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FOLIATIONS OF M^3 DEFINED BY $\mathbb{R}^2$-ACTIONS (*)

by J.L. ARRAUT and M. CRAIZER

1. Introduction.

Let M be a C^r, $r=\infty$ or $r=\omega$, m-dimensional closed orientable manifold and $\mathcal{F}$ a C^r p-dimensional foliation of M. Define the rank of the foliated manifold $(M,\mathcal{F})$ as the maximum number of C^r commuting vector fields, l.i. at each point, that are tangent to $\mathcal{F}$, and denote it by $\operatorname{rank}(M,\mathcal{F})$. When $p=m$, this definition gives $\operatorname{rank}(M)$, the rank of M in the sense of Milnor, see [7]. Observe that $\operatorname{rank}(M,\mathcal{F})=p$ if and only if $\mathcal{F}$ is the underlying foliation of a C^r locally free action Φ of $\mathbb{R}^p$ on M. We say in this case that Φ is tangent to $\mathcal{F}$. Note also that $\operatorname{rank}(M,\mathcal{F})\leqslant\operatorname{rank}(M)$.

Let $\Phi:\mathbb{R}^p\times M\to M$ be a C^r action. For each $x\in M$ the map $\Phi_x:\mathbb{R}^p\to M$ defined by $\Phi_x(v)=\Phi(v,x)$, and also its image $\Phi_x(\mathbb{R}^p)$, will be called the *orbit* of x by Φ. To each $v\in\mathbb{R}^p$ is associated a flow $\Phi^v:\mathbb{R}\times M\to M$, defined by $\Phi^v(t,x)=\Phi(tv,x)$. Let $\{v_j\}$, $1\leqslant j\leqslant p$, be an ordered base of $\mathbb{R}^p$. The ordered set $\{X_j\}$, where X_j is the vector field of M tangent to the flow Φ^{v_j}, is called a base of *infinitesimal generators* of Φ. The canonical infinitesimal generators of Φ are those associated to the canonical base of $\mathbb{R}^p$.

From now on, we shall assume that $m=3$ and $p=2$. Our aim is to give a geometric characterization of the foliated manifolds of rank 2. A substantial part of this work had already been done by G. Chatelet,

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H. Rosenberg, R. Roussarie and D. Weil. They proved the following properties of a foliated manifold $(M, \mathcal{F})$ of rank 2 :

T.0 ([9]). — M is a torus bundle over the circle. In other words, M is up to diffeomorphism, the manifold M_F obtained from $T^2 \times \mathbb{R}$ by identifying (x, t) with $(F(x), t+1)$, for some $F \in SL(2; \mathbb{Z})$.

T.1 ([4]). — Suppose that $\mathcal{F}$ is a foliation by planes. Then M is diffeomorphic to T^3 and $\mathcal{F}$ is conjugated to a foliation given by the suspension of two commuting diffeomorphisms of T^1.

T.2 ([4]). — Suppose that $\mathcal{F}$ has no compact leaves, but it is not a foliation by planes. Then F has the two eigenvalues equal to $+1$, and $\mathcal{F}$ is a foliation by cylinders, conjugated to the suspension of the foliation $T^1 \times \{\theta\}$, $\theta \in T^1$, of T^2, by a diffeomorphism that leaves it invariant.

T.3 ([4]). — Suppose that $\mathcal{F}$ has a compact leaf L. Then the manifold obtained by cutting M along L is diffeomorphic to $T^2 \times [0,1]$. If L' is another compact leaf then $L \cup L'$ bounds a manifold which is also diffeomorphic to $T^2 \times [0,1]$.

Let N be a compact orientable 3-manifold. Denote by $G^r(N)$, $r = \infty$ or $r = \omega$, the set of C^r 2-dimensional foliations $\mathcal{F}$ of N, tangent to the border if $\partial N \neq \phi$, which are transversally orientable, its leaves are tori, cylinders or planes and satisfy the restrictions imposed by T.1, T.2 and T.3, and by $G^r_o(N)$ the foliations of $G^r(N)$ that have at least one compact leaf. We can summarize the contribution of the mentioned mathematicians to the characterization problem as follows :

1.1. If $\mathrm{rank}(M, \mathcal{F}) = 2$, then M is diffeomorphic to M_F, for some $F \in SL(2, \mathbb{Z})$, and $\mathcal{F} \in G^r(M)$. Besides, if $\mathcal{F} \in G^r(M) \backslash G^r_o(M)$, then $\mathrm{rank}(M, \mathcal{F}) = 2$.

The second statement in 1.1 follows easily from T.1 and T.2. In this paper we give a criterion to decide which foliations $\mathcal{F} \in G^r_o(M_F)$ are such that $\mathrm{rank}(M_F, \mathcal{F}) = 2$.

For $\mathcal{F} \in G^r_o(M_F)$ the characterization is workout by first cutting the manifold along a compact leaf, obtaining, by T.3, a tube $T^2 \times [0,1]$. Next, proving that the induced foliation on this tube, that we keep calling $\mathcal{F}$, is the underlying foliation of infinitelly many C^r $\mathbb{R}^2$-actions. Finally, finding the obstructions for one such action to be compatible with the glueing map

$F : T^2 \times \{0\} \to T^2 \times \{1\}$. We consider two different cases. The first one is when the union of the compact leaves of $\mathcal{F}$ has non-empty interior; for $r = \omega$ this is equivalent to say that all leaves are compact. In this case we prove :

THEOREM 1.2. — *Let $\mathcal{F} \in G_o^r(M_F)$, with $r = \infty$ or $r = \omega$, be such that the union of its compact leaves has non-empty interior. Then* $\operatorname{rank}(M_F, \mathcal{F}) = 2$.

To talk about the second case we need some preliminaries. Consider in $T^2 \times [0,1] = \mathbb{R}^2/\mathbb{Z}^2 \times [0,1]$ the coordinates (x,y,z) with $(x,y) \in \mathbb{R}^2$ and $z \in [0,1]$. Denote by $s(x,y)$ the segment $(x,y) \times [0,1] \subset T^2 \times [0,1]$ and identify $s(0,0)$ with $[0,1]$. Let $\mathcal{F}$ be a C^∞ foliation of $T^2 \times [0,1]$ whose compact leaves are just $T^2 \times \{0\}$ and $T^2 \times \{1\}$. Clearly $\mathcal{F}$ is transversal to $s(0,0)$ in a neighborhood of 0 and 1, therefore the holonomy of $\mathcal{F}$ at the leaf $T^2 \times \{j\}$, $j = 0,1$, is given by a representation

$$\chi_j : \pi_1(T^2 \times \{j\}, (0,0)) \to \mathcal{D}([0,1], j)$$

where $\mathcal{D}([0,1], j)$ denotes the group of germs of local C^∞ diffeomorphisms of $[0,1]$ at j. Since $\mathcal{F}$ has no compact leaves in $T^2 \times (0,1)$, it follows that at least one of the two germs $\chi_0(e_1)$ or $\chi_0(e_2)$ has 0 as its unique fixed point, where e_1 and e_2 denote the canonical generators of $\pi_1(T^2 \times \{0\}, (0,0))$. If $f \in \chi_0(e_1)$ has 0 as the only fixed point, it follows from Theorem A.2 of appendix A, that there exists $\delta > 0$ and a C^1 vector field ξ on $[0,\delta)$ such that $\xi^1 = f$, where ξ^t is the time t of the flow associated with ξ. Moreover, if $g \in \chi_0(e_2)$, it follows from A.3 that $g = \xi^T$ for some $T \in \mathbb{R}$. Write $\alpha = (T,-1)$. If every $f \in \chi_0(e_1)$ has a fixed point besides 0 we will have $g = \xi^1$, by A.2, and by [6] $f = \xi^0$. Write in this case $\alpha = (1,0)$. We call the straight line $\boldsymbol{\alpha}$, generated by α, the *principal direction* of $\mathcal{F}$.

Example 1.3. — Let $\mu : [0,1] \to \mathbb{R}$ be a non-increasing C^∞ function such that $\mu(z) = +1, (-1)$, in a neighborhood of $0, (1)$. This function will remain fixed through the paper. Let $\lambda : [0,1] \to \mathbb{R}$ be a C^∞ function such that $\lambda(z) = 0$ if and only if $z = 0$ or $z = 1$, $\alpha = (\alpha_1, \alpha_2) \neq (0,0)$, and $i \in \{0,1\}$. The foliation of $T^2 \times [0,1]$ defined by the equation

$$\lambda\alpha_2 dx - \lambda\alpha_1 dy + \mu^{1-i} dz = 0 \tag{1.1}$$

will be denoted by $\mathcal{F}(\lambda, \alpha, i)$. The vector fields $\mu^{1-i}\partial/\partial x - \lambda\alpha_2\partial/\partial z$ and $\mu^{1-i}\partial/\partial y + \lambda\alpha_1\partial/\partial z$ are tangent to this foliation and through them we can compute the generators $\chi_0(e_1)$ and $\chi_0(e_2)$. It is easy to see that the principal direction of $\mathcal{F}(\lambda, \alpha, i)$ is $\boldsymbol{\alpha}$ and that the leaves inside $T^2 \times (0,1)$ are planes if and only if α_1 and α_2 are l.i. over Q, and cylinders otherwise. To sense the role of i it would be convenient, at this moment, to draw pictures of (1.1) for $i = 0, 1$.

T.4 ([3] and [8]). — Let $\mathcal{F}$ be C^∞ foliation of $T^2 \times [0,1]$ whose compact leaves are just $T^2 \times \{0\}$ and $T^2 \times \{1\}$, and with principal direction $\boldsymbol{\alpha}$. Then, there exists $i \in \{0,1\}$, and fixing $\alpha \in \boldsymbol{\alpha}$ there exists λ such that $\mathcal{F}$ is C^0 conjugated to $\mathcal{F}(\lambda, \alpha, i)$ by a homeomorphism homotopic to the identity. The number i is called the type of $\mathcal{F}$.

Assume now that the union $\mathcal{C}$ of the compact leaves of $\mathcal{F}$ is a nowhere dense set. $M_F - \mathcal{C}$ is an open and dense set that, by T.3, can be decomposed into a countable number of open connected components R_j, $j \in \sum$, each one diffeomorphic to $T^2 \times (0,1)$. Observe that, since $\mathcal{F}$ is transversally orientable, only a finite number of $R_j's$ are of type 0. If one cuts M_F along a compact leaf, one obtains a natural linear order on $\sum$. We call a subset $[j_1, j_2] = \{j \in \sum;\ j_1 \leqslant j \leqslant j_2\}$ an interval. Let $[j_1, j_2]$ be an interval such that $\mathcal{F}|_{R_j}$ has principal direction $\boldsymbol{\alpha}$ $\forall_j \in [j_1, j_2]$. Then $S = c\ell(\bigcup_{j \in [j_1, j_2]} R_j)$ is called a *simple tube* with principal direction $\boldsymbol{\alpha}$. A *maximal simple tube* is a simple tube that is not properly contained in any other simple tube. One can decompose M_F into a countable number of maximal simple tubes S_j, with $j \in \Delta$, where Δ is a set endowed with the order inherited from $\sum$.

Let Φ be a locally free action of $\mathbb{R}^2$ on $T^2 \times [0,1]$ which has $T^2 \times \{0\}$ and $T^2 \times \{1\}$ as orbits. For each $v \in \mathbb{R}^2$, the restriction of the flow Φ^v to $T^2 \times \{j\}$, $j = 0, 1$, defines a unique asymptotic direction $u_j \in H_1(T^2 \times \{j\}) = \mathbb{R}^2$. The linear map $A_\Phi : \mathbb{R}^2 \to \mathbb{R}^2$, given by $A_\Phi(u_0) = u_1$, will be called the *continuation map* of Φ.

One can assume w.l.o.g., that the canonical infinitesimal generators $\{X_1, X_2\}$ of Φ are constant when restricted to $\partial(T^2 \times [0,1])$ i.e., $X_\ell|_{T^2 \times \{j\}} = X_{1\ell}(j)\partial/\partial x + X_{2\ell}(j)\partial/\partial y$, with $X_{k\ell}(j) \in \mathbb{R}$, for $j = 0, 1$ and $k, \ell = 1, 2$. It follows that the matrix of A_Φ in the canonical base of $\mathbb{R}^2$ is $A_\Phi = X(1)X(0)^{-1}$, where $X(j) = [X_{k\ell}(j)]$. It is clear now that for one such action to project into a C^∞ $\mathbb{R}^2$-action on M_F it is necessary that $A_\Phi = F$. This shows that the presence of the continuation map in the characterization problem is inevitable. In section 4 we prove its main properties :

THEOREM 1.4. — *Let* Φ *be a* C^∞ $\mathbb{R}^2$*-action tangent to* $\mathcal{F} \in G^\infty(T^2 \times [0,1])$. *Assume that* $T^2 \times [0,1]$ *is a simple tube of* $\mathcal{F}$ *with principal direction* $\boldsymbol{\alpha}$ *and* n *components of type* 0. *Then*

1.4.1. $A_\Phi(\alpha) = \alpha$, $\forall \alpha \in \boldsymbol{\alpha}$.

1.4.2. A_Φ *preserves orientation* $\Longleftrightarrow$ n *is even.*

Let Φ be a $\mathbb{R}^2$-action tangent to $\mathcal{F} \in G^\infty(T^2 \times [0,1])$ that has $T^2 \times \{a_j\}$, $0 = a_0 < a_1 < a_2 < \dots < a_n = 1$, as orbits. If Φ_j denotes the restriction of Φ to $T^2 \times [a_{j-1}, a_j]$, $1 \leqslant j \leqslant n$, then it is clear that $A_\Phi = A_{\Phi_n} \circ \dots \circ A_{\Phi_1}$. We enunciate now Theorems 1.5 and 1.6, which together with 1.2 characterize the elements of $G_0^r(M_F)$ of rank 2.

THEOREM 1.5. — *Let* $\mathcal{F} \in G_0^\infty(M_F)$ *and assume that the union of its compact leaves is a nowhere dense set. Let* $M_F = \bigcup_{j \in \Delta} S_j$ *be the decomposition in maximal simple tubes,* $\boldsymbol{\alpha}^j$ *the principal direction of* $\mathcal{F}|_{S_j}$, *and* n_j *the number of components of type 0 contained in* S_j.

1.5.1. *Suppose that* $\Delta = \{1\}$. *Then,* $\mathrm{rank}(M_F, \mathcal{F}) = 2$ *if and only if* $F(\alpha) = \alpha$, $\forall \alpha \in \boldsymbol{\alpha}^1$.

1.5.2. *Suppose* $\Delta = \{1,2\}$. *Then,* $\mathrm{rank}(M_F, \mathcal{F}) = 2$ *if and only if* $(-1)^{n_2} \det(F(\alpha), \beta) > 0$ *for any* $\alpha \in \boldsymbol{\alpha}^1$ *and* $\beta \in \boldsymbol{\alpha}^2$ *with* $\det(\alpha, \beta) > 0$.

1.5.3. Suppose $\mathrm{card}(\Delta) \geqslant 3$. *Then,* $\mathrm{rank}(M_F, \mathcal{F}) = 2$.

An example of a nontrivial application of this theorem is the following : If $\mathcal{F} \in G_0^\infty(T^3)$ has exactly 2 compact leaves, bounding 2 components of type 0 with different principal directions, then $\mathrm{rank}(T^3, \mathcal{F}) = 1$.

In the C^ω case, the hypothesis that the union of the compact leaves is a nowhere dense set implies that $\mathcal{F}$ has a finite number of compact leaves. Besides, the analyticity of the holonomy at the compact leaves implies that the principal direction of each component is the same. Hence $\Delta = \{1\}$.

THEOREM 1.6. — *Let* $\mathcal{F} \in G_0^\omega(M_F)$ *be a foliation with a finite number of compact leaves and principal direction* $\boldsymbol{\alpha}^1$. *Then,* $\mathrm{rank}(M_F, \mathcal{F}) = 2$ *if and only if* $F(\alpha) = \alpha$, $\forall \alpha \in \boldsymbol{\alpha}^1$.

Remark 1.7. — If $F \in SL(2; \mathbb{Z})$ admits an eigenspace with irrational slope associated to the eigenvalue $+1$, then $F = I$. Using this fact one obtains sharper informations in some cases of Theorems 1.5 and 1.6. For example, if $\Delta = \{1\}$ and the open leaves of $\mathcal{F}$ are planes, then necessarily $M_F = T^3$.

Section 2 is devoted to the construction of C^∞ $\mathbb{R}^2$-actions tangent to any given $\mathcal{F} \in G^\infty(T^2 \times [0,1])$. The construction of analytic $\mathbb{R}^2$-actions for the foliations $\mathcal{F} \in G_0^\omega(M_F)$ which satisfy the hypothesis of 1.6 is done separately in section 3. The proof of 1.2, 1.5 and 1.6 are given in section 5. For the proofs of these theorems we needed some results on the structure of foliations $\mathcal{F} \in G^r(T^2 \times [0,1])$. Since we did not find them written anywhere, we decided to include appendices A, B and C, containing the statements and the proofs of them. We would like to call attention on Theorem B.2, which talks about the immersion of certain diffeomorphisms of [0,1], with a countable number of fixed points, in C^1-flows. This theorem allowed us to prove Theorem B.1, a C^1-conjugation theorem of simple tubes to models, which in turn is essential for the proof of Theorem 1.4.

The notion of principal direction for foliations of $T^2 \times [0,1]$ whose compact leaves are just $T^2 \times \{0\}$ and $T^2 \times \{1\}$, is already present in [3]. In [11], it became clear that this notion is essential for the understanding of $\mathbb{R}^2$-actions in the neighborhood of a compact leaf. Here, it is present in the continuation map, and was also used in [1] and [15]. We make use of this oportunity to thank N.C. Saldanha for helpful conversations, S. Matsumoto for suggesting us the way to construct the analytic actions of section 3, and the referee for valuable suggestions for the presentation of this paper.

2. Actions on tubes.

Let $\mathcal{F}$ be any element of $G^\infty(T^2 \times [0,1])$ and $A^\infty(T^2 \times [0,1], \mathcal{F})$ the set of C^∞ $\mathbb{R}^2$-actions on $T^2 \times [0,1]$ with underlying foliation $\mathcal{F}$. In this section we construct some elements of this set.

Let $\Phi \in A^\infty(T^2 \times [0,1], \mathcal{F})$ and $X = \{X_1, X_2\}$ its frame of canonical infinitesimal generators. Assume that X_1 and X_2 restricted to $T^2 \times \{j\}$, $j = 0,1$, are constant vector fields i.e., $X_\ell|_{T^2 \times \{j\}} = X_{1\ell}(j)\partial/\partial x + X_{2\ell}(j)\partial/\partial y$ with $X_{k\ell}(j) \in \mathbb{R}$, $k,\ell = 1,2$. Write $X(j) = [X_{k\ell}(j)]$ and call $X(0)$ the *initial condition frame* and $X(1)$ the *terminal condition frame*. For this kind of actions, it is clear that, $X(1) = A_\Phi X(0)$, where A_Φ is the continuation map of Φ, defined in the introduction.

LEMMA 2.1. — *Let $\mathcal{F} \in G^\infty(T^2 \times [0,1])$ be a foliation whose compact leaves are just $T^2 \times \{0\}$ and $T^2 \times \{1\}$, with principal direction $\boldsymbol{\alpha}$ and type i. If $A \in GL(2;\mathbb{R})$ is such that :*

i) $A\alpha = \alpha, \ \forall \alpha \in \boldsymbol{\alpha}$.

ii) $(-1)^{1-i}\det A > 0$.

Then, fixing $\alpha \in \boldsymbol{\alpha}$ and taking β such that $\det(\alpha,\beta) > 0$, there exists $\Psi \in A^\infty(T^2 \times [0,1], \mathcal{F})$ such that its canonical frame $X = \{X_1, X_2\}$ of infinitesimal generators satisfy : $X(0) = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix}$ and $X(1) = AX(0)$. In particular $A_\Psi = A$.

Proof. — Let $\mathcal{F}_0$ be the foliation of $T^2 \times [0,1]$, defined by the kernel of the 1-form $\lambda\alpha_2 dx - \lambda\alpha_1 dy + \mu^{1-i}dz$, to which $\mathcal{F}$ is conjugated by H, according to Theorem A.1. Note that the vector fields $\alpha_1\partial/\partial x + \alpha_2\partial/\partial y$, $E_1 = \mu^{1-i}\partial/\partial x - \lambda\alpha_2\partial/\partial z$ and $E_2 = \mu^{1-i}\partial/\partial x + \lambda\alpha_1\partial/\partial z$ generate $T_p\mathcal{F}_0$ at every $p \in T^2 \times [0,1]$. Define $\gamma = A\beta$. By i) and ii), β and $(-1)^{1-i}\gamma$ are on the same side of the line $\boldsymbol{\alpha}$. Let $Y_1(z) = \alpha_1\partial/\partial x + \alpha_2\partial/\partial y$ and $Y_2(z) = Y_{12}(z)E_1 + Y_{22}(z)E_2$, where $(Y_{12}(z), Y_{22}(z))$, $0 \leqslant z \leqslant 1$, is a C^∞ path in $\mathbb{R}^2$ which coincides with (β_1,β_2) near 0 and with $(-1)^{1-i}(\gamma_1,\gamma_2)$ near 1 and which do not cross $\boldsymbol{\alpha}$. $Y = \{Y_1, Y_2\}$ is a frame of $T\mathcal{F}_0$. From A.1.2, one knows that Y_2 is smooth in $T^2 \times (0,1)$, C^1 at $T^2 \times \{0\}$ and, if the open leaves of $\mathcal{F}_0$ are planes, it is also C^1 at $T^2 \times \{1\}$. In any case $[Y_1, Y_2]$ is well defined and null. Define now $X_\ell = H^* Y_\ell$, $\ell = 1,2$. X_1 and X_2 are smooth in $T^2 \times (0,1)$ because Y_1, Y_2 and H are. It follows from A.1.1, that in a neighborhood U_0 of $T^2 \times \{0\}$, X_1 and X_2 are the liftings of $\alpha_1\partial/\partial x + \alpha_2\partial/\partial y$ and $\beta_1\partial/\partial x + \beta_2\partial/\partial y$ on $T^2 \times \{0\}$ to $\mathcal{F}|_{U_0}$, and in a neighborhood U_1 of $T^2 \times \{1\}$ they are the liftings of $\alpha_1\partial/\partial x + \alpha_2\partial/\partial y$ and $\gamma_1\partial/\partial x + \gamma_2\partial/\partial y$ on $T^2 \times \{1\}$ to $\mathcal{F}|_{U_1}$. Therefore X_1 and X_2 are C^∞ in all $T^2 \times [0,1]$. We conclude that $X = \{X_1, X_2\}$ is a C^∞ commutative frame of $T\mathcal{F}$ that defines an element $\Psi \in A^\infty(T^2 \times [0,1], \mathcal{F})$. The assertion about the infinitesimal generators of Ψ is clear from the construction.

2.2. Let $\mathcal{F} \in G^\infty(T^2 \times [0,1])$ be a foliation without components of type 0. It is shown in [4] that $\mathcal{F}$ is C^∞-conjugated to a foliation $\mathcal{F}'$ of $T^2 \times [0,1]$, transversal to the segments $s(x,y) = (x,y) \times [0,1]$, and the conjugation can be taken to be the identity in a neighborhood of $\partial(T^2 \times [0,1])$. Given any pair of l.i. constant vector fields on $T^2 \times \{0\}$, their liftings to $\mathcal{F}$ define an element $\Phi \in A^\infty(T^2 \times [0,1], \mathcal{F})$. It is worth noting that the restriction of the infinitesimal generators of Φ to $T^2 \times \{0\}$ coincide with their restriction to $T^2 \times \{1\}$, or in other words : $A_\Phi = I$.

2.3. Let $\mathcal{F}$ be any element of $G^\infty(T^2 \times [0,1])$. We shall construct some elements of $A^\infty(T^2 \times [0,1], \mathcal{F})$. Denote by U_j, $1 \leqslant j \leqslant n$, the closure of the components of type 0 of $\mathcal{F}$ and by $\boldsymbol{\alpha}^j$ the principal direction of $\mathcal{F}|_{U_j}$. We can assume w.l.o,g., that $U_j = T^2 \times [a_j, b_j]$, with $0 = b_0 \leqslant a_1 < b_1 \leqslant a_2 < b_2 \leqslant ... \leqslant a_n < b_n \leqslant a_{n+1} = 1$. For those j, $0 \leqslant j \leqslant n$, such that $b_j < a_{j+1}$, define $V_j = T^2 \times [b_j, a_{j+1}]$, and for those j with $b_j = a_{j+1}$, define $V_j = \varnothing$. One obtains a decomposition

$$T^2 \times [0,1] = V_0 \cup U_1 \cup V_1 \cup U_2 \cup V_2 \cup ... \cup U_n \cup V_n.$$

Fix $\alpha \in \boldsymbol{\alpha}^1$ and take β such that $\det(\alpha, \beta) > 0$. Let $D_1 = \alpha_1 \partial/\partial x + \alpha_2 \partial/\partial y$ and $D_2 = \beta_1 \partial/\partial x + \beta_2 \partial/\partial y$. Given matrices A_j, $1 \leqslant j \leqslant n$, with $\det A_j < 0$ and such that $A_j(\alpha) = \alpha$, $\forall \alpha \in \boldsymbol{\alpha}^j$, one constructs, using 2.1, actions Ψ_j, tangent to $\mathcal{F}|_{U_j}$, with initial condition frame $A_{j-1} \circ ... \circ A_1 \circ D$. The terminal condition frame of Ψ_j is $A_j \circ ... \circ A_1 \circ D$ and the continuation map is $A_{\Psi_j} = A_j$. For $0 \leqslant j \leqslant n$ one constructs, using 2.2, actions Φ_j, tangent to $\mathcal{F}|_{V_j}$ with initial condition frame $A_j \circ ... \circ A_1 \circ A_0 \circ D$, where $A_0 = I$. The terminal condition frame of Φ_j is also $A_j \circ ... \circ A_1 \circ D$ and $A_{\Phi_j} = I$. Observe that these actions have been constructed in such a way that the terminal condition frame of a tube is the initial condition frame of the next. Besides, in a neighborhood of every leaf $T^2 \times \{b_j\}$ and $T^2 \times \{a_j\}$ the infinitesimal generators are liftings of constant vector fields. Therefore the action $\Phi = \Phi(A_1, ..., A_n)$ whose restriction to V_j is Φ_j and to U_j is Ψ_j, is an element of $A^\infty(T^2 \times [0,1], \mathcal{F})$, with initial condition frame D, terminal condition frame $A_n \circ ... \circ A_1 \circ D$ and continuation map $A_\Phi = A_n \circ ... \circ A_1$.

Remarks 2.4. — Let $(T^2 \times [0,1], \mathcal{F})$ be a simple tube with principal direction $\boldsymbol{\alpha}$, n components of type 0, and $\Phi = \Phi(A_1, ..., A_n)$ any one of the actions constructed in 2.3. Denote by $\{X_0, Y_0\}$ the frame of canonical infinitesimal generators of Φ. The vector field X_0 has the following properties :

2.4.1. $\quad X_0|_{T^2 \times \{a_j\}} = X_0|_{T^2 \times \{b_j\}} = \alpha_1 \partial/\partial x + \alpha_2 \partial/\partial y, \quad \forall j$

2.4.2. If α_1 and α_2 are l.d. over Q, then the orbits of X_0 are all closed with the same period. If $m\alpha_1 = n\alpha_2$, with m, and n integers without common factors, then this period is $\dfrac{n}{\alpha_1} = \dfrac{m}{\alpha_2}$.

2.4.3. If α_1 and α_2 are l.i. over Q, then there exists a C^1 diffeomorphism H of $T^2 \times [0,1]$ which conjugates X_0 with $\alpha_1\partial/\partial x + \alpha_2\partial/\partial y$.

2.4.1 and 2.4.2 follow easily from the construction of Φ. To see 2.4.3, let $H_j : U_j \to U_j$, $1 \leqslant j \leqslant n$, be the C^1-diffeomorphism given by Theorem A.1 and $K_j : V_j \to V_j$, $0 \leqslant j \leqslant n$, be the C^1.-diffeomorphism given by Theorem B.1. Properties A.1.1 and B.1.1 show that H_j and K_j can be pasted together to define a global C^1-diffeomorphism H such that $H_*X_0 = \alpha_1\partial/\partial x + \alpha_2\partial/\partial y$. The following two lemmas will be useful in the proof of the sufficiency part of Theorem 1.5.

LEMMA 2.5. — *Let $T^2 \times [0,1]$ be a simple tube of a foliation $\mathcal{F} \in G^\infty(T^2 \times [0,1])$ with principal direction $\boldsymbol{\alpha}$ and n components of type 0. If $A \in GL(2;\mathbb{R})$ satisfies :*

i) $A(\alpha) = \alpha \quad \forall \alpha \in \boldsymbol{\alpha}$

ii) $(-1)^n \det A > 0$.

Then, there exists a C^∞ $\mathbb{R}^2$-action Φ on $T^2 \times [0,1]$, with underlying foliation $\mathcal{F}$, and whose continuation map is A.

Proof. — Choose a component R of type i, that we can assume, w.l.o.g., to be $T^2 \times [a,b]$, with $0 \leqslant a < b \leqslant 1$ and put $E_1 = T^2 \times [0,a]$, $E_2 = T^2 \times [b,1]$. Let Ψ_j, $j = 1,2$, be a C^∞ $\mathbb{R}^2$-action on E_j, tangent to $\mathcal{F}|_{E_j}$, and constructed by the procedure given in 2.3, and B_j its continuation map. It is clear that $B_2^{-1} \circ A \circ B_1^{-1}(\alpha) = \alpha$ and that $\det(B_2^{-1} \circ A \circ B_1^{-1})(-1)^{1-i} > 0$. Hence, by 2.1, there is a C^∞ action Φ_1 on R, tangent to $\mathcal{F}|_R$, whose continuation map is $B_2^{-1} \circ A \circ B_1^{-1}$. As in 2.3, the actions Ψ_1, Φ_1, Ψ_2 can be pasted together to define a C^∞ $\mathbb{R}^2$-action Φ, tangent to $\mathcal{F}$, and with $A_\Phi = B_2 \circ (B_2^{-1} \circ A \circ B_1^{-1}) \circ B_1 = A$.

LEMMA 2.6. — *Let $\mathcal{F} \in G^\infty(T^2 \times [0,1])$ be such that $T^2 \times [0,1] = S_1 \cup W \cup S_2$, where S_j, $j = 1,2$, is a simple tube with principal direction $\boldsymbol{\alpha}^j$, n_j components of type 0, and W has no components of type 0. Assume $\boldsymbol{\alpha}^1 \neq \boldsymbol{\alpha}^2$ and take $\alpha \in \boldsymbol{\alpha}^1$ and $\beta \in \boldsymbol{\alpha}^2$ with $\det(\alpha,\beta) > 0$. If $C \in GL(2;\mathbb{R})$ satisfies*

i) $(-1)^{n_2} \det(C(\alpha),\beta) > 0$

ii) $(-1)^{n_1+n_2} \det C > 0$.

Then, there exists a C^∞ $\mathbb{R}^2$*-action* Φ *on* $T^2 \times [0,1]$*, tangent to* $\mathcal{F}$*, such that its continuation map* A_Φ *is equal to* C*.*

Proof. — We can assume that $S_1 = T^2 \times [0,a]$, $W = T^2 \times [a,b]$ and $S_2 = T^2 \times [b,1]$, with $0 < a \leqslant b < 1$. Define linear isomorphisms A and B of R^2 by :

$$\begin{array}{ll} A(\alpha) = \alpha & B(\alpha) = C(\alpha) \\ A(C^{-1}(\beta)) = \beta & B(\beta) = \beta. \end{array}$$

We have $C = B \circ A$ and hypothesis i) becomes $\det(B(\alpha), B(\beta)).(-1)^{n_2} > 0$; which says that B preserves orientation if and only if n_2 is even. By 2.5 there exists a C^∞ $\mathbb{R}^2$-action Φ_2 on S_2, tangent to $\mathcal{F}|_{S_2}$ and with continuation map B. Now, from $A = B^{-1} \circ C$ and $\det(\alpha, \beta) > 0$, we obtain $\det(A(\alpha), A(\beta)).(-1)^{n_1} = \det(B^{-1} \circ C(\alpha), B^{-1} \circ C(\beta)).(-1)^{n_1} = \det(C(\alpha), C(\beta)).(-1)^{n_1+n_2} > 0$. Again by 2.5 there exists a C^∞ $\mathbb{R}^2$-action Φ_1 on S_1, tangent to $\mathcal{F}|_{S_1}$ and with continuation map A. The initial condition frame of Φ_1 is $D = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix}$ and its terminal condition frame is $A \circ D$. Next, let Ψ be an action on W, constructed according to 2.2, tangent to $\mathcal{F}|_W$, with initial condition frame $A \circ D$. Since $A_\Psi = I$, the terminal condition frame of Ψ is also $A \circ D$. Finally, choose infinitesimal generators for Φ_2 in such a way that its initial condition frame be $A \circ D$. The $\mathbb{R}^2$-action Φ, whose restriction to S_j is Φ_j and to W is Ψ, is the required action.

3. Construction of analytic actions.

Let $\mathcal{F} \in G_0^\omega(M_F)$ be a foliation with k compact leaves and principal direction $\boldsymbol{\alpha}$ with $F(\alpha) = \alpha$, $\forall \alpha \in \boldsymbol{\alpha}$. We shall construct a locally free analytic action of $\mathbb{R}^2$ on M_F with underlying foliation $\mathcal{F}$. Fix an $\alpha = (\alpha_1, \alpha_2) \in \boldsymbol{\alpha}$.

3.1. By cutting M_F along the compact leaves we obtain k submanifolds diffeomorphic to $T^2 \times [0,1]$ that we denote by $W_1, ..., W_k$. It follows from A.1 that, for each $1 \leqslant \ell \leqslant k$, there exists a homeomorphism $H_\ell : W_\ell \to T^2 \times \left[\frac{\ell - 1}{k}, \frac{\ell}{k}\right]$, which is a C^1 diffeomorphism when restricted to $H_\ell^{-1}\left(T^2 \times \left[\frac{\ell-1}{k}, \frac{\ell}{k}\right)\right)$, such that $(H_\ell)_*\mathcal{F}$ is given by

$$\lambda_\ell \alpha_2 dx - \lambda_\ell \alpha_1 dy + \mu^{1-i_\ell} dz = 0$$

for some $i_\ell \in \{0,1\}$ and a continuous function $\lambda_\ell : \left[\frac{\ell-1}{k}, \frac{\ell}{k}\right] \to \mathbb{R}$ that is C^1 in $\left[\frac{\ell-1}{k}, \frac{\ell}{k}\right)$. We denote by H the homeomorphism of M_F whose restriction to W_ℓ is H_ℓ, $1 \leqslant \ell \leqslant k$. Let $\phi : M_F \to M_F$ be the translation $(x,y,z) \to \left(x, y, z + \frac{1}{2k}\right)$. Then $\phi_* H_* \mathcal{F}$ is a foliation transversal to $H_*\mathcal{F}$ and whose compact leaves are given by $z = \frac{\ell}{k} - \frac{1}{2k}$, $1 \leqslant \ell \leqslant k$. Take $J : M_F \to M_F$ of class C^ω, C^0 near to H, and whose restriction to each $H^{-1}\left(T^2 \times \left[\frac{\ell-1}{k}, \frac{\ell}{k}\right)\right)$ is C^1 near to H_ℓ, $1 \leqslant \ell \leqslant k$. Then $\mathcal{F}_1 := J_*\mathcal{F}$ and $\mathcal{G}_1 = \phi_* J_* \mathcal{F}$ are transversal analytic foliations. Hence $\mathcal{G} := J^*(\mathcal{G}_1)$ is transversal to $\mathcal{F}$.

3.2. Assume that the open leaves of $\mathcal{F}$ are cylinders. Then, the leaves of $\mathcal{F}_1 \cap \mathcal{G}_1$ are circles. By making a change of coordinates of the form $(x,y,z) \to (A(x,y),z)$, for some $A \in SL(2,\mathbb{Z})$, if necessary, we can assume that $\alpha = e_1$. Consider the vector field E_1 on $T^2 \times [0,1]$ given by $E_1(p) = e_1$, $\forall p \in T^2 \times [0,1]$. Since $F(e_1) = e_1$, this vector can be projected onto M_F. Let X_1 be the vector field on M_F tangent to $\mathcal{F}_1 \cap \mathcal{G}_1$ and whose projection on the direction of E_1 is unitary. It is clear that the orbits of X_1 are closed with period 1. Therefore the same holds for the vector field $X := J^*(X_1)$. Let s be an X-invariant analytic metric on M_F. It can be obtained by averaging any analytic metric with respect to the flow of X. Take Y to be the vector field tangent to $\mathcal{F}$ s-unitary and s-orthogonal to X. Y is then an analytic vector field which commutes with X.

3.3. Suppose now that the open leaves of $\mathcal{F}$ are planes. By T.1, in the introduction, $F = \mathrm{id}$ and $M_F = T^3$. Let $\mathcal{T}$ be the foliation by tori of T^3 given by $dx = 0$ and $\mathcal{H} = J^*(\mathcal{T})$. It is clear that $\mathcal{F}$ and $\mathcal{G}$ are transversal to $\mathcal{H}$. The vector field X tangent to $\mathcal{F} \cap \mathcal{G}$ defined in 3.1 can also be defined here, although its orbits are not closed any more. Let T_0 be a fixed leaf of $\mathcal{H}$ and $R : T_0 \to T_0$ be the return map of X. Let K be a compact leaf of $\mathcal{F}$ and L a compact leaf of $\mathcal{G}$ such that in the region V between K and L all leaves of $\mathcal{F}$ are open. Denote by $U(K)$ a small neighborhood of K. By [5], $\mathcal{G}|_{U(K)}$ can be written as $\Pi^*(\mathcal{G} \cap K)$, where $\Pi : U(K) \to K$ is some C^ω projection. Hence by C.1, there exists an analytic diffeomorphism A sending $\mathcal{F} \cap \mathcal{G}|_{U(K)}$ to the linear foliation defined by $\alpha_2 dx - \alpha_1 dy = 0$. Therefore, $R|_{T_0 \cap U(K)}$ is analytically conjugated to a rotation and thus can be embedded in an analytic S^1-action. The same fact is true near L. We can extend this action in an unique way to $T_0 \cap V$ preserving the foliations $\mathcal{F}$ and $\mathcal{G}$. Proceeding in the same way with the other regions we conclude that there is an analytic S^1-action, Ψ_t, such that R embeds in it. Take now an analytic Riemannian metric m_1 on T_0 and consider the Riemannian metric m obtained by averaging m_1 with respect to Ψ_t

$$m = \int_{S_1} \Psi_t^* m_1 \, dt.$$

It is clear that m is analytic and R-invariant. Take Z the vector field on T_0 tangent to $\mathcal{F} \cap T_0$ and m-unitary. Next, take Y tangent to $\mathcal{H} \cap \mathcal{F}$, invariant by X and such that $Y|_{T_0} = Z$. Y is then an analytic vector field on T^3 that commutes with X.

4. The continuation map.

This section is dedicated to prove Theorem 1.4. Let Φ be a C^∞ $\mathbb{R}^2$-action tangent to $\mathcal{F} \in G^\infty(T^2 \times [0,1])$. Assume that $(T^2 \times [0,1], \mathcal{F})$ is a simple tube with principal direction $\boldsymbol{\alpha}$ and n components of type 0. Fix $\alpha = (\alpha_1, \alpha_2) \neq (0,0)$ in $\boldsymbol{\alpha}$.

4.1 Let Φ_0 be one of the $\mathbb{R}^2$-actions, tangent to $\mathcal{F}$, defined in 2.4 and X_0 and Y_0 its canonical infinitesimal generators. Recall that $X_0|_{T^2 \times \{0\}} = \alpha_1 \partial/\partial x + \alpha_2 \partial/\partial y$.

Define the C^∞ function $V : T^2 \times [0,1] \to \mathbb{R}^2$ by

$$D_1\Phi(0,p).V(p) = X_0(p),$$

and for each $p \in T^2 \times [0,1]$ let

$$U(p) = \lim_{T \to \infty} \frac{1}{T} \int_0^T V(X_0^t(p))dt$$

where X_0^t is the flow defined by X_0. If $\alpha_1 m = \alpha_2 n$, with m and n integers without common factors, then, by 2.4.2, the orbits of X_0^t are closed with period $T_0 = \dfrac{n}{\alpha_1} = \dfrac{m}{\alpha_2}$. Therefore

$$U(p) = \frac{1}{T_0}\int_0^{T_0} V(X_0^t(p))dt$$

is a function of class C^∞. If α_1 and α_2 are l.i. over Q, then by 2.4.3, there exists a C^1 diffeomorphism H of $T^2 \times [0,1]$ conjugating X_0 with the vector field $\alpha_1\partial/\partial x + \alpha_2\partial/\partial y$. Write $H(p) = (x(p), y(p), z(p))$. Then, by Birkhoff's theorem

$$U(p) = \int_{T^2\times\{z(p)\}} V(H^{-1}(q))d\mu(q),$$

where μ is the Lebesgue measure of $T^2 \times \{z(p)\}$. Hence U is of class C^1.

Let W be the closure of a connected component of the complement of the union of the compact leaves of $\mathcal{F}$. Fix $y \in W$. Let $L(y)$ denote the leaf of $\mathcal{F}$ containing y and $\Phi_y : \mathbb{R}^2 \to L(y)$, the covering map $\Phi_y(v) = \Phi(v, y)$. For any $a \in \mathbb{R}^2$, denote by $C_t(a)$, $t \in \mathbb{R}$, the lifting of the X_0-orbit of $\Phi_y(a)$ with initial point $C_0(a) = a$. We have then

$$\int_0^T V(X_0^t(\Phi_y(a)))dt = C_T(a) - a, \quad \forall T \in \mathbb{R}. \tag{4.1}$$

4.2. Assume that α_1 and α_2 are l.d. over Q, as in 2.4.2. Then

$$U(y) = \frac{1}{T_0}\int_0^{T_0} V(X_0^t(y))dt = \frac{1}{T_0}C_{T_0}(0)$$

which implies that $\Phi_y(T_0U(y)) = \Phi_y(C_{T_0}(0)) = y$ and hence $T_0U(y)$ belongs to the isotropy group $\mathcal{I}(y)$ of Φ at the point y. Now, $\mathcal{I}(y)$ is constant along $L(y)$ and since $L(y)$ accumulates on ∂W, it follows that $T_0U(y) \in \mathcal{I}(\partial W)$. Finally, since $U(y)$ is continuous and $\mathcal{I}(\partial W)$ is discrete, one concludes that $T_0U(y)$ is constant in W.

4.3. Assume now that α_1 and α_2 are l.i. over Q. For $z \in L(y)$, write $z = \Phi_y(a)$ for some $a \in R^2$. One has, by (4.1),

$$U(z) - U(y) = \lim_{T\to\infty} \frac{1}{T}(C_T(a) - a - C_T(0)).$$

Let $\delta = \inf\{|D_1\Phi(v)|;\ v \in \mathbb{R}^2,\ |v| = 1\}$. Then

$$|C_T(a) - a - C_T(0)| \leqslant \frac{1}{\delta}\ d(X_0^T(z), X_0^T(y)), \quad \forall T \in \mathbb{R}$$

where d is the metric on the leaf $L(y)$. Take now a linear combination Z_0 of X_0 and Y_0 such that $Z_0^1(y) = z$. Then $Z_0^1(X_0^T(y)) = X_0^T(z)$, $\forall T \in \mathbb{R}$, and hence

$$d(X_0^T(z), X_0^T(y)) \leqslant ||Z_0||_\infty$$

where $||Z_0||_\infty = \sup\{|Z_0(y)|;\ y \in W\}$. These facts together show that $U(z) = U(y)$. Since each leaf of $\mathcal{F}$ inside W is dense in W, it follows that $U(y)$ is constant in W.

4.4. It is now easy to conclude that U is constant in the simple tube $T^2 \times [0,1]$. In fact, by 4.2 and 4.3, U is constant in each component W_σ of the complement of the compact leaves of F, $\cup W_\sigma$ is an open and dense subset of the simple tube and U is of class C^1.

If $y_0 \in T^2 \times \{0\}$, then $D_1\Phi(0, y_0)V(y_0) = X_0(y_0)$. Similarly, if $y_1 \in T^2 \times \{1\}$, $D_1\Phi(0, y_1)V(y_1) = X_0(y_1)$. Since $V(y_0) = U(y_0) = U(y_1) = V(y_1)$, and $X_0(y_0)$ and $X_0(y_1)$ are defined by $\alpha_1\partial/\partial x + \alpha_2\partial/\partial y$, one concludes from the definition of A_Φ that

$$A_\Phi(\alpha_1, \alpha_2) = (\alpha_1, \alpha_2).$$

This proves 1.4.1.

If two compact leaves T_0 and T_1 are sufficiently near to each other, then the continuation map associated to them is near the identity, and hence preserves orientation. This observation reduces the proof of 1.4.2 to the proof of the following :

LEMMA 4.5. — *Assume that the compact orbits of Φ are just $T^2 \times \{0\}$ and $T^2 \times \{1\}$. Then A_Φ preserves orientation if and only if the underlying foliation $\mathcal{F}$ of Φ is of type 1.*

Take $\beta = (\beta_1, \beta_2)$ with $\det(\alpha, \beta) > 0$ and choose $A \in GL(2, \mathbb{R})$ such that $A\alpha = \alpha$ and $(-1)^{1-i}\det A > 0$. By 2.1 there exists $\Psi \in A^\infty(T^2 \times [0,1], \mathcal{F})$ such that its canonical frame of infinitesimal generators $Z_0 = \{X_0, Y_0\}$ satisfies : $X_0(0) = \alpha_1\partial/\partial x + \alpha_2\partial/\partial y$, $Y_0(0) = \beta_1\partial/\partial x + \beta_2\partial/\partial y$, $X_0(1) = AX_0(0)$, and $Y_0(1) = AY_0(0)$. Of course $A_\Psi = A$. Assume w.l.o.g., that Φ acts by translations on $T^2 \times \{0\}$ and $T^2 \times \{1\}$, and let $Z = \{X, Y\}$ be its frame of infinitesimal generators determined by the initial condition $X(0) = X_0(0)$ and $Y(0) = Y_0(0)$. Since Ψ and Φ are both tangent to $\mathcal{F}$ one can write $X = aX_0 + bY_0$, $Y = cX_0 + dY_0$, where a, b, c and d are smooth functions on $T^2 \times [0, 1]$, which restrict to constants a_j, b_j, c_j and d_j on $T^2 \times \{j\}, j = 0, 1$, with $a_0 = 1, b_0 = 0, c_0 = 0, d_0 = 1$. We know that $ad - bc \neq 0$ at every point of $T^2 \times [0, 1]$. Since $a_0d_0 - b_0c_0 = 1 > 0$ we conclude that $a_1d_1 - b_1c_1 > 0$. It is clear that $A_\Phi = A_1 \circ A$, where $A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$. Therefore $\mathrm{sgn}(\det A_\Phi) = \mathrm{sgn}(\det A)$. Since $(-1)^{1-i}\det A > 0$ we obtain $(-1)^{i-1}\det A_\Phi > 0$, which proves the lemma.

5. Proof of the characterization theorems.

In this section we prove Theorems 1.2, 1.4, and 1.5. Each one of them refers to a foliation $\mathcal{F} \in G_0^r(M_F)$, with $r = \infty$ or $r = \omega$. To start the proofs we cut M_F along a compact leaf L obtaining a C^r-foliation of $T^2 \times [0, 1]$, tangent to the boundary, that we keep calling $\mathcal{F}$. A convenient choice of L will help. We recall a statement, already used in section 2, that permits to paste actions whose underlying foliations are restrictions of $\mathcal{F}$ to contiguous tubes.

5.1. Given $\mathcal{F} \in G^\infty(T^2 \times [0, 1])$ with $T^2 \times \{b\}$ as a compact leaf, let Φ_1 be a C^∞ action defined on $T^2 \times [a, b]$, tangent to $\mathcal{F}$, and Φ_2 a C^∞ action defined on $T^2 \times [b, c]$, tangent to $\mathcal{F}$, $0 \leqslant a < b < c \leqslant 1$. Assume that in a neighborhood of $T^2 \times \{b\}$ the infinitesimal generators of Φ_1 and Φ_2 are the liftings to $\mathcal{F}$ of a pair of constant vector fields on $T^2 \times \{b\}$. Then, the action Φ on $T^2 \times [a, c]$, that restricts to Φ_1, and to Φ_2 is also C^∞.

5.2. *Proof of Theorem 1.2.* — Since the union of the compact leaves of $\mathcal{F}$ has non-empty interior, we take two compact leaves T_0 and T_1 such that the submanifold of $T^2 \times [0, 1]$ bounded by them has only compact leaves. We can assume w.l.o.g. that $T_0 = T^2 \times \{a\}$, $T_1 = T^2 \times \{1\}$, for some $0 \leqslant a < 1$, and also that each leaf of $\mathcal{F}$ in the tube $T^2 \times [a, 1]$ is of the form $T^2 \times \{z\}$, $z \in [a, 1]$. Define an action Φ_1 on $T^2 \times [0, a]$ by 2.3 and denote its continuation map by B. On $T^2 \times [a, 1]$, define Φ_2 by

$$d\Phi_2(x, y, z)v = A(z)v$$

where $v \in \mathbb{R}^2$, $(x,y,z) \in T^2 \times [a,1]$, and $A : [a,1] \to GL(2;\mathbb{R})$ is a C^∞-function with $A(z) = B$, for z near a, and $A(z) = F \circ B^{-1}$, for z near 1. By 5.1 the action Φ on $T^2 \times [0,1]$ whose restriction to $T^2 \times [0,a]$ is Φ_1 and to $T^2 \times [a,1]$ is Φ_2, is C^∞. Besides, $A_\Phi = F \circ B^{-1} \circ B = F$, hence Φ can be projected to M_F and this completes the proof for $r = \infty$.

When $r = \omega$, all leaves are compact and it is simple to show that $\mathcal{F}$ is C^ω conjugated to the foliation $T^2 \times \{z\}$, $z \in [0,1]$. Define the C^ω action Φ on $T^2 \times [0,1]$ by

$$d\Phi(x,y,z)v = A(z)v$$

$\forall (x,y,z) \in T^2 \times [0,1]$, $v \in \mathbb{R}^2$, where $A : [0,1] \to GL(2;\mathbb{R})$ is C^ω, $A(0) = I$ and $A(1) = F$. By conjugating back one obtains the desired C^ω action.

Assume now that $\mathcal{F} \in G_0^\infty(M_F)$ and that the union of the compact leaves is a nowhere dense set. Decompose $M_F = \bigcup_{j \in \Delta} S_j$ into maximal simple tubes, as explained in the introduction. For $j \in \Delta$, denote by $\boldsymbol{\alpha}^j$ the principal direction of S_j and by n_j the number of components of type 0 contained in S_j. If Φ is a smooth $\mathbb{R}^2$-action with underlying foliation $\mathcal{F}$, we will assume w.l.o.g., that the infinitesimal generators of Φ on $T^2 \times \{0\}$ and $T^2 \times \{1\}$ are constant. Under this assumption, as was explained in the introduction, the continuation map A_Φ coincides with F.

5.3. *Proof of Theorem 1.5.* — We keep in mind that since F preserves orientation, and $\mathcal{F}$ is transversally orientable, then the total number of components of type 0 is even.

5.3.1. Suppose that $\Delta = \{1\}$, or in other words that $(T^2 \times [0,1], \mathcal{F})$ is a simple tube with principal direction $\boldsymbol{\alpha}^1$. If Φ is a smooth $\mathbb{R}^2$-action with underlying foliation $\mathcal{F}$ then, by 4.1, $A_\Phi(\alpha) = \alpha$, $\forall \alpha \in \boldsymbol{\alpha}^1$, and recall that $F = A_\Phi$. Now assume that $F(\alpha) = \alpha$, $\forall \alpha \in \boldsymbol{\alpha}^1$. Since n is even and $\det(F) > 0$, we can apply Lemma 2.5 to F and obtain a smooth $\mathbb{R}^2$-action Φ, tangent to $\mathcal{F}$, and such that $A_\Phi = F$.

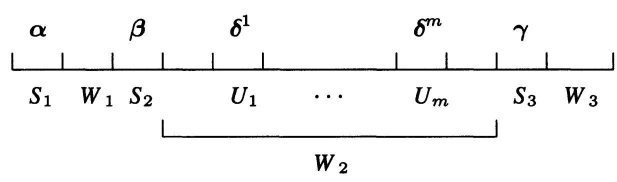
5.3.2. Suppose that $\Delta = \{1,2\}$ and let $\alpha \in \boldsymbol{\alpha}^1$ and $\beta \in \boldsymbol{\alpha}^2$ with $\det(\alpha,\beta) > 0$. If A and B are the continuation maps of $\Phi|_{S_1}$ and $\Phi|_{S_2}$ then $B \circ A = A_\Phi = F$. It follows from 4.1 that $\det(B(\alpha), B(\beta))(-1)^{n_2} > 0$ and that $B(\alpha) = F(\alpha)$ and $B(\beta) = \beta$. We conclude that $\det(F(\alpha),\beta)(-1)^{n_2} > 0$.

Now assume that $\det(F(\alpha),\beta)(-1)^{n_2} > 0$. Since n_1+n_2 is even and $\det F > 0$, we can apply Lemma 2.6 to F and obtain a smooth $\mathbb{R}^2$-action Φ, tangent to $\mathcal{F}$, and such that $A_\Phi = F$. Therefore, Φ projects to an action on M_F.

5.3.3. Suppose that $\operatorname{card}(\Delta) \geqslant 3$. Cut M_F through a compact leaf L which belongs to the boundary of a maximal simple tube S, with principal direction $\boldsymbol{\alpha}$, and such that Δ, with the induced linear order, has a first element that we denote by 1. S becomes S_1. If Δ has a last element denote it by 3 and let $\boldsymbol{\gamma}$ be the principal direction of S_3. It is clear that S_3 is not contiguous to S_1 and that $F(\boldsymbol{\alpha}) \neq \boldsymbol{\gamma}$. If Δ has not a last element we can choose an element, and denote it by 3, such that S_3 is not contiguous to S_1, its principal direction $\boldsymbol{\gamma} \neq F(\boldsymbol{\alpha})$, and $W_3 = \bigcup_{3<j} S_j$ does not contain components of type 0. Next, choose an element in Δ and denote it by 2, with $1 < 2 < 3$, such that $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$, where $\boldsymbol{\beta}$ is the principal direction of S_2, and that $W_1 = \bigcup_{1<j<2} S_j$ does not contain components of type 0. Finally let $W_2 = \bigcup_{2<j<3} S_j$. Observe that none of the $S_j's$ is empty but any of the $W_j's$, $j = 1,2,3$, could be. In any case we have $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$ and $F(\boldsymbol{\alpha}) \neq \boldsymbol{\gamma}$.

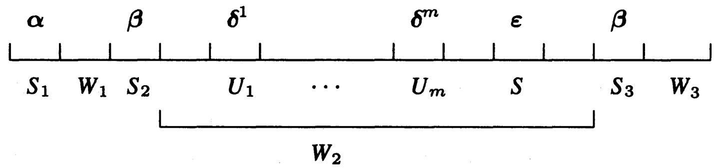
Let U_ℓ, $1 \leqslant \ell \leqslant m$ be the components of type 0 contained in W_2 and $\boldsymbol{\delta}^\ell$ the principal direction in U_ℓ. The total number of this components is even and equal to $n_1 + n_2 + n_3 + m$. We will distinguish two cases :

I. $\boldsymbol{\gamma} \neq \boldsymbol{\beta}$. In this case the diagram

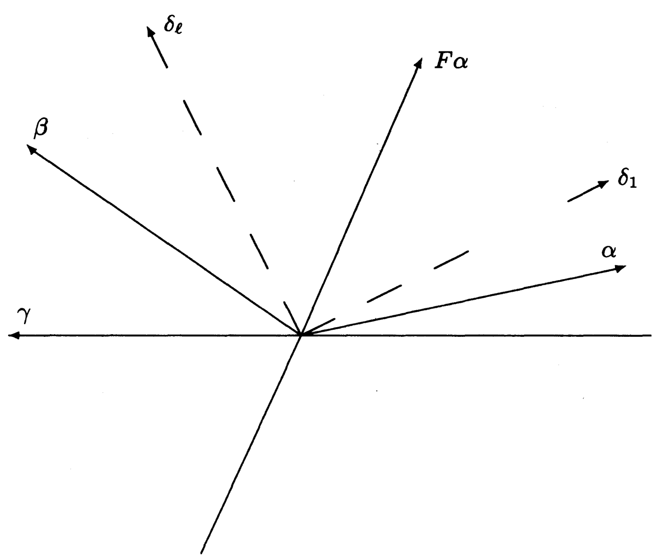


summarize the situation.

II. $\boldsymbol{\gamma} = \boldsymbol{\beta}$. In this case $W_2 \neq \varnothing$ and we can choose a maximal simple tube $S \subset W_2$ with principal direction $\boldsymbol{\varepsilon} \neq \boldsymbol{\beta}$ such that either S is to the right of U_m, and therefore do not contain components of type 0, or $S \supset U_m$ and then $\boldsymbol{\varepsilon} = \boldsymbol{\delta}^m$.

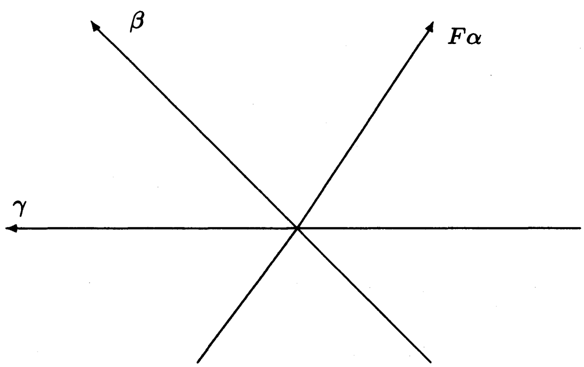


We now proceed to prove case I. Decompose W_2 as we did with $T^2 \times [0,1]$ in 2.3. We can assume w.l.o.g., that each component of the boundary of each tube that received a name is a product $T^2 \times \{pt\}$. Since there are no components of type 0 in V_j, $0 \leqslant j \leqslant m$, define, as in 2.2, an action on each V_j tangent to $\mathcal{F}|_{V_j}$ and with continuation map I. Do the same thing on W_3. Next choose $\alpha \in \boldsymbol{\alpha}$, $\beta \in \boldsymbol{\beta}$, $\gamma \in \boldsymbol{\gamma}$ and $\delta^\ell \in \boldsymbol{\delta}^\ell$, $1 \leqslant \ell \leqslant m$, such that $\det(\alpha, \beta) > 0$, $\det(\beta, \gamma) > 0$, $\det(\delta^\ell, \beta) > 0$ if $\boldsymbol{\delta}^\ell \neq \boldsymbol{\beta}$, and $\delta^\ell = \beta$ if $\boldsymbol{\delta}^\ell = \boldsymbol{\beta}$.



The idea is to construct, using 2.5, smooth actions on the tubes U_ℓ, $1 \leqslant \ell \leqslant m$, and S_3, with pre-assigned continuation maps D_ℓ and C, respectively, chosen in such a way that $S_1 \cup W_1 \cup S_2$ and the map

$(C \circ D_m \circ ...D_1)^{-1} \circ F$ satisfy the hypothesis of Lemma 2.6. Define D_ℓ by $D_\ell(\delta^\ell) = \delta^\ell$ and $D_\ell(\beta) = -\beta$ if $\boldsymbol{\delta}^\ell \neq \boldsymbol{\beta}$ or $D_\ell(\alpha) = -\boldsymbol{\alpha}$ if $\boldsymbol{\delta}^\ell = \boldsymbol{\beta}$. If $D = D_m \circ ... \circ D_1$, it is clear that $(-1)^m \det D > 0$ and $D(\beta) = \pm\beta$. Apply Lemma 2.5 to the pair $(\mathcal{F}|_{U_\ell}, D_\ell)$ to obtain an action tangent to $\mathcal{F}|_{U_\ell}$ and continuation map D_ℓ. The fact that $\mathbb{R}^2\backslash(\boldsymbol{\gamma} \cup F(\boldsymbol{\alpha}))$ has four connected components allows one to define a map C with $C(\gamma) = \gamma$ and such that, for any parity value of n_2 and n_3, one has both $(-1)^{n_3}\det C > 0$ and $(-1)^{n_2}\det(F\alpha, C \circ D\beta) > 0$. Again by 2.5 there is an action tangent to $\mathcal{F}|_{S_3}$ and continuation map C. Consider now the map $(C \circ D)^{-1} \circ F$. From $(-1)^{m+n_3}\det(C)\det(D) > 0$ and $(-1)^{n_2}\det(F\alpha, C \circ D\beta)$ one obtains $(-1)^{n_2}\det((C \circ D)^{-1} \circ F\alpha, \beta) > 0$. We also have $(-1)^{n_1+n_2}\det((C \circ D)^{-1} \circ F) > 0$. Therefore we can apply lemma 2.6 to the pair $(\mathcal{F}|_{S_1 \cup W_1 \cup S_2}, (C \circ D)^{-1} \circ F)$ to obtain an action whose continuation map is $(C \circ D)^{-1} \circ F$. Using 5.1 we paste together the actions defined above and obtain a C^∞ action Φ on $T^2 \times [0,1]$, tangent to $\mathcal{F}$ and with continuation map F. Φ can of course be projected to M_F.



The proof of case II is completely similar. We will just say how to choose the pre-assigned continuation maps on the tube W_2, where the change is necessary. If S with principal direction $\boldsymbol{\varepsilon}$ is to the right of U_m, define D_ℓ, $1 \leqslant \ell \leqslant m$, exactly as in case I. Next, choose $\varepsilon \in \boldsymbol{\varepsilon}$, such that $\det(\varepsilon, \beta) > 0$ and define a linear map E by $E(\varepsilon) = \varepsilon$, $E(\beta) \notin F(\boldsymbol{\alpha}) \cup \boldsymbol{\beta}$ and $\det E > 0$. Define C as in case I and use $C \circ E \circ D$ instead of $C \circ D$. If $S \supset U_m$ define D_ℓ $1 \leqslant \ell \leqslant m-1$ as above and D_m by $D_m(\delta^m) = \delta^m$ and $D_m(\beta) \notin F(\boldsymbol{\alpha}) \cup \boldsymbol{\beta}$ with $\det D_m < 0$. This completes the proof of 5.3.3.

5.4. *Proof of Theorem 1.6.* — The sufficiency part was done in section 3. The necessity part follows from 4.1.

6. Appendices.

A. Foliations of $T^2 \times [0,1]$ whose only compact leaves are $T^2 \times \{0\}$ and $T^2 \times \{1\}$.

As we mentioned in the introduction, T.4, every smooth foliation on $T^2 \times [0,1]$, tangent to the boundary and with no compact leaves in the interior, is topologically equivalent to a foliation defined by the kernel of the 1-form

$$\lambda \alpha_2 dx - \lambda \alpha_1 dy + \mu^{1-i} dz.$$

Now, we enunciate a sharper result, which says that the equivalence can have some differentiability. We shall use the notation $s(x,y) = (x,y) \times [0,1]$.

THEOREM A.1. — *Let $\mathcal{F}$ be a C^∞ foliation of $T^2 \times [0,1]$, tangent to the boundary, with no compact leaves in the interior, with principal direction $\boldsymbol{\alpha}$ and type $i \in \{0,1\}$. Then, fixing $\alpha \in \boldsymbol{\alpha}$ there exists a continuous function $\lambda : [0,1] \to \mathbb{R}$, and a homeomorphism H of $T^2 \times [0,1]$, isotopic to the identity, conjugating $\mathcal{F}$ with the foliation $\mathcal{F}(\lambda, \alpha, i)$. Moreover*

A.1.1. *There exists neighborhoods U_0 of $T^2 \times \{0\}$ and U_1 of $T^2 \times \{1\}$ such that H preserves each set $s(x,y) \cap U_0$ and $s(x,y) \cap U_1$, $\forall (x,y) \in T^2$, and is equal to the identity map when restricted to the sets $T^2 \times \{0\}$, $T^2 \times \{1\}$, $s(0,0) \cap U_0$ and $s(0,0) \cap U_1$.*

A.1.2. *λ is C^∞ in (0,1), C^1 at 0 and if the open leaves of $\mathcal{F}$ are planes it is also C^1 at 1. Correspondingly, H is a C^∞ diffeomorphism of $T^2 \times (0,1)$, C^1 at $T^2 \times \{0\}$ and, if the open leaves of $\mathcal{F}$ are planes, H is a C^1 diffeomorphism of $T^2 \times [0,1]$.*

To prove A.1 we need some preliminaries.

THEOREM A.2 ([12],[13],[14]). — *Let $f : [a,b) \to [a,c)$ be a diffeomorphism having a as its unique fixed point. There exists a unique*

C^1 vector field ξ on $[a,b)$ such that $\xi^1 = f$. Moreover ξ is C^∞ in (a,b).

THEOREM A.3 ([5],[6]). — *If $g : [a,b) \to [a,d)$ is another diffeomorphism having a as its unique fixed point and such that $g \circ f = f \circ g$, then there exists $T \in \mathbb{R}$ such that $g = \xi^T$.*

LEMMA A.4. — *Let $\eta = \beta(x)\partial/\partial x$ be any vector field defined on (a,b) such that $\eta^1 = f$. Then $\lim_{x\to a} \beta(x) = 0$.*

Proof. — Assume, w.l.o.g., that $f(x) < x$, $\forall x \in (a,b)$. Fix $x_0 \in (a,b)$. By Lemma 2.5 of [12] there exists a constant $A > 0$ such that

$$\frac{(f^n)'(x)}{(f^n)'(y)} \leqslant A$$

$\forall x, y \in [f(x_0), x_0]$ and $\forall n \in N$. Since

$$\int_{f(x_0)}^{x_0} (f^n)'(x)dx = f^n(x_0) - f^{n+1}(x_0)$$

converges to 0 as $n \to \infty$, we conclude that

$$\lim_{x\to a} (f^n)'(x) = 0$$

$\forall x \in [f(x_0), x_0]$. From this and the equation $\beta(f^n(x)) = (f^n)'(x)\beta(x)$ it is easy to conclude that $\lim_{x\to a} \beta(x) = 0$.

LEMMA A.5. — *Let η be any vector field on (a,b) such that $\eta^S = \xi^S$ and $\eta^T = \xi^T$ for some pair of rationally independent real numbers S and T. Then $\xi = \eta$.*

Proof. — The set $\{t \in \mathbb{R}; \eta^t = \xi^t\}$ is a closed subgroup of $\mathbb{R}$ containing S and T. Hence it is equal to $\mathbb{R}$.

We begin now the proof of A.1. Assume w.l.o.g, that $\alpha_2 \neq 0$. Take $\varepsilon > 0$ small and write $s_0 = \{(0,0,z);\ 0 \leqslant z < 2\varepsilon\}$ and $s_1 = \{(0,0,z);\ 1 - 2\varepsilon < z \leqslant 1\}$. Denote by $\chi_j : \pi_1(T^2 \times \{j\}) \times s_j \to s_j$, $j = 0, 1$, the holonomy maps of $T^2 \times \{j\}$. Since $\alpha_2 \neq 0$, $\chi_0(e_1)$ has 0 as its unique fixed point. By A.2, there exists a C^1 vector field $\xi_0 = \lambda_0(z)\partial/\partial z$, C^∞ out of 0, such that $\chi_0(e_1) = \xi_0^{-\alpha_2}$, and by A.3, $\chi_0(e_2) = \xi^{\alpha_1}$ for some $\alpha_1 \in \mathbb{R}$. It is now easy to show that there exists a neighborhood U_0 of $T^2 \times \{0\}$ and a C^1 diffeomorphism $H_0 : U_0 \to H_0(U_0)$ conjugating $\mathcal{F}|_{U_0}$ with the foliation defined by

$$\lambda_0(z)(\alpha_2 dx - \alpha_1 dy) + dz = 0.$$

The torus $T_0 = H_0^{-1}(T^2 \times \{0\})$ is transversal to $\mathcal{F}$ and we can construct, in a similar way, a transversal torus T_1 near $T^2 \times \{1\}$.

Let $A = S^1 \times \{0\} \times [0,1]$. It is clear that A is transversal to $\mathcal{F}$ in a neighborhood of the boundary and if we choose ε small enough it will also be transversal to T_0 and T_1.

In Theorem 1.2 of [10] it is proved that if the open leaves of $\mathcal{F}$ are planes then A is isotopic to an anulus A_0 such that A_0 is transverse to $\mathcal{F}$ and $A = A_0$ in a neighborhood of ∂A. The same proof, given there, works when the open leaves of $\mathcal{F}$ are cylinders if we assume that the generators of $\pi_1(A)$ and of π_1 (Cylinder) do not define the same element in $\pi_1(T^2 \times [0,1])$, but this is precisely the case here because $\alpha_2 \neq 0$. Therefore, w.l.o.g., we can assume that A itself is transversal to $\mathcal{F}$.

Let $C_0 = T_0 \cap A$ and $C_1 = T_1 \cap A$. The foliation $\mathcal{F} \cap A$ induces the holonomy maps $P_0 : s_0 \backslash \{0\} \to C_0$, $P_1 : s_1 \backslash \{1\} \to C_1$ and $P : C_0 \to C_1$. The fact that $P_0 \circ \chi_0(e_1) = P_0$ implies that the vector field $\zeta_0 = (P_0)_* \xi_0$ is well defined on C_0, and also that $\zeta_0^{-\alpha_2} = \text{id}$. One can check that $\zeta_0^{\alpha_1}$ is exactly the Poincaré map of $\mathcal{F} \cap T_0$ with respect to the transversal circle C_0. Notice, that these observations imply that $\mathcal{F} \cap T_0$ is a C^∞ linearizable foliation of T_0.

Let $\zeta_1 = P_*(\zeta_0)$. It is clear that $\zeta_1^{-\alpha_2} = \text{id}$ and that $\zeta_1^{\alpha_1}$ is the Poincaré map of $\mathcal{F} \cap T_1$ with respect to the transversal circle C_1. Hence the vector field $\xi_1 := P_1^*(\zeta_1)$ satisfies ${\xi_1}^{-\alpha_2} = \chi_1(e_1)$ and ${\xi_1}^{\alpha_1} = \chi_1(e_2)$. Writing $\xi_1 = \lambda_1(z)\partial/\partial z$, we can conclude from A.4 that $\lim_{z\to 1} \lambda_1(z) = 0$, and therefore ξ_1 can be extended continuosly to 1. If the open leaves of $\mathcal{F}$ are planes, we can use Lemma A.5 to conclude that ξ_1 is equal to the vector field given by A.2, and hence is C^1 at $z = 1$.

It is now easy to show that there exists a neighborhood U_1 of $T^2 \times \{1\}$ and a homeomorphism $H_1 : U_1 \to H_1(U_1)$ conjugating $\mathcal{F}|_{U_1}$ and the foliation given by the equation

$$\lambda_1(z)(\alpha_2 dx - \alpha_1 dy) + (-1)^{1-i} dz = 0.$$

Moreover, H_1 is C^∞ out of $T^2 \times \{1\}$ and, if the open leaves of $\mathcal{F}$ are planes, it is C^1 at $T^2 \times \{1\}$.

Let $\lambda : [0,1] \to \mathbb{R}$ be any function extending λ_0 and λ_1 such that λ is C^∞ in (0,1) and $\lambda(z) = 0 \Longleftrightarrow z = 0$ or $z = 1$. Denote $t(\varepsilon)$ the time required for the flow of $\lambda(z)\partial/\partial z$ to go from $z = \varepsilon$ to $z = 1 - \varepsilon$. Define now $Z_0 = H_0^*(\partial/\partial x - \alpha_2\lambda_0\partial/\partial z)$ and $Z_1 = H_1^*(\partial/\partial x - (-1)^{1-i}\alpha_2\lambda_1\partial/\partial z)$. Adapting the proof of lemma 3 of [3], we can show that there exists a C^∞ vector field Z defined in W, the region between T_0 and T_1, tangent to $\mathcal{F}$ such that $Z = Z_0$ in $W \cap U_0$, $Z = Z_1$ in $W \cap U_1$ and $Z^{t(\varepsilon)}(T_0) = T_1$.

Let $H = H_0$ in U_0, $H = H_1$ in U_1 and extend H to W as follows : for $p \in W$ $p = Z^t(q)$ with $q \in U_0$, $H(p) := Y_0^t(H(q))$ where $Y_0 = \partial/\partial x - \alpha_2\lambda(z)\mu(z)\partial/\partial z$. One can easily verify that this definition does not depend on $q \in U_0$ and that H coincides with H_0 at $U_0 \cap W$ and with H_1 at $U_1 \cap W$. This completes the proof of A.1.

B. Simple tubes whose open leaves are planes.

THEOREM B.1. — *Let $(T^2 \times [0,1], \mathcal{F})$ be a simple tube with principal direction α of irrational slope and without components of type 0. Fixing $(0,0) \neq (\alpha_1, \alpha_2) \in \alpha$, there exists a C^1 function $\lambda : [0,1] \to \mathbb{R}$ and a C^1 diffeomorphism H of $T^2 \times [0,1]$, isotopic to the identity, conjugating $\mathcal{F}$ with the foliation given by*

$$(b.1) \qquad \lambda\alpha_2 dx - \lambda\alpha_1 dy + dz = 0$$

moreover :

B.1.1. *There exists neighborhoods U_0 of $T^2 \times \{0\}$ and U_1 of $T^2 \times \{1\}$ such that H preserves each segment $s(x,y) \cap U_0$ and $s(x,y) \cap U_1$ for all $(x,y) \in T^2$, and is equal to the identity when restricted to $T^2 \times \{0\}$, $T^2 \times \{1\}$, $s(0,0) \cap U_0$ and $s(0,0) \cap U_1$.*

Any foliation $\mathcal{F} \in G^r(T^2 \times [0,1])$ without components of type 0 is conjugated to the foliation obtained by the suspension of a pair of commuting diffeomorphisms of $[0,1]$. Hence, the proof of B.1 can be reduced to the proof of a theorem on diffeomorphisms of the interval.

Let $f : [a,b] \to [a,b]$ be a C^∞ diffeomorphism and denote by P the set of fixed points of f. Assume that $a, b \in P$ and that $\text{int}(P) = \varnothing$.

Decompose $[a,b]\backslash P$ into its connected components $I_\gamma = (a_\gamma, b_\gamma)$, $\gamma \in \Gamma$, where Γ is countable set. Denote by $C^\infty(f|_{I_\gamma})$ the centralizer of $f|_{I_\gamma}$.

THEOREM B.2. — *Suppose that $C^\infty(f|_{I_\gamma})$ is not isomorphic to $\mathbb{Z}$, $\forall\gamma \in \Gamma$. Then there exists a C^1 vector field $\xi = \lambda(z)\partial/\partial z$ on $[a,b]$ such that $\xi^1 = f$.*

We now show how B.1 follows from B.2. Assume w.l.o.g., that $\mathcal{F}$ is transversal to each segment $s(x,y)$, $(x,y) \in T^2$. Next, observe that any foliation given by (b.1) is also transversal to each segment $s(x,y)$ and its holonomy with respect to $s(0,0) = [0,1]$

$$\chi : \pi_1(T^2,(0,0)) \to \mathrm{Diff}[0,1]$$

satisfies $\chi(e_1) = \xi^{-\alpha_2}$ and $\chi(e_2) = \xi^{\alpha_1}$, where $\xi = \lambda(z)\partial/\partial z$. Therefore, in order to prove B.1 it is enough to show the existence of a C^1 function $\lambda : [0,1] \to \mathbb{R}$ such that $f = \chi_{\mathcal{F}}(e_1) = \xi^{-\alpha_2}$ and $g = \chi_{\mathcal{F}}(e_2) = \xi^{\alpha_1}$, where $\xi = \lambda(z)\partial/\partial z$.

Let P be the set of fixed points of f. Since the open leaves of $\mathcal{F}$ are planes and the union of the compact leaves is a rare set it is clear that $\mathrm{int}(P) = \varnothing$. Decompose $[0,1]\backslash P$ into its connected components I_γ, $\gamma \in \Gamma$, with Γ a countable set. Since $\boldsymbol{\alpha}$ has irrational slope it follows that $C^\infty(f|_{I_\gamma}) \neq \mathbb{Z}$; $\forall\gamma \in \Gamma$. We can therefore apply B.2 to find a C^1 vector field $\xi = \lambda(z)\partial/\partial z$ on $[0,1]$ such that $f = \xi^{-\alpha_2}$. By A.3 $g = \xi^T$ for some $T \in \mathbb{R}$. Since $(T,\alpha_2) \in \boldsymbol{\alpha}$ then necessarily $T = \alpha_1$. This shows that $g = \xi^{\alpha_1}$ and completes the proof of B.1.

The proof of B.2 is based in the following lemma :

LEMMA B.3. — *Let $f : [a,b] \to [a,b]$ be a diffeomorphism whose fixed points are a and b. Suppose that $C^\infty(f)$ is not isomorphic to $\mathbb{Z}$. Then there exists a C^1 vector field $\xi = \lambda(z)\partial/\partial z$ such that $f = \xi^1$. Moreover,*

$$|\lambda'(z)| \leqslant |\lambda'(a)| + C\rho(b-a)^2 \tag{b.2}$$

$\forall z \in [a,b]$, where $\rho = \dfrac{\log f'(a)}{f'(a)-1}$, and C is a constant which only depends on the norm of $\log(f')$ in the C^1-topology.

Proof. — Let ξ be a C^1 vector field defined in a neighborhood of a such that $\xi^1 = f$, see A.2. Extend it to $[a,b)$ using iterations of f.

By the same argument, there exists a C^1 vector field η on $(a,b]$ such that $\eta^1 = f$. Let $g \in \mathcal{C}^\infty(f)$. By A.3, $g = \xi^T = \eta^T$ for some $T \in \mathbb{R}$. By hypothesis there exists some $g \in \mathcal{C}^\infty(f)$ that yields an irrational T, and by A.5, $\xi = \eta$. The estimate (b.2) is based on the estimate

$$(b.3) \qquad |\lambda(z)| \leqslant C_2\rho\Delta(z)|z-a|$$

$\forall x \in [a,b]$, proved in [14], 3.12. Here $\Delta(z) = |f(z) - z|$ and C_2 is a constant depending only on the norm of $\log(f')$ in the C^1-topology.

From $f_*\xi = \xi$ one obtains $\lambda(f(z)) = f'(z)\lambda(z)$, and derivating it, we obtain $\lambda'(f(z))f'(z) = f''(z)\lambda(z) + f'(z)\lambda'(z)$. Writing $\phi(z) = (\log(f'))'(z)$, this last equality becomes

$$\lambda'(f(z)) - \lambda'(z) = -\lambda(z).\phi(z).$$

Now, adding this expression along the orbit of z we obtain

$$\lambda'(f^i(z)) - \lambda'(z) = -\sum_{j=0}^{i-1} \lambda(f^i(z))\phi(f^j(z)).$$

Therefore, $|\lambda'(f^i(z)) - \lambda'(z)| \leqslant ||\phi|| \sum_{j=0}^{i-1} |\lambda(f^j(z))|$, and hence, by (b.3)

$$|\lambda'(f^i(z)) - \lambda'(z)| \leqslant C\rho|b-a| \sum_{j=0}^{\infty} \Delta(f^j(z))$$

where C depends only on the norm of $\log(f')$ in the C^1-topology. Taking limits with $i \to \infty$, we conclude that

$$|\lambda'(a) - \lambda'(z)| \leqslant C\rho(b-a)^2$$

which proves (b.2).

Proof of B.2. — For each $\gamma \in \Gamma$, denote by $\lambda_\gamma \partial/\partial z$ the vector field on I_γ given by B.3. Define $\lambda : [a,b] \to \mathbb{R}$ by $\lambda(z) = 0$ if $z \in P$ and $\lambda(z) = \lambda_\gamma(z)$ if $z \in I_\gamma$. It is clear that $\xi = \lambda(z)\partial/\partial z$ satisfies $\xi^1 = f$, hence it remains to be proved that λ is a C^1-function. If $z \in [a,b]\backslash P$

it is clear that λ is C^1 at z. If z is an isolated point of P, then $\lambda'_-(z) = \log f'_-(z) = \log f'_+(z) = \lambda'_+(z)$, where $-$ and $+$ as indices denote left and right lateral derivatives, respectively. Hence, λ is C^1 at such a point.

For any $z \in P$ let $P_-(z) = [a,z] \cap P$ and $P_+(z) = [z,b] \cap P$. Take now z to be an accumulation point of P. We assume, w.l.o.g., that z is an accumulation point of P_+. Then clearly $f'(z) = 1$. We shall now calculate $\lambda'(z)$. If $y \in (z,b)$ is not in P, then $y \in I_\gamma = (a_\gamma, b_\gamma)$, for some $\gamma \in \Gamma$. By the Mean Value theorem and (b.2),

$$|\lambda(y)| \leqslant \sup_{t \in I_\gamma} |\lambda'(t)|(y - a_\gamma) \leqslant [|\lambda'_+(a_\gamma)| + C\rho(b_\gamma - a_\gamma)^2](y - z),$$

hence

$$\left| \frac{\lambda(y) - \lambda(z)}{y - z} \right| \leqslant |\log f'(a_\gamma)| + C\rho(b_\gamma - a_\gamma)^2.$$

When y tends to z, a_γ and b_γ tend also to z and ρ tends to 1. Therefore $\lambda'_+(z) = 0$. If z is also an accumulation point of P_-, then the same argument shows that $\lambda'_-(z) = 0$. If not, then there exists $\gamma_0 \in \Gamma$ such that $I_{\gamma_0} = (a_{\gamma_0,z})$. Hence $\lambda'_-(z) = \log f'_-(z) = 0$. This shows that $\lambda'(z) = 0$. Take now a sequence $\{z_n\}$ converging z. Assume for the moment that $z_n > z$, $\forall_n$. If $z_n \in P$, then $\lambda'(z_n) = 0$, as we have shown above. If $z_n \notin P$, then $z_n \in I_{\gamma_n}$, for some $\gamma_n \in \Gamma$. By (b.2),

$$|\lambda'(z_n)| \leqslant |\log f'(a_{\gamma_n})| + C\rho_n(b_{\gamma_n} - a_{\gamma_n})^2.$$

Now, when n tends to ∞, a_{γ_n} and b_{γ_n} tend to z and ρ_n tends to 1. Therefore $\lambda'(z_n)$ converges to 0. The case $z_n < z$, $\forall n$, is treated in an analogous way. We have then proved that λ is C^1 on $[a,b]$, which completes the proof of B.2.

C. Analytic foliations near a compact leaf.

In this appendix we prove a conjugation theorem for analytic foliations in the neighborhood of a compact leaf.

THEOREM C.1. — *Let $\mathcal{F} \in G_0^\omega(M_F)$ be a foliation with a finite number of compact leaves, principal direction $\boldsymbol{\alpha}$, and whose open leaves are planes. If K is a compact leaf, then there exists a neighborhood $U_0(K)$ of K and an analytic diffeomorphism $H : U(K) \to T^2 \times (-\varepsilon, \varepsilon)$, conjugating $\mathcal{F}|_{U(K)}$ with the foliation given by*

$$(c.1) \qquad \lambda\alpha_2 dx - \lambda\alpha_1 dy + dz = 0$$

on $T^2 \times (-\varepsilon, \varepsilon)$, *where* $\lambda : (-\varepsilon, \varepsilon) \to \mathbb{R}$ *is analytic. Moreover, if* $\Pi : U(K) \to K$ *is some* C^ω *projection,* H *sends the fibers of this projection to the sets* $s(x,y) \cap U(K)$, $(x,y) \in T^2$.

To prove C.1 we need the following lemma :

LEMMA C.2. — *Let* f *be a* C^ω *diffeomorphism of* $\mathbb{R}$ *such that* $f(0) = 0$. *If the centralizer* $C^\omega(f)$ *is not isomorphic to* $\mathbb{Z}$, *then* f *is the time 1 of an analytic flow.*

Proof. — Take $f \in \mathrm{Diff}_0^\omega(\mathbb{R})$. By A.2, f is the time 1 of a C^∞ flow ϕ^t. Our task is to show that, for each t, ϕ^t is in fact analytic. We know that $G = \{t \in \mathbb{R}; \phi^t \text{ is analytic}\}$ is an additive subgroup of $\mathbb{R}$ that, by hypothesis, is not isomorphic to $\mathbb{Z}$. Hence g is dense in $\mathbb{R}$. For any $s \in \mathbb{R}$, take then a sequence $\{t_n\} \subset G$ converging to s. Then $\{\phi_{t_n}\}$ is converging to ϕ_s in the C^0-topology. Since each ϕ_{t_n} is analytic, so is ϕ_s. This proves C.2.

Proof of C.1. — Let $\chi : \pi_1(K) \times s(0,0) \to s(0,0)$ be the holonomy map. By hypothesis, the open leaves of $\mathcal{F}$ are planes and hence the centralizer of $\xi(e_1)$ is not isomorphic to $\mathbb{Z}$. Therefore, by C.2, $\chi(e_1) = \xi^{-\alpha_2}$, for some analytic vector field $\xi = \lambda(z)\partial/\partial z$. By A.3, $\chi(e_2) = \xi^{\alpha_1}$. With this information at hand, we can complete the proof as we did in appendix A.

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