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BRELOT'S AXIOMATIC THEORY OF THE DIRICHLET PROBLEM AND HUNT'S THEORY

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Since the axiomatic theory of the Dirichlet problem has been developed, by Doob, Brelot, and Bauer, it has been a very natural question to wonder whether it would come into the general probabilistic theory of potentials of Hunt. We show here that the answer is positive, under very mild assumptions. This paper, however, does not involve probabilistic considerations, not even at some places where they might simplify the proofs.

Our hypotheses are those of Brelot, as they appear for instance in [3]. Namely, we assume Brelot's axioms 1-2-3, the existence of a countable base of open sets for the basic space E, the existence of a positive potential, and the harmonicity of the constant functions. On the other hand, we obviously are not using the full strength of Brelot's axiomatics; our reasoning would probably apply without essential modifications to the latest version of Bauer's axiomatic approach (see [2]).

While we shall use freely the results from Brelot's papers, which are sufficiently well known, we shall refrain from doing the same thing with M^me^ Hervé's thesis [5]. Indeed, we shall give a complete proof of the construction of the kernel (2),

(1) Cet article a été rédigé pendant un séjour de l'auteur à l'Université de Washington, sous les auspices de la National Science Foundation, grant P-11 008; il a été publié sous la forme d'un « technical report » en Juin 1963, en Anglais, et il n'a pas semblé utile de le traduire en Français pour cette publication définitive (qui comporte quelques améliorations de détail).

(2) Let $\mathcal{B}^+$ be the cone of all positive (possibly infinite) Borel measurable functions on E. A *kernel* V on E is a function $f \rightsquigarrow Vf$ from $\mathcal{B}^+$ to $\mathcal{B}^+$ which is additive,

in the most economical way possible; this paper thus will almost be self contained.

In order to simplify things as much as possible, we shall only consider *finite* superharmonic functions. This will not hamper the generality, as almost all the superharmonic functions we shall consider will in fact be bounded and continuous.

Notations. $\mathcal{S}$, $\mathcal{S}^+$ are the cones of (finite) superharmonic and positive superharmonic functions in E ([3]). The cones $\mathcal{S}(W)$, $\mathcal{S}^+(W)$ are defined in the same way for any open set $W \subset E$.

$\mathcal{C}$, $\mathcal{C}_{\mathcal{K}}$, $\mathcal{C}_0$, $\mathcal{C}_b$ respectively are the spaces of continuous functions, continuous functions with compact support, continuous functions tending to 0 at infinity, continuous bounded functions in E. The last two spaces are Banach spaces for the uniform norm. Similar notations $\mathcal{C}(A)$ ... are used for any locally compact subspace $A \subset E$. If A is compact, all four spaces are identical.

Let f be a function on E, and A be any subset of E. Then $f|_A$ is the restriction of f to A. The boundary of A in E will be denoted by A^*.

Let W be any regular open set, and f any (sufficiently regular) function defined on E. We denote by $H_W f$ ([4]) the function equal to f on $E \setminus W$, to the (generalized) Dirichlet solution associated with f in W. If f is superharmonic, the same is true for $H_W f$. More generally, we recall that if W is an arbitrary open set, if f is a superharmonic function in E, g a superharmonic function in W such that:

$$\liminf_{y \to x,\ y \in W} g(y) \geqslant f(x)$$

for every $x \in W^*$, then the function equal to f in $E \setminus W$, to $\min(f, g)$ in W, is superharmonic in E.

positively homogeneous, and such that $V(\lim_n f_n) = \lim_n Vf_n$ for any increasing sequence (f_n) of elements of $\mathcal{B}^+$. A kernel V is submarkovian (markovian) if V1 is smaller than (equal to) 1. The function Vf may be defined also when f is universally measurable and positive, or universally measurable and such that $V|f|$ is finite. A more detailed account may be found in the seminar volume (8).

([3]) M^me^ HERVÉ uses the notation $\mathcal{S}^+$ as we do (except for the restriction of finiteness), but uses $\mathcal{S}$ for *differences* of superharmonic functions.

([4]) The standard notation of Brelot and M^me^ HERVÉ would be H_W^f; ours emphasizes the fact that H_W is a kernel.

The *specific order* for (finite) functions ($\prec$) is the order defined by the cone $\mathcal{S}^+$. The ordinary order for functions ($\leqslant$) will be called the *natural* order in the sequel.

1. M[me] Hervé's partition theorem.

Let W be an open set in E, g belong to $\mathcal{S}^+$, and f belong to $\mathcal{S}(W)$ (more generally, we might take for f any finite function defined on a subset of E which contains W) (5). We follow M[me] Hervé, and say that g is a W-*majorant* of f if the function $g - f$ is superharmonic in W (not necessarily positive). The set of all W-majorants of f is not affected if one adds to f any function which is harmonic in W (and quite arbitrary out of W).

Let g and h be two W-majorants of f; then $\min(g, h)$ still W-majorizes f. Let then F be any set of functions, and let G be the set of all functions $g \in \mathcal{S}^+$ which are W-majorants of every $f \in F$. If G is not empty, it contains a smallest element g_0 (for the natural order; g_0 is the lower semi-continuous regularized function of $\inf\limits_{g \in G} g$). Then we have the basic:

Theorem 1 (M[me] Hervé) (6). — *Every* $g \in G$ *majorizes* g_0 *in the specific sense.*

To prove this result, we must take a regular open set U and check the relation $(g - g_0) \geqslant H_U(g - g_0)$, or

$$(g - H_U g)|_U \geqslant (g_0 - H_U g_0)|_U.$$

These functions obviously are $(W \cap U)$-majorants of the functions $f|_U$, $f \in F$. So the theorem will be a consequence of the following:

Lemma. — *The function* $(g_0 - H_U g_0)|_U$ *is (for the natural order in* U) *the smallest* $(W \cap U)$-*majorant of the functions* $f|_U$, $f \in F$.

(5) A function f which possesses a W-majorant is equal, in W, to a difference of superharmonic functions. We shall often use the fact that a finite function h, which is equal in W to a difference of elements of $\mathcal{S}(W)$, is superharmonic in W if and only if the inequality $H_U h \leqslant h$ holds for any regular open set $U \subset W$. This is applied below to the functions $g_0 - f$, and $g - g_0$.

(6) See [5], p. 456, theorem 12.1.

Proof. — Let $k \in \mathcal{S}^+(U)$ be a $(W \cap U)$-majorant of all function $f|_U$, $f \in F$. Let h be the function:

$$\begin{array}{ll} \min\ (k + H_U g_0,\ g_0) & \text{in } U \\ g_0 & \text{in } E \setminus U \end{array}$$

h belongs to $\mathcal{S}^+$, by the positivity of k and the inequality $\liminf_{y \to x,\, y \in U} H_U g_0(y) \geq g_0(x)$ for $x \in U^*$. The lemma will be proved if we show that h is greater than g_0, and this will follow if we prove that h is a W-majorant of every $f \in F$. Now we have:

$$(h - f)|_W = \begin{array}{ll} \min(k + H_U g_0 - f,\ g_0 - f) & \text{in } W \cap U \\ g_0 - f & \text{in } W \setminus U \end{array}$$

$g_0 - f$ being superharmonic in W, $k + H_U g_0 - f$ superharmonic in $W \cap U$ and majorizing $g_0 - f$ at the boundary of $W \cap U$ in W, this function is superharmonic in W and the theorem is proved.

DEFINITION (Mme Hervé). — *Let f be a finite potential,* W *be any open set in* E, *and* K *any compact subset of* E. *We shall denote by f_W the smallest* W-*majorant of f, by f_K the function* $f - f_{E \setminus K}$.

We shall follow Mme Hervé, and prove that the set function $K \rightsquigarrow f_K$ is induced by a kernel. The finite potential f will be kept fixed in the sequel.

2. Properties of the set functions f_W and f_K.

1) f being a W-majorant of itself, the theorem implies $f_W \prec f$. Both functions f_W and f_K thus are *potentials*.

2) f_W being a W-majorant of f, and f a W-majorant of f_W, *the function $f - f_W$ is harmonic in* W; *the function f_K is harmonic out of* K.

3) f_K is (*specifically*) *the greatest specific minorant of f which is harmonic out of* K. Let indeed h be much a minorant; $f - h$ is a $(E \setminus K)$-majorant of f, thus $f - h \succ f - f_K$ and $h \prec f_K$.

4) *The function f_W is harmonic out of $\overline{W}$; the function $f - f_K$ is harmonic in* K^0. Let indeed U be a regular open set such

that $\overline{U} \cap \overline{W} = \emptyset$; the set of W-majorants of f is closed under the operation H_U; it is thus saturated in $E \setminus \overline{W}$, and its infimum is harmonic in that set.

5) $W_1 \subset W_2 \Longrightarrow f_{W_1} \prec f_{W_2}$; $\quad K_1 \subset K_2 \Longrightarrow f_{K_1} \prec f_{K_2}$;
$W \subset K \Longrightarrow f_W \prec f_K$; $\quad K \subset W \Longrightarrow f_K \prec f_W$.

The first two properties are obvious. The third one follows from the harmonicity of $f - f_K$ in K^0 (4) which implies that f_K is a W-majorant of f. To prove the fourth property, we remark that $f_W + f_{E \setminus K} - f$ is superharmonic in W and $E \setminus K$, and thus in E; on the other hand, it majorizes $-f$, and f is a potential: the minimum principle ([3], p. 93, proposition 9) implies that this function is positive. We thus have $f \prec f_W + f_{E \setminus K}$, or $f_K \prec f_W$.

6) $K \subset W \Longrightarrow f_K = (f_W)_K$.
Because $f_K \prec f_W$ (5), thus $(f_K)_K \prec (f_W)_K \prec f_K$, and $(f_K)_K = f_K$ (3).

7) *Let* W *be the union of an increasing sequence of open sets* W_n; *then* $f_W = \operatorname{Max}_n f_{W_n}$. *Let* K *be the intersection of a decreasing sequence of compact sets* K_n; *then* $f_K = \operatorname{Min}_n f_{K_n}$
The second sentence obviously follows from the first one, which we prove now. Let g be $\operatorname{Max}_n f_{W_n}$; we only have to show that $g \prec f_{W_n}$, or that g is a W-majorant of f, or that $f - g$ is harmonic in W. Let U be an open set with a compact closure $\overline{U} \subset W$; it follows from 6 that $f_{\overline{U}} = g_{\overline{U}}$. Now $f - f_{\overline{U}}$ and $g - g_{\overline{U}}$ are harmonic in U (4), so $f - g$ is harmonic in U. This open set being arbitrary, $f - g$ is harmonic in W.

8) $f_W = \operatorname{Max}_{K \subset W} f_K$; $\quad f_K = \operatorname{Min}_{W \supset K} f_W$.
Easy corollary of 5, and 7.

9) $(f_{K_1})_{K_2} = f_{K_1 \cap K_2}$; $\quad (f_{W_1})_{W_2} = f_{W_1 \cap W_2}$.
The first relation is obvious from (3). The second one follows by taking an increasing sequence of compact sets $K_{in} (n \in N)$ such that $W_i = \bigcup_n K_{in}^0$ $(i = 1,2)$ and passing to the limit using 5,7.

10) $f_{W_1 \cup W_2} + f_{W_1 \cap W_2} = f_{W_1} + f_{W_2}$; $\quad f_{K_1 \cup K_2} + f_{K_1 \cap K_2} = f_{K_1} + f_{K_2}$.
We need only prove the first equality, as the second one is an obvious consequence and it and the relation $f_K + f_{E \setminus K} = f$.

Let h be the function $f_{W_1 \cup W_2} + f_{W_1 \cap W_2} - f_{W_1} - f_{W_2}$. From 9, we get the equality $f_{W_1 \cup W_2} - f_{W_2} = f_{W_1 \cup W_2} - (f_{W_1 \cup W_2})_{W_2}$; the first member is thus harmonic in W_2. In the same way

$$f_{W_1} - f_{W_1 \cap W_2} = f_{W_1} - (f_{W_1})_{W_2}$$

is harmonic in W_2; h itself is thus harmonic in W_2, and symmetrically in W_1.

$f_{W_1 \cup W_2} + f_{W_1 \cap W_2} - f_{W_1}$ is a W_2-majorant of f_{W_2}, and thus a specific majorant of f_{W_2}. So we have:

$$f_{W_1 \cup W_2} + f_{W_1 \cap W_2} \succ f_{W_1} + f_{W_2}.$$

To prove that we have equality, we remark that, h being harmonic in $W_1 \cup W_2$, $f_{W_1} + f_{W_2} - f_{W_1 \cap W_2}$ is a $(W_1 \cup W_2)$-majorant (and thus a specific majorant) of $f_{W_1 \cup W_2}$, and we get the reversed inequality. We are now able to prove the following:

THEOREM 2 (Mme Hervé) (7). *Let f be a finite potential. There is one and only one positive kernel* V *which possesses the following properties:*

a) *for every positive bounded Borel measurable function g, the function Vg is superharmonic, positive, harmonic in the complement of the closed support of g.*

b) $V(1) = f$.

If the function f is continuous, and the function g is Borel measurable and bounded, then Vg is continuous.

Proof. — We shall first establish the existence part. Let x be any element of E; it follows from properties 5,8,10 that the set function $K \rightsquigarrow f_K(x)$ is increasing, right continuous and *additive*. It may thus be extended to all subsets of E as a capacity (8), and the additivity implies that the restriction of this capacity to the Borel sets is a measure V_x. Define, for any positive Borel function f:

$$Vf(x) = \int_E f(y)\, dV_x(y)$$

(7) See [5], pp. 490-493, and specially theorem 20.2.

(8) Mme Hervé ([5], p. 464) constructs her kernels with the help of a theorem from Bourbaki's Integration (chapter IV, § 4, no 10). The general capacitability theorem of Choquet (see for instance part II of [3]) leads to the same result, and it has the advantage of being much better known than Bourbaki's theorem (and of having independent interest).

The function Vf is Borel measurable when f is the indicator of a compact set K, and thus also for any Borel function f. This implies that the mapping $f \rightsquigarrow Vf$ is a kernel. Let A be any Borel set; the equality $V(\chi_A) = \operatorname{Max}_{K \subset A} V(\chi_K) = \operatorname{Max}_{K \subset A} f_K$ shows that the function $V(\chi_A)$ is superharmonic, positive, and a specific minorant of f. Let g be measurable, positive, bounded by 1; then g is the limit of an increasing sequence of finite linear combinations of characteristic functions of sets; Vg thus is positive, superharmonic, and a specific minorant of f. So Vg is continuous if f is continuous.

If A is a Borel set, $V(\chi_A)$ is harmonic in the complement of $\overline{A}$. The above approximation procedure shows that Vg is harmonic in the complement of the support of g.

Let V' be another kernel with the same properties — more precisely, we shall only assume that property a) holds when g belongs to $\mathcal{C}^+_{\mathcal{K}}$. This first implies, by an increasing passage to the limit, that $V'g$ is superharmonic whenever g is positive and lower semicontinuous. Let K be a compact set, (K_n) be a decreasing sequence of compact neighbourhoods of K such that $\bigcap_n K_n = K$, and let W be the complement of K. Let (f_n) be a decreasing sequence of positive continuous functions, such that $f_n = 1$ on K, $f_n = 0$ out of K_n; finally, set $f'_K = V'(\chi_K)$. This function is harmonic out of K, and possesses the property that $H_U f'_K \leq f'_K$ for any regular open set U, because of the relation $f'_K = \inf_n V'f_n$. On the other hand,

$$f'_K = f - V'(\chi_W)$$

is a difference of superharmonic functions, and thus is superharmonic. The relation $f'_K \prec f$ then implies $f'_K \prec f_K$ (property 3), or $V'(\chi_K) \prec f_K$. This being true for compact sets is true for open sets too (property 8). The inequalities

$$V'(\chi_K) \prec f_K, \qquad V'(\chi_W) \prec f_W$$

cannot be strict, as one gets an equality by adding them. So we have $V'(\chi_K) = f_K$ for any compact K, and the kernels V and V' are identical.

3. Construction of the resolvent.

We suppose now that the potential f is bounded and continuous. The kernel V associated with f defines a bounded positive operator in the Banach space $\mathcal{C}_b$. We shall use Hunt's method (or rather the modification of it Lion gave in [7]) to get a submarkovian resolvent (9) $(V^\lambda)_{\lambda \geqslant 0}$ such that $V^0 = V$. We only sketch the proofs.

Let us remark first that V satisfies the *weak principle of the positive maximum.* Namely, if g is any function in $\mathcal{C}_b$, and S is the closed set $\{x: g(x) \geqslant 0\}$, we have:

$$\operatorname*{Max}_{x \in E} Vg(x) = \operatorname*{Max}_{x \in S} Vg(x)$$

provided the first member is positive.

In order to prove it, let us call m the second member, and assume that m is $\geqslant 0$. The function Vg^+ being harmonic in $E \setminus S$, the function $m - Vg$ is superharmonic in $E \setminus S$. The function $V|g|$ is a potential, and we have $m - Vg \geqslant - V|g|$. Now $m - Vg$ is positive at every boundary point of $E \setminus S$. The result then follows from the minimum principle (Brelot [3] p. 93).

Let us assume now that λ is a strictly positive number, and that we have been able to find an operator V^λ in $\mathcal{C}_b$ such that:

$$V - V^\lambda = \lambda V V^\lambda = \lambda V^\lambda V$$

we may check the following points:

a) V^λ *is a positive operator.*

Let us show indeed that $g \leqslant 0$ implies $V^\lambda g \leqslant 0$. If this were not true, $V^\lambda g = V(g - \lambda V^\lambda g)$ would have a strictly positive maximum, and thus take a strictly positive value on the set $\{g - \lambda V^\lambda g \geqslant 0\}$, which is impossible, g being negative.

b) V^λ *is a kernel* (it is majorized by V, which is a kernel).

c) $\|\lambda V^\lambda\|$ *is smaller than* 1.

It is sufficient to check the inequality $\lambda V^\lambda 1 \leqslant 1$. Remark that 1 majorizes $\lambda V^\lambda 1$ on the set $\{1 - \lambda V^\lambda 1 \geqslant 0\}$; now

(9) We also refer the reader to [7] for the definition of a resolvent, and a statement of the Hille-Yosida theorem.

$\lambda V^\lambda 1 = V[\lambda(1 - \lambda V^\lambda 1)]$; so the positive maximum principle implies that 1 majorizes $\lambda V^\lambda 1$ everywhere.

To conclude the proof of the existence of a resolvent, we make a few easy remarks. First, $V^\lambda = V(I + \lambda V)^{-1}$ exists for small λ; the set of all $\lambda > 0$ such that V^λ exists is open (as the complement of the spectrum of $-V$). Whenever the operators V^λ and V^μ exist, they satisfy the resolvent equation $V^\lambda - V^\mu = -(\lambda - \mu)V^\lambda V^\mu$. A simple Cauchy sequence argument then shows that the set of all $\lambda > 0$ such that V^λ exists is also closed in $R_+ \setminus \{0\}$. So V^λ exists for all $\lambda > 0$.

Our aim in this paper is the construction of the semi-group (P_t) which admits (V^λ) as its resolvent. Unfortunately, the denseness conditions of the Hille-Yosida theorem are not satisfied here, and we shall need more complicated considerations.

We recall [10] that a positive measurable function u is said to be *supermedian* with respect to the resolvent (V^λ) if it satisfies the inequality $\lambda V^\lambda u \leqslant u$ for every $\lambda > 0$. We have the following result, which will be improved later on:

LEMMA. — *Every positive superharmonic function u is supermedian with respect to the resolvent* (V^λ).

Proof. — It is sufficient to check the inequality $\lambda V^\lambda u \leqslant u$ in the case of a bounded positive superharmonic function u. The function $V^\lambda u = V[u - \lambda V^\lambda u]$ may be written as $\alpha . V(v - w)$, where α is a constant, and the functions v and w take their values in $[0,1]$. The functions Vv, Vw are specific minorants of f, and therefore continuous. The function $\lambda V^\lambda u$ thus is continuous, and its absolute value is majorized by a potential; it is subharmonic in the open set $\{u - \lambda V^\lambda u < 0\} = W$, and majorized on $E \setminus W$ by the superharmonic function u. The minimum principle then implies that it is majorized by u on E.

4. Construction of new kernels.

Let W be a *regular* open set. We define a new kernel:

$$T_W g = Vg - H_W Vg \quad (g \text{ measurable and bounded}).$$

[19] See the seminar volume [8].

If g is positive and bounded, Vg is superharmonic, continuous and bounded, so $T_W g$ is positive, continuous, bounded, equal to 0 on $E \setminus W$. If we identify the space $\mathcal{C}_0(W)$ to the set of all continuous functions on E which vanish on $E \setminus W$, we may consider the operator T_W as an operator on $\mathcal{C}_0(W)$, and thus as a kernel on W. We shall be rather sloppy in distinguishing these two interpretations of T_W.

The operator T_W satisfies the weak principle of the positive maximum on $\mathcal{C}_0(W)$; one may therefore find a submarkovian resolvent $(T_W^\lambda)_{\lambda \geqslant 0}$ on $\mathcal{C}_0(W)$, such that $T_W^0 = T_W$. The measures $T_W^\lambda(x, dy)$ will equivalently be considered as measures on W, or as measures on E concentrated on W (with $T_W^\lambda(x, dy) = 0$ for $x \in E \setminus W$). The positive superharmonic functions in W are supermedian functions with respect to the resolvent (T_W^λ).

We now define a new resolvent by setting:

$S_W^\lambda g = \frac{1}{\lambda} H_W g + T_W^\lambda(g - H_W g)$ (g Borel measurable and bounded, $\lambda > 0$). We get this time a *markovian* resolvent on E; the measures $\lambda S_W^\lambda(x, dy)$ are concentrated on $\overline{W}$ for $x \in \overline{W}$, and are unit masses at x for $x \in E \setminus W$. If g is harmonic, we have $\lambda S_W^\lambda g = g$. The positive superharmonic functions on E are supermedian with respect to this resolvent, which carries $\mathcal{C}_0(E)$ into $\mathcal{C}_0(E)$, We shall show that, if f has been properly chosen, one may apply the Hille-Yosida theorem to (S_W^λ) on $\mathcal{C}_0(E)$.

Any probabilist will recognize in (T_W^λ) the resolvent of a process killed at the complement of W, and in (S_W^λ) the resolvent of a process stopped at the complement of W.

5. Choice of f; construction of the local semi-groups.

Let (p_n) be a sequence of bounded and continuous potentials, which separates the points of E [11]. For each n, let a_n be a constant such that $a_n - p_n$ is subharmonic and positive, and let q_n be the positive subharmonic function:

$$q_n = (a_n - p_n) + (a_n - p_n)^2.$$

[11] See [3] p. 97, or [5] p. 438.

We choose now a sequence of positive constants b_n such that the series $\Sigma b_n q_n$ converges to a (continuous and) bounded function q. Let c be a constant such that $c - q$ is positive. *We take for f the potential part of the positive superharmonic function $c - q$.* We begin the construction of the semi-groups with the following lemma:

LEMMA. — *The weak limit of the measures $\lambda S_W^\lambda(x, dy)$ is ε_x for every $x \in E$, as λ tends to $+\infty$.*

Proof. — There is no problem at all for $x \in E \setminus W$, because the measure $\lambda S_W^\lambda(x, dy)$ remains equal to ε_x. For $x \in W$, let μ be any weak limit point of the measures $\lambda S_W^\lambda(x, dy)$ as $\lambda \to \infty$. The lemma will be proved if we show that $\mu = \varepsilon_x$. The passages to the limit below are justified because the measures involved are concentrated on the compact set $\overline{W}$.

a) We have $\mu(h) \leqslant h(x)$ for any continuous function h, which is supermedian with respect to (S_W^λ). As a particular case, we have $\mu(h) \leqslant h(x)$ for any superharmonic positive continuous function h.

b) $\mu(h) = h(x)$ for any harmonic function h.

c) We have $f = V1$, and therefore $T_W 1 = f - H_W f$. The resolvent equation implies that $\lim_{\lambda \to \infty} \lambda T_W^\lambda(f - H_W f) = f - H_W f$, and thus also $\lim_{\lambda \to \infty} \lambda S_W^\lambda(f - H_W f) = f - H_W f$. We thus have $\mu(f - H_W f) = f(x) - H_W f(x)$, and finally $\mu(f) = f(x)$.

Let us use now the definition of f: (*c*) and the relation $\mu(1) = 1$ imply $\mu(q) = q(x)$; using (*a*), we find that $\mu(q_n) = q_n(x)$ for every n, and thus:

$$\mu(a_n - p_n) = a_n - p_n(x); \quad \mu[(a_n - p_n)^2] = (a_n - p_n(x))^2$$

and:

$$[\mu(p_n)]^2 = \mu(p_n^2) = p_n^2(x)$$

It then follows from Schwarz' inequality that μ is concentrated on the set $\{y : p_n(y) = p_n(x)\}$; the sequence (p_n) separating points, we must have $\mu = \varepsilon_x$.

This result would allow us to construct the stopped semigroup and processes on $\overline{W}$. We shall not try to do it here, however, and rather concentrate our interest on the resolvent (T_W^λ).

Let us consider the operators (T^λ_W) as operators on $\mathcal{C}_0(W)$, and remark that the lemma and the explicit relation between the measures $\lambda S^\lambda_W(x, dy)$ and T^λ_W imply (using the relation $H_W g = 0$ for $g \in \mathcal{C}_0(W)$):

$$\lim_{\lambda \to \infty} \lambda T^\lambda_W g = g \text{ in the weak topology of } \mathcal{C}_0(W)\ ({}^{12}) \text{ for } g \in \mathcal{C}_0(W).$$

The image of the resolvent T^λ_W is thus (weakly) dense in $\mathcal{C}_0(W)$. We may now apply the theorem of Hille-Yosida and get a submarkovian semi-group (P^W_t), strongly continuous on $\mathcal{C}_0(W)$, which admits (T^λ_W) for its resolvent. As above, we shall consider the kernels $P^W_t(x, dy)$ as kernels on W, or as kernels on E such that $P^W_t(x, dy) = 0$ for $x \in E \setminus W$.

6. The definitive semi-group.

The following property is obvious from a stochastic point of view.

Lemma. — *Let* W *and* W′ *be two regular open sets such that* $\overline{W} \subset W'$. *Then we have* $P^W_t g \leqslant P^{W'}_t g$ *for every positive Borel function* g.

Proof. — It is sufficient (as we have $P^W_t g = P^W_t(g.\chi_W)$) to check this inequality when g is positive, continuous, with compact support in W. The functions $t \rightsquigarrow P^W_t g$, $t \rightsquigarrow P^{W'}_t g$ being continuous, it is sufficient to verify that the function:

$$\lambda \rightsquigarrow \int_0^\infty e^{-\lambda t}(P^{W'}_t g - P^W_t g)\, dt = T^\lambda_{W'} g - T^\lambda_W g$$

is completely monotone. Using the formula:

$$\frac{d^n}{d\lambda^n} T^\lambda_U g = n!(-1)^n [T^\lambda_U]^{n+1} g \quad (U = W \text{ or } W')$$

which follows from the resolvent equation, we find that we must only check the inequality:

$$T^\lambda_W g \leqslant T^\lambda_{W'} g \quad (g \in \mathcal{C}^+_K(W))$$

The resolvent equation implies the identity:

$$T^\lambda_{W'} - T^\lambda_W = (I - \lambda T^\lambda_{W'})(T_{W'} - T_W)(I - \lambda T^\lambda_W)$$

(12) We recall that the conjugate space of $\mathcal{C}_0(W)$ is the space of all bounded Radon measures on W.

It is thus enough to show that the function $(T_{W'}-T_W)(I-\lambda T_W^\lambda)g$ is supermedian with respect to (T_W^λ), or is superharmonic and positive in W′. Let us set:

$$k = (I - \lambda T_W^\lambda)g; \qquad k' = k + \varepsilon \quad (\varepsilon > 0).$$

We want to whow that $(T_{W'} - T_W)k$ is positive and superharmonic in W′. Now we have:

$$(T_{W'} - T_W)k = T_{W'}k - [T_{W'}k - H_W T_{W'}k] = \begin{cases} T_{W'}k & \text{in } W' \setminus W \\ H_W T_{W'}k & \text{in } W'. \end{cases}$$

Let s be this function, and let s' be the function obtained by replacing k by k' in its definition. The function k being equal to 0 in $W' \setminus W$, the function $T_{W'}k'$ is superharmonic in a neighborhood of W^*, while the second function $H_W T_{W'}k'$ is harmonic inside W, and has the same boundary value at W^* as $T_{W'}k'$. To prove the superharmonicity of s', it is thus sufficient to establish the inequality:

$$T_{W'}k' \geqslant H_W T_{W'}k' \text{ in } W.$$

Now the function $T_{W'}k' - H_W T_{W'}k'$ is equal to

$$T_W k' = T_W \varepsilon + T_W k = T_W \varepsilon + T_W^\lambda g \geqslant 0.$$

The function s' is thus superharmonic. As it vanishes at the boundary of W′, it is positive in W′, and letting $\varepsilon \to 0$ one gets the same result for s.

We are now ready to prove our main result, which implies that we are in the general set-up of Hunt's theory of potentials:

THEOREM 3. — *There exists a submarkovian semi-group of kernels,* (P_t), *on the space* E, *which possesses the following properties:*

a) Its potential kernel is equal to V.

b) The function $P_t g$ *is continuous for every* $g \in \mathcal{C}_{\mathcal{K}}(E)$.

c) $\lim_{t \to 0} P_t g = g$ *uniformly on compact sets for every* $g \in \mathcal{C}_{\mathcal{K}}(E)$.

d) The positive superharmonic functions on E *are exactly the excessive functions* (13) *with respect to* (P_t).

(13) Let f be a positive, universally measurable function on E; f is excessive if the inequality $P_t f \leqslant f$ holds for every t, and $\lim_{t\to 0} P_t f$ is equal to f. The main results concerning these functions may be found in [6] and in the seminar volume [8].

We do not state in fact the most important feature of (P_t) : the possibility of constructing good Markov processes with continuous sample functions. This would rather be attained by a stochastic constructions starting with the construction of stopped and killed processes, than using the hypotheses of theorem 3, but we do not want to go into probabilistic considerations.

Proof of theorem 3.

Let (W_n) be a sequence of regular open sets such that $\overline{W}_n \subset W_{n+1}, \bigcup_n W_n = E$ — such a sequence exists, according to a theorem of M^me^ Hervé [14]. We define, for every Borel measurable positive function g:

$$P_t g = \lim_n P_t^{W_n} g$$

Leg (g_k) be a sequence of positive measurable functions which increases to a function g; the convergence in the preceding formula being monotone, we have:

$$P_t g = \sup_n P_t^{W_n} g = \sup_n \sup_k P_t^{W_n} g_k = \sup_k \sup_n P_t^{W_n} g_k = \sup_k P_t g_k$$

P_t therefore is a kernel.

It is easily checked that:

$$P_s P_t = P_{s+t}, \quad P_0 = I$$
$$\int_0^\infty e^{-\lambda t} P_t \, dt = V^\lambda (\lambda \geqslant 0)$$

Let g be a positive continuous function: the functions $P_t^{W_n} g$ are continuous, so $P_t g$ is lower semi-continuous. In the same way, the functions $t \rightsquigarrow P_t^{W_n} g$ are continuous, so the function $t \rightsquigarrow P_t g(x)$ is lower semi-continuous for every $x \in E$.

Let H be the image of the resolvent (V^λ) operating on $\mathcal{C}_b$. The closure of H is a Banach subspace of $\mathcal{C}_b$, and the theorem of Hille-Yosida implies that a strongly continuous semigroup Q_t may be found on $\overline{H}$, such that

$$V^\lambda h = \int_0^\infty e^{-\lambda t} Q_t h \, dt \quad (h \in \overline{H}).$$

The function $f = V1$ belongs to H; the functions $Q_t f$ are

[14] See [5], p. 441, proposition 7.1.

thus continuous on E, and the function $t \rightsquigarrow Q_t f_{W_n}$ is continuous in t.

On the other hand the functions $t \rightsquigarrow P_t f$ are continuous and decreasing (f being excessive with respect to the semigroup $(P_t^{W_n})$). So the function $t \rightsquigarrow P_t f$ is lower semicontinuous and decreasing, i.e. right continuous. The functions $t \rightsquigarrow Q_t f$, $t \rightsquigarrow P_t f$ have the same Laplace transform, and are right continuous: they are identical. The functions $P_t f$ are therefore continuous on E, and converge uniformly to f as t tends to 0.

Let g be any continuous function such that $0 \leqslant g \leqslant f$; the functions $P_t g$, $P_t(f - g)$ are lower semi-continuous, and their sum is continuous, they are therefore continuous. In the same way, the function $t \rightsquigarrow P_t g$ is continuous. This applies to any continuous function g with compact support (λg^+ and λg^- being majorized by f for λ small enough).

Let g be a continuous function with compact support such that $0 \leqslant g \leqslant f$; according to a result of Mme Hervé (15), one may find two continuous potentials p_1 and p_2 such that $p_1 - p_2$ is arbitrarily close to g in the uniform norm. Replacing p_i by $\min(p_i, f)$ $(i = 1, 2)$ if necessary, we may assume that these potentials are majorized by f. The functions $P_t p_i$ are therefore continuous, and increase to p_i as $t \to 0$. Using Dini's lemma, we find that $P_t p_i$ converges to p_i uniformly on compact sets. The same is thus true for $P_t g$ and g.

Let s be any superharmonic and positive function. We have seen that s is supermedian with respect to every semigroup $(P_t^{W_n})$; this implies that s is supermedian with respect to (P_t). On the other hand, the measures $P_t(x, dy)$ converge weakly to ε_x as t tends to 0. The function s being lower semicontinuous, we have:

$$s(x) \leqslant \liminf_{t \to 0} P_t s(x).$$

The reversed inequality being true, s is excessive with respect to (P_t). The converse follows from the fact that every excessive function is the limit of an increasing sequence of potentials $Vg_n (g_n \geqslant 0)$, and these potentials are superharmonic functions.

(15) See [3] p. 97, or [5] p. 439.

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