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ANOSOV FLOWS AND NON-STEIN SYMPLECTIC MANIFOLDS

by Yoshihiko MITSUMATSU (*)

1. Introduction.

In her paper [14], McDuff constructed symplectic structures with contact type boundaries on compact 4-manifolds $W^4 = M^3 \times [0,1]$ where M^3 is the unit cotangent bundle $S^1(T^*\Sigma_g)$ of closed hyperbolic surfaces Σ_g. In the case of closed manifolds, we already have many examples of closed symplectic manifolds which do not admit Kähler structures, *e.g.*, Kodaira and Thurston's nilpotent 4-manifolds [16] and so on [14]. The obstructions to Kähler structures from symplectic structures are usually found in the Hodge theory.

McDuff's example shows that in the category of compact manifolds with boundaries the symplectic geometry admits a different feature of convex boundaries from that of the complex geometry, *i.e.*, disconnected convex boundaries can happen. (The greater part of [14] is rather devoted to show some similarities between the two geometries, using the arguments on J-holomorphic spheres.)

In Section 2 of the present article, we interpret and simplify McDuff's method of construction, *i.e.*, finding and somehow joining two contact structures with opposite orientations on a single 3-manifold M, in terms of the *linking pairing* on the dual of unimodular 3-dimensional Lie algebras. Then we obtain a slightly wider class of 3-manifolds M such that $M \times [-1,1]$ admits such convex symplectic structures, which includes all compact quotients of $\widetilde{\mathrm{SL}(2;\mathbb{R})}$ and of some solvable Lie group by their discrete cocompact lattices (Theorems 1, 2).

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In the subsequent section, we generalize the construction a little bit further, however apart from the linking pairing, so that we can start from any closed 3-manifold which admits Anosov flows with smooth invariant volumes (Theorem 3).

As to the category of open manifolds, in [4] Eliashberg and Gromov formulated the notion of *complete convexity* for open symplectic manifolds. This notion has its origin in *Stein manifolds*, which satisfy it. They also showed that contact type boundaries can always be completed as convex ends. Using this method they modified McDuff's examples into completely convex open symplectic manifolds which can not be raised to Stein complex manifolds. Of course, this method works for our examples as well. Especially in the case of our main examples in Section 2, this completion is explained in terms of the linking pairing.

In Section 1, the notions of convexities in symplectic and complex geometry are reviewed.

To close this article, we raise in the final section some remarks and problems around bi-contact structures whose existence under the presence of Anosov flows is the main observation from a dynamical point of view.

Laudenbach recently informed the author that the simplification of McDuff's construction in terms of Lie algebra has already been done by E. Ghys around 1990–1991 and such phenomenon that non-connected convex boundary can happen has been well-known since then. It is described in [12]. Independently, H. Geiges has similar results with some generalization to higher dimensions [6]. The author would like to express his further gratitude to Étienne Ghys for his numerous suggestions.

1. Review of symplectic convexity.

We review the concept of symplectic convexity which was introduced in [4]. Bennequin's exposition [1] is also a nice reference for topics around the symplectic, contact, and holomorphic convexity.

1.1 Stein manifolds and convexity.

Among a large variety of equivalent definitions the notion of Stein manifold, one of the most convenient for us might be a theorem of Remmert and Stein; a complex manifold W is a *Stein* manifold iff it can be holomorphically and properly embedded into $\mathbb{C}^N$ for some N.

A theorem of Grauert says this is equivalent to the existence of a proper s.p.s.h. function bounded below. Here a smooth function is said to be *strictly pluri-subharmonic* (*s.p.s.h.*) if it is always strictly subharmonic whenever restricted to any local holomorphic curve.

As such a function φ can be perturbed into a s.p.s.h. Morse function, a Stein manifold W of $\dim_{\mathbb{C}} W = n$ is homotopically equivalent to a CW-complex of dimension at most n. Also, this function φ provides us a Kähler structure $\omega = -\mathrm{d}J^*\mathrm{d}\varphi$ on W, where J is the natural almost complex structure on TW and J^* is its dual on T^*W. Then the gradient vector field $Z = -\nabla\varphi$ of $-\varphi$ is complete and satisfies the relation

$$\mathcal{L}_Z\omega = -\omega$$

and the compactness condition (*i.e.*, the forward orbit of any compact subset remains relatively compact). As we will review soon later, the notion of *complete convexity* of symplectic manifolds was extracted from this situation. If we start from an embedding of a Stein manifold W into $\mathbb{C}^N$, we can take $\varphi(x) = \frac{1}{4}\|x\|^2$ as a s.p.s.h. function on W and the resulting symplectic structure ω is nothing but the restriction to W of the standard symplectic structure on $\mathbb{C}^N$.

The existence of a proper s.p.s.h. function also implies that a Stein manifold is approximated from inside by a sequence of s.p.c. submanifolds. A relatively compact connected open submanifold D of a complex manifold W and its boundary $M = \partial D$ are said to be *strictly pseudo-convex* (*s.p.c.*) if there exists s.p.s.h. function φ on a neighbourhood of $\bar{D}$ with 0 its regular value such that $D = \varphi^{-1}(-\infty, 0)$ and $\partial D = \varphi^{-1}(0)$. A s.p.c. boundary M has a natural contact structure $\xi = TM \cap J(TM)$. In general, on a level hypersurface $M = \varphi^{-1}(0)$ of a regular value 0 of some smooth function φ, the 1-form $\alpha = J^*\,\mathrm{d}\varphi_{|M}$ defines a hyperplane field ξ and the *Levi form* $L = \mathrm{d}\alpha(\,\cdot, J\cdot)$ defines a symmetric pairing on ξ. The fact that φ is s.p.s.h. implies that L is positive definite and also that ξ is a contact plane field. As is well-known in the theory of complex analytic spaces, any s.p.c. domain can be blown down to some Stein space and therefore has connected boundadry. See [15] or [9].

1.2. Convexity in symplectic geometry.

A vector field Z on a symplectic manifold (W, ω) (or on its open subset) satisfying

$$\mathcal{L}_Z\omega = -\omega$$

is said to be *contracting.* Under the relation $\iota_Z\omega = -\alpha$, a contracting field is equivalent to a primitive α (which is called *Liouville form*), of the symplectic form $\omega = d\alpha$. If a hypersurface M of W is transverse to a contracting vector field Z, $\alpha_{|M}$ is a contact 1-form on M. Such a hypersurface M is said to be of *contact type* (see [17]). The boundary of a symplectic manifold is said to be *of contact type* or *convex* if there exists a contracting vector field on some neighbourhood of the boundary which is inward transverse to the boundary.

An end of an open symplectic manifold is said to be *completely convex* if there exists a contracting vector field on some neighbourhood of the end which is complete and whose backward orbit of some compact subset fills up the end. As is shown in [4], any convex boundary can be completed as a completely convex end, because, a contracting field Z defines a product collar neighbourhood of the boundary such that the symplectic structure is expanding exponentially along the flow generated by $-Z$, therefore we can stack and stack again the collar neighbourhood infinitely many times to obtain the completion. If there exists a globally defined contracting complete vector field and whose forward orbit of any compact subset remains relatively compact, the symplectic manifold is said to be *completely convex.* Surely any Stein manifold admits such a contracting field.

In [4], Eliashberg and Gromov modified McDuff's example by this completion procedure to obtain an example of completely convex symplectic manifold which can not be raised to a Stein structure. If we further impose the existence of Morse like function which is compatible with the contracting field Z, the manifold is called *Weinstein.* As it follows from [2] all Weinstein structure admits a compatible Stein structure. Therefore we will work only with the complete convexity.

2. Main examples.

2.1. The linking pairing on 3-dimensional Lie algebras.

Let $\boldsymbol{g}$ be a 3-dimensional unimodular Lie algebra corresponding to a Lie group G. Here we regard elements of $\boldsymbol{g}$ (resp. of the dual $\boldsymbol{g}^*$) as left invariant vector fields (resp. 1-forms) on G. We fix an invariant volume element vol in $\overset{3}{\wedge}\boldsymbol{g}^*$, or may fix a co-compact discrete subgroup Γ of G, instead of choosing vol $\in \overset{3}{\wedge}\boldsymbol{g}^*$. The *linking pairing* $\mathrm{LK}(\alpha,\beta) \in \mathbb{R}$ for α,

$\beta \in \boldsymbol{g}^*$ is defined by

$$\alpha \wedge \mathrm{d}\beta = \mathrm{LK}(\alpha, \beta) \cdot \mathrm{vol} \quad \text{or} \quad \mathrm{LK}(\alpha, \beta) = \int_{G/\Gamma} \alpha \wedge \mathrm{d}\beta.$$

Remarks.

1) For 3-dimensional Lie groups, the existence of co-compact discrete subgroups of G is equivalent to the unimodularity.

2) If $\boldsymbol{g}$ is unimodular, *i.e.*, $\mathrm{vol} \in \overset{3}{\wedge} \boldsymbol{g}^*$ is adjoint-invariant, then, for any $\alpha, \beta, \gamma \in \boldsymbol{g}^*$ and $X \in \boldsymbol{g}$, we have $\mathcal{L}_X(\alpha \wedge \beta \wedge \gamma) = 0$ and thus $\mathrm{d}(\alpha \wedge \beta) = 0$. Therefore the unimodularity implies that this bilinear form LK is symmetric and adjoint-invariant.

3) Among 3-dimensional unimodular Lie algebras, only $\mathrm{psl}(2;\mathbb{R})$ and a solvable one (see Example 3) below), have non-definite linking pairing LK.

4) The pairing LK is non-degenerate only for $\mathrm{psl}(2;\mathbb{R})$ or $\mathrm{su}(2)$, the simple Lie algebras, and of course essentially coincides with the Killing form. In the solvable or nilpotent cases, LK and the Killing form have in general different ranks.

5) The unimodularity also implies that this linking is essentially defined on the space of exact 2-forms $B^2(\boldsymbol{g}) = \mathrm{d}(\boldsymbol{g}^*) \in \overset{2}{\wedge}(\boldsymbol{g}^*)$ by

$$\mathrm{lk}(\mathrm{d}\alpha, \mathrm{d}\beta) = \mathrm{LK}(\alpha, \beta)$$

and is then non-degenerate. The choice of the volume element vol gives a bijective correspondence between $\boldsymbol{g}$ and $\overset{2}{\wedge}(\boldsymbol{g}^*)$, through which the pairing lk is in fact interpreted as an asymptotic linking of corresponding vector fields. This is the reason why we call LK the linking pairing.

Examples.

1) The Lie algebra $\mathrm{su}(2)$ has a basis $\langle X_1, X_2, X_3 \rangle$ and its dual basis $\langle X_1^*, X_2^*, X_3^* \rangle$ for $\mathrm{su}(2)^*$ which satisfy

$$[X_i, X_{i+1}] = X_{i+2}, \quad \mathrm{d}X_{i+2}^* = X_{i+1}^* \wedge X_i^*, \quad i = 1, 2, 3 \ (\mathrm{mod}\ 3).$$

Therefore $\langle X_1^*, X_2^*, X_3^* \rangle$ is ortho-normal *w.r.t.* LK. Thus, $\mathrm{su}(2)$ has (positive) definite linking pairing. Every non-trivial element $\alpha \in \mathrm{su}(2)^*$ defines essentially the same contact structure on $\mathrm{SU}(2)$.

2) As to the other simple Lie algebra $\mathrm{psl}(2;\mathbb{R})$, the dual basis $\langle h^*, \ell^*, k^* \rangle$ to

$$\left\langle h = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \ell = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad k = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle$$

is orthogonal and satisfy

$$\mathrm{LK}(h^*,h^*) = -1, \quad \mathrm{LK}(\ell^*,\ell^*) = -1, \quad \mathrm{LK}(k^*,k^*) = 1,$$

for vol $= h^* \wedge \ell^* \wedge k^*$, and thus LK is non-definite. The negative (resp. positive) 1-form (*e.g.*, ℓ^* (resp. k^*)) defines a negative (resp. positive) contact structure on $\widetilde{\mathrm{SL}(2;\mathbb{R})}/\Gamma$. Any point on the light cone $\mathcal{L} = \{\alpha \in \boldsymbol{g}^*; \mathrm{LK}(\alpha,\alpha) = 0\}$ except for the origin is dual to a parabolic element in $\mathrm{psl}(2;\mathbb{R})$ and defines an Anosov weak (un)stable foliation.

3) Fix a unimodular hyperbolic matrix $A \in \mathrm{SL}(2;\mathbb{Z})$, trace$A > 2$, and take semi-direct products S_A of Lie groups and Γ_A of their co-compact lattices

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{R}^2 & \longrightarrow & S_A & \longrightarrow & \mathbb{R} & \to & 0 \\ & & \cup & & \cup & & \cup & & \\ 0 & \to & \mathbb{Z}^2 & \longrightarrow & \Gamma_A & \longrightarrow & \mathbb{Z} & \to & 0 \end{array}$$

defined by the action $\exp(\tau \log A)$ of $\tau \in \mathbb{R}$ on $\mathbb{R}^2$. Then we obtain a solvable Lie group S_A. Changing the coordinate according to the eigen decomposition of the automorphism A, we see that its Lie algebra s_A is, independently of the choice of A, always isomorphic to the solvable Lie algebra solv :

$$0 \to \mathbb{R}^2 \longrightarrow \mathrm{solv} \longrightarrow \mathbb{R} \to 0,$$

the semi-direct product defined by the action $\left(\begin{smallmatrix} \tau & 0 \\ 0 & -\tau \end{smallmatrix}\right)$ of $\tau \in \mathbb{R}$ on $\mathbb{R}^2$. We can take a basis $\langle X, Y, T\rangle$ of solv according to the above semi-direct product decomposition, so that $[T,X] = X$, $[T,Y] = -Y$, and $[X,Y] = 0$ hold. Then the self-linking of $\alpha = xX^* + yY^* + \tau T^* \in \mathrm{solv}$ is given by $\mathrm{LK}(\alpha,\alpha) = xy$. Thus LK is non-definite and degenerate.

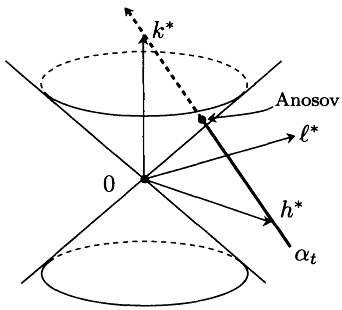
2.2. Constructions of convex non-Stein manifolds.

Following McDuff's construction, we join two invariant contact forms $\alpha_{-1} = h^*$ and $\alpha_1 = k^*$ on $M = \mathrm{PSL}(2;\mathbb{R})/\Gamma$ by a smooth curve α_t in $\mathrm{psl}(2;\mathbb{R})^*$, in order to obtain a smooth 1-form $\alpha = \{\alpha_t\}$ on $M \times [-1,1]$ such that $\omega = \mathrm{d}\alpha$ defines a symplectic structure with contact type boundaries. The linking pairing tells us that we simply have to join them by the segment.

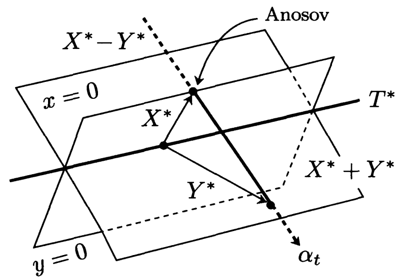
THEOREM 1. — *Let $\boldsymbol{g}$ be one of two Lie algebras* $\mathrm{psl}(2;\mathbb{R})$ *or* solv *and $M = G/\Gamma$ be a quotient of corresponding Lie group by a co-compact lattice. Then, any smooth curve α_t, $-1 \leq t \leq 1$, in $\boldsymbol{g}^*$ which satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{LK}(\alpha_t,\alpha_t) > 0, \quad \mathrm{LK}(\alpha_{-1},\alpha_{-1}) < 0 < \mathrm{LK}(\alpha_1,\alpha_1),$$

defines a symplectic strucuture $\omega = \mathrm{d}\alpha$ *with contact type boundaries on* $M \times [-1,1]$. *For example, in the case of* $\boldsymbol{g} = \mathrm{psl}(2;\mathbb{R})$ (*resp.* solv), $\alpha_t = \frac{1}{2}\{(1+t)k^* + (1-t)h^*\}$, (*resp.* $\alpha_t = X^* + tY^*$) *defines such a structure.*



psl(2, ℝ)*



solv*

Lemma. — *For any 3-dimensional unimodular Lie algebra* $\boldsymbol{g}$, *we have the followings:*

1) $\mathrm{d}_2 : \overset{2}{\wedge}\boldsymbol{g}^* \longrightarrow \overset{3}{\wedge}\boldsymbol{g}^*$ *vanishes.*

2) $\dfrac{\mathrm{d}}{\mathrm{d}t}(\alpha_t \wedge \mathrm{d}\alpha_t) = 2\Big(\dfrac{\mathrm{d}}{\mathrm{d}t}\alpha_t\Big) \wedge \mathrm{d}\alpha_t.$

The following proposition follows immediately from this lemma, which assures the non-degeneracy of the 2-form ω under the condition of the theorem. This will complete the proof of Theorem 1. □

Proposition 1. — *For any fixed volume element* $\mathrm{d}\,\mathrm{vol} \in \overset{3}{\wedge}\boldsymbol{g}^*$, *we have*

$$\omega^2 = \frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{LK}(\alpha_t,\alpha_t)\cdot \mathrm{d}t \wedge \mathrm{d}\,\mathrm{vol}.$$

As was mentioned in Section 1.2 we complete the two boundary components $M \times \{\pm 1\}$ as completely convex ends to obtain a complete convex symplectic structure on $M \times \mathbb{R}$. This is done in an explicit way in terms of the linking pairing LK.

Theorem 2. — *Any smooth curve* $\alpha_t : \mathbb{R} \to \boldsymbol{g}^*$ *which satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{LK}(\alpha_t,\alpha_t) > 0, \qquad \lim_{t\to\pm\infty}\mathrm{LK}(\alpha_t,\alpha_t) \to \pm\infty$$

defines a complete convex symplectic structure $\omega = \mathrm{d}\alpha$ *on* $M \times \mathbb{R}$. *For example, in the case of* $\boldsymbol{g} = \mathrm{psl}(2;\mathbb{R})$ (*resp.* solv),

$$\alpha_t = \tfrac{1}{2}\{(1+t)k^* + (1-t)h^*\},$$

(*resp.* $\alpha_t = X^* + tY^*$) *defines such a structure. This cannot be raised to a Stein manifold because of* $H_3(M \times \mathbb{R}) \neq 0$.

Remarks.

1) The curve α_t hits the light cone $\mathcal{L}$ at some point which defines an Anosov foliation. As we will observe in the next section, any Anosov flow on a 3-manifold gives rise to a pair of contact structures with opposite orientations.

2) Theoretically Theorem 2 is of no interest, because, up to symplectomorphism, the structure only depends on the germ of the curve α_t around the point of transverse intersection with the light cone and does not depend on any prolongation of it.

To prove Theorem 2, it suffices to see the behaviour in the t-direction of the contracting vector field Z of $\omega = \mathrm{d}\alpha$ ($i.e.$, $\iota_Z\omega = -\alpha$). The followings do it, where 3) is deduced from Lemma 2) above, while 4) is an independent fact. □

LEMMA.

3) *One has*

$$-\,\mathrm{d}t(Z) = \left\{\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{LK}(\alpha_t,\alpha_t)\right\}^{-1}\mathrm{LK}(\alpha_t,\alpha_t).$$

4) *For any smooth function which satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}f(t) > 0 \quad \textit{and} \quad \lim_{t\to\pm\infty} f(t) \to \pm\infty,$$

the vector field

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}f(x)\right)^{-1} f(x)\,\frac{\mathrm{d}}{\mathrm{d}t}$$

on the real line $\mathbb{R}$ *is complete.*

3. Volume preserving Anosov flows.

We generalize the construction in the previous section to closed oriented 3-manifolds which admit nice Anosov flows.

THEOREM 3. — *If a closed oriented 3-manifold M admits a smooth Anosov flow ϕ which preserves a smooth volume $d\,\mathrm{vol}$ of M, then $M\times[-1,1]$ carries a symplectic structure with contact type boundaries, and thus $M\times\mathbb{R}$ carries a complete convex symplectic structure which cannot be raised to a Stein manifold.*

It suffices to construct a symplectic structure with a Liouville form α on $M\times[-1,1]$ corresponding to a contracting field which assures the boundaries of contact type.

First we take an arbitrary Riemannian metric g_0. Let

$$TM = T\phi \oplus E^{uu} \oplus E^{ss}$$

be the Anosov decomposition, *i.e.*,

$$T\phi = \{\text{tangent vector along the flow lines}\},$$

$$E^{uu} = \{v \in TM; \|T\phi_t(v)\| \leq A\cdot \mathrm{e}^{Bt}\|v\|,\ t\leq 0\},$$

$$E^{ss} = \{v \in TM; \|T\phi_t(v)\| \geq A^{-1}\cdot \mathrm{e}^{Bt}\|v\|,\ t\geq 0\},$$

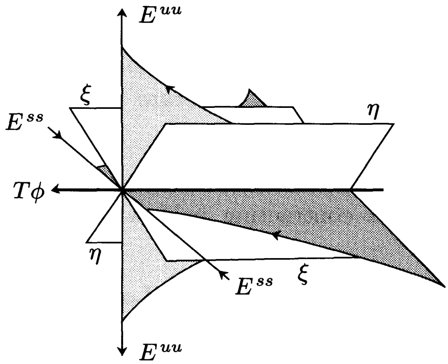
for some constants $A\geq 1$ and $B>0$. We call E^{ss} (E^{uu}) *strong* (*un*)*stable direction.*

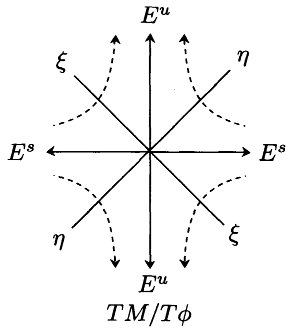
The C^r-section theorem (see [11]) says that weakly (un)stable two 2-plane fields $E^u = T\phi\oplus E^{uu}$ and $E^s = T\phi\oplus E^{ss}$ have their differentiability at least of class C^1, so that we obtain two codimension 1 foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ with their tangent bundles E^u and E^s respectively. The following lemma is a kind of folklore.

LEMMA. — *For some Riemannian metric, we can assume the constant A in the definition of the Anosov splitting to be 1, i.e., we can assume that any non-zero tangent vector in the strong unstable direction begins growing bigger immediately along the flow.*

We call such a metric *adapted.* This is given by replacing any metric g_0 with an average $g = T^{-1}\int_0^T \phi^* g_0$ for sufficiently large T. This lemma and the following proposition is independent of the presence of smooth invariant volume.

PROPOSITION 2. — *For any Anosov flow ϕ on M^3 with an adapted Riemannian metric, let ξ be a plane field defined by rotating E^u around $T\phi$ as the axis by an angle of $45°$, and η be another one by $-45°$. Then the pair (ξ,η) of those plane fields defines a bi-contact structure.*





A *bi-contact structure* (ξ, η) on a 3-manifold M is defined to be a transverse pair of contact plane fields ξ and η with different orientations, *i.e.*, ξ and η are defined by 1-forms α and β respectively such that $\alpha \wedge \mathrm{d}\alpha$ and $\beta \wedge \mathrm{d}\beta$ are volume forms of M of opposite orientations.

As it can be seen from the figures, neither of the two plane fields ξ and η is preserved by the flow, while they are tangent to it. Therefore they are non-integrable and define contact structures. Because the rotational movements of $T\phi_t(\xi)$ and $T\phi_t(\eta)$ around the axis $T\phi$ are in the opposite directions, the two contact structures have different orientations.

Proof of Theorem 3. — A smooth invariant volume $\mathrm{d\,vol}$ gives rise to a transverse invariant volume $\iota_{\mathrm{d/dt}}(\mathrm{d\,vol})$ on the normal bundle $TM/T\phi$ to the flow. Independently, applying the previous lemma, take any adapted metric h_0 on $TM/T\phi$. Then, we obtain a unique C^1 metric h on $TM/T\phi$ such that $E^u \oplus E^s$ is an orthogonal decomposition *w.r.t.* h , $h_{|E^u} = h_{0|E^u}$, and $\mathrm{vol}_h = \iota_{\mathrm{d/dt}}(\mathrm{d\,vol})$. Remark that this C^1-metric h is adapted. Then take C^1 1-forms ω^u and ω^s which define weak (un)stable foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ respectively and satisfy $\|\omega^u\|_h \equiv \|\omega^s\|_h \equiv 1$.

Now we C^1-approximate ω^u and ω^s by C^∞ 1-forms Ω^u and Ω^s respectively. Then, the contact structures ξ and η are defined by the C^∞ 1-forms

$$\alpha^\xi = \tfrac{1}{2}(\Omega^u + \Omega^s), \quad \alpha^\eta = \tfrac{1}{2}(\Omega^u - \Omega^s)$$

and we define a 1-form $\alpha = \{\alpha_t\}$ on $M \times [-1, 1]$ by

$$\alpha_t = (1-t)\alpha^\xi + (1+t)\alpha^\eta = \Omega^u - t\Omega^s.$$

The symplectic 2-form ω on $M \times [-1,1]$ is defined to be $\omega = \mathrm{d}\alpha$ as before. A direct calculation shows

$$\omega^2 = \mathrm{d}t \wedge \{2(\alpha^\eta \wedge \mathrm{d}\alpha^\eta - \alpha^\xi \wedge \mathrm{d}\alpha^\xi) + \mathrm{d}(\Omega^u \wedge \Omega^s) - 2t\Omega^s \wedge \mathrm{d}\Omega^s\}.$$

In the process of C^1-approximation of ω^u and ω^s by smooth forms Ω^u and Ω^s, the first term remains close to a volume form coming from C^1 contact structures. On the other hand, thanks to the invariance of the volume and the integrability, the second and the third ones can be made arbitrarily small. Thus we obtain a symplectic structure on $M \times [-1,1]$ with contact type boundaries. □

Remark. — If we impose the smoothness on the strong Anosov splitting as well as on the invariant volume, a theorem of Ghys (see [7]) tells us that there exists no such Anosov flow other than the examples 1) and 2) in Section 2.1.

On the other hand, the Dehn surgery method to produce new manifolds carrying Anosov flows, which was initiated by Handel and Thurston in [10] and developed by Goodman in [8], shows us that there exist much more graph manifolds which admit Anosov flows other than quotients of 3-dimensional Lie groups. Some of them admit invariant smooth volumes, so that we can apply Theorem 3, while the others do not. Recently Foulon introduced the idea of *contact Anosov flow* to establish a wider class of Anosov flows with invariant smooth volumes. For the moment, his result [5] seems to give us the widest such class, to which we can apply our theorem.

4. Bi-contact structures and projectively Anosov flows.

Even though it is nearly trivial, Proposition 2 in the previous section seems to have some independent interest. Therefore we close this article by raising some remarks and problems around bi-contact structures. More detailed discussions will find their place in a forthcoming paper.

The converse to Proposition 2 is not true in general. A bi-contact structure (ξ, η) provides us a vector field X as the intersection of ξ and η and a flow ϕ_t generated by X. If we apply the following lemma with a help of the compactness of M to the time one map of the flow which is induced on the oriented projectified S^1-bundle $S^1(TM/T\phi)$ (*i.e.*, the associated $(\mathbb{R}^2 \setminus \{0\})/\mathbb{R}_+$-bundle of the set of oriented lines) of the normal bundle, we obtain an almost converse of Proposition 2, *i.e.*, we can find two invariant continuous plane fields

$$E^u = \lim_{t\to+\infty}(T\phi_t)_*\xi = \lim_{t\to+\infty}(T\phi_t)_*\eta$$

and

$$E^s = \lim_{t\to-\infty}(T\phi_t)_*\xi = \lim_{t\to-\infty}(T\phi_t)_*\eta$$

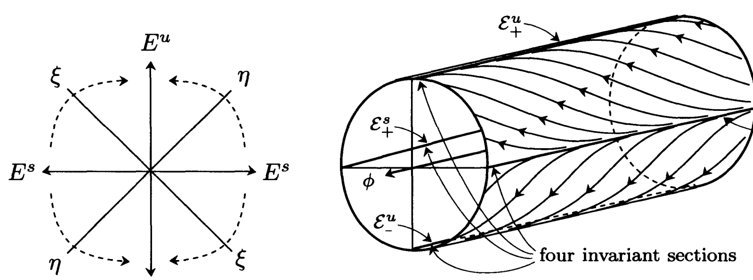
like Anosov weak splitting, by the standard argument to find an invariant section under a fiber contraction. This is the first half of Proposition 3 below. However, the flow thus obtained is not necessarily Anosov.

LEMMA. — *For any $\varepsilon > 0$, any positive integer k, and any*

$$A_i = \begin{pmatrix} a_i, & b_i \\ c_i, & d_i \end{pmatrix} \in \mathrm{SL}(2;\mathbb{R}), \quad i = 1, \dots, k,$$

which satisfy $a_i, b_i, c_i, d_i \geq \varepsilon$, we have

$$\mathrm{Tr}(A_1 \cdots A_k) \geq (1 + 2\varepsilon^2)^{k/2}.$$



$TM/T\phi$ $\quad$ $S^1(TM/T\phi)$

A flow ϕ_t on a closed 3-manifold M is said to be *projectively Anosov* (*pA* for short), iff the oriented projectified S^1-bundle $S^1(TM/T\phi)$ admits four continuous sections $\mathcal{E}^u_\pm$ and $\mathcal{E}^s_\pm$ which are invariant under the projectified action of $T\phi_t$ and any other orbit not contained in $\mathcal{E}^u_\pm$ nor in $\mathcal{E}^s_\pm$ is attracted to $\mathcal{E}^u_\pm$ as $t \to \infty$ and to $\mathcal{E}^s_\pm$ as $t \to -\infty$. Necessarily, $\mathcal{E}^u_\pm$ and $\mathcal{E}^s_\pm$ are antipodal two by two. We can show the latter half of the following proposition in the same way as we did in the previous section by passing through adapted metrics.

PROPOSITION 3. — *If M^3 admits a bi-contact structure, the vector field defined by their intersection generates a projectively Anosov flow. Conversely, if M admits a projectively Anosov flow, there exists a bi-contact structure whose intersection is tangent to the flow.*

The followings are some examples of pA flows which are not Anosov. We call such a flow *essential pA* or *EpA*.

Examples.

1) Any Anosov flow can be deformed around its closed orbit into a new EpA flow in such a way that the first return map is contracting in a small neighbourhood of the periodic point.

2) Any T^2-bundle over S^1 admits EpA flows, constructed as follows. A pair $\Pi = (\xi_0, \eta_0)$ of contact plane fields with opposite orientations on a 3-manifold is called *pre-bi-contact* (*pbC* for short) structure if there exists a common smooth non-singular Legendrian vector field X_Π. Its singular set Σ_Π is defined to be $\{x \mid {\xi_0}_x = {\eta_0}_x\}$, which easily turns out to be smooth tori transverse to X_Π. A pbC structure with non-empty singular set is called *essential pbC* (*EpbC* for short) structure.

Any EpbC structures can be modified into a bi-contact structure as follows. Let α_0 and β_0 be contact 1-forms which define ξ_0 and η_0 respectively and γ be a non singular 1-form such that $\gamma(X_\Pi) > 0$. Then it is easy to see that for sufficiently small $\epsilon > 0$ a new pair of contact 1-forms $(\alpha, \beta) = (\alpha_0, \beta_0 + \epsilon\gamma)$ defines a bi-contact structure. On T^3 with its standard coordinate (x, y, z), a pair of contact 1-forms

$$(\alpha_0, \beta_0) = \big(\cos(2\pi z)\,\mathrm{d}x + \sin(2\pi z)\,\mathrm{d}y, \cos(2\pi z)\,\mathrm{d}x - \sin(2\pi z)\,\mathrm{d}y\big)$$

defines the most typical EpbC structure $\Pi = (\xi_0, \eta_0)$ with its singular set $\Sigma_\Pi = \{z \in \frac{1}{4}\mathbb{Z}\}$, and the common Legendrian vector field is given by $X_\Pi = \mathrm{d}/\mathrm{d}z\cdot$ Fortunately, the resulting bi-contact structure (ξ, η) has a real analytic Anosov like splitting E^u and E^s which gives rise to analytic foliations $\mathcal{F}^u$ and $\mathcal{F}^s$. The singular set Σ_Π coincides with the union of compact leaves of these foliations. See also remarks below.

Similar construction works for any T^2-bundle on S^1.

Remarks and problems.

1) Neither T^2 nor Nil-manifolds admits Anosov flows, because their fundamental groups do not grow exponentially. Therefore, EpA flows on such manifolds can not be obtained by the deformation method as in Example 1) above.

Moreover, pA flows obtained by the modification from EpbC structures have compact leaves in the weak Anosov-like splitting with hyperbolic holonomies (in some sense), which never happen in Anosov flows.

2) In general the weak splitting $E^u + E^s$ (or equivalently, the invariant sections $\mathcal{E}^u_\pm$ and $\mathcal{E}^s_\pm$) of a pA flow does not have the differentiability of class C^1. Therefore, in spite of the fact that the plane fields E^u and E^s are integrable, we do not know whether the unique integrability holds or not.

3) If we assume the splitting of class C^1, we see that weak (un)stable foliation does not have Reeb components. Especially S^3 admits no pA flows with C^1-splitting. As to the general case, nothing about the (non)existence on S^3 is known.

4) The concept of (E)pA diffeomorphism on T^2 is defined likewise. Only known EpA diffeomorphisms are obtained by the deformation method from Anosov diffeomorphisms. It is conjectured that there exists no pA diffeomorphism in any isotopy class which is represented by a non-hyperbolic matrix in $\mathrm{SL}(2;\mathbb{Z})$.

5) If we try to follow the similar construction of symplectic structures in Section 3 starting from the EpA flow on T^3 above, the symplectic structure degenerates exactly on the compact leaves. This degeneration seems rather essential, while it is totally unclear whether the construction works for Anosov flows in general or no. The symplectic fillability (see [3]) for Anosov flows in general is unknown.

6) A related problem is to know which 3-manifold can (not) be realized as the boundary of a convex symplectic 4-manifold, especially of those with disconnected boundary or of type $M^3 \times [0,1]$. Reference [14] says it is impossible to realize $S^3 \coprod$ (other components).

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