h_{λ^*} to be a non-zero vector in $(V_{\lambda^*})^H$ and q_{λ^*} to be a non-zero vector in $(V_{\lambda^*})^U$. Then

$$k[X^*]^H = \bigoplus_{\lambda \in \Gamma} \langle h_{\lambda^*} \rangle, \quad k[X^*]^U = \bigoplus_{\lambda \in \Gamma} \langle q_{\lambda^*} \rangle.$$

It follows immediately from Eq. (4) that $\langle\theta(B)h_{\lambda^*}\rangle = V_{\lambda^*}$. Therefore h_{λ^*} has a non-trivial projection on $(V_{\lambda^*})^U$ in the weight decomposition of V_{λ^*} relative T. Hence, we may assume this projection equals q_{λ^*}, *i.e.*

$$h_{\lambda^*} = q_{\lambda^*} + \sum_{\nu<\lambda^*} a_{\nu\lambda} e_\nu, \tag{5}$$

where $\{e_\nu\}$ are other weight vectors in V_{λ^*} and $a_{\nu\lambda} \in k$. Here '$<$' denotes the standard partial order on the set of weights, *i.e.* $\alpha < \beta$ if and only if $\beta - \alpha$ is a sum of positive roots. Since $\{q_{\lambda^*} \mid \lambda \in \Gamma\}$ are weight vectors, the multiplication in $k[X^*]^U$ looks as follows

$$q_{\lambda^*} q_{\mu^*} = b_{\lambda+\mu} q_{\lambda^*+\mu^*}. \tag{6}$$

The multiplication in $k[X^*]^H$ is more complicated, because $\{h_{\lambda^*} \mid \lambda \in \Gamma\}$ are not weight vectors in the respective G-modules. But, one knows that $V_{\lambda^*} \cdot V_{\mu^*} \subset k[X^*]$ is a homomorphic image of $V_{\lambda^*} \otimes V_{\mu^*}$ (the dot here denotes the multiplication in $k[X^*]$). Therefore one sees by properties of tensor products, if $V_\nu \subset V_{\lambda^*} \cdot V_{\mu^*} \subset k[X^*]$ and $\nu \neq \lambda^* + \mu^*$, then $\nu < \lambda^* + \mu^*$. Thus, making compare (5) and (6), we get

$$h_{\lambda^*} h_{\mu^*} = b_{\lambda+\mu} h_{\lambda^*+\mu^*} + \sum_{\nu<\lambda^*+\mu^*} c^\nu_{\lambda\mu} h_\nu, \tag{7}$$

where $b_{\lambda+\mu} \in k \setminus \{0\}, c^\nu_{\lambda\mu} \in k$. The existence of a deformation is a formal consequence of these equalities, see [Po2], prop.9. (It is evident that $k[X^*]^U$ is the "limit" of $k[X^*]^H$, when all $c^\nu_{\lambda\mu}$ tend to zero.)

(ii) $k[X^*]^U$ is isomorphic to the semigroup algebra of Γ^*. Therefore $k[X^*]^U$ is a polynomial algebra if and only if Γ^* is a free semigroup. Let $\mu_1, \dots, \mu_r$ be the free generators of Γ^* and $q_i \in V^U_{\mu_i} \setminus \{0\}$ be free generators of $k[X^*]^U$. Let us show $h_i \in V^H_{\mu_i} \subset k[X^*]^H$, $i = 1, \dots, r$ freely generate $k[X^*]^H$. We argue by induction. Assume it is already verified that $h_\nu \in k[h_1, \dots, h_r]$ for any $\nu \in \Gamma^*$ such that $\nu < \lambda$. Suppose $\lambda = \sum_{i=1}^r c_i\mu_i$. Then $h_\lambda - \alpha \prod_{i=1}^r h_i^{c_i} \in k[X^*]^H$ and it follows from Eq. (7) that under a suitable choice of α one could eliminate the component of weight λ in this expression. In this case $h_\lambda - \alpha \prod_{i=1}^r h_i^{c_i} \in \bigoplus_{\nu<\lambda} \langle h_\nu\rangle$, and it lies in $k[h_1, \dots, h_r]$ by the induction hypothesis. Hence $h_\lambda \in k[h_1, \dots, h_r]$ as well. □

THEOREM 3. — *Let X be an affine normal spherical G-variety. Suppose $x \in X$ lies in the dense G-orbit and $H := G_x$. Then the restriction homomorphism $\varphi \in k[X \times X^*] \mapsto \varphi(x, \cdot) \in k[X^*]$ induces an isomorphism:*

$$\tau : k[X \times X^*]^G \cong k[X^*]^H.$$

Proof. — Since $\overline{G(\{x\} \times X^*)} = X{\times}X^*$, τ is injective. Conversely, let us take any $f \in k[X^*]^H$ and show that it comes from a G-invariant regular function on $X{\times}X^*$. Clearly f gives rise to a regular G-invariant function $\tilde{f}$ on the open subvariety $Gx \times X^*$ by

$$\tilde{f}(gx, y^*) = f(g^{-1} \circ y^*),$$

and it is the only candidate for the required regular function on the whole $X{\times}X^*$. Obviously, $\tilde{f} \in k(X{\times}X^*)^G$. Suppose $\tilde{f}$ is not regular. Then it has the divisor of poles $(\tilde{f})_\infty$ whose support lies in the complement of $Gx{\times}X^*$, *i.e.*

$$\text{supp}(\tilde{f})_\infty = \bigcup_i (D_i \times X^*), \tag{8}$$

where $\{D_i\}$ are divisors in $X \setminus Gx$. By proposition 1(i), $\tilde{f} \in k(X{\times}X^*)^{\tilde{G}}$ as well, hence $\text{supp}(\tilde{f})_\infty$ should be θ-invariant. This clearly contradicts Eq. (8). Thus $\tilde{f}$ must be regular. □

COROLLARY 2. — *$k[X{\times}X^*]^G$ is a deformation of $k[X^*]^U$ (and of $k[X]^U$). If $k[X]^U$ is a polynomial algebra, then $k[X{\times}X^*]^G$ is a polynomial one as well.*

Proof. — It immediately follows from Theorem 3 and Proposition 3. □

There is a famous sufficient condition of freeness of algebra of U-invariants (see *e.g.*[Li], 1.1). We sketch the proof for convenience of the reader.

PROPOSITION 4. — *Suppose X is an affine normal spherical G-variety such that any invertible regular function on X is constant and the divisor class group $Cl(X)$ is trivial. Then $k[X]^U$ is a polynomial algebra.*

Proof. — Since T normalizes U, it naturally acts on $X/\!\!/U$. Under our assumption T has an open orbit in $X/\!\!/U$, *i.e.* $X/\!\!/U$ appears to be a toric variety. Moreover, $X/\!\!/U$ is normal, invertible regular functions on $X/\!\!/U$ are constant, and $\text{Cl}(X/\!\!/U) = 0$ as well. Therefore we may conclude $X/\!\!/U \cong \mathbb{A}^r$. □

Our next purpose is to establish a relationship between free generators of $k[X]^U$ and $k[X{\times}X^*]^G$. Keep the previous notation.

PROPOSITION 5. — *Suppose $f_1, \ldots, f_r$ are free (T-homogeneous) generators of $k[X]^U$ and $V_i = \langle Gf_i \rangle \subset k[X]$. Take a non-zero function*

$g_i \in (V_i \otimes V_i^*)^G \subset k[X\times X^*]^G$, $i = 1,\dots,r$. Then $g_1,\dots,g_r$ freely generate $k[X\times X^*]^G$.

Proof. — Since f_i is T-homogeneous, V_i is an irreducible G-module. Therefore $\dim(V_i \otimes V_i^*)^G = 1$ and $\langle g_i \rangle$ is well-defined. Put $h_i = g_i(x,\cdot) \in V_i^* \subset k[X^*]^H$. By Proposition 3 (proof of part (ii)) $h_1,\dots,h_r$ freely generate $k[X^*]^H$. Then the assertion follows by Theorem 3. □

Spherical G-modules enjoy a nice property, which is not observed generally.

THEOREM 4. — *Suppose V is a spherical G-module. Then G-module $V \oplus V^*$ has properties (EQ) and (FA).*

Proof. — Property (FA) follows from Proposition 4 and Corollary 2, *i.e.* it remains only to prove (EQ). We wish to show that $\pi_G = \pi_{G,V\oplus V^*} : V\oplus V^* \to (V\oplus V^*)/\!\!/G$ is equidimensional. The fiber $\mathfrak{N}_G(V\oplus V^*) := \pi_G^{-1}(\pi_G(0))$ is called the *null-cone.* It is well-known that property (EQ) for G-modules is equivalent to equality $\dim \mathfrak{N}_G(V\oplus V^*) = \dim V\oplus V^* - \dim(V\oplus V^*)/\!\!/G$ (see *e.g.*[VP2]). Our idea is to prove it for a particular class of spherical G-modules and then to reduce the problem to this special case.

(a) First, assume that there is $v \in V$ such that $U_v = \{e\}$, *i.e.* action $(U : V)$ is *locally free.* Let us denote by $r = r_G(V)$ common dimension of varieties $V/\!\!/U$ and $(V\oplus V^*)/\!\!/G$. Then $r = \dim V - \dim U$ and $\dim \mathfrak{N}_G(V\oplus V^*) \geq 2\dim V - r = \dim V + \dim U$. On the other hand, we can estimate $\dim \mathfrak{N}_G(V\oplus V^*)$ by using formulas due to Schwarz. Since $k[V]^G = k$, the origin is the only closed G-orbit in V. Hence the zero weight subspace V^T of V is trivial, because all G-orbits, intersecting V^T are closed. Therefore by [Sch1], prop. 2.10

$$\dim \mathfrak{N}_G(V\oplus V^*) \leq \dim G - \dim B + \frac{1}{2}\dim(V\oplus V^*) = \dim V + \dim U.$$

Thus $\dim \mathfrak{N}_G(V\oplus V^*) = 2\dim V - r$ and we are done.

(b) Now assume that $U_v \neq \{e\}$ for any $v \in V$. Then it follows from [P1] that stabilizer in general position S for the action $(G : V\oplus V^*)$ is not trivial, and even $S' \neq \{e\}$. By [LR], we have $(V\oplus V^*)/\!\!/G \cong (V\oplus V^*)^S/\!\!/N_G(S)$ and by [P6], 3.4 property (EQ) for $(N_G(S) : (V\oplus V^*)^S)$ implies that for $(G : V\oplus V^*)$. Since the component group does not affect the null-cone, it suffices to handle the action $(N_G(S)^0 : (V\oplus V^*)^S)$.

Put $K = N_G(S)^0/S^0$. This is a connected reductive group and the action of a maximal unipotent subgroup of K on V^S is already locally free [P1]. Thus, taking into account (a), it suffices to show that V^S is a spherical K-module. And this follows from the general statement, which is implicit in [P5], 1.9 that $c_G(V) = c_K(V^S)$. □

Remarks. — 1. One could attempt to establish (EQ) in the following way. Let $x \in V$ be a point in the dense G-orbit and $H = G_x$. Then G/H is quasi-affine, in other words, H is an *observable* spherical subgroup of G. Since $\overline{Gx} = V$, we have $T_x(Gx) = V$. Therefore $V \cong \mathfrak{g}/\mathfrak{h}$ as H-modules and the representation $(H : V^*)$ is nothing but the coisotropy representation of $H \subset G$. According to [P2], sect. 3, the action $(H : V^*)$ has properties (EQ) and (FA). This provides also a different proof of property (FA) for $(G : V \oplus V^*)$ without deformation arguments. Consider the quotient morphism $\pi_{H,V^*} : V^* \to V^*/\!\!/H$ and its null-cone $\mathfrak{N}_H(V^*)$. By Theorem 3, $\mathfrak{N}_G(V \oplus V^*) \cap (x + V^*) = x + \mathfrak{N}_H(V^*)$. This immediately implies that irreducible components of $\mathfrak{N}_G(V \oplus V^*)$ whose intersection with $x + V^*$ is non-empty are of dimension $2 \dim V - \dim(V \oplus V^*)/\!\!/G$. So the problem is how to deal with the other ones.

2. The quotient morphism $\pi_{G,X \times X^*}$ is not necessarily equidimensional for arbitrary spherical varieties, see (4.3).

Now let us come back to X and understand what one can say about the action $(G^\theta : X)$. In *spherical* case the subvariety D defined in sect. 1 coincides with $X \times X^*$ and by Theorem 2 there is a finite surjective morphism $\varrho : X/\!\!/G^\theta \to X \times X^*/\!\!/G$. If X is normal, then birationality of ϱ is sufficient to guarantee that it is an isomorphism, but this is not always the case. The situation for $(G^\theta : X)$ is not fully understood, at least it clearly behaves worse than the doubled action. Below I give without proofs several results about it.

1. $k[X]^{G^\theta}$ is a deformation of a subalgebra of $k[X]^U$.
2. Let $\varphi_1, \dots, \varphi_p$ be a basis of the semigroup of the dominant weights $\mathcal{X}_+$ (only the part related with the semisimple part of G is uniquely determined). Define $\mathcal{X}_+^{\text{ev}}$ to be the subset of $\mathcal{X}_+$ consisting of all weights with even coordinates (in additive notation) and $\Gamma^{\text{ev}} = \Gamma \cap \mathcal{X}_+^{\text{ev}}$. (Obviously, this definition does not depend on a choice of basis.) Then $X/\!\!/G^\theta$ is an affine space if and only if Γ^{ev} is a free semigroup.
3. "$k[X]^U$ is a polynomial algebra" does not imply "$k[X]^{G^\theta}$ is a polynomial algebra" (see 4.4).

4. If $(U : X)$ is locally free, then degree of ϱ is a power of 2. [For then $S \subset T$. Hence $\psi(G) \cap S = S$ and $\psi(S) = \{a^2 \mid a \in S\}$. Then the assertion follows by theorem 2(v).] Apparently this is always true.

I think that for any spherical G-module V the action $((G^\theta)^0 : V)$ has properties (EQ) and (FA). At least this is true for all spherical modules related with Hermitian symmetric spaces (*cf.* example in introduction and 4.5).

3. Around Poincaré series.

In this section, we consider several phenomena related with affine conical varieties, not necessarily spherical. First, we recall well known results on Poincaré series of graded algebras (see *e.g.*[VP2], §3).

Suppose X is an affine cone with vertex, *i.e.* X is equipped with an action of the multiplicative group k^* such that there exists a point in X lying in the closure of every k^*-orbit. Then $k[X]$ is $\mathbb{N}$-graded, $k[X] = \bigoplus_{i \geq 0} k[X]_i$, and $k[X]_0 = k$. The formal power series $F(X; z) = \sum_{i\geq 0} \dim k[X]_i z^i$ is called the Poincaré series of X or $k[X]$. Since $k[X]$ is noetherian, $F(X, z)$ is the Taylor expansion at $z = 0$ of a rational function in z. If $F(X, z) = P(z)/Q(z)$, where P, Q are polynomials in z, then the integer $q(X) := \deg P - \deg Q$ is called the *degree* of $k[X]$. A G-variety X is said to be a *G-cone*, if it is a cone with vertex and the G-action respects the graded structure of $k[X]$. In other words, actions of G and k^* commute. In this case, one can also consider the Poincaré series $F(X/\!\!/G; z)$ of the algebra $k[X]^G$ and its degree $q(X/\!\!/G)$. These definitions are obviously transferred on multi-graded algebras, see *e.g.*[P5], sect. 2. For instance, if X is a G-cone then $k[X \times X^*]^G$ is bi-graded and has the Poincaré series

$$F(X \times X^* /\!\!/ G; z_1, z_2) = \sum_{n,m\geq 0} \dim(k[X]_n \otimes k[X^*]_m)^G z_1^n z_2^m,$$

whose degree $q(X \times X^* /\!\!/ G) = (q, q^*)$ is a vector with equal components. Here q (resp. q^*) is degree relative to z_1 (resp. z_2).

The algebra of U-invariants always possesses a multi-graded structure, even if X is not conical. It originates from the action of a maximal torus that normalizes U. Namely,

$$k[X]^U = \bigoplus_{\lambda \in \mathcal{X}_+} k[X]^U_\lambda \ ,$$

where

$$k[X]^U_\lambda = \{f \in k[X]^U \mid t \cdot f = \lambda(t)f, \ \text{ for any } \ t \in T\}.$$

We set $\Gamma = \{\lambda \in \mathcal{X}_+ \mid k[X]^U_\lambda \neq 0\}$. It is compatible with using of Γ in sect. 2. Then the Poincaré series is defined by

$$F(X/\!\!/U; y_1, \dots, y_p) = \sum_{n_1,\dots,n_p} \dim k[X]^U_\lambda y_1^{n_1} \dots y_p^{n_p},$$

where $p = \dim T, \lambda = \sum\limits_i n_i\varphi_i \in \Gamma$, and $\varphi_1, \dots, \varphi_p$ are the fundamental weights (relative to T and U). One can set (just formally) $y_i = e^{\varphi_i}$, $i = 1, \dots, p$. In this notation, which will be used thereafter,

$$F(X/\!\!/U; \varphi_1, \dots, \varphi_p) = \sum_{\lambda \in \Gamma} \dim k[X]^U_\lambda e^\lambda .$$

Next, assume that X is a G-cone. One can then define a refined version of multi-grading on $k[X]^U$. We set $k[X]^U_{n,\lambda} = k[X]_n \cap k[X]^U_\lambda$ and

$$F(X/\!\!/U; z, \varphi_1, \dots, \varphi_p) = \sum_{\lambda \in \Gamma, n \geq 0} \dim k[X]^U_{n,\lambda} z^n e^\lambda .$$

It is a rational function in z and y_i's, and we shall denote by $q(X/\!\!/U)$ degree of $F(X/\!\!/U; z, \varphi_1, \dots, \varphi_p)$ relative to z. Set

$$F_\lambda(X/\!\!/U; z) := \sum_{n \geq 0} \dim k[X]^U_{n,\lambda} z^n .$$

Obviously this is the Poincaré series of the isotypic component of the weight λ in $k[X]^U$. The following assertion, which relates the Poincaré series of the isotypic components and of algebras of G-invariants, is an easy consequence of Schur's lemma (cf. [P1], §3).

PROPOSITION 6. — *Suppose X, Y are affine G-cones. Then*

(1) $F(X \times Y/\!\!/G; z_1, z_2) = \sum\limits_\lambda F_\lambda(X/\!\!/U; z_1) F_{\lambda^*}(Y/\!\!/U; z_2)$;

(2) $F(X \times X^*/\!\!/G; z_1, z_2) = \sum\limits_\lambda F_\lambda(X/\!\!/U; z_1) F_\lambda(X/\!\!/U; z_2)$.

Proof. — 1. $k[X \times Y] = k[X] \otimes k[Y]$ and $(V_\lambda \otimes V_\mu)^G \neq 0$ if and only if $\mu = \lambda^*$.

2. $F_{\lambda^*}(X^*/\!\!/U; z) = F_\lambda(X/\!\!/U; z)$. □

This proposition is especially useful when the algebras of U-invariants on both X and Y are polynomial, for then one can easily find explicit expression of F_λ's.

Example. — Consider $G = Sp_6$ and its 2nd and 3rd fundamental modules, *i.e.* $X = V_{\varphi_2}, Y = V_{\varphi_3}$. The degrees and the weights of free generators of algebras of U-invariants are found in [B1]. By applying Proposition 6, one obtains the following expression for $F(V_{\varphi_2} \oplus V_{\varphi_3} /\!\!/ Sp_6; z_1, z_2)$ (we omit simple but bulky computations):

numerator: $1 + z_1^6 z_2^6$;

denominator: $(1-z_1^2)(1-z_1^3)(1-z_2^4)(1-z_1^2z_2^4)(1-z_1^4z_2^4)(1-z_1^3z_2^4)(1-z_1^6z_2^4)$.

These formulas give us a strong evidence in favor of the fact that $k[V_{\varphi_2} \times V_{\varphi_3}]^{Sp_6}$ is a hypersurface.

Let X be a spherical G-variety. Then $\dim k[X]^U_\lambda \leq 1$ for any $\lambda \in \mathcal{X}_+$, that is $F(X/\!\!/U; \varphi_1, \dots, \varphi_p) = \sum_{\lambda \in \Gamma} e^\lambda$ and everything is determined by the structure of Γ. Moreover, if X is a spherical G-cone, an easy argument yields a relationship between Poincaré series of $k[X]^U$ and $k[X \times X^*]^G$. It was the point of departure in my considerations on doubled varieties.

THEOREM 5. — *Let X be an affine spherical G-cone. Then*

$$F(X \times X^* /\!\!/ G; z_1, z_2) = F(X/\!\!/U; z_1 z_2, 0, \dots, 0).$$

Proof. — By definition, one has

$$F(X/\!\!/U; z, 0, \dots, 0) = \sum_{\lambda \in \Gamma} F_\lambda(X/\!\!/U; z).$$

By the sphericality condition, $F_\lambda(X/\!\!/U; z)$ is a monomial for any $\lambda \in \Gamma$. More precisely, V_λ occurs only once in $k[X]$ and if $V_\lambda \subset k[X]_{d(\lambda)}$, then $F_\lambda(X/\!\!/U; z) = z^{d(\lambda)}$. Therefore by Proposition 6

$$F(X \times X^* /\!\!/ G; z_1, z_2) = \sum_{\lambda \in \Gamma} z_1^{d(\lambda)} z_2^{d(\lambda)} = F(X/\!\!/U; z_1 z_2, 0, \dots, 0). \qquad \square$$

COROLLARY 3. — *Suppose $k[X]^U$ is a polynomial algebra and degrees of free generators of $k[X]^U$ are $d_1, \dots, d_r$. Then*

$$F(X \times X^* /\!\!/ G; z_1, z_2) = \prod_{i=1}^{r} \frac{1}{1 - (z_1 z_2)^{d_i}}. \tag{9}$$

$\square$

Remark. — The assertion of Corollary 3 follows from the theory developed in sect. 2 as well, but this method looks also very attractive.

However, Eq. (9) does not imply immediately that $k[X \times X^*]^G$ is a polynomial algebra (see [VP2], 3.10, Ex. 1^o).

Remarkably that one can achieve almost a similar result for G-varieties of complexity one. Namely, I am going to prove that under a mild assumption the Poincaré series of $k[X]^U$ fully determines that of $k[X \times X^*]^G$. The reason of success in Theorem 5 is that $k[X]^U$ is essentially the semigroup algebra of Γ in the spherical case. Fortunately, one can also produce a description of multiplicities in Γ in the case of complexity one. The exact formulation includes some technical conditions which are automatically satisfied *e.g.*when X is a prehomogeneous G-module.

PROPOSITION 7 [P6], 1.9. — *Let X be an affine unirational factorial variety acted upon by G. Suppose $c_G(X) = 1$, $k[X]^G = k$, and $k[X]$ does not contain non-constant invertible functions. Then*

(1) $m_\lambda := \dim k[X]^U_\lambda < \infty$ *for any* $\lambda \in \Gamma$;

(2) *there is a unique* $\mu \in \Gamma$ *such that*

(i) $m_\mu = 2$;

(ii) *if* $\omega \in \Gamma$, $\omega - l\mu \in \Gamma$, *and* $\omega - (l+1)\mu \notin \Gamma$ $\quad (l \in \mathbb{N})$, *then* $m_\omega = l+1$.

DEFINITION. — *The character μ that satisfies all the conditions of the proposition is said to be* remarkable.

Consider the subset $\Theta = \{\lambda \in \Gamma \mid \lambda - \mu \notin \Gamma\}$. Then Γ is the disjoint union of $\Theta + l\mu$, $l = 0, 1, \ldots$. Moreover, by Proposition 7, one has $m_\gamma = l + 1$ if and only if $\gamma \in \Theta + l\mu$. Therefore one can perform the following computation:

$$F(X/\!\!/U; \varphi_1, \ldots, \varphi_p) = \sum_{\lambda \in \Gamma} m_\lambda e^\lambda = \sum_{\lambda \in \Theta} e^\lambda + 2 \sum_{\lambda \in \Theta} e^{\lambda+\mu} + \ldots$$
$$+ (l+1) \sum_{\lambda \in \Theta} e^{\lambda + l\mu} + \ldots = \sum_{\lambda \in \Theta} e^\lambda (1 + 2e^\mu + \ldots)$$
$$= \frac{\sum_{\lambda \in \Theta} e^\lambda}{(1 - e^\mu)^2}.$$

Thus, one obtains the following

COROLLARY 4. — *Under the assumptions of Proposition 7 the Poincaré series of the algebra of U-invariants depends only on Θ and μ.*

More precisely,

$$F(X/\!\!/U;\varphi_1,\dots,\varphi_p)=\frac{\sum_{\lambda\in\Theta}e^{\lambda}}{(1-e^{\mu})^2}.\qquad\square$$

Consider a more interesting setting. Let X be a G-cone. Then one can introduce the $\hat{G}$-action on X, where $\hat{G}=G\times k^*$. It is immediate that $c_G(X)-1\le c_{\hat{G}}(X)\le c_G(X)$. Suppose X satisfies all the assumptions of Proposition 7 and its complexity with respect to $\hat{G}$-action equals one as well. The latter condition means that for the remarkable weight μ, the subspace $k[X]^U_\mu$, which is two-dimensional, belongs to a sole graded piece, say $k[X]_{n_0}$, of $k[X]$ relative to k^*-action. Since $\dim k[X]^U_\lambda=1$ for $\lambda\in\Theta$, the integer $d(\lambda)$ such that $k[X]^U_\lambda\subset k[X]_{d(\lambda)}$ is also well-defined. Let us observe that $k[X]$ does not contain non-constant invertible functions whenever X is a cone. Therefore in the notation just introduced, one immediately gets

PROPOSITION 8. — *Let X be an affine unirational factorial G-cone. Suppose $c_G(X)=c_{\hat{G}}(X)=1$, and $k[X]^G=k$. Then*

$$F(X/\!\!/U;z,\varphi_1,\dots,\varphi_p)=\frac{\sum_{\lambda\in\Theta}z^{d(\lambda)}e^{\lambda}}{(1-z^{n_0}e^{\mu})^2}.\qquad\square$$

This implies the following counterpart of Theorem 5 in the case of actions of complexity one.

THEOREM 6. — *Under the assumptions of Proposition 8, the following relation holds:*

$$F(X\times X^*/\!\!/G;z_1,z_2)=\frac{1+(z_1z_2)^{n_0}}{1-(z_1z_2)^{n_0}}F(X/\!\!/U;z_1z_2,0,\dots,0).$$

Proof. — We write $F_\lambda(z)$ instead of $F_\lambda(X/\!\!/U;z)$. The easy observation is that for $\gamma=\lambda+l\mu\in\Theta+l\mu$, we have $F_\gamma(z)=(l+1)z^{d(\lambda)+ln_0}$. Then

$$\begin{aligned}F(X\times X^*/\!\!/G;z_1,z_2)&=\sum_{\lambda\in\Gamma}F_\lambda(z_1)F_\lambda(z_2)\\&=\sum_{l=0}^{\infty}\sum_{\lambda\in\Theta}F_{\lambda+l\mu}(z_1)F_{\lambda+\mu}(z_2)\\&=\sum_{l=0}^{\infty}\sum_{\lambda\in\Theta}(l+1)z_1^{d(\lambda)+ln_0}(l+1)z_2^{d(\lambda)+ln_0}\end{aligned}$$

$$= \sum_{l=0}^{\infty}(l+1)^2(z_1z_2)^{ln_0}\sum_{\lambda\in\Theta}(z_1z_2)^{d(\lambda)}$$

$$= \frac{1+(z_1z_2)^{n_0}}{(1-(z_1z_2)^{n_0})^3}\sum_{\lambda\in\Theta}(z_1z_2)^{d(\lambda)}.$$

By Proposition 8,

$$F(X/\!\!/U; z_1z_2, 0, \ldots, 0) = \frac{\sum_{\lambda\in\Theta}(z_1z_2)^{d(\lambda)}}{(1-(z_1z_2)^{n_0})^2},$$

hence the assertion. □

Remark. — It follows from Theorems 5 and 6 that $q = q^* = q(X/\!\!/U)$ in the cases under consideration. A proof of this equality for any doubled action will appear in a forthcoming paper.

Example. — We set $G = SL_2 \times k^*$ and $X = R(3)$ is the space of 4-dimensional irreducible representation of SL_2, where k^* acts by homotheties. In this case $\hat{G} = SL_2 \times (k^*)^2$. Obviously, both copies of k^* act identically on $R(3)$ but the second copy is needed in order to retain the information about (usual) degree of a polynomial, when one substitutes 0's in $F(X/\!\!/U)$. Let φ (resp. ϖ) be the fundamental weight of SL_2 (resp. of the connected center of G). The algebra of U-invariants is generated by 4 functions modulo a single relation. The weights of generators are $3\varphi+\varpi$, $2\varphi+2\varpi$, $3\varphi+3\varpi$, 4ϖ and that of the relation is $6\varphi+6\varpi$. By a general result for actions of complexity one [P6], 2.4, μ is the weight of any generating relation. That is, $\mu = 6\varphi+6\varpi$. Since ϖ is responsible for usual degree of a polynomial, this also means that $n_0 = 6$. Thus,

$$F(R(3)/\!\!/U; z, \varphi, \varpi) = \frac{1-z^6e^{6\varphi+6\varpi}}{(1-ze^{3\varphi+\varpi})(1-z^2e^{2\varphi+2\varpi})(1-z^3e^{3\varphi+3\varpi})(1-z^4e^{4\varpi})}$$

and

$$F(R(3)\times R(3)^*/\!\!/G; z_1, z_2) = \frac{1+(z_1z_2)^6}{\prod_{i=1}^4(1-(z_1z_2)^i)}.$$

4. Examples and applications.

We retain the previous notation, in particular, S is the stabilizer in general position for the doubled action and the mapping ϱ is defined before Theorem 2.

(4.1) *Invariants of 4-tuples of subspaces and orthogonal invariants of pairs of subspaces.*

Let $G = SL_n = SL(V)$ and denote by $C(\varphi_i)$, $i = 1, \dots, n-1$ the closure of G-orbit of highest weight vectors in V_{φ_i}, where φ_i is ith fundamental weight of SL_n. Then $C(\varphi_i)$ is the affine cone over Grassmanian of i-dimensional subspaces of V in the Plücker embedding. It is easy to see that $C(\varphi_i)^* = C(\varphi_{n-i})$. Define X to be $C(\varphi_i) \times C(\varphi_j)$, $i < j, \quad i + j \neq n$. It is known that X is spherical [Li] and $\mathrm{Cl}(X) = 0$ [VP1]. Therefore by Corollary 2 and Proposition 4, $k[X \times X^*]^G$ is a polynomial algebra. Since $X \times X^* = C(\varphi_i) \times C(\varphi_{n-i}) \times C(\varphi_j) \times C(\varphi_{n-j})$, this algebra gives SL_n-invariants of the configuration of 4-tuples of subspaces. In this approach one gets only 4-tuples that consist of 2 pairs of subspaces of complementary dimension. Making use a description of $k[X]^U$ [Li] and our Proposition 5, one can easily describe the generators of $k[X \times X^*]^G$, in this case $\dim X \times X^* /\!\!/ G = \min\{i, n-j\} + 2$. Algebras of invariants for arbitrary 4-tuples described by R. Howe and R. Huang [HH][(2)].

In this case $G^\theta = SO_n$. Having replaced X to X^* (if necessary), one may assume that $i+j < n$. Then $S = SL_{j-i} \times SL_{n-i-j} \times T_{i-1}$, where index near T shows dimension of torus [P3], table 1. Therefore by Theorem 2(v), Proposition 2, and Richardson's lemma [Lu], Lemma 1, we get a natural isomorphism $\varrho : X /\!\!/ SO_n \xrightarrow{\sim} X \times X^* /\!\!/ SL_n \cong \mathbb{A}^{i+2}$.

(4.2) Let $V_1, \dots, V_{r+1}$ be k-vector spaces and $\underline{m} = (m_1, \dots, m_r)$ a sequence of positive integers. Define $X(\underline{m})$ to be the set of sequences $(u_1, \dots, u_r)$, where $u_i \in \mathrm{Hom}(V_i, V_{i+1})$ such that $\mathrm{rk}(u_i) \leq m_i$ and $u_{i+1} \circ u_i = 0$. This is a variety of complexes introduced by Eisenbud and Buchsbaum. The group $G = \prod_{i=1}^{r+1} GL(V_i)$ acts naturally on $X(\underline{m})$. Assume $m_i + m_{i-1} \leq n_i := \dim V_i$ ($m_0 = m_{r+1} := 0$). Then X is spherical and $r_G(X(\underline{m})) = \sum m_i$ [B2, 1.3]. Moreover, Brion indicates explicitly a representative y in the open B-orbit in $X(\underline{m})$. Then (y, y^*) is a generic point in our sense and $S = G_{(y,y^*)}$. In this case $S = \prod_{i=1}^{r} T_{m_i} \times \prod_{i=1}^{r+1} GL_{n_i - m_i - m_{i-1}}$. Therefore by the same reason as in the previous example, we get a natural isomorphism

$$X(\underline{m}) /\!\!/ \prod_{i=1}^{r+1} O(V_i) \cong X(\underline{m}) \times X(\underline{m})^* /\!\!/ \prod_{i=1}^{r+1} GL(V_i) (\cong \mathbb{A}^{\sum m_i}).$$

(2) I would like to thank F. Knop for providing me this reference.

(4.3) Now we shall show that $\pi_{G,X\times X^*}$ not always has (EQ) for arbitrary spherical G-varieties, even if $X/\!\!/U \cong \mathbb{A}^r$ and G is semisimple. Recall that H is defined at the beginning of sect. 2.

LEMMA 1. — *If $\pi_G = \pi_{G,X\times X^*}$ has (EQ), then $\pi_H = \pi_{H,X^*}$ has (EQ) as well.*

Proof. — Take an arbitrary point $\xi \in X^*/\!\!/H \cong X \times X^*/\!\!/G$. By Theorem 3 $\pi_G^{-1}(\xi) \cap (\{x\}\times X^*) = \{x\}\times\pi_H^{-1}(\xi)$. Since $\dim(\pi_G^{-1}(\xi)\cap\{gx\}\times X^*) = \dim(\pi_G^{-1}(\xi)\cap\{x\}\times X^*)$ for any $g\in G$ and $\dim Gx = X$, we get $\dim\pi_G^{-1}(\xi) \geq \dim\pi_H^{-1}(\xi) + \dim X$. □

Example. — By Lemma 1, it suffices to find spherical X such that π_{H,X^*} has not (EQ). We let G to be SL_3. Take 2 regular dominant linearly independent weights λ_1 and λ_2. Let $v_i \in V_{\lambda_i}$ be a *lowest* weight vector relative to U. Put $X := \overline{SL_3(v_1+v_2)} \subset V_{\lambda_1}\oplus V_{\lambda_2}$. The general theory of such varieties developed in [VP1]. By using it, the reader can easily restore the proof of all facts stated below.

In this case, the stabilizer H of a point of the open G-orbit in X contains a maximal unipotent subgroup. Hence we may assume that $H \supset U$ and then $k[X^*]^U = k[X^*]^H$. Clearly $(U : X^*)$ has (EQ) if and only if $(U : X)$ has. So we consider the latter action. Here $\dim X = 5$ and $\dim X/\!\!/U = 2$. The algebra $k[X]^U$ is freely generated by (the restrictions on X of) the coordinates of v_1 and v_2 in the weight decompositions of V_{λ_1} and V_{λ_2}. The plane $P = \langle v_1, v_2\rangle$ lies in X and $P\cap\mathfrak{N}_U(X) = \{0\}$. Let $w\in W$ be the reflection corresponding to a simple root and $\overline{w}$ be a representative of w in $N_G(T)$. Then $\overline{w}(P) \subset \mathfrak{N}_U(X)$, because $w\lambda_i \neq \lambda_i$, and it is easy to see that $\dim U\cdot\overline{w}(P) = 4$. That is, $\dim\mathfrak{N}_U(X) = 4 > 5-2$.

(4.4) *Example* (of a semisimple group G acting on X such that $X/\!\!/U$ is an affine space, but $X/\!\!/G^\theta$ is not).

It suffices to give an example of a free semigroup $\Gamma\subset\mathcal{X}_+$ such that Γ^{ev} is not free, because for any Γ there is a spherical variety such that Γ is its weight semigroup [VP1]. Let $\operatorname{rk} G\geq 2$ and Γ generated by φ_1 and $\varphi_1+2\varphi_2$. Then Γ^{ev} has 3 generators.

(4.5) *Example.* $G = SL_n\times T_1$, $V = \{$space of symmetric $n\times n$-matrices$\}$. The action is defined by $(g,A)\mapsto gAg^t$, $g\in SL_n, A\in V$ and T_1 acts by homotheties. Then $G^\theta = SO_n\times\mathbb{Z}_2$ and it is easy to see that $V/\!\!/G^\theta$ is not affine space, if $n\geq 3$, although $V/\!\!/SO_n = \mathbb{A}^n$. Indeed, $k[V]^{SO_n}$

is freely generated by the coefficients of the characteristic polynomial of a matrix. That is, $k[V]^{SO_n} = k[\sigma_1, \dots, \sigma_n]$, where $\det(\lambda I - A) = \lambda^n - \sigma_1(A)\lambda^{n-1} + \sigma_2(A)\lambda^{n-2} - \dots$. The non-unit element of $\mathbb{Z}_2$ send σ_i into $(-1)^i\sigma_i$, hence the assertion. It is also not difficult to check that S is finite, $|S| = n2^{n-1}$, and $|\psi(S)| = n$. Therefore $\deg \varrho = 2^{n-1}$. This may be confirmed by the following argument. Consider the sequence of regular mappings:

$$\Delta/\!\!/SO_n \to \Delta/\!\!/G^\theta \xrightarrow{\varrho} (V \oplus V^*)/\!\!/G,$$

where the left mapping is of degree 2. The degrees of free generators of $k[\Delta]^{SO_n}$ are $1, 2, \dots, n$ ($\deg \sigma_i = i$) and that of $k[V \oplus V^*]^G$ are $2, 4, \dots, 2n$. Since the restriction mapping $k[V \oplus V^*]^G \to k[\Delta]^{SO_n}$ respects degrees of polynomials, degree of the finite mapping $\Delta/\!\!/SO_n \to (V \oplus V^*)/\!\!/G$ is 2^n.

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