

# ANNALES DE L'INSTITUT FOURIER

CARME LESEDUARTE

JAUME LLIBRE

## **The full periodicity kernel of the trefoil**

*Annales de l'institut Fourier*, tome 46, n° 1 (1996), p. 219-262

[http://www.numdam.org/item?id=AIF\\_1996\\_\\_46\\_1\\_219\\_0](http://www.numdam.org/item?id=AIF_1996__46_1_219_0)

© Annales de l'institut Fourier, 1996, tous droits réservés.

L'accès aux archives de la revue « Annales de l'institut Fourier » (<http://annalif.ujf-grenoble.fr/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

## THE FULL PERIODICITY KERNEL OF THE TREFOIL

by C. LESEDUARTE and J. LLIBRE (\*)

---

### 1. Introduction and main results.

Let  $E$  be a topological space. We shall study some properties of the set of periods for a class of continuous maps from  $E$  into itself. We need some notation.

The set of natural numbers, real numbers and complex numbers will be denoted by  $\mathbb{N}$ ,  $\mathbb{R}$  and  $\mathbb{C}$  respectively. For a map  $f : E \rightarrow E$  we use the symbol  $f^n$  to denote  $f \circ f \circ \dots \circ f$  ( $n \in \mathbb{N}$  times),  $f^0$  or «id» denotes the identity map of  $E$ . Then, for a point  $x \in E$  we define the *orbit* of  $x$ , denoted by  $\text{Orb}_f(x)$ , as the set  $\{f^n(x) : n = 0, 1, 2, \dots\}$ . We say  $x$  is a *fixed point* of  $f$  if  $f(x) = x$ . We say  $x$  is a *periodic point of  $f$  of period  $k \in \mathbb{N}$*  (or simply a  *$k$ -point*) if  $f^k(x) = x$  and  $f^i(x) \neq x$  for  $1 \leq i < k$ . In this case we say the orbit of  $x$  is a *periodic orbit of period  $k$*  (or simply a  *$k$ -orbit*). Note that if  $x$  is a  $k$ -point, then  $\text{Orb}_f(x)$  has exactly  $k$  elements, each of which is a  $k$ -point. We denote by  $\text{Per}(f)$  the set of periods of all periodic points of  $f$ .

A *connected finite regular graph* (or just a *graph* for short) is a pair consisting of a connected Hausdorff space  $E$  and a finite subspace  $V$ , whose elements are called *vertices*, such that the following conditions hold:

(1)  $E \setminus V$  is the disjoint union of a finite number of open subsets  $e_1, \dots, e_k$ , called *edges*. Each  $e_i$  is homeomorphic to an open interval of the real line.

(2) The boundary,  $\text{Cl}(e_i) \setminus e_i$ , of the edge  $e_i$  consists of two distinct vertices, and the pair  $(\text{Cl}(e_i), e_i)$  is homeomorphic to the pair  $([0,1], (0,1))$ .

If  $v$  and  $e$  are the number of vertices and edges respectively of  $E$ , then the *Euler characteristic of  $E$* , is  $\chi(E) = v - e$ . A vertex which belongs to

---

(\*) The authors have been partially supported by a DGYCIT Grant n° PB 93-0860.  
Key words: Periods – Full periodicity – Kernel.  
Math. classification: 54H20.

the boundary of at least three different edges is called a *branching point* of  $E$ . A vertex which belongs to a unique edge is called an *endpoint*.

An  $E$  map is a continuous self-map of  $E$  having fixed all branching points of  $E$ .

We say an  $E$  map  $f$  has *full periodicity* if  $\text{Per}(f) = \mathbb{N}$ . The set  $K \subset \mathbb{N}$  is the *full periodicity kernel* of  $E$  if it satisfies the following two conditions:

(1) If  $f$  is an  $E$  map and  $K \subset \text{Per}(f)$ , then  $\text{Per}(f) = \mathbb{N}$ .

(2) If  $S \subset \mathbb{N}$  is a set such that for every  $E$  map  $f$ ,  $S \subset \text{Per}(f)$  implies  $\text{Per}(f) = \mathbb{N}$ , then  $K \subset S$ .

Note that, for a given  $E$ , if there is a full periodicity kernel, then it is unique.

From now on the topological space  $E$  will denote one of the following spaces:

$$\begin{aligned} \mathbf{I}_i &= \{z \in \mathbb{C} : z^i \in [0, 1]\}, \quad i = 2, 3, \dots, 6. \\ \mathbf{O} &= \{z \in \mathbb{C} : |z + i| = 1\}, \\ \mathbf{O}_1 &= \mathbf{O} \cup \{z \in \mathbf{I}_2 : \text{Re } z \geq 0\}, \\ \mathbf{O}_2 &= \mathbf{O} \cup \mathbf{I}_2, \\ \mathbf{O}_3 &= \mathbf{O} \cup \{z \in \mathbf{I}_4 : \text{Im } z \geq 0\}, \\ \mathbf{O}_4 &= \mathbf{O} \cup \mathbf{I}_4, \\ \infty &= \mathbf{O} \cup \{z \in \mathbb{C} : |z - i| = 1\}, \\ \infty_1 &= \infty \cup \{z \in \mathbf{I}_2 : \text{Re } z \geq 0\}, \\ \infty_2 &= \infty \cup \mathbf{I}_2, \\ \mathbf{T} &= \{z \in \mathbb{C} : z = \cos(3\theta) e^{i\theta}, 0 \leq \theta \leq 2\pi\}. \end{aligned}$$

The spaces  $\mathbf{I}_2, \mathbf{I}_3, \mathbf{I}_4, \mathbf{I}_5, \mathbf{I}_6, \mathbf{O}, \mathbf{O}_1, \mathbf{O}_2, \mathbf{O}_3, \mathbf{O}_4, \infty, \infty_1, \infty_2$  and  $\mathbf{T}$  are called the *interval* or the  $\mathbf{I}$ , the *3-od* or *3-star* or the  $\mathbf{Y}$ , the *4-od* or the *4-star*, the *5-od* or *5-star*, the *6-od* or *6-star*, the *circle*, the *sigma*, the *alpha*, the *circle with three whiskers*, the *circle with four whiskers*, the *eight*, the *eight with one whiskers*, the *eight with two whiskers* and the *trefoil* respectively.

The spaces  $\mathbf{I}_3, \mathbf{I}_4, \mathbf{I}_5, \mathbf{I}_6, \mathbf{O}_1, \mathbf{O}_2, \mathbf{O}_3, \mathbf{O}_4, \infty, \infty_1, \infty_2$  and  $\mathbf{T}$  have exactly one branching point, namely  $\mathbf{0} = 0 \in \mathbb{C}$ . We also denote by  $\mathbf{0}$  the  $0 \in \mathbf{O}$ .

The full periodicity kernel of  $\mathbf{I}_2, \mathbf{I}_3, \mathbf{I}_4, \mathbf{I}_5, \mathbf{I}_6, \mathbf{O}, \mathbf{O}_1, \mathbf{O}_2$  and  $\infty$  are known and presented in the following theorem.

THEOREM 1.1. — *The following statements hold:*

- (a) *The set  $\{3\}$  is the full periodicity kernel of  $\mathbf{I}_2$ .*
- (b) *The set  $\{2, 3, 4, 5, 7\}$  is the full periodicity kernel of  $\mathbf{I}_3$ .*
- (c) *The set  $\{2, 3, 4, 5, 6, 7, 10, 11\}$  is the full periodicity kernel of the  $\mathbf{I}_4$ .*
- (d) *The set  $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 16, 17, 18, 21, 23\}$  is the full periodicity kernel of the  $\mathbf{I}_5$ .*
- (e) *The full periodicity kernel of the  $\mathbf{I}_6$  is the set*  
 $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18, 21, 22, 23, 28, 29\}$ .
- (f) *The set  $\{1, 2, 3\}$  is the full periodicity kernel of  $\mathbf{O}$ .*
- (g) *The set  $\{2, 3, 4, 5, 7\}$  is the full periodicity kernel of  $\mathbf{O}_1$ .*
- (h) *The set  $\{2, 3, 4, 5, 6, 7, 10, 11\}$  is the full periodicity kernel of  $\mathbf{O}_2$ .*
- (i) *The set  $\{2, 3, 4, 5, 6, 7, 8, 10, 11\}$  is the full periodicity kernel of  $\infty$ .*
  - Theorem 1.1 (a) is due to Sharkovskii [Sh] (see also [LY]),
  - Theorem 1.1 (b) was shown by Mumbrú [M] (see also [ALM1]),
  - Theorem 1.1 (c) has been proved by Alsedà and Moreno [AM] and independently by Leseduarte and Llibre [LL2],
  - Statements (d) and (e) of Theorem 1.1 are due to Alsedà and Moreno [AM],
  - Theorem 1.1 (f) is due to Block [Bc1] (see also [LR]),
  - Theorem 1.1 (g) has been proved by Llibre, Paraños and Rodríguez [LPR1] (see also [LL1]),
  - Statements (h) and (i) of Theorem 1.1 are due to Leseduarte and Llibre [LL2].

Our main goal in this paper is to characterize the full periodicity kernel of  $\mathbf{O}_3$ ,  $\mathbf{O}_4$ ,  $\infty_1$ ,  $\infty_2$  and  $\mathbf{T}$ . Thus, our main results are the following:

THEOREM 1.2. — *The full periodicity kernel of  $\mathbf{O}_3$  is the set*

$$\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 16, 17, 18, 21, 23\}.$$

THEOREM 1.3. — *The full periodicity kernel of  $\mathbf{O}_4$  is the set*

$$\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18, 21, 22, 23, 28, 29\}.$$

THEOREM 1.4. — *The full periodicity kernel of  $\infty_1$  is the set*

$$\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16, 17, 18, 21, 23\}.$$

THEOREM 1.5. — *The full periodicity kernel of  $\infty_2$  is the set*  
 $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 21, 22, 23, 28, 29\}$ .

THEOREM 1.6. — *The full periodicity kernel of  $\mathbf{T}$  is the set*  
 $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 21, 22, 23, 28, 29\}$ .

Theorems 1.2, 1.3, 1.4, 1.5 and 1.6 are proved in Sections 6, 7, 10, 11 and 12 respectively. Sections from 2 to 5 present preliminary definitions and results that are necessary for proving these five main theorems. In Section 13 we compare our results on the full periodicity kernel with related results of Blokh. Finally, in Section 14 we comment that full periodicity implies positive topological entropy for continuous self-maps on a graph.

The tools for studying the set of periods and the full periodicity kernel change strongly when we consider maps with some discontinuity points, see for instance [ALMT].

## 2. Intervals and basic intervals.

From now on we shall talk about the whiskers and the circles of  $E$ . A *circle* of  $E$  is the closure of a connected component of  $E \setminus \{0\}$  which is homeomorphic to  $\mathbf{O}$ . A *whiskers* of  $E$  is the closure of a connected component of  $E \setminus \{0\}$  which is homeomorphic to  $\mathbf{I}_2$ .

A *closed* (respectively *open*, *half-open* or *half-closed*) interval  $J$  of  $E$  is a subset of  $E$  homeomorphic to the closed interval  $[0, 1]$  (respectively  $(0, 1)$ ,  $[0, 1)$ ). Notice that an interval cannot be a single point.

Let  $J$  be a closed interval of  $E$ , and let  $h : [0, 1] \rightarrow J$  be a homeomorphism. Then  $h(0) = a$  and  $h(1) = b$  are called the *endpoints* of  $J$ . If  $a$  and  $b$  belong to the same whiskers of  $E$ , then  $J$  will be denoted by  $[a, b]$  or  $[b, a]$ . We take an orientation, that we call *counterclockwise*, in each circle of  $E$ . If  $a$  and  $b$  belong to the same circle of  $E$ , then we write  $[a, b]$  to denote the closed interval from  $a$  counterclockwise to  $b$ .

Note that it is possible that two different intervals of a circle of  $E$  have the same endpoints. But two different points of a whiskers of  $E$  always determine a unique closed interval.

Now we define a special class of subintervals of  $E$ . Let  $Q = \{q_1, q_2, \dots, q_n\}$  be a finite subset of  $E$  containing  $\mathbf{0}$ . For each pair  $q_i, q_j$  such

that  $q_i \neq q_j$  we say that the interval  $[q_i, q_j]$  (respectively  $[q_j, q_i]$ ) is *basic* if and only if  $(q_i, q_j) \cap Q = \emptyset$  (respectively  $(q_j, q_i) \cap Q = \emptyset$ ). The set of all these basic intervals is called the *set of basic intervals associated to  $Q$* .

### 3. Loops and $f$ -graphs.

Let  $f : E \rightarrow E$  be an  $E$  map. If  $K$  and  $J$  are closed intervals of  $E$ , then we say that  $K$   $f$ -covers  $J$  or  $K \rightarrow J$  (or  $J \leftarrow K$ ), if there is a closed subinterval  $M$  of  $K$  such that  $f(M) = J$ . If  $K$  does not  $f$ -cover  $J$  we write  $K \not\rightarrow J$ .

A *path of length  $m$*  is any sequence  $J_0 \rightarrow J_1 \rightarrow \dots \rightarrow J_{m-1} \rightarrow J_m$ , where  $J_0, J_1, \dots, J_m$  are closed subintervals of  $E$  (in general, basic intervals). Furthermore, if  $J_0 = J_m$ , then this path is called a *loop of length  $m$* . Such a loop will be called *non-repetitive* if there is no integer  $i$ ,  $0 < i < m$ , such that  $i$  divides  $m$  and  $J_{j+i} = J_j$  for all  $j$ ,  $0 \leq j \leq m - i$ . We say that we *add* or we *concatenate* the loop  $J_0 \rightarrow J_1 \rightarrow \dots \rightarrow J_{m-1} \rightarrow J_0$  to the loop  $K_0 \rightarrow K_1 \rightarrow \dots \rightarrow K_{n-1} \rightarrow K_0$  if they have a common vertex  $J_0 = K_0$  and we form the new loop  $J_0 \rightarrow J_1 \rightarrow \dots \rightarrow J_{m-1} \rightarrow K_0 \rightarrow K_1 \rightarrow \dots \rightarrow J_0$ . A loop which cannot be formed by adding two loops will be called *elementary*.

Let  $Q$  be a finite subset of  $E$  containing  $\mathbf{0}$ . An  $f$ -graph of  $Q$  is a graph with the basic intervals associated to  $Q$  as vertices, and such that if  $K$  and  $J$  are basic intervals and  $K$   $f$ -covers  $J$ , then there is an arrow from  $K$  to  $J$ . Note that the  $f$ -graph of  $Q$  is unique up to labeling of the basic intervals. Hence from now on we shall talk about *the  $f$ -graph* of  $Q$  (or just *the  $f$ -graph* for short). The next three lemmas are well-known in one dimensional dynamics, see for instance [ALM2]. We only prove the third one because we will use its proof later.

LEMMA 3.1. — *Let  $f$  be an  $E$ -map and let  $K, J, L$  be closed subintervals of  $E$ . If  $L \subset J$  and  $K$   $f$ -covers  $J$ , then  $K$   $f$ -covers  $L$ .*

LEMMA 3.2. — *Let  $f$  be an  $E$  map and let  $J$  be a closed subinterval of  $E$  such that  $J$   $f$ -covers  $J$ . Then  $f$  has a fixed point in  $J$ .*

LEMMA 3.3. — *Let  $f$  be an  $E$  map and let  $J_0, J_1, \dots, J_{n-1}$  be closed subintervals of  $E$  such that  $J_i \rightarrow J_{i+1}$  for  $i = 0, 1, \dots, n - 2$  and  $J_{n-1} \rightarrow J_0$ . Then there exists a fixed point  $x$  of  $f^n$  in  $J_0$  such that  $f^i(x) \in J_i$  for  $i = 1, 2, \dots, n - 1$ .*

*Proof.* — We shall use backward induction. Let  $K_{n-1} \subset J_{n-1}$  be a closed interval such that  $f(K_{n-1}) = J_0$ , and suppose we have constructed  $K_i \subset J_i$  for some  $i > 0$ ,  $i \leq n - 1$  such that  $f(K_i) = K_{i+1}$  if  $i < n - 1$  and  $f(K_i) = J_0$  if  $i = n - 1$ . Then, by Lemma 3.1,  $J_{i-1}$   $f$ -covers  $K_i$  and therefore there exists an interval  $K_{i-1} \subset J_{i-1}$  such that  $f(K_{i-1}) = K_i$ . Let  $g$  be as follows:

$$g = f|_{K_{n-1}} \circ \cdots \circ f|_{K_1} \circ f|_{K_0}.$$

Then  $K_0 \subset J_0$  and  $g(K_0) = J_0$ . Consequently  $f^n(K_0) = J_0$ . By continuity of  $f^n$  and Lemma 3.2  $f^n$  has a fixed point  $x \in K_0 \subset J_0$ , such that  $f^i(x) \in K_i \subset J_i$  for  $i = 1, 2, \dots, n - 1$ . □

Let  $J$  be an interval of  $E$ . Then  $\text{Int}(J)$  and  $\text{Cl}(J)$  denote the interior and the closure of  $J$  respectively.

**PROPOSITION 3.4.** — *Let  $f$  be an  $E$  map having a  $k$ -orbit  $P$ . Consider the set of basic intervals associated to  $P' = P \cup \{\mathbf{0}\}$ . Let  $J_0 \rightarrow J_1 \rightarrow \cdots \rightarrow J_{m-1} \rightarrow J_m = J_0$  be a non-repetitive loop of length  $m$  of the  $f$ -graph of  $P'$  such that at least one  $J_i$  does not contain  $\mathbf{0}$ . If  $m \neq 2k$ , then  $m \in \text{Per}(f)$ .*

*Proof.* — By Lemma 3.1  $J_0$   $f^m$ -covers  $J_0$ . Then by Lemma 3.2 there exists  $x \in J_0$  such that  $f^m(x) = x$ . If  $x$  has period  $m$  we are done. So suppose that  $x$  has period  $s$ ,  $0 < s < m$ . Thus  $s$  divides  $m$ .

It is not possible that  $x = \mathbf{0}$  because  $\mathbf{0}$  is a fixed point and some  $f^i(x) \in J_i$  with  $J_i \cap \{\mathbf{0}\} = \emptyset$ .

If  $x \in \text{Int}(J_0)$ , then  $\text{Orb}_f(x) \cap P = \emptyset$ . So each  $f^i(x)$  is exactly in one basic interval, and consequently the loop is repetitive (because  $s < m$  and  $s$  divides  $m$ ). Hence,  $x$  must be a point of  $P$ . So  $\text{Orb}_f(x) \subset P$ . Without loss of generality we can assume that  $s = k$ .

Let  $K_0 \subset J_0$  be the interval constructed in the proof of Lemma 3.3, then  $f^i(x) \in f^i(K_0) \subset J_i$  for  $i = 0, 1, \dots, m$ . Since  $x = f^s(x) \in f^s(K_0) \subset J_s$  it follows that  $J_0$  and  $J_s$  have a common endpoint  $x$ .

Assume that  $J_0 = J_s$ . Both sets  $K_0$  and  $f^s(K_0)$  are contained in  $J_0$  and contain  $x$ , an endpoint of  $J_0$ . Therefore  $L = K_0 \cap f^s(K_0)$  is an interval (in fact it is either  $K_0$  or  $f^s(K_0)$ ). Clearly  $f^i(L) \subset f^i(K_0) \subset J_i$ ,  $f^i(L) \subset f^{s+i}(K_0) \subset J_{s+i}$ , and  $f^i(L)$  is an interval for  $0 \leq i \leq s$ . Thus  $J_i = J_{s+i}$  for  $i = 0, 1, \dots, s - 1$ .

Repeating this process we get  $J_i = J_{s+i}$  for  $i = 0, 1, \dots, m - s$ . Hence the loop is repetitive because  $s$  divides  $m$ , a contradiction with the assumptions. So  $J_0 \neq J_s$ .

If  $J_q = J_{q+s}$  for some  $0 < q < m - s$ , then the above arguments prove that  $J_{q+i} = J_{q+s+i}$  for  $i = 0, 1, \dots, s - 1$ . Repeating this process we obtain that  $J_i = J_{s+i}$  for  $i = 0, 1, \dots, m - s$  and so the loop is repetitive, a contradiction with the assumptions. Therefore we can assume that  $J_q \neq J_{q+s}$  for  $0 \leq q < m - s$ .

Since  $x$  is a periodic point of period  $s$ , it follows that  $J_0 = J_{2s}$  and  $J_s = J_{3s}$ . By the above arguments we get  $J_m = J_0 = J_{2s} = J_{4s} = \dots$  and  $J_s = J_{3s} = J_{5s} = \dots$ . In particular  $m$  must be even. Furthermore  $J_i = J_{2s+i}$  for  $0 \leq i \leq 2s - 1$ . Hence  $2s = 2k$  divides  $m$ . Since  $m \neq 2k$  the loop is repetitive, in contradiction with the hypotheses.  $\square$

Under the assumptions of Proposition 3.4 and if  $m = 2k$ , we can prove that  $m \in \text{Per}(f)$  if  $E$  is different from  $\infty$  and  $\mathbf{T}$ . Unfortunately we do not know under the same assumptions if  $m \in \text{Per}(f)$  when  $m = 2k$  and  $E$  is either  $\infty$  or  $\mathbf{T}$ . But this is not important for the rest of the paper.

### 4. Q-linear maps.

Let  $G = \mathbf{I}_i$ , for  $i = 2, 3, \dots, 6$ . It is easy to see that any tree  $G$  has a metric  $\mu$  such that if  $x, y \in G$  and  $z \in [x, y]$ , then  $\mu(x, y) = \mu(x, z) + \mu(z, y)$ , this metric is called the *taxicab metric*.

Let  $f$  be an  $E$  map and let  $Q = \{q_1, q_2, \dots, q_m\}$  be an invariant subset of  $E$  under  $f$  such that  $\mathbf{0} \in Q$ . We assume that there are points of  $Q$  in each connected component of  $E \setminus \{\mathbf{0}\}$ . Let  $E_Q$  be the minimal connected subgraph of  $E$  containing all the basic intervals associated to  $Q$ . Clearly  $E_Q$  is homeomorphic to  $E$ . We say that  $f$  is *Q-linear* if the following conditions hold:

(1)  $E_Q = E$ ; in particular the endpoints of  $E$  are points of  $Q$ ;

(2) for any basic interval  $J$  associated to  $Q$ ,  $f(J)$  is an interval formed by the union of basic intervals of  $Q$ ;

(3)  $f|_J : J \rightarrow f(J)$  is a linear homeomorphism with respect to the taxicab metric, i.e.  $f|_J$  is a homeomorphism satisfying that for any  $x, y, z \in J$  such that  $\mu(x, y) = \mu(x, z) + \mu(z, y)$  we have that

$$\mu(f(x), f(y)) = \mu(f(x), f(z)) + \mu(f(z), f(y)).$$



We say an  $E$  map  $g$  is a  $Q$ -linearization of  $f$  if the following conditions hold:

- (1)  $g|_Q = f|_Q$ ;
- (2)  $g$  is  $Q$ -linear;
- (3) the  $g$ -graph of  $Q$  is a subgraph of the  $f$ -graph of  $Q$ .

Suppose that  $f$  is an  $E$  map having a  $k$ -orbit  $P$  such that  $P$  has points in each connected component of  $E \setminus \{0\}$ . Set  $P' = P \cup \{0\}$ . Clearly, if  $E \in \{O, \infty, T\}$  then  $E_{P'} = E$ . Assume now that  $E \notin \{O, \infty, T\}$ . For each whiskers  $W$  of  $E$  we consider the endpoint  $q \in W$  of  $E$  and the point  $p \in W$ ,  $p \in P$  such that  $(p, q) \cap P = \emptyset$ . Let  $E'$  be the new topological space obtained by shrinking the interval  $[p, q]$  to the point  $p$ . Note that  $E'$  is homeomorphic to  $E$ . We define the  $E$  map  $h : E' \rightarrow E'$  by  $h(x) = f(x)$  if  $f(x) \in E'$  and  $h(x) = p$  otherwise. Of course  $P$  is a  $k$ -orbit for  $h$ ,  $\text{Per}(h) \subset \text{Per}(f)$  and the endpoint of  $W$  belongs to  $P$ . Therefore we can assume that  $E_{P'} = E$ . In particular, we can talk about the  $P'$ -linearization of  $f$  in the above way.

*In the rest of this section we assume that  $f$  is an  $E$  map having a  $k$ -orbit  $P$  and consider the set of basic intervals associated to  $P' = P \cup \{0\}$ .*

LEMMA 4.1. — *Let  $K$  and  $J$  be basic intervals and let  $g$  be a  $P'$ -linearization of  $f$ . If  $x \in \text{Int}(J)$ ,  $g(x) \neq 0$  and  $g(x) \in K$ , then  $J$   $g$ -covers  $K$ .*

*Proof.* — Let  $a, b$  be the endpoints of  $J$ . Since  $J$  is a basic interval associated to  $P'$ , its endpoints have image in  $P'$  and so  $\{f(a), f(b)\} \cap \text{Int}(K) = \emptyset$ . By  $P'$ -linearity, since  $g(x) \in K$ ,  $g(x) \neq 0$  and  $x \in \text{Int}(J)$ , there exists an interval  $L \subset J$  such that  $g(L) = K$ . So  $J$   $g$ -covers  $K$ .  $\square$

Let  $J$  be a basic interval. If  $0 \in J$ , then  $J$  will be called a *branching interval*; otherwise  $J$  will be called a *non-branching interval*.

The following proposition is the converse result of Proposition 3.4 for  $P'$ -linear maps.

PROPOSITION 4.2. — *Let  $g$  be a  $P'$ -linearization of  $f$ . If  $g$  has an  $m$ -point for  $m \notin \{1, 2, 3, 4, 5, 6, k\}$ , then there exists a non-repetitive loop of length  $m$  through the  $g$ -graph such that at least one basic interval of the loop does not contain  $0$ .*

*Proof.* — Let  $x$  be a periodic point of period  $m$  for  $g$ . Then  $\text{Orb}_g(x) \cap P' = \emptyset$ , so for each  $i$ ,  $0 \leq i < m$ , there exists a unique

basic interval  $J_i$  containing  $g^i(x)$ . Since  $g$  is  $P'$ -linear, by Lemma 4.1,  $J_0 \rightarrow J_1 \rightarrow \dots \rightarrow J_{m-1} \rightarrow J_m = J_0$  is a loop of the  $g$ -graph. First we shall show that this loop is non-repetitive.

Since  $g$  is  $P'$ -linear, we can define by backward induction on  $i$ , a collection of subintervals  $K_i$  of  $J_i$  such that  $g : K_i \rightarrow K_{i+1}$  is one-to-one and onto, where  $K_m = J_m = J_0$ . Suppose now the loop is repetitive, then there exists  $s$ ,  $0 < s < m$ , such that  $s$  divides  $m$  and  $J_i = J_{i+s}$  for  $0 \leq i \leq m - s$ . We take  $s$  the smallest number in such a way. We claim that  $K_i \subset K_{i+s}$  for  $0 \leq i \leq m - s$ . To prove the claim consider  $K_{m-s} \subset J_{m-s} = J_m = K_m$  and by backward induction, suppose  $K_{i+1} \subset K_{i+s+1}$  and  $K_i \not\subset K_{i+s}$ . So, there is  $a \in K_i$  such that  $a \notin K_{i+s}$ , and  $g(a) \in K_{i+1} \subset K_{i+s+1}$ . Since  $K_{i+s} \rightarrow K_{i+s+1}$ , there exists  $b \in K_{i+s}$  (and so  $b \neq a$ ) such that  $g(b) = g(a)$ . This is a contradiction with the fact that  $g$  is  $P'$ -linear and  $g|_{J_i}$  is one-to-one. Hence the claim is proved.

Thus  $g^s(K_0) = K_s \supset K_0$  and by Lemma 3.2,  $g^s$  has a fixed point  $y \in K_0$ . Since  $m$  is divisible by  $s$ ,  $g^m(y) = y$ . Note that  $x \neq y$  because  $x$  has period  $m$ , and  $y$  has period  $s < m$ . Hence the map  $g^m : K_0 \rightarrow K_m$  is linear and has at least two fixed points. Therefore  $g^m|_{K_0}$  must be the identity map and so  $K_0 = K_m = J_m = J_0$ . Then we get  $K_0 = K_s = K_{2s} = \dots = K_m$  because  $K_0 \subset K_s \subset K_{2s} \subset \dots \subset K_m = K_0$ . Now consider the linear map  $g^s : K_0 \rightarrow K_s = K_0$  which has a fixed point. Since  $g^s|_{K_0}$  is one-to-one and onto, we have two possibilities.

- *Case 1:*  $g^s|_{K_0} = \text{id}$ .

Then  $g^s(x) = x$  but  $x$  has period  $m > s$ , a contradiction.

- *Case 2:*  $g^s|_{K_0} \neq \text{id}$  and  $g^{2s}|_{K_0} = \text{id}$ .

Let  $x_0 \in K_0 = J_0$  be a  $k$ -point for  $f$  such that  $\text{Orb}_f(x_0) \subset P$ . Then  $g^{2s}(x_0) = x_0$ . Moreover  $x_0$  is an endpoint of  $K_0$  and so  $k = 2s$ . On the other hand, since  $g^{2s}(x) = x$  and  $x$  has period  $m > s$  we have  $2s = m$ . So  $k = m$ , a contradiction with the hypotheses. In short we have proved that the loop  $J_0 \rightarrow J_1 \rightarrow \dots \rightarrow J_{m-1} \rightarrow J_m = J_0$  is non-repetitive.

Suppose that all the basic intervals of the non-repetitive loop of length  $m$  contain  $\mathbf{0}$ . Therefore  $\text{Orb}_g(x)$  is contained in the branching intervals. Since  $m > 6$ , there is a basic interval  $J_i$  containing at least two points of  $\text{Orb}_g(x)$ . Let  $u, v \in \text{Orb}_g(x) \cap J_i$  such that  $(\mathbf{0}, v) \cap \text{Orb}_g(x) = \emptyset$ , and  $(u, v) \cap \text{Orb}_g(x) = \emptyset$ . Since the loop is non-repetitive, there is  $J_j \neq J_i$  such that  $J_j \cap \text{Orb}_g(x) \neq \emptyset$ . Let  $z \in J_j \cap \text{Orb}_f(x)$  such that  $(z, \mathbf{0}) \cap P' = \emptyset$ . Therefore there is  $r$ ,  $0 < r < m$  such that  $g^r(u) = z$  and  $g^{m-r}(z) = u$ .

On the other hand  $g^r|_{[u, \mathbf{0}]}$  is linear and so  $g^r|_{[u, \mathbf{0}]} = [z, \mathbf{0}]$ . Furthermore  $v \in (u, \mathbf{0})$  and so  $g^r(v) \in (z, \mathbf{0})$  in contradiction with the fact that  $(z, \mathbf{0}) \cap \text{Orb}_g(x) = \emptyset$ .  $\square$

**COROLLARY 4.3.** — *Let  $g$  be a  $P'$ -linearization of  $f$ . If  $m \in \text{Per}(g)$  and  $m \notin \{2, 3, 4, 5, 6, k, 2k\}$ , then  $m \in \text{Per}(f)$ .*

*Proof.* — Both  $E$  maps  $f$  and  $g$  have points of periods 1 and  $k$ . If  $m \notin \{1, 2, 3, 4, 5, 6, k\}$ , then by Proposition 4.2 there exists a non-repetitive loop in the  $g$ -graph of length  $m$  such that at least one of its basic intervals does not contain  $\mathbf{0}$ . Therefore, since the  $g$ -graph of  $P'$  is a subgraph of the  $f$ -graph of  $P'$  and  $m \neq 2k$ , by Proposition 3.4,  $f$  has a periodic point of period  $m$ .  $\square$

**Remark 4.4.** — Suppose that  $f$  is  $P'$ -linear. Then each branching interval  $f$ -covers exactly one branching interval, and perhaps some non-branching intervals. Moreover each non-branching interval  $f$ -covers either zero or two branching intervals.

### 5. Preliminary results in $I_2, I_3, I_4, I_5, I_6, O, O_1, O_2, O_3$ and $O_4$ .

We need to introduce some orderings in the set of natural numbers, adding or removing some few elements.

The *Sharkovskii ordering*  $>_s$  on the set  $\mathbb{N}_s = \mathbb{N} \cup \{2^\infty\}$  is given by

$$\begin{aligned} 3 &>_s 5 >_s 7 >_s \cdots >_s \\ 2 \cdot 3 &>_s 2 \cdot 5 >_s 2 \cdot 7 >_s \cdots >_s \\ 2^2 \cdot 3 &>_s 2^2 \cdot 5 >_s 2^2 \cdot 7 >_s \cdots >_s \\ 2^n \cdot 3 &>_s 2^n \cdot 5 >_s 2^n \cdot 7 >_s \cdots >_s \\ 2^\infty &>_s \cdots >_s 2^n >_s \cdots >_s 2^4 >_s 2^3 >_s 2^2 >_s 2 >_s 1. \end{aligned}$$

We shall use the symbol  $\geq_s$  in the natural way. The symbol  $2^\infty$  ensures the existence of supremum of every subset with respect to the ordering  $>_s$ . For  $n \in \mathbb{N}_s$  we denote

$$S(n) = \{k \in \mathbb{N} : n \geq_s k\}.$$

So

$$S(2^\infty) = \{2^i : i = 0, 1, 2, \dots\}.$$

Now we state the Sharkovskii Theorem [Sh] (see also [St], [BGMY] and [ALM2]).

THEOREM 5.1 (Interval Theorem).

(a) If  $f$  is an interval map, then  $\text{Per}(f) = S(n)$  for some  $n \in \mathbb{N}_s$ .

(b) If  $n$  is an element of  $\mathbb{N}_s$  then there exists an interval map  $f$  such that  $\text{Per}(f) = S(n)$ .

If we want to get a similar result for the space  $\mathbf{Y}$ , we need two new orderings. The *green ordering*  $>_g$  on  $\mathbb{N} \setminus \{2\}$  is given by

$$\begin{aligned} 5 &>_g 8 >_g 4 >_g 11 >_g 14 >_g 7 >_g 17 >_g 20 >_g 10 >_g \cdots >_g \\ 3 \cdot 3 &>_g 3 \cdot 5 >_g 3 \cdot 7 >_g \cdots >_g \\ 3 \cdot 2 \cdot 3 &>_g 3 \cdot 2 \cdot 5 >_g 3 \cdot 2 \cdot 7 >_g \cdots >_g \\ 3 \cdot 2^2 \cdot 3 &>_g 3 \cdot 2^2 \cdot 5 >_g 3 \cdot 2^2 \cdot 7 >_g \cdots >_g \\ 3 \cdot 2^3 &>_g 3 \cdot 2^2 >_g 3 \cdot 2 >_g 3 \cdot 1 >_g 1. \end{aligned}$$

The *red ordering*  $>_r$  on  $\mathbb{N} \setminus \{2, 4\}$  is given by

$$\begin{aligned} 7 &>_r 10 >_r 5 >_r 13 >_r 16 >_r 8 >_r 19 >_r 22 >_r 11 >_r \cdots >_r \\ 3 \cdot 3 &>_r 3 \cdot 5 >_r 3 \cdot 7 >_r \cdots >_r \\ 3 \cdot 2 \cdot 3 &>_r 3 \cdot 2 \cdot 5 >_r 3 \cdot 2 \cdot 7 >_r \cdots >_r \\ 3 \cdot 2^2 \cdot 3 &>_r 3 \cdot 2^2 \cdot 5 >_r 3 \cdot 2^2 \cdot 7 >_r \cdots >_r \\ 3 \cdot 2^3 &>_r 3 \cdot 2^2 >_r 3 \cdot 2 >_r 3 \cdot 1 >_r 1. \end{aligned}$$

For  $n \in \mathbb{N} \setminus \{2\}$  denote

$$G(n) = \{k \in \mathbb{N} : n \geq_g k\},$$

for  $n \in \mathbb{N} \setminus \{2, 4\}$  denote

$$R(n) = \{k \in \mathbb{N} : n \geq_r k\}$$

and additionally

$$G(3 \cdot 2^\infty) = R(3 \cdot 2^\infty) = \{1\} \cup \{3n : n \in S(2^\infty)\}.$$

We also denote

$$\mathbb{N}_g = (\mathbb{N} \setminus \{2\}) \cup \{3 \cdot 2^\infty\} \quad \text{and} \quad \mathbb{N}_r = (\mathbb{N} \setminus \{2, 4\}) \cup \{3 \cdot 2^\infty\}.$$

The following theorem is due to Alsedà, Llibre and Misiurewicz [ALM1] for  $\mathbf{I}_3$  maps.

THEOREM 5.2 ( $\mathbf{I}_3$  Theorem).

(a) If  $f$  is an  $\mathbf{I}_3$  map, then  $\text{Per}(f) = S(n_s) \cup G(n_g) \cup R(n_r)$  for some  $n_s \in \mathbb{N}_s, n_g \in \mathbb{N}_g$  and  $n_r \in \mathbb{N}_r$ .

(b) If  $n_s \in \mathbb{N}_s, n_g \in \mathbb{N}_g$  and  $n_r \in \mathbb{N}_r$ , then there exists an  $\mathbf{I}_3$  map  $f$  such that  $\text{Per}(f) = S(n_s) \cup G(n_g) \cup R(n_r)$ .

Let  $\mathbf{I}_n$  be the  $n$ -od space define as the set  $\{z \in \mathbb{C} : z^n \in [0, 1]\}$ . In order to obtain a generalization of the Sharkovskii Theorem for  $\mathbf{I}_n$  we need to define partial ordering  $\leq_n$  for  $n \geq 1$ . The ordering  $\geq_1$  is the ordering  $\geq_s$ . If  $n > 1$  then the ordering  $\leq_n$  is defined as follows. Let  $m, k$  be positive integers.

- *Case 1:*  $k = 1$ . Then  $m \leq_n k$  if and only if  $m = 1$ .
- *Case 2:*  $k$  is divisible by  $n$ . Then  $m \leq_n k$  if and only if either  $m = 1$  or  $m$  is divisible by  $n$  and  $m/n >_s k/n$ .
- *Case 3:*  $k > 1$ ,  $k$  not divisible by  $n$ . Then  $m \leq_n k$  if and only if either  $m = 1$ ,  $m = k$ , or  $m = ik + jn$  for some integers  $i \geq 0, j \geq 1$ .

From the definition we have that  $\leq_2$  is the Sharkovskii ordering. A set  $Z$  is an *initial segment* of  $\leq_n$  if whenever  $k$  is an element of  $Z$  and  $m \leq_n k$ , then  $m$  also belongs to  $Z$ ; i.e.  $Z$  is closed under  $\leq_n$  predecessors. The following result of Baldwin [Ba] is a generalization of the Sharkovskii Theorem and the  $\mathbf{I}_3$  Theorem for arbitrary continuous self-maps of  $\mathbf{I}_n$ .

THEOREM 5.3 ( $n$ -od Theorem).

(a) Let  $f$  be a continuous self-map of  $\mathbf{I}_n$ . Then  $\text{Per}(f)$  is a nonempty union of initial segments of  $\{\leq_p : 1 \leq p \leq n\}$ .

(b) If  $Z$  is a nonempty finite union of initial segments of  $\{\leq_p : 1 \leq p \leq n\}$ , then there is a continuous self-map of  $\mathbf{I}_n$   $f$  such that  $f(\mathbf{0}) = \mathbf{0}$  and  $\text{Per}(f) = Z$ .

The  $n$ -od Theorem has been extended by Baldwin and Llibre in [BL] to continuous maps on a tree having all their branching points fixed.

We define the *Block ordering*  $>_0$  on  $\mathbb{N}$  as the converse of the natural ordering on  $\mathbb{N} \setminus \{1\}$  and we add the 1 as the smallest element; i.e.  $2 >_0 3 >_0 4 >_0 \dots >_0 1$ . For  $n \in \mathbb{N}$ , we denote

$$B(n) = \{k \in \mathbb{N} : n \geq_0 k\}.$$

Sharkovskii Theorem has been generalized by Block to the circle maps having fixed points in [Bc2].

THEOREM 5.4 (Circle Theorem).

(a) *If  $f$  is a circle map having fixed points, then  $\text{Per}(f) = S(n_s) \cup B(n_b)$  for some  $n_s \in \mathbb{N}_s$  and  $n_b \in \mathbb{N}$ .*

(b) *If  $n_s \in \mathbb{N}_s$  and  $n_b \in \mathbb{N}$ , then there exists a circle map  $f$  having fixed points such that  $\text{Per}(f) = S(n_s) \cup B(n_b)$ .*

In [LPR2], [LPR3] the Sharkovskii Theorem has been extended to connected graphs  $G$  with zero Euler characteristic having all branching points fixed. Given a graph  $G$ , let  $e(G)$  and  $b(G)$  the number of its endpoints and branching points respectively.

THEOREM 5.5 (Graph Theorem). — *Let  $G$  be a connected graph such that  $b(G) \neq 0$  and  $\chi(G) = 0$ .*

(a) *Let  $f : G \rightarrow G$  be a continuous map with all branching points fixed. Then  $\text{Per}(f)$  is a nonempty finite union of initial segments of  $\{\leq_p : 0 \leq p \leq e(G) + 2\}$ .*

(b) *If  $Z$  is a nonempty finite union of initial segments of*

$$\{\leq_p : 0 \leq p \leq e(G) + 2\},$$

*then there is a continuous map  $f : G \rightarrow G$  with all the branching points fixed such that  $\text{Per}(f) = Z$ .*

We note that if  $G$  is a connected graph such that  $\chi(G) = 0$  and  $b(G) = 0$ , then  $G$  is homeomorphic to  $\mathbf{O}$ . The set of periods for continuous self-maps on  $\mathbf{O}$  wich have fixed points is characterized in the Circle Theorem.

## 6. The full periodicity kernel of $\mathbf{O}_3$ .

The objective of this section is to prove Theorem 1.2.

Since  $\mathbf{I}_5$  is homeomorphic to  $\{z \in \mathbf{O}_3 : \text{Im } z \geq -1\}$ , we can consider  $\mathbf{I}_5 = \{z \in \mathbf{O}_3 : \text{Im } z \geq -1\}$ . Let  $f$  an  $\mathbf{I}_5$  map. We shall extend  $f$  to an  $\mathbf{O}_3$  map  $\bar{f}$  as follows. We define  $\bar{f}(z) = f(z)$  if  $z \in \mathbf{I}_5$  and  $f$  restricted to  $\text{Cl}(\mathbf{O}_3 \setminus \mathbf{I}_5)$  is any homeomorphism between  $\text{Cl}(\mathbf{O}_3 \setminus \mathbf{I}_5)$  and the unique closed interval in  $\mathbf{I}_5$  having  $f(1 - i)$  and  $f(-1 - i)$  as endpoints. Of course  $\text{Per}(f) = \text{Per}(\bar{f})$ . From Theorem 1.1 (d) it follows that  $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 16, 17, 18, 21, 23\}$  is a subset of the full periodicity kernel of  $\mathbf{O}_3$ . Then, to prove Theorem 1.2 it is sufficient to show the following proposition.

PROPOSITION 6.1. — *Let  $f$  be an  $\mathbf{O}_3$  map. Then the following statements hold:*

(a) *If  $7 \in \text{Per}(f)$ , then*

$$\mathbb{N} \setminus \{2, 3, 4, 5, 6, 8, 9, 10, 11, 13, 14, 16, 17, 18, 21, 23, 28\} \subset \text{Per}(f).$$

(b) *If  $13 \in \text{Per}(f)$ , then  $28 \in \text{Per}(f)$ .*

*Proof.* — Note that  $\chi(\mathbf{O}_3) = 0$ ,  $b(\mathbf{O}_3) = 1$  and  $e(\mathbf{O}_3) = 3$ . From the Graph Theorem, the set of periods of  $f$  is a nonempty finite union of initial segments of  $\{\leq_p; 0 \leq p \leq 5\}$ . Now we shall compute the periods forced by the periods 7 and 13 in the orderings of  $\mathbf{O}_3$ .

From the definition of the orderings  $\leq_p$ , we have that

$$7 \geq_0 n \text{ for each } n \in \mathbb{N} \setminus \{2, 3, 4, 5, 6\};$$

$$7 \geq_s n \text{ for each } n \in \mathbb{N} \setminus \{3, 5\};$$

$$7 \geq_3 n \text{ for each } n \in \mathbb{N} \setminus \{2, 4, 5, 8, 11, 14\};$$

$$7 \geq_4 n \text{ for each } n \in \mathbb{N} \setminus \{2, 3, 5, 6, 9, 10, 13, 14, 17, 21\};$$

$$7 \geq_5 n \text{ for each } n \in \mathbb{N} \setminus \{2, 3, 4, 6, 8, 9, 11, 13, 14, 16, 18, 21, 23, 28\}.$$

Therefore, if  $7 \in \text{Per}(f)$  then

$$\mathbb{N} \setminus \{2, 3, 4, 5, 6, 8, 9, 10, 11, 13, 14, 16, 17, 18, 21, 23, 28\} \subset \text{Per}(f)$$

and statement (a) holds.

On the other hand,  $13 >_p 28$  for  $0 \leq p \leq 5$ . Consequently if  $13 \in \text{Per}(f)$ , then  $28 \in \text{Per}(f)$  and statement (b) holds.  $\square$

## 7. The full periodicity kernel of $\mathbf{O}_4$ .

The objective of this section is to prove Theorem 1.3.

Since  $\mathbf{I}_6$  is homeomorphic to  $\{z \in \mathbf{O}_4 : \text{Im } z \geq -1\}$ , we can consider  $\mathbf{I}_6 = \{z \in \mathbf{O}_4 : \text{Im } z \geq -1\}$ . Let  $f$  an  $\mathbf{I}_6$  map. We shall extend  $f$  to an  $\mathbf{O}_4$  map  $\bar{f}$  as follows. We define  $\bar{f}(z) = f(z)$  if  $z \in \mathbf{I}_6$  and  $\bar{f}$  restricted to  $\text{Cl}(\mathbf{O}_4 \setminus \mathbf{I}_6)$  is any homeomorphism between  $\text{Cl}(\mathbf{O}_4 \setminus \mathbf{I}_6)$  and the unique closed interval of  $\mathbf{I}_6$  having  $f(1 - i)$  and  $f(-1 - i)$  as endpoints. Of course  $\text{Per}(f) = \text{Per}(\bar{f})$ . From Theorem 1.1 (e) it follows that  $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18, 21, 22, 23, 28, 29\}$  is a subset of the full periodicity kernel of  $\mathbf{O}_4$ . Then, Theorem 1.3 is a corollary of the following proposition.

PROPOSITION 7.1. — *Let  $f$  be an  $\mathbf{O}_4$  map. Then the following statements hold:*

(a) *If  $7 \in \text{Per}(f)$ , then*

$$\{2, 3, 4, 5, 6, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18, 21, 22, 23, 28, 29, 35\} \subset \text{Per}(f).$$

(b) *If  $11 \in \text{Per}(f)$ , then  $35 \in \text{Per}(f)$ .*

*Proof.* — Since  $\chi(\mathbf{O}_4) = 0$ ,  $b(\mathbf{O}_4) = 1$  and  $e(\mathbf{O}) = 4$ , by the Graph Theorem it follows that  $\text{Per}(f)$  is a nonempty union of initial segments of  $\{\leq_p: 0 \leq p \leq 6\}$ . We have that  $7 \geq_6 n$  for each  $n \in \mathbb{N} \setminus \{2, 3, 4, 5, 8, 9, 10, 11, 14, 15, 16, 17, 21, 22, 23, 28, 29, 35\}$ . Therefore, from the proof of Proposition 6.1 we obtain that if  $7 \in \text{Per}(f)$ , then  $\mathbb{N} \setminus \{2, 3, 4, 5, 6, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18, 21, 22, 23, 28, 29, 35\} \subset \text{Per}(f)$  and statement (a) follows.

On the other hand,  $11 >_p 35$  for  $0 \leq p \leq 6$ . Consequently if  $11 \in \text{Per}(f)$ , then  $35 \in \text{Per}(f)$  and statement (b) holds.  $\square$

### 8. The unfolding of $\infty_1, \infty_2$ and $\mathbf{T}$ .

If we identify the endpoints of the segment  $[0, 1]$  to the point  $\mathbf{0}$ , then we obtain a space homeomorphic to  $\mathbf{O}$ .

We represent the cartesian product  $\mathbf{O} \times \mathbf{O}$  (the torus) as the square  $[0, 1] \times [0, 1]$  identifying the points  $(x, 0)$  and  $(x, 1)$  for all  $x \in [0, 1]$ , and the points  $(0, y)$  and  $(1, y)$  for all  $y \in [0, 1]$ . Thus the graph of an  $\mathbf{O}$  map  $f$  is the subset  $\{(x, f(x)) : x \in \mathbf{O}\}$  of  $\mathbf{O} \times \mathbf{O}$ , and it can be represented as in Figure 8.1.

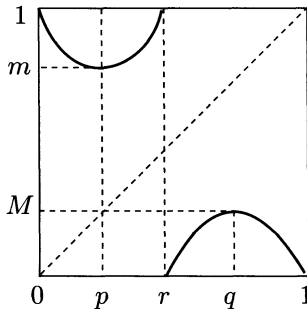


Figure 8.1. The graph of a  $\mathbf{O}$  map.



Roughly speaking, we think the graph of an  $\mathbf{O}$  map like the graph of an interval map  $g$  from  $[0, 1]$  into itself with the above identifications. This allows us to talk about local or absolute maximum or minimum for an  $\mathbf{O}$  map in the same way as for interval maps. Thus, for instance, in the points  $p$  and  $q$  the  $\mathbf{O}$  map represented in Figure 8.1 has a local minimum and maximum with values  $m$  and  $M$  respectively.

Let  $f$  be a  $P'$ -linear  $\mathbf{O}$  map such that  $f(\mathbf{0}) = \mathbf{0}$  and each basic interval associated to  $P'$  does not  $f$ -cover itself. Therefore the graph of  $f$  does not touch the diagonal except at  $\mathbf{0}$ . Let  $V = [a, b]$  a closed subinterval of  $\mathbf{O}$  such that  $f(a) = f(b) = \mathbf{0}$ ,  $f(c) \neq \mathbf{0}$  for all  $c \in (a, b)$ . Then we say that  $V$  is an *upper* (respectively *down*) *subinterval* according with they contain more local minima (respectively maxima) than local maxima (respectively minima) of  $f$ . Since  $f$  is  $P'$ -linear these upper and down subintervals are well-defined. Thus for instance the subinterval  $[0, r]$  is an upper subinterval of the map  $f$  of Figure 8.1.

*In the rest of this section we shall consider  $E \in \{\infty_1, \infty_2, \mathbf{T}\}$  and  $f$  will be a  $P'$ -linear map such that each basic interval associated to  $P'$  does not  $f$ -cover itself and  $k$  will be the period of  $P$ . We identify  $\mathbf{O}$  with a circle of  $E$  and  $\mathbf{0} \in \mathbf{O}$  with  $\mathbf{0} \in E$ .*

Let  $V = [a, b]$  be a closed subinterval of  $E$  contained in a circle or in a whiskers of  $E$  such that  $f(a) = f(b) = \mathbf{0}$ ,  $f(c) \neq \mathbf{0}$  for all  $c \in (a, b)$  and  $f(V) \not\subseteq \mathbf{O}$ . Then in a similar way as for  $\mathbf{O}$  maps, we can say as above that  $V$  is an *upper* or *down subinterval*.

Let  $V \subset E$  be contained in a whiskers or in a circle of  $E$ . We say that  $V$   *$f$ -covers*  $\mathbf{O}$  (or  $V \rightarrow \mathbf{O}$ , or  $\mathbf{O} \leftarrow V$ ) if one of the following statements holds:

- (1) There exists  $[a, b] \subset V$  with  $f(a) = f(b) = \mathbf{0}$ ,  $f(c) \neq \mathbf{0}$  for all  $c \in (a, b)$  and  $f([a, b]) = \mathbf{O}$ .
- (2) The set  $V$  is a circle of  $E$  such that  $f(V) = \mathbf{O}$  and  $f(x) \neq \mathbf{0}$  for all  $x \neq \mathbf{0}$ .

Moreover, if (1) occurs with  $V = [a, b]$  or (2) occurs, then we say that  $V$  is a *crossing subset* of  $\mathbf{O}$ . If  $V$  does not  $f$ -cover  $\mathbf{O}$ , then we write  $V \rightarrow \mathbf{O}$ .

*Remark 8.1.* — In a similar way as in Lemma 3.1, if  $K$  and  $L$  are closed subintervals of  $E$  such that  $L \subset \mathbf{O}$ ,  $K \rightarrow \mathbf{O}$  and  $\mathbf{0} \notin \text{Int}(L)$ , then  $K \rightarrow L$ .

In this section we also assume that  $E$  has no crossing subsets of  $\mathbf{O}$ . Then following ideas of [LPR3] and [LL1] we define the *unfolding* of  $\infty_1$  as the graph  $\infty_1^* = G_1 \cup G_2 \cup G_3$  where

$$\begin{aligned} G_1 &= \{(z, t) \in \mathbb{C} \times \mathbb{R} : t = 0 \text{ and } |z - i| = 1 \text{ or } z \in \mathbf{I}_2, \operatorname{Re} z \geq 0\}, \\ G_2 &= \{(z, t) \in \mathbb{C} \times \mathbb{R} : t = 0, |z + i| = 1\}, \\ G_3 &= \{(z, t) \in \mathbb{C} \times \mathbb{R} : t = |\operatorname{Im} z|, |z + i| = 1\}. \end{aligned}$$

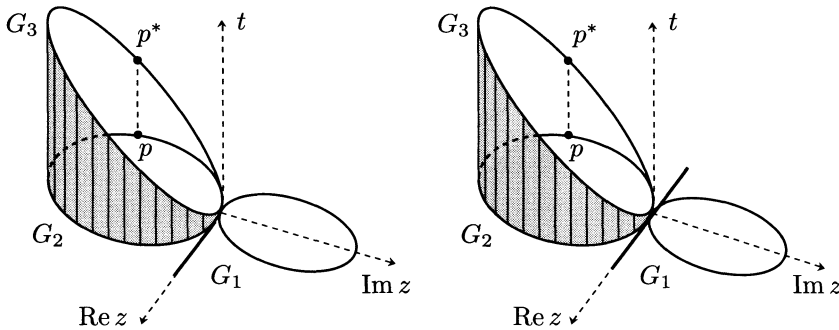


Figure 8.2. The unfoldings of  $\infty_1$  and  $\infty_2$ .

Define the *unfolding* of  $\infty_2$  as the graph  $\infty_2^* = G_1 \cup G_2 \cup G_3$  where

$$\begin{aligned} G_1 &= \{(z, t) \in \mathbb{C} \times \mathbb{R} : t = 0 \text{ and } |z - i| = 1 \text{ or } z \in \mathbf{I}_2\}, \\ G_2 &= \{(z, t) \in \mathbb{C} \times \mathbb{R} : t = 0, |z + i| = 1\}, \\ G_3 &= \{(z, t) \in \mathbb{C} \times \mathbb{R} : t = |\operatorname{Im} z|, |z + i| = 1\}. \end{aligned}$$

Define the *unfolding* of the trefoil as the graph  $\mathbf{T}^* = G_1 \cup G_2 \cup G_3$  where

$$\begin{aligned} G_1 &= \{(z, t) \in \mathbb{C} \times \mathbb{R} : t = 0, z = \cos(3\theta) e^{i\theta}, \frac{1}{6}\pi \leq \theta \leq \frac{11}{6}\pi\}, \\ G_2 &= \{(z, t) \in \mathbb{C} \times \mathbb{R} : t = 0, z = \cos(3\theta) e^{i\theta}, -\frac{1}{6}\pi \leq \theta \leq \frac{1}{6}\pi\}, \\ G_3 &= \{(z, t) \in \mathbb{C} \times \mathbb{R} : t = \operatorname{Re} z, z = \cos(3\theta) e^{i\theta}, -\frac{1}{6}\pi \leq \theta \leq \frac{1}{6}\pi\}. \end{aligned}$$

Clearly in the three cases  $G_2$  and  $G_3$  are homeomorphic to  $\mathbf{O}$ , moreover  $G_1 \cup G_2$  is homeomorphic to  $E$ , so we identify  $\mathbf{O}$  with  $G_2$  and  $G_1 \cup G_2$  with  $E$  (see Figures 8.2 and 8.3). Consider the projection  $\pi : E^* \rightarrow E$  defined by  $\pi(z, t) = (z, 0)$ . We denote by  $p^*$  the unique point of  $G_3$  such that  $\pi(p^*) = p$ .

Since  $f$  is  $P'$ -linear,  $f$  has finitely many local extrema; and consequently finitely many upper and down subintervals. Moreover from the fact that there are no crossing subsets of  $\mathbf{O}$ , it follows that there exists a finite «partition» of  $E$  into upper subintervals, down subintervals and subintervals with image in  $\text{Cl}(E \setminus \mathbf{O})$ . Now for the given  $E$  map  $f$  we define  $f^* : E \rightarrow E^*$  as follows. If  $p \in E$  then  $f^*(p)$  is either  $f(p)^*$  if  $f(p) \in \mathbf{O}$  and  $p$  belongs to an upper subinterval, or  $f(p)$  otherwise. Clearly  $f^*$  is well-defined. We remark that  $f = \pi \circ f^* : E \rightarrow E$ . Define  $F = f^* \circ \pi : E^* \rightarrow E^*$ . In the rest of this section we shall study the relationship between the periods of  $f$  and  $F$ .

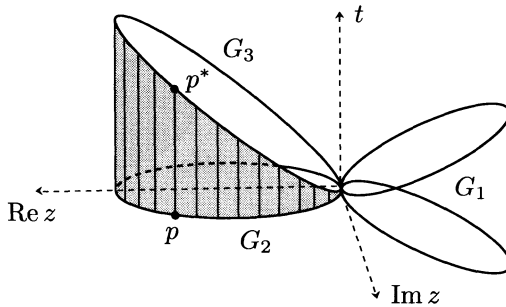


Figure 8.3. The unfolding of the trefoil.

LEMMA 8.2. — Assume that there are no crossing subsets of  $\mathbf{O}$ . If  $q \in E^*$  is a periodic point of  $F$  of period  $n$ , then  $p = \pi(q)$  is a fixed point of  $f^n$ .

Proof. — Since

$$q = F^n(q) = (f^* \circ \pi)^n(q) = f^* \circ (\pi \circ f^*)^{n-1} \circ \pi(q) = f^*(f^{n-1}(p)),$$

we get that  $p = \pi(q) = f^n(p)$ . □

LEMMA 8.3. — Assume that there are no crossing subsets of  $\mathbf{O}$ . Then the following statements hold:

- (a) If  $p = \pi(q)$  is a periodic point of  $f$  of period  $n$ , then  $p = \pi(F^n(q))$ .
- (b) Si  $p \in G_1$  is a periodic point of  $f$  of period  $n$ , then  $p$  is a fixed point of  $F^n$ .

Proof. — Statement (a) follows from the equalities

$$p = \pi(q) = f^n(\pi(q)) = (\pi \circ f^*)^n(\pi(q)) = \pi \circ (f^* \circ \pi)^n(q) = (\pi \circ F^n)(q).$$

If  $p$  is a periodic point of  $f$  of period  $n$ , we have that

$$p = f^n(p) = f^n(\pi(p)) = (\pi \circ f^*)^n(\pi(p)) = \pi \circ (f^* \circ \pi)^n(p) = (\pi \circ F^n)(p).$$

Since  $p \in G_1$ , we get that  $F^n(p) = p$ , and statement (b) is proved.  $\square$

PROPOSITION 8.4. — *Suppose that there are no crossing subsets of  $\mathbf{O}$ . Then the following statements hold:*

- (a) *If  $q$  is an  $n$ -point for  $F$ , then  $p = \pi(q)$  is an  $n$ -point for  $f$ .*
- (b) *If  $p$  is an  $n$ -point for  $f$  and  $p \in G_1$ , then  $p$  is an  $n$ -point for  $F$ .*

*Proof.* — We prove (a). Let  $q$  be an  $n$ -point for  $F$ . By Lemma 8.2,  $p = \pi(q)$  is a fixed point of  $f^n$ . Therefore, there is a divisor  $s$  of  $n$  such that  $p$  is an  $s$ -point for  $f$ . If  $s = n$ , then we are done. So, assume that  $s < n$ . By Lemma 8.3 (a),  $p = \pi(F^s(q))$ . Since  $s < n$ ,  $F^s(q) = p'$  with  $p' \neq q$ , and of course  $p'$  belongs to the  $F$ -periodic orbit of  $q$ . Then

$$\begin{aligned} q &= F^n(q) = (f^* \circ \pi)^n(q) = (f^* \circ \pi)^{n-1} \circ f^*(\pi(q)) \\ &= (f^* \circ \pi)^{n-1} \circ f^*(p) = (f^* \circ \pi)^{n-1} \circ f^*(\pi(F^s(q))) \\ &= (f^* \circ \pi)^n(F^s(q)) = F^n(p') = p', \end{aligned}$$

which is a contradiction. Hence  $s = n$  and (a) is proved.

Now we show (b). Let  $p$  be an  $n$ -point for  $f$  and  $p \in G_1$ . By Lemma 8.3 (b),  $p = F^n(p)$ . Again, there is a divisor  $s$  of  $n$  such that  $p$  is an  $s$ -point for  $F$ . If  $s = n$ , then we are done. So, assume that  $s < n$ . Then  $F^s(p) = p$ . By Lemma 8.2, since  $p \in G_1$  we get that  $p = f^s(p)$ , a contradiction. Then the lemma follows.  $\square$

## 9. More results in $\infty$ , $\infty_1$ , $\infty_2$ and $\mathbf{T}$ .

Now we add some results for  $P'$ -linear maps which we will use for the computation of the full periodicity kernel of  $\infty_1$ ,  $\infty_2$  and  $\mathbf{T}$ . The following proposition follows from Section 13 of [LL2].

PROPOSITION 9.1. — *Let  $f$  be an  $\infty$  map. The following statements hold:*

- (a) *If  $7 \in \text{Per}(f)$ , then*

$$\text{Per}(f) \supset \{2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 17, 21\}.$$

- (b) If  $11 \in \text{Per}(f)$ , then  $35 \in \text{Per}(f)$ .
- (c) If  $13 \in \text{Per}(f)$ , then  $28 \in \text{Per}(f)$ .

If  $U$  is a finite subset of  $E$ , we shall denote by  $\text{Card}(U)$  the cardinal of  $U$ .

PROPOSITION 9.2. — *Let  $E \in \{\mathbf{I}_3, \mathbf{I}_4, \mathbf{I}_5, \mathbf{I}_6, \mathbf{O}_2, \mathbf{O}_3, \mathbf{O}_4, \infty_1, \infty_2, \mathbf{T}\}$ . Let  $f$  be an  $E$  map having a  $k$ -orbit  $P$ . Suppose that  $f$  is  $P'$ -linear. Assume that each basic interval is  $f$ -covered by some basic interval different from itself and that there is a basic interval  $J_0$  such that  $J_0 \rightarrow J_0$ . Then  $\{n \in \mathbb{N} : n \geq k + 3\} \setminus \{2k\} \subset \text{Per}(f)$ .*

*Proof.* — We denote by  $S$  the set of all basic intervals associated to  $P'$ . Notice that  $\text{Card}(S) = k$  if  $E$  is any  $n$ -star,  $\text{Card}(S) = k + 1$  if  $E \in \{\mathbf{O}_2, \mathbf{O}_3, \mathbf{O}_4\}$ ,  $\text{Card}(S) = k + 2$  if  $E \in \{\infty_1, \infty_2\}$ , and  $\text{Card}(S) = k + 3$  if  $E = \mathbf{T}$ . Since each basic interval is  $f$ -covered by some basic interval we get that  $f(E) = E$ .

Set  $K_i = f^i(J_0)$  for  $i \geq 0$ . Note that each  $K_i$  is a connected set and  $\text{Card}(K_1 \cap P) \geq 2$ . We consider two cases.

- *Case 1:*  $E \in \{\mathbf{I}_3, \mathbf{I}_4, \mathbf{I}_5, \mathbf{I}_6, \mathbf{O}_2, \mathbf{O}_3, \mathbf{O}_4\}$ .

From the fact that  $P$  is a periodic orbit and  $f(E) = E$ , it follows that there exists an integer  $r$  such that  $K_0 \subsetneq K_1 \subsetneq \dots \subsetneq K_r = E$ , and  $\text{Card}(K_i \cap P) \geq i + 1$  for  $i < r$ . Since  $P$  has period  $k$  we have that  $r \leq \text{Card}(K_{r-1} \cap P) \leq k$ . Since each basic interval is  $f$ -covered by some basic interval different from itself, for each  $J_i \in S$ ,  $J_i \subset K_i \setminus K_{i-1}$  there exists  $J_{i-1} \in S$ ,  $J_{i-1} \subset K_{i-1} \setminus K_{i-2}$  such that  $J_{i-1} \rightarrow J_i$ . By hypothesis there exists  $M \in S$ ,  $M \neq J_0$  such that  $M \rightarrow J_0$ . Hence there is a loop of length  $\ell \leq r + 1 \leq k + 1$  containing  $J_0$ . By construction, this loop is formed by pairwise different basic intervals and so is non-repetitive. The above loop of length  $\ell$  together with the loop  $J_0 \rightarrow J_0$  give us a non-repetitive loop of length  $n$  for each  $n \geq k + 1$  containing  $J_0$ .

We claim that at least one of the intervals of the above loop of length  $n$  does not contain  $\mathbf{0}$ . If  $\mathbf{0} \notin J_0$ , then we are done. So suppose that  $\mathbf{0} \in J_0$ . Since  $J_0 \rightarrow J_0$ ,  $f(\mathbf{0}) = \mathbf{0}$  and  $f$  is  $P'$ -linear we get that the basic intervals different from  $J_0$  of  $K_1$  do not contain  $\mathbf{0}$  (see Remark 4.4). So the claim is proved. Hence by Proposition 3.4 the result follows.

- *Case 2:*  $E \in \{\infty_1, \infty_2, \mathbf{T}\}$ .

From the facts that  $P$  is a periodic orbit,  $K_i$  is a connected

set and  $f(E) = E$ , we have that there exists an integer  $r$  such that  $K_0 \subsetneq K_1 \subsetneq \dots \subsetneq K_r = E'$ , and one of the following statements holds:

- (1)  $E' = E$ ;
- (2)  $E \in \{\infty_1, \infty_2\}$ ,  $E'$  is homeomorphic to one of the spaces  $\mathbf{I}_3, \mathbf{I}_4, \mathbf{I}_5, \mathbf{I}_6$ , and  $E \setminus \text{Int}(E')$  is formed by two basic intervals  $J_1, J_2$  contained in different circles of  $E$  such that  $J_1 \rightleftharpoons J_2$ ;
- (3)  $E = \mathbf{T}$ ,  $E'$  is homeomorphic to one of the spaces  $\mathbf{O}_2, \mathbf{O}_3, \mathbf{O}_4$ , and  $E \setminus \text{Int}(E')$  is formed by two basic intervals  $J_1, J_2$  contained in different circles of  $E$  such that  $J_1 \rightleftharpoons J_2$ ;
- (4)  $E = \mathbf{T}$ ,  $E'$  is homeomorphic to one of the spaces  $\mathbf{I}_3, \mathbf{I}_4, \mathbf{I}_5, \mathbf{I}_6$ , and  $E \setminus \text{Int}(E')$  is formed by three basic intervals  $J_1, J_2, J_3$  contained in different circles of  $E$  such that  $J_1 \rightarrow J_2 \rightarrow J_3 \rightarrow J_1$ .

First we suppose that statement (1) holds. We remark that if  $r \leq k$ , then the result follows as in Case 1. So, since  $\text{Card}(S) \in \{k + 2, k + 3\}$ , we can assume that  $r \in \{k + 1, k + 2\}$ . In the same way as in Case 1 we obtain a loop of length  $\ell \leq r + 1 \leq k + 3$  containing  $J_0$  and consequently  $\{n \in \mathbb{N} : n \geq k + 3\} \setminus \{2k\} \subset \text{Per}(f)$ .

Finally we assume that statement (2), (3) or (4) holds. Note that  $P \subset E'$ . Consider the  $E'$  map  $g$  defined as  $g = f|_{E'}$ . Clearly  $g$  is well-defined because  $f$  is  $P'$ -linear. Of course  $g$  is either an  $\mathbf{I}_i$  map for  $i = 3, \dots, 6$ , or an  $\mathbf{O}_j$  map for  $j = 2, 3, 4$ . Moreover  $\text{Per}(g) \subset \text{Per}(f)$ . Then the result follows as in Case 1. □

The next lemmas will be used in Sections 10, 11 and 12.

**LEMMA 9.3.** — *Set  $E \in \{\infty_1, \infty_2\}$ . Let  $f$  be an  $E$  map having a  $k$ -orbit  $P$ . Suppose that  $f$  is  $P'$ -linear. Then each basic interval  $J$  contained in a whiskers of  $E$  is  $f$ -covered by some basic interval different from itself.*

*Proof.* — Let  $p \neq \mathbf{0}$  be the endpoint of the whiskers of  $E$  containing  $J$ . Since  $f$  is  $P'$ -linear, we have that  $p \in P$ . Moreover, from the facts that  $\mathbf{0}$  is a fixed point,  $f$  is  $P'$ -linear and  $f(E)$  is connected, it follows that each basic interval of  $E$  contained in the whiskers of  $E$  is  $f$ -covered by some basic interval.

Suppose that  $J \rightarrow J$ , otherwise we are done. Since  $[p, \mathbf{0}]$  is a whiskers of  $E$ , we can consider a total ordering  $<$  on  $[p, \mathbf{0}]$  such that  $\mathbf{0}$  is the largest element and  $p$  the smallest one. Set  $J = [p_j, p_k]$ , with  $p \leq p_j < p_k \leq \mathbf{0}$ . Now, since  $f$  is  $P'$ -linear we can consider two cases.

- *Case 1:*  $p \leq f(p_j) < p_j < p_k$  and  $f(p_k) \notin [p, p_k]$ .

If there are no basic intervals  $K \neq J$  such that  $K \rightarrow J$ , then  $f(P \cap [p, p_j]) \subset P \cap [p, p_j]$  with  $P \cap [p, p_j] \neq \emptyset$ . This is a contradiction because  $P$  is a periodic orbit not contained into  $[p, \mathbf{0}]$ .

- *Case 2:*  $p \leq f(p_k) \leq p_j < p_k$  and  $f(p_j) \notin [p, p_k]$ .

Then  $p_k < \mathbf{0}$ , and clearly

$$f([p_k, \mathbf{0}]) \supset [f(p_k), f(\mathbf{0})] \supset [f(p_k), \mathbf{0}] \supset [p_j, \mathbf{0}] \supset J.$$

Therefore, there is a basic interval  $J_1 \subset [p_k, \mathbf{0}]$  which  $f$ -covers  $J$  and  $J_1 \neq J$ . □

LEMMA 9.4. — *Set  $E \in \{\infty_1, \infty_2\}$ . Let  $f$  be an  $E$  map having a  $k$ -orbit  $P$ . Suppose that  $f$  is  $P'$ -linear. If  $J_0$  is a closed subinterval of  $E$  with endpoints elements of  $P'$  and contained in a whiskers of  $E$ , then there is a loop of length  $k$  in the  $f$ -graph containing  $J_0$  formed by closed subintervals of  $E$ .*

*Proof.* — Let  $J_0 = [x, y]$  with  $x, y \in P'$  and  $[x, y]$  contained in a whiskers of  $E$ . For each  $i$ ,  $0 < i \leq k$ , we define  $J_i$  recursively as the closed subinterval with endpoints  $f^i(x)$  and  $f^i(y)$  such that  $J_{i-1} \rightarrow J_i$ . Then  $J_k = J_0$  because  $J_0$  is contained in a whiskers. Then we have the loop  $J_0 \rightarrow J_1 \rightarrow \dots \rightarrow J_k = J_0$  of length  $k$ . Of course, in general the intervals  $J_i$  are not basic and the loop can be repetitive or non-repetitive. □

LEMMA 9.5. — *Set  $E \in \{\infty_1, \infty_2, \mathbf{T}\}$ . Let  $f$  be an  $E$  map having a  $k$ -orbit  $P$ . Suppose that  $f$  is  $P'$ -linear. Let  $J$  and  $K$  be basic intervals (eventually  $J = K$ ) such that  $J$   $f^m$ -covers  $K$  for some  $m \geq 1$ . Then there is a path of length  $m$  starting at  $J$  and ending at  $K$ .*

*Proof.* — If  $m = 1$  it is trivial. So suppose that  $m > 1$ . For  $1 < i \leq m$ , given  $J_i \in S$ ,  $J_i \subset f^i(J)$ , since  $f$  is  $P'$ -linear, we can select  $J_{i-1} \in S$  such that  $J_{i-1} \subset f^{i-1}(J)$  and  $J_{i-1} \rightarrow J_i$ . Then, by induction assumption, the path  $J_0 = J \rightarrow J_1 \rightarrow \dots \rightarrow J_{m-1} \rightarrow J_m = K$  proves the lemma. □

LEMMA 9.6. — *Set  $E \in \{\infty_1, \infty_2, \mathbf{T}\}$ . Let  $f$  be an  $E$  map having a  $k$ -orbit  $P$  with  $k \in \{7, 11, 13\}$ . Suppose that  $f$  is  $P'$ -linear and that each basic interval associated to  $P'$  is  $f$ -covered by some different basic interval. Let  $J$  and  $K$  be basic intervals. Then at least one of the following statements holds:*

(a) If  $E = \infty_1$ , then

$$\mathbb{N} \setminus \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16, 17, 18, 21, 23\} \subset \text{Per}(f).$$

If  $E = \infty_2$ , then

$$\mathbb{N} \setminus \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 21, 22, 23, 28, 29\} \subset \text{Per}(f).$$

If  $E = \mathbf{T}$ , then

$$\mathbb{N} \setminus \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 21, 22, 23, 28, 29\} \subset \text{Per}(f).$$

(b) There is a path of length  $m$  starting at  $J$  and ending at  $K$ , where  $1 \leq m \leq k + 1$  if  $E \in \{\infty_1, \infty_2\}$  and  $1 \leq m \leq k + 2$  if  $E = \mathbf{T}$ .

*Proof.* — Since each basic interval is  $f$ -covered by some basic interval, we get that  $f(E) = E$ . Set  $K_i = f^i(J)$  for  $i \geq 0$ . Moreover since  $P$  is a periodic orbit, there is an integer  $r \geq 1$  such that

$$\bigcup_{i=0}^r K_i = \bigcup_{i=0}^{r+1} K_i = E' \neq \bigcup_{i=0}^{r-1} K_i,$$

and one of the following statements holds:

(1)  $E' = E$ ;

(2)  $E \in \{\infty_1, \infty_2\}$ ,  $E'$  is homeomorphic to one of the spaces  $\mathbf{I}_3, \mathbf{I}_4, \mathbf{I}_5, \mathbf{I}_6$ , and  $E \setminus E'$  is formed by two basic intervals  $J_1, J_2$  contained in different circles of  $E$  such that  $J_1 \rightleftarrows J_2$ ;

(3)  $E = \mathbf{T}$ ,  $E'$  is homeomorphic to one of the spaces  $\mathbf{O}_2, \mathbf{O}_3, \mathbf{O}_4$ , and  $E \setminus E'$  is formed by two basic intervals  $J_1, J_2$  contained in different circles of  $E$  such that  $J_1 \rightleftarrows J_2$ ;

(4)  $E = \mathbf{T}$ ,  $E'$  is homeomorphic to one of the spaces  $\mathbf{I}_3, \mathbf{I}_4, \mathbf{I}_5, \mathbf{I}_6$ , and  $E \setminus E'$  is formed by three basic intervals  $J_1, J_2, J_3$  contained in different circles of  $E$  such that  $J_1 \rightarrow J_2 \rightarrow J_3 \rightarrow J_1$ .

We denote by  $S$  the set of all basic intervals associated to  $P'$ . Since  $\text{Card}(S) = k + 2$  if  $E \in \{\infty_1, \infty_2\}$  and  $\text{Card}(S) = k + 3$  if  $E = T$ , we get that  $r \leq k + 1$  if  $E \in \{\infty_1, \infty_2\}$  and  $r \leq k + 2$  if  $E = T$ . Clearly  $P \subset E'$ .

First we suppose that statement (2), (3) or (4) holds. Then we consider the  $E'$  map  $g = f|_{E'}$ . Since  $\bigcup_{i=0}^r K_i = \bigcup_{i=0}^{r+1} K_i = E'$ ,  $g$  is well-defined. Of course  $P$  is a  $k$ -orbit for  $g$ . Then from the  $n$ -odd Theorem and the Graph Theorem statement (a) of Lemma 9.6 holds and we are done.



Finally suppose that  $E' = E$ . Therefore  $J$   $f^m$ -covers  $K$ , for some  $1 \leq m \leq k + 1$  if  $E \in \{\infty_1, \infty_2\}$  and  $1 \leq m \leq k + 2$  if  $E = T$ . Thus by Lemma 9.5 there is a path of length  $m$  starting at  $J$  and ending at  $K$  and statement (b) of Lemma 9.6 follows.  $\square$

*From now on we shall denote by  $C_1$  and  $C_2$  two different circles of  $E$ .*

LEMMA 9.7. — *Set  $E \in \{\infty_1, \infty_2, \mathbf{T}\}$ . Let  $f$  be an  $E$  map having a  $k$ -orbit  $P$  with  $k \in \{7, 11, 13\}$ . Suppose that  $f$  is  $P'$ -linear and that each basic interval associated to  $P'$  is  $f$ -covered by some different basic interval. If  $C_1 \rightleftharpoons C_2$  then at least one of the following statements holds:*

(a) *If  $E = \infty_1$ , then*

$$\mathbb{N} \setminus \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16, 17, 18, 21, 23\} \subset \text{Per}(f).$$

*If  $E = \infty_2$ , then*

$$\mathbb{N} \setminus \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 21, 22, 23, 28, 29\} \subset \text{Per}(f).$$

*If  $E = \mathbf{T}$ , then*

$$\mathbb{N} \setminus \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 21, 22, 23, 28, 29\} \subset \text{Per}(f).$$

(b)  $\text{Per}(f) \supset \{n \in \mathbb{N} : n \geq 2k + 1\}$  if  $E \in \{\infty_1, \infty_2\}$ .

$\text{Per}(f) \supset \{n \in \mathbb{N} : n \geq 2k + 1 \text{ odd}\} \cup \{n \in \mathbb{N} : n \geq 2k + 4 \text{ even}\}$   
if  $E = \mathbf{T}$ .

*Proof.* — By hypotheses we have  $C_1 \rightleftharpoons C_2$ . We claim that there are two basic intervals  $L \subset C_1$  and  $M \subset C_2$  such that  $L \rightleftharpoons M$ . Now we prove the claim. If there is a 2-orbit  $\{x, y\}$  with  $x \in C_1$  and  $y \in C_2$ , then we consider the basic intervals  $L \subset C_1$  and  $M \subset C_2$  containing  $x$  and  $y$  respectively. From the linearity of  $f$  we get that  $L \rightleftharpoons M$ . Now we suppose that there are no 2-orbits  $\{x, y\}$  with  $x \in C_1$  and  $y \in C_2$ . Since  $k$  is not even and  $C_1 \rightleftharpoons C_2$ , without loss of generality we can assume that there is a closed subinterval  $K \subset C_1$  such that  $K$  is a crossing subset of  $C_2$ . Let  $K_2 \subset C_2$  be a minimal closed subinterval  $f$ -covering  $K$ . Let  $K_1 \subset C_1$  be a minimal closed subinterval  $f$ -covering  $K_2$ . In particular  $K_1 \rightleftharpoons K_2$ . Since there are no 2-orbits  $\{x, y\}$  with  $x \in C_1$  and  $x \in C_2$ , from Lemma 3.3 it follows that  $\mathbf{0} \in K_1 \cap K_2$  and the branching intervals  $L \subset K_1$ ,  $M \subset K_2$  verify  $L \rightleftharpoons M$ . So the claim is proved.

First we suppose that  $L$  or  $M$   $f^k$ -covers itself. Without loss of generality we can assume that  $L$   $f^k$ -covers  $L$ . Then, by Lemma 9.5 there exists a loop of length  $k$  containing  $L$ . Therefore the above loop of length  $k$  together with the loop  $L \rightleftarrows M$  give us a loop  $\gamma$  of length  $n$  for each  $n > k$  odd and each  $n \geq 2k + 2$  even. Note that  $\gamma$  is non-repetitive because  $k$  is not multiple of 2. We claim that we can construct  $\gamma$  containing some non-branching interval. Now we prove the claim. If  $\mathbf{0} \notin L$  or  $\mathbf{0} \notin M$ , then we are done. So suppose that  $\mathbf{0} \in L \cap M$ . Then the only branching intervals  $f$ -covered by  $L$  and  $M$  are  $L$  and  $M$  (see Remark 4.4). Hence  $\gamma$  contains some non-branching interval and the claim is proved. By Proposition 3.4 the result follows.

Now we can assume that  $L$  and  $M$  do not  $f^k$ -cover itself. Thus, since  $P$  has period  $k$ , we get that  $L$   $f^k$ -covers  $J$  for each  $J \in S$  with  $J \subset C_1$  and  $M$   $f^k$ -covers  $J$  for each  $J \in S$  with  $J \subset C_2$ .

Without loss of generality we have three possibilities for the basic intervals  $L$  and  $M$ : either  $\mathbf{0} \in L \cap M$ ; or  $\mathbf{0} \in L$  and  $\mathbf{0} \notin M$ ; or  $\mathbf{0} \notin L \cup M$ . If  $\mathbf{0} \in L \cap M$ , then without loss of generality we can assume that there is a basic interval  $M_1 \subset C_2 \setminus \text{Int}(M)$  such that  $L \rightarrow M_1$ . Moreover, since  $f$  is  $P'$ -linear,  $\mathbf{0} \notin M_1$ . If  $\mathbf{0} \in L$  and  $\mathbf{0} \notin M$ , then since  $f(\mathbf{0}) = \mathbf{0}$  and  $L \rightarrow M$ , we have that there is a basic interval  $M_1 \subset C_2 \setminus \text{Int}(M)$  such that  $L \rightarrow M_1$ . Finally, since  $k \geq 7$ , if  $\mathbf{0} \notin L \cup M$ , then again we can assume that there is a basic interval  $M_1 \subset C_2 \setminus \text{Int}(M)$  such that  $L \rightarrow M_1$ .

In short, we get that there is  $M_1 \in S$  such that  $M_1 \subset C_2 \setminus \text{Int}(M)$ ,  $L \rightarrow M_1$  and  $\mathbf{0} \notin M$  or  $\mathbf{0} \notin M_1$ . Therefore  $M$   $f^k$ -covers  $M_1$ . From Lemma 9.5 there is a path  $M \rightarrow \dots \rightarrow M_1$  of length  $k$ . If Lemma 9.6 (a) holds, then Lemma 9.7 (a) follows, and we are done. Otherwise, from Lemma 9.6 (b) we can assume that there is a path  $M_1 \rightarrow \dots \rightarrow M$  of length  $m \leq k + 1$  if  $E \in \{\infty_1, \infty_2\}$  and  $m \leq k + 2$  if  $E = \mathbf{T}$ . We can suppose that  $m$  is the shortest length of all paths from  $M_1$  to  $M$ . This path of length  $m$  together with the path  $M \rightarrow L \rightarrow M_1$  and the path  $M \rightarrow \dots \rightarrow M_1$  of length  $k$  give us two loops of lengths  $m + 2$  and  $k + m$ . Note that both loops contain  $M$  and  $M_1$ .

First suppose that  $m$  is odd. Then the loop  $M \rightarrow L \rightarrow M_1 \rightarrow \dots \rightarrow M$  of length  $m + 2$  and the loop  $M \rightleftarrows L$  allow us to construct a non-repetitive loop of length  $n$  for each  $n \geq k + 2$  odd if  $E \in \{\infty_1, \infty_2\}$  and  $n \geq k + 4$  odd if  $E = \mathbf{T}$ . This loop contains  $M$  and  $M_1$ . On the other hand, the loops  $M \rightarrow \dots \rightarrow M_1 \rightarrow \dots \rightarrow M$  of length  $k + m$  and the loop  $M \rightleftarrows L$  allow us to construct a non-repetitive loop of length  $n$  for each  $n \geq 2k + 2$  even

if  $E \in \{\infty_1, \infty_2\}$  and  $n \geq 2k + 4$  even if  $E = \mathbf{T}$ . This loop contains  $M$  and  $M_1$ . Since  $\mathbf{0} \notin M$  or  $\mathbf{0} \notin M_1$ , by Proposition 3.4 the result follows.

Finally suppose that  $m$  is even. The loop  $M \rightarrow L \rightarrow M_1 \rightarrow \dots \rightarrow M$  of length  $m + 2$  and the loop  $M \rightleftharpoons L$  give us a non-repetitive loop of length  $n$  for each  $n \geq k + 3$  even containing  $M$  and  $M_1$ . Moreover, the loop  $M \rightarrow \dots \rightarrow M_1 \rightarrow \dots \rightarrow M$  of length  $k + m \leq 2k + 1$  and the loop  $M \rightleftharpoons L$  give us a non-repetitive loop of length  $n$  for each  $n \geq 2k + 1$  odd containing  $M$  and  $M_1$ . Since  $\mathbf{0} \notin M$  or  $\mathbf{0} \notin M_1$ , from Proposition 3.4, statement (b) of Lemma 9.7 holds and we are done.  $\square$

The following lemma will be used in the rest of this section and in Section 12.

LEMMA 9.8. — *Set  $E \in \{\infty_1, \infty_2, \mathbf{T}\}$ . Let  $f$  be an  $E$  map having a  $k$ -orbit  $P$ . Suppose that  $f$  is  $P'$ -linear. Let  $\beta : J = K_0 \rightarrow K_1 \rightarrow \dots \rightarrow K_r \rightarrow J$  and  $\gamma : J = M_0 \rightarrow M_1 \rightarrow \dots \rightarrow M_s \rightarrow J$  be two different loops in the  $f$ -graph having a common basic interval  $J$ . Then the loop obtained by concatenating  $\beta$  and  $\gamma$  contains some non-branching interval.*

*Proof.* — If some basic interval of  $\beta$  or  $\gamma$  does not contain  $\mathbf{0}$ , then we are done. So suppose that all basic intervals of  $\beta$  and  $\gamma$  are branching intervals. From Remark 4.4,  $J$   $f$ -covers exactly one branching interval, and perhaps some non-branching interval. Hence  $K_1 = M_1$ . Since all basic intervals are branching intervals, repeating this argument we get that  $\beta$  and  $\gamma$  are the same loop, in contradiction with the hypotheses.  $\square$

The following lemma follows from ideas of [Ba] and [LPR2], [LPR3].

LEMMA 9.9. — *Let  $E \in \{\infty_1, \infty_2\}$ . Let  $f$  be an  $E$  map having a  $k$ -orbit  $P$  with  $k \in \{7, 11, 13\}$ . Suppose that  $f$  is  $P'$ -linear. Assume that there are  $p_1, \dots, p_t$  points of  $P$  for some  $t \in \{3, 4\}$  such that*

$$[\mathbf{0}, p_1] \rightarrow [\mathbf{0}, p_2] \rightarrow \dots \rightarrow [\mathbf{0}, p_t] \rightarrow [\mathbf{0}, p_1],$$

$$(\mathbf{0}, p_i) \cap P = \emptyset \text{ for } i = 1, \dots, t$$

(i.e.  $[\mathbf{0}, p_i]$  is a branching interval), and if  $i \neq j$ , then  $[\mathbf{0}, p_i]$  and  $[\mathbf{0}, p_j]$  are contained in different components of  $E \setminus \{\mathbf{0}\}$ . Then

$$\{n = ki + tj, i \geq 0, j \geq 1\} \subset \text{Per}(f).$$

*Proof.* — Without loss of generality we can assume that  $[\mathbf{0}, p_1]$  is contained in a whiskers  $W$  of  $E$ . By Lemma 9.4 there is a loop of length  $k$

containing  $[0, p_1]$ . This loop together with the loop  $[0, p_1] \rightarrow [0, p_2] \rightarrow \dots \rightarrow [0, p_1]$  of length  $t$  give us a loop of length  $n$  for each  $n = ki + tj$ ,  $i \geq 1, j \geq 1$ . Since  $k$  is not divisible by  $t$  and from Lemma 9.8, we get that the loop of length  $n$  is non-repetitive and at least one of its intervals does not contain  $0$ . Consequently  $\{n = ki + tj, i \geq 1, j \geq 1\} \subset \text{Per}(f)$ . Now we need to prove that all multiple of  $t$  also belongs to  $\text{Per}(f)$ .

From the hypotheses,  $(0, p_{j+1}] \subset f(0, p_j]$  for  $0 < j < t$  and  $(0, p_1] \subset f(0, p_t]$ . Thus  $f^j(0, p_1] \subset f^{t+j}(0, p_1]$  for all  $j$ . On the other hand, since  $k$  is not divisible by  $t$ ,  $f^i(0, p_1]$  contains elements of different components of  $E \setminus \{0\}$  and so there must be an integer  $i$  such that  $0 \in f^i(0, p_1]$ . We fix the least such  $i$ . Consider two cases.

- *Case 1:  $i > t$ .*

Let  $r$  be the largest positive integer such that  $i > rt$ . From the facts that  $[0, p_1] \subset W$ ,  $W$  is an interval and  $f$  is  $P'$ -linear, we have that  $f^{(r-1)t}(0, p_1] = (0, u] \subset W$  and  $f^{rt}(0, p_1] = f^t(0, u] = (0, v] \subset W$  for some  $u, v \in P$  with  $u \in (0, v)$ . Then there exists  $a \in (0, u]$  such that  $f^t(a) = v$ . Since  $0 \in f^i(0, p_1]$  and  $r$  is the largest positive integer such that  $i > rt$ , by the minimality of  $i$  it follows that there exists  $b \in (0, v]$  such that  $f^t(b) = 0$ . Note that  $b \notin (0, u]$  because  $0 \notin f^{rt}(0, p_1] = f^t(0, u]$ . Since  $W$  is an interval, we get  $f^t[0, a] \supset [0, a] \cup [a, b]$ ,  $f^t[a, b] \supset [0, a] \cup [a, b]$  and  $f^t(a) \neq a$ . Then by well-known results for interval maps (see Proposition 1.2.9 of [ALM2]) we obtain that  $f^t$  has points of all periods in  $[0, b]$ , and consequently  $f$  has periodic points of each multiple of  $t$ .

- *Case 2:  $i \leq t$ .*

Since  $f^t(0, p_1] \supset [0, p_1]$ , there is  $a \in (0, p_1)$  such that  $f^t(a) = p_1$  and  $f^t(y) \neq p_1$  for all  $y \in (0, a)$ . Moreover, from the linearity of  $f$  it follows that  $f^t|_{[0, a]}$  is linear. Since  $0 \in f^i(0, p_1]$  and  $i \leq t$ , there exists  $b \in (0, p_1)$  such that  $f^t(b) = 0$ . Note that  $b \notin (0, a)$  because  $f^t$  is linear in  $[0, a]$ ,  $f^t(0) = 0$  and  $f^t(a) = p_1$ . Hence  $f^t[0, a] \supset [0, a] \cup [a, b]$  and  $f^t[a, b] \supset [0, a] \cup [a, b]$ . Thus the proof follows as in Case 1. □

LEMMA 9.10. — *Let  $E \in \{\infty_1, \infty_2\}$ . Let  $f$  be an  $E$  map having a  $k$ -orbit  $P$  with  $k \in \{7, 11, 13\}$  such that  $f$  is  $P'$ -linear. Suppose that there are  $t$  closed subintervals  $K_1, \dots, K_t$  with  $t \in \{2, 3, 4\}$  such that  $K_j$  is contained in the closure of a connected component  $R_j$  of  $E \setminus \{0\}$  for each  $j = 1, \dots, t$ , and if  $K_i \neq K_j$  then  $R_i \neq R_j$ . Assume that  $R_1$  is a whiskers of  $E$ . If  $K_1 \rightarrow \dots \rightarrow K_t \rightarrow K_1$ , then*

$$\text{Per}(f) \supset \{n = ki + tj : i \geq 1, j \geq 1\}.$$

*Proof.* — If there is a  $t$ -orbit  $\{x, \dots, f^{t-1}(x)\}$  such that  $x \in K_1, \dots, f^{t-1}(x) \in K_t$ , then let  $J_1 \subset K_1, \dots, J_t \subset K_t$  be the basic intervals containing  $x, \dots, f^{t-1}(x)$  respectively. From the linearity of  $f$  we have that  $J_1 \rightarrow \dots \rightarrow J_t \rightarrow J_1$ . If there are no  $t$ -orbits in  $K_1 \cup \dots \cup K_t$ , then by Lemma 3.3 there exist  $t$  branching intervals  $J_1 \subset K_1, \dots, J_t \subset K_t$  such that  $J_1 \rightarrow \dots \rightarrow J_t \rightarrow J_1$ .

Since  $J_1$  is contained in a whiskers of  $E$ , fom Lemma 9.4 there is a loop of length  $k$  containing  $J_1$ . This loop together with the loop  $J_1 \rightarrow \dots \rightarrow J_t \rightarrow J_1$  give us a loop  $\gamma$  of length  $n$  for each  $n = ki + tj$  with  $i \geq 1, j \geq 1$ . The loop  $\gamma$  is non-repetitive because  $k$  is not divisible by  $t$  and by Lemma 9.8 at least one of its intervals does not contain  $0$ . Hence by Proposition 3.4 the result holds. □

### 10. The full periodicity kernel of $\infty_1$ .

The goal of this section is to prove Theorem 1.4.

Since  $\mathbf{I}_5$  is homeomorphic to  $\{z \in \infty_1 : -1 \leq \text{Im } z \leq 1\}$ , we can consider  $\mathbf{I}_5 = \{z \in \infty_1 : -1 \leq \text{Im } z \leq 1\}$ . Set  $A = \{z \in \infty_1 : \text{Im } z \geq 1\}$  and  $B = \{z \in \infty_1 : \text{Im } z \leq -1\}$ . Let  $f$  be a  $\mathbf{I}_5$  map. We shall extend  $f$  to an  $\infty_1$  map  $\bar{f}$  as follows. We define  $\bar{f}(z) = f(z)$  if  $z \in \mathbf{I}_5$ ,  $\bar{f}|_A$  is any homeomorphism between  $A$  and the unique closed interval in  $\mathbf{I}_5$  having  $f(1+i)$  and  $f(-1+i)$  as endpoints; and finally  $\bar{f}|_B$  is any homeomorphism between  $B$  and the unique closed interval in  $\mathbf{I}_5$  having  $f(1-i)$  and  $f(-1-i)$  as endpoints. Of course  $\text{Per}(f) = \text{Per}(\bar{f})$ . From Theorem 1.1 (d), the set  $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 16, 17, 18, 21, 23\}$  is a subset of the full periodicity kernel of  $\infty_1$ . Then to prove Theorem 1.4 it is sufficient to show the following two propositions.

PROPOSITION 10.1. — *Let  $f$  be an  $\infty_1$  map such that*

$$\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16, 17, 18, 21, 23\} \subset \text{Per}(f).$$

*Then  $\text{Per}(f) = \mathbb{N}$ .*

PROPOSITION 10.2. — *There is an  $\infty_1$  map  $g$  such that*

$$\text{Per}(g) = \mathbb{N} \setminus \{12\}.$$

Proposition 10.1 will be a corollary of the following proposition.

PROPOSITION 10.3. — *Let  $f$  be an  $\infty_1$  map. Then the following statements hold:*

(a) *If  $7 \in \text{Per}(f)$ , then*

$$\text{Per}(f) \supset \mathbb{N} \setminus \{2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 16, 17, 18, 21, 23, 28, 35\}.$$

(b) *If  $11 \in \text{Per}(f)$ , then  $35 \in \text{Per}(f)$ .*

(c) *If  $13 \in \text{Per}(f)$ , then  $28 \in \text{Per}(f)$ .*

*In the rest of this section we fix the  $\infty_1$  map  $f$  having a  $k$ -orbit  $P$  with  $k \in \{7, 11, 13\}$  and the set of the basic intervals  $S$  associated to  $P'$ .*

This fixed  $\infty_1$  map will be called the *standard  $\infty_1$  map*.

LEMMA 10.4. — *Let  $f$  be the standard  $\infty_1$  map. If the periodic orbit  $P$  does not have points in each connected component of  $\infty_1 \setminus \{0\}$  then Proposition 10.3 holds.*

*Proof.* — Let  $E'$  be the union of the closures of the connected components of  $\infty_1 \setminus \{0\}$  having points of  $P$ . Of course  $E' \subset \infty_1$ . Then we define the map  $g : E' \rightarrow E'$  as follows. For  $z \in E'$ ,  $g(z) = f(z)$  if  $f(z) \in E'$ ; and  $g(z) = 0$  otherwise. Notice that  $g$  is either an  $\mathbf{I}_2$ ,  $\mathbf{O}$ ,  $\mathbf{O}_1$  or  $\infty$  map. Moreover  $\text{Per}(g) \subset \text{Per}(f)$ . Hence from the Interval Theorem, the Circle Theorem, the Graph Theorem and Proposition 9.1 the result follows. □

Remark 10.5. — From Lemma 10.4 we can assume that the periodic orbit  $P$  has points into each connected component of  $\infty_1 \setminus \{0\}$ . Furthermore, by Corollary 4.3 in what follows we can suppose that the standard map  $f$  will be  $P'$ -linear.

LEMMA 10.6. — *Let  $f$  be the standard  $\infty_1$  map. Suppose that there is a basic interval  $J$  such that no basic interval of  $S \setminus \{J\}$   $f$ -covers  $J$ . Then Proposition 10.3 follows.*

*Proof.* — By Lemma 9.3 each basic interval of the whiskers of  $\infty_1$  is  $f$ -covered by some different basic interval. Therefore  $J$  is contained in a circle of  $\infty_1$ . Consider the map  $g = f|_{\infty_1 \setminus \text{Int}(J)}$ . Clearly  $g$  is well-defined because  $f$  is  $P'$ -linear and no basic interval of  $S \setminus \{J\}$   $f$ -covers  $J$ . Moreover  $g$  is either a  $\mathbf{O}_2$  or  $\mathbf{O}_3$  map such that  $\text{Per}(g) = \text{Per}(f)$ . Hence from the Graph Theorem the lemma follows. □

*Remark 10.7.* — From Lemma 10.6 we can assume that each basic interval is  $f$ -covered by some different basic interval. On the other hand, Proposition 9.2 shows that if there exists some basic interval which  $f$ -covers itself, then Proposition 10.3 holds. So, from now on, we can suppose that each basic interval does not  $f$ -cover itself.

**LEMMA 10.8.** — *Let  $f$  be the standard  $\infty_1$  map. Identify  $\mathbf{O}$  with a circle of  $\infty_1$ . If there are no crossing subsets of  $\mathbf{O}$ , then Proposition 10.3 holds.*

*Proof.* — With the notation of Section 8, by Proposition 8.4 (b) we have that  $k \in \text{Per}(F)$ . Since  $P$  has elements on each component of  $\infty_1 \setminus \{\mathbf{0}\}$  and there are no crossing subsets of  $\mathbf{O}$ , we get that  $F(\infty_1^*)$  is homeomorphic to  $\mathbf{O}_2$  or  $\mathbf{O}_3$ . So from the Graph Theorem we obtain that if  $7 \in \text{Per}(F)$ , then  $\text{Per}(F) \supset \mathbb{N} \setminus \{2, 3, 4, 5, 6, 8, 9, 10, 11, 13, 14, 16, 17, 18, 21, 23, 28\}$ ; if  $11 \in \text{Per}(F)$ , then  $35 \in \text{Per}(F)$ ; and if  $13 \in \text{Per}(F)$ , then  $28 \in \text{Per}(F)$ . Now from Proposition 8.4(a) the result follows.  $\square$

*Remark 10.9.* — If there are no subsets of  $\infty_1$   $f$ -covering  $C_1$  or  $C_2$ , from Lemma 10.8, Proposition 10.3 holds. So from now on we can assume that there are crossing subsets of  $C_1$  and  $C_2$ .

*Proof of Proposition 10.3.* — Denote by  $W$  and  $p_1$  the whiskers and the endpoint of  $\infty_1$  respectively. From Remark 10.7 each basic interval does not  $f$ -cover itself. Therefore  $f^{k-1}(p_1) \notin W$ . Without loss of generality we can assume that  $f^{k-1}(p_1) \in C_1$ . Moreover, from the fact that  $f(\mathbf{0}) = \mathbf{0}$ , it follows that there are two closed subintervals  $M_1, M_2$  contained in  $C_1$  such that  $\mathbf{0} \notin \text{Int}(M_1)$ ,  $\mathbf{0} \notin \text{Int}(M_2)$ ,  $\text{Int}(M_1) \cap \text{Int}(M_2) = \emptyset$ , and  $M_1 \rightarrow W \leftarrow M_2$ .

If  $W \rightarrow C_1$ , then since  $\mathbf{0} \notin \text{Int}(M_1)$ , from Remark 8.1  $W \rightleftharpoons M_1$ , and from Lemma 9.10 the result holds. So, in what follows we can suppose that  $W \nrightarrow C_1$ .

By Remarks 10.7 and 10.9, and since  $W \nrightarrow C_1$  we have that  $C_2 \rightarrow C_1$ . If  $C_1 \rightarrow C_2$  from Lemma 9.7 the result follows. So from now on we can suppose that  $C_1 \nrightarrow C_2$ .

By Remarks 10.7 and 10.9 we get that  $W \rightarrow C_2$ , then there exist closed subintervals  $K_1, K_2 \subset C_2$  such that  $W \rightarrow K_1 \rightarrow M_1 \rightarrow W$  and  $W \rightarrow K_2 \rightarrow M_2 \rightarrow W$ . If  $k \in \{11, 13\}$  by Lemma 9.10 the result holds. If  $k = 7$ , from Lemma 9.10 we have that  $\text{Per}(f) \supset \{n = 7i + 3j : i \geq 1, j \geq 1\}$ . So we need to prove that  $15 \in \text{Per}(f)$ . Concatenating the loops  $W \rightarrow K_1 \rightarrow M_1 \rightarrow W$  and  $W \rightarrow K_2 \rightarrow M_2 \rightarrow W$  we obtain a non-repetitive loop of length  $3i$  for  $i \geq 1$ . If  $\mathbf{0} \notin W \cap K_1 \cap K_2 \cap M_1 \cap M_2$  then the

result holds. Otherwise, there are three branching intervals  $J_1 \subset W$ ,  $J_2 \subset C_2$  and  $J_3 \subset C_1$  such that  $J_1 \rightarrow J_2 \rightarrow J_3 \rightarrow J_1$ . From Lemma 9.9 the result holds.  $\square$

*Proof of Proposition 10.2.* — Let  $Z$  be the union of the following initial segments:  $\{n \leq_5 11\}$ ,  $\{n \leq_s 8\}$ ,  $\{n \leq_3 6\}$  and  $\{n \leq_5 14\}$ . This is  $Z = \mathbb{N} \setminus \{7, 9, 12, 13, 17, 18, 22, 23, 28\}$ . From the  $n$ -od Theorem there exists an  $\mathbf{I}_5$  map  $f_1$  such that  $\text{Per}(f_1) = Z$ . We shall extend  $f_1$  to an  $\infty_1$  map  $g$  as follows. As in the beginning of this section, consider  $\infty_1 = \mathbf{I}_5 \cup A \cup B$ . Let  $g(z) = f_1(z)$  if  $z \in \mathbf{I}_5$ . Then we need to define  $g|_{A \cup B}$ .

We choose seven points  $a_i$ ,  $i = 1, 2, \dots, 7$ , in  $A \cup B$  as follows. We consider  $A$  as the union of the following intervals which have pairwise disjoint interiors:  $J_1 = [1 + i, a_1]$ ,  $J_2 = [a_1, a_3]$ ,  $J_3 = [a_3, a_5]$ ,  $J_4 = [a_5, a_7]$  and  $J_5 = [a_7, -1 + i]$ . Set  $B$  as the union of the following intervals which have pairwise disjoint interiors:  $J_6 = [1 - i, a_2]$ ,  $J_7 = [a_2, a_4]$ ,  $J_8 = [a_4, a_6]$  and  $J_9 = [a_6, -1 - i]$ . Set  $P = \{a_i : i = 1, 2, \dots, 7\}$ . Define  $g(a_i) = a_{i+1}$  for  $i = 1, 2, \dots, 6$  and  $g(a_7) = a_1$ . Clearly  $P$  is a 7-orbit. Let  $g$  be restricted to each  $J_i$  the unique linear map with respect to the taxicab metric such that the only elementary loops in the  $g|_{A \cup B}$ -graph are  $J_1 \rightleftarrows J_6$ ,  $J_1 \rightarrow J_6 \rightarrow J_2 \rightarrow J_7 \rightarrow J_3 \rightarrow J_8 \rightarrow J_4 \rightarrow J_1$  and  $J_1 \rightarrow J_6 \rightarrow J_2 \rightarrow J_7 \rightarrow J_3 \rightarrow J_8 \rightarrow J_4 \rightarrow J_9 \rightarrow J_5 \rightarrow J_1$ . Since  $\mathbf{I}_5$  is an invariant set under  $g$ , from Propositions 3.4 and 4.2 we get that  $\text{Per}(g|_{A \cup B}) = \{n \in \mathbb{N} : n \geq 7 \text{ odd}\} \cup \{n \in \mathbb{N} : n \geq 16 \text{ even}\}$ . Of course  $\text{Per}(g) = \text{Per}(f_1) \cup \text{Per}(g|_{A \cup B}) = \mathbb{N} \setminus \{12\}$ .  $\square$

## 11. The full periodicity kernel of $\infty_2$ .

The goal of this section is to prove Theorem 1.5.

Since  $\mathbf{I}_6$  is homeomorphic to  $\{z \in \infty_2 : -1 \leq \text{Im } z \leq 1\}$ , we can consider  $\mathbf{I}_6 = \{z \in \infty_2 : -1 \leq \text{Im } z \leq 1\}$ . Set  $A = \{z \in \infty_2 : \text{Im } z \geq 1\}$  and  $B = \{z \in \infty_2 : \text{Im } z \leq -1\}$ . Let  $f$  be an  $\mathbf{I}_6$  map. We shall extend  $f$  to an  $\infty_2$  map  $\bar{f}$  as follows. We define  $\bar{f}(z) = f(z)$  if  $z \in \mathbf{I}_6$ ;  $\bar{f}|_A$  is any homeomorphism between  $A$  and the unique closed interval in  $\mathbf{I}_6$  having  $f(1+i)$  and  $f(-1+i)$  as endpoints; and finally  $\bar{f}|_B$  is any homeomorphism between  $B$  and the unique closed interval of  $\mathbf{I}_6$  having  $f(1-i)$  and  $f(-1-i)$  as endpoints. Of course  $\text{Per}(f) = \text{Per}(\bar{f})$ . From Theorem 1.1 (e),  $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18, 21, 22, 23, 28, 29\}$  is a subset of the full periodicity kernel of  $\infty_2$ . On the other hand,  $\infty_1 \subset \infty_2$ . Let



$f$  be an  $\infty_1$  map. We shall extend  $f$  to an  $\infty_2$  map as follows. Set  $\bar{f}(z) = f(z)$  if  $z \in \infty_1$  and  $f(z) = \mathbf{0}$  otherwise. Of course  $\text{Per}(f) = \text{Per}(\bar{f})$ . By Theorem 1.4, 12 belongs to the full periodicity of  $\infty_2$ . Then, to prove Theorem 1.5 it is sufficient to show the following proposition.

PROPOSITION 11.1. — *Let  $f$  be an  $\infty_2$  map. Then the following statements hold:*

(a) *If  $7 \in \text{Per}(f)$ , then*

$$\mathbb{N} \setminus \{2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 21, 22, 23, 28, 29, 35\} \subset \text{Per}(f).$$

(b) *If  $11 \in \text{Per}(f)$ , then  $35 \in \text{Per}(f)$ .*

*In the rest of this section we fix the  $\infty_2$  map  $f$  having a  $k$ -orbit  $P$  with  $k \in \{7, 11\}$  and the set of the basic intervals  $S$  associated to  $P'$ .*

*This fixed  $\infty_2$  map will be called the standard  $\infty_2$  map.*

LEMMA 11.2. — *Let  $f$  be the standard  $\infty_2$  map. If the periodic orbit  $P$  does not have points in each connected component of  $\infty_2 \setminus \{\mathbf{0}\}$ , then Proposition 11.1 holds.*

*Proof.* — Let  $E'$  be the union of the closures of the connected components of  $\infty_2 \setminus \{\mathbf{0}\}$  having points of  $P$ . Of course  $E' \subset \infty_2$ . Then we define the map  $g : E' \rightarrow E'$  as follows. For  $z \in E'$ ,  $g(z) = f(z)$  if  $f(z) \in E'$ ; and  $g(z) = \mathbf{0}$  otherwise. Notice that  $g$  is either a  $\mathbf{I}_2$ ,  $\mathbf{O}$ ,  $\mathbf{O}_1$ ,  $\mathbf{O}_2$ ,  $\infty$  or  $\infty_1$  map. Moreover  $\text{Per}(g) \subset \text{Per}(f)$ . Hence from the Interval Theorem, the Graph Theorem, Proposition 9.1 and Proposition 10.3 the result follows.  $\square$

Remark 11.3. — From Lemma 11.2 we can assume that the periodic orbit  $P$  has points in each connected component of  $\infty_2 \setminus \{\mathbf{0}\}$ . Furthermore, by Corollary 4.3 in what follows we can suppose that the standard map  $f$  will be  $P'$ -linear.

LEMMA 11.4. — *Let  $f$  be the standard  $\infty_2$  map. Suppose that there is a basic interval  $J$  such that there are no basic intervals of  $S \setminus \{J\}$   $f$ -covering  $J$ . Then Proposition 11.1 holds.*

*Proof.* — By Lemma 9.3 we get that  $J$  is contained in a circle of  $\infty_2$ . Consider the map  $g = f|_{\infty_2 \setminus \text{Int}(J)}$ . Clearly  $g$  is well-defined because  $f$  is  $P'$ -linear. Moreover  $g$  is either an  $\mathbf{O}_3$  or  $\mathbf{O}_4$  map such that  $\text{Per}(g) \subset \text{Per}(f)$ . Hence from the Graph Theorem the lemma follows.  $\square$

*Remark 11.5.* — From Lemma 11.4 we can assume that each basic interval is  $f$ -covered by some different basic interval. On the other hand, Proposition 9.2 shows that if there exists some basic interval  $f$ -covering itself, then Proposition 11.1 holds. So, from now on we can assume that each basic interval does not  $f$ -cover itself.

LEMMA 11.6. — *Let  $f$  be the standard  $\infty_2$  map. Identify  $\mathbf{O}$  with a circle of  $\infty_2$ . If there are no crossing subsets of  $\mathbf{O}$  then Proposition 11.1 holds.*

*Proof.* — With the notation of Section 8, by Proposition 8.4 (b) we have that  $k \in \text{Per}(F)$ . Since  $P$  has elements in each component of  $\infty_2 \setminus \{\mathbf{0}\}$  and there are no crossing subsets of  $\mathbf{O}$ , we get that  $F(\infty_2^*)$  is homeomorphic to  $\mathbf{O}_3$  or  $\mathbf{O}_4$ . So from the Graph Theorem we obtain that if  $7 \in \text{Per}(F)$ , then  $\mathbb{N} \setminus \{2, 3, 4, 5, 6, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18, 21, 22, 23, 28, 29, 35\} \subset \text{Per}(F)$ , and if  $11 \in \text{Per}(F)$ , then  $35 \in \text{Per}(F)$ . Now by Proposition 8.4 (a) the result follows. □

*Remark 11.7.* — If there are no subsets of  $\infty_2$   $f$ -covering  $C_1$  or  $C_2$ , from Lemma 11.6, Proposition 11.1 holds. So from now on we can assume that there are crossing subsets of  $C_1$  and  $C_2$ .

*Proof of Proposition 11.1.* — Denote by  $W_1, W_2$  the whiskers of  $\infty_2$  and by  $p_1 \in W_1, p_2 \in W_2$  its endpoints. From Remark 11.5 each basic interval does not  $f$ -cover itself. Therefore  $f^{k-1}(p_1) \notin W_1$  and  $f^{k-1}(p_2) \notin W_2$ . We consider two cases.

- *Case 1:*  $f^{k-1}(p_1) \in W_2$ .

Consequently  $W_2 \rightarrow W_1$ . If  $W_1 \rightarrow W_2$ , since the whiskers are subintervals of  $E$ , from Lemma 9.10 the result follows. Hence, from now on we will assume that  $W_1 \nrightarrow W_2$ . In particular  $f^{k-1}(p_2) \notin W_1$ . Without loss of generality we can suppose that  $f^{k-1}(p_2) \in C_1$ . Moreover since  $f(\mathbf{0}) = \mathbf{0}$ , it follows that there are two closed subintervals  $M_1, M_2$  contained in  $C_1$  such that  $\text{Int}(M_1) \cap \text{Int}(M_2) = \emptyset, \mathbf{0} \notin \text{Int}(M_1), \mathbf{0} \notin \text{Int}(M_2)$  and  $M_1 \rightarrow W_2 \leftarrow M_2$ .

If  $W_2 \rightarrow C_1$ , then from Remark 8.1  $W_2 \rightleftharpoons M_1$ , and by Lemma 9.10 the result holds. So from now on we will assume that  $W_2 \nrightarrow C_1$ .

If  $W_1 \rightarrow C_1$ , then from Remark 8.1  $W_1 \rightarrow M_1$ . Thus we consider the loop  $W_1 \rightarrow M_1 \rightarrow W_2 \rightarrow W_1$  and from Lemma 9.10 the result follows. So from now on we can assume that  $W_1 \nrightarrow C_1$ .

Since each basic interval does not  $f$ -cover itself,  $C_1 \not\rightarrow C_1$ . Then from Remark 11.7 we have that  $C_2 \rightarrow C_1$ . Hence there is a closed subinterval  $K \subset C_2$  such that  $K \rightarrow M_1$ . If  $W_2 \rightarrow C_2$ , then we consider the loop  $W_2 \rightarrow K \rightarrow M_1 \rightarrow W_2$  and by Lemma 9.10 the result holds. Thus we can assume that  $W_2 \not\rightarrow C_2$ .

If  $C_1 \rightarrow C_2$ , from Lemma 9.7 we are done. So we can assume that  $C_1 \not\rightarrow C_2$ .

By Remarks 11.5 and 11.7 we get that  $W_1 \rightarrow C_2$ . Therefore there are closed subintervals  $K_1 \subset W_1$ ,  $K_2 \subset W_2$  and  $L_1, L_2 \subset C_2$  such that  $K_1 \rightarrow L_1 \rightarrow M_1 \rightarrow K_2 \rightarrow K_1$  and  $K_1 \rightarrow L_2 \rightarrow M_2 \rightarrow K_2 \rightarrow K_1$ . If  $k = 11$ , then statement (b) of Proposition 11.1 follows from Lemma 9.10. If  $k = 7$ , by Lemma 9.10 we obtain that  $\text{Per}(f) \supset \{n = 7i + 4j : i \geq 1, j \geq 1\}$ . So we need to prove that  $\{20, 24\} \subset \text{Per}(f)$ . With the loops  $K_1 \rightarrow L_1 \rightarrow M_1 \rightarrow K_2 \rightarrow K_1$  and  $K_1 \rightarrow L_2 \rightarrow M_2 \rightarrow K_2 \rightarrow K_1$  we obtain a non-repetitive loop of length  $4i$  for  $i \geq 1$ . If  $\mathbf{0} \notin K_1 \cap K_2 \cap L_1 \cap L_2 \cap M_1 \cap M_2$  then the result holds. Otherwise, there are four branching intervals  $J_1 \subset W_1, J_2 \subset C_2, J_3 \subset C_1$  and  $J_4 \subset W_2$  such that  $J_1 \rightarrow J_2 \rightarrow J_3 \rightarrow J_4 \rightarrow J_1$ . Therefore statement (a) of Proposition 11.1 follows from Lemma 9.9 and we are done.

- *Case 2:  $f^{k-1}(p_1) \notin W_2$ .*

Without loss of generality we can assume that  $f^{k-1}(p_1) \in C_1$ . Moreover, since  $f(\mathbf{0}) = \mathbf{0}$ , there exist two closed subintervals  $M_1, M_2 \subset C_1$  such that  $\text{Int}(M_1) \cap \text{Int}(M_2) = \emptyset, \mathbf{0} \notin \text{Int}(M_1), \mathbf{0} \notin \text{Int}(M_2)$  and  $M_1 \rightarrow W_1 \leftarrow M_2$ .

If  $W_1 \rightarrow C_1$ , then from Remark 8.1 we have  $W_1 \rightleftharpoons M_1$  and the result follows from Lemma 9.10. Hence from now on we can suppose that  $W_1 \not\rightarrow C_1$ .

First assume that  $C_2 \rightarrow C_1$ . Then there is a closed subinterval  $K \subset C_2$  such that  $K \rightarrow M_1$ . If  $C_1 \rightarrow C_2$  the result follows from Lemma 9.7. So we can suppose that  $C_1 \not\rightarrow C_2$ . If  $W_1 \rightarrow C_2$ , then there is a closed subinterval  $L \subset W_1$  such that  $L \rightarrow K \rightarrow M_1 \rightarrow L$ . So from Lemma 9.10 the result follows. Hence we can assume that  $W_1 \not\rightarrow C_2$ . From Remarks 11.5 and 11.7 we obtain that  $W_2 \rightarrow C_2$ .

If  $C_1 \rightarrow W_2$  or  $C_2 \rightarrow W_2$ , by above arguments the proposition follows. So we can assume that  $C_1 \not\rightarrow W_2$  and  $C_2 \not\rightarrow W_1$ . In particular  $f^{k-1}(p_2) \notin C_1 \cup C_2$ . Moreover, since  $f^{k-1}(p_2) \notin W_2$  we get that  $f^{k-1}(p_2) \in W_1$ . Consequently the rest of the proof follows as in Case 1.

Finally assume that  $C_2 \rightarrow C_1$ . From Remarks 11.5 and 11.7 we have that  $W_2 \rightarrow C_1$ . If  $C_1 \rightarrow W_2$  or  $W_1 \rightarrow W_2$  by above arguments the proposition holds. So, in particular we can assume that  $f^{k-1}(p_2) \notin C_1 \cup W_1$ . Therefore  $f^{k-1}(p_2) \in C_2$ . Moreover since  $f(\mathbf{0}) = \mathbf{0}$ , there are closed subintervals  $L_1, L_2 \subset C_2$  such that  $L_1 \rightarrow W_2 \leftarrow L_2$ ,  $\text{Int}(L_1) \cap \text{Int}(L_2) = \emptyset$ ,  $\mathbf{0} \notin \text{Int}(L_1)$  and  $\mathbf{0} \notin \text{Int}(L_2)$ . If  $W_2 \rightarrow C_2$  or  $C_1 \rightarrow C_2$  by above arguments the result follows. Hence from Remarks 11.5 and 11.7 we can suppose that  $W_1 \rightarrow C_2$ . Now the proposition follows as before. □

### 12. The full periodicity kernel of the trefoil.

The goal of this section is to prove Theorem 1.6.

Let  $p_1$  and  $p_2$  be the two endpoints of  $\infty_2$ . Let  $A$  be a graph homeomorphic to  $\mathbf{I}$  with endpoints  $p_1$  and  $p_2$  such that  $\infty_2 \cup A$  is homeomorphic to  $\mathbf{T}$ . Thus in this section we shall consider  $\mathbf{T} = \infty_2 \cup A$ . Let  $f$  be an  $\infty_2$  map, we shall extend  $f$  to a  $\mathbf{T}$  map  $\bar{f}$  as follows. We define  $\bar{f}(z) = f(z)$  if  $z \in \infty_2$  and  $\bar{f}|_A$  is any homeomorphism between  $A$  and a closed subinterval in  $\infty_2$  having  $f(p_1)$  and  $f(p_2)$  as endpoints. Of course  $\text{Per}(f) = \text{Per}(\bar{f})$ . From Theorem 1.5  $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 21, 22, 23, 28, 29\}$  is a subset of the full periodicity kernel of  $\mathbf{T}$ . Then, to prove Theorem 1.6 it is sufficient to show the following proposition.

PROPOSITION 12.1. — *Let  $f$  be a  $\mathbf{T}$  map. Then the following statements hold:*

(a) *If  $7 \in \text{Per}(f)$ , then*

$$\mathbb{N} \setminus \{2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14,$$

$$15, 16, 17, 18, 21, 22, 23, 28, 29, 35\} \subset \text{Per}(f).$$

(b) *If  $11 \in \text{Per}(f)$ , then  $35 \in \text{Per}(f)$ .*

*In the rest of this section we fix the  $\mathbf{T}$  map  $f$  having a  $k$ -orbit  $P$  with  $k \in \{7, 11\}$  and the set of the basic intervals  $S$  associated to  $P'$ .*

This fixed  $\mathbf{T}$  map will be called the *standard  $\mathbf{T}$  map*.

LEMMA 12.2. — *Let  $f$  be the standard  $\mathbf{T}$  map. If the periodic orbit  $P$  does not have points in each connected component of  $\mathbf{T} \setminus \{\mathbf{0}\}$ , then Proposition 12.1 holds.*

*Proof.* — Let  $E'$  be the union of the closures of the connected components of  $\mathbf{T} \setminus \{0\}$  having points of  $P$ . Of course  $E' \subset \mathbf{T}$ . We define the map  $g : E' \rightarrow E'$  as follows. For  $z \in E'$ ,  $g(z) = f(z)$  if  $f(z) \in E'$ ; and  $g(z) = 0$  otherwise. Notice that  $g$  is either an  $\mathbf{O}$  or an  $\infty$  map. Moreover  $\text{Per}(g) \subset \text{Per}(f)$ . Hence from the Circle Theorem and Proposition 9.1 the result follows.  $\square$

*Remark 12.3.* — From Lemma 12.2 we can assume that the periodic orbit  $P$  has points into each connected component of  $\mathbf{T} \setminus \{0\}$ . Furthermore, by Corollary 4.3 in what follows we can suppose that the standard  $\mathbf{T}$  map  $f$  will be  $P'$ -linear.

LEMMA 12.4. — *Let  $f$  be the standard  $\mathbf{T}$  map. If there is a basic interval  $J$  such that no basic intervals of  $S \setminus \{J\}$   $f$ -cover  $J$ , then Proposition 12.1 holds.*

*Proof.* — Consider the map  $g = f|_{\mathbf{T} \setminus \text{Int}(J)}$ . Clearly  $g$  is well-defined because  $f$  is  $P'$ -linear. Moreover  $g$  is either an  $\infty_1$  or an  $\infty_2$  map such that  $\text{Per}(g) \subset \text{Per}(f)$ . Hence from Propositions 10.3 and 11.1 the lemma follows.  $\square$

*Remark 12.5.* — From Lemma 12.4 we can assume that each basic interval is  $f$ -covered by some different basic interval. On the other hand, Proposition 9.2 shows that if there exists some basic interval  $f$ -covering itself, then Proposition 12.1 holds. So, from now on we can assume that each basic interval does not  $f$ -cover itself.

LEMMA 12.6. — *Let  $f$  be the standard  $\mathbf{T}$  map. Identify  $\mathbf{O}$  with a circle of  $\mathbf{T}$ . If there are no crossing subsets of  $\mathbf{O}$ , then Proposition 12.1 holds.*

*Proof.* — With the notation of Section 8, by Proposition 8.4 (b) we have that  $k \in \text{Per}(F)$ . Since  $P$  has elements in each component of  $\mathbf{T} \setminus \{0\}$  and there are no crossing subsets of  $\mathbf{O}$ , we get that  $F(\mathbf{T}^*)$  is homeomorphic to  $\infty_1$  or  $\infty_2$ . So from Propositions 10.3 and 11.1 we obtain that if  $7 \in \text{Per}(F)$ , then  $\mathbb{N} \setminus \{2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 21, 22, 23, 28, 29, 35\} \subset \text{Per}(F)$ , and if  $11 \in \text{Per}(F)$ , then  $35 \in \text{Per}(F)$ . Now by Proposition 8.4 (a) the result follows.  $\square$

*From now on we shall denote by  $C_1, C_2$  and  $C_3$  the three circles of  $\mathbf{T}$ .*

*Remark 12.7.* — If there are no crossing subsets of  $C_1, C_2$  or  $C_3$ , from Lemma 12.6, Proposition 12.1 holds. So from now on we can assume that there are crossing subsets of  $C_1, C_2$  and  $C_3$ .

LEMMA 12.8. — *Let  $f$  be the standard  $\mathbf{T}$  map with  $k = 11$ . Let  $J_1, J_2, J_3$  and  $J$  be basic intervals such that  $J_i \subset C_i$  for  $i = 1, 2, 3$ ,  $J \subset C_2$  and  $J \neq J_2$ . Suppose that there are three loops  $J_1 \rightarrow J_2 \rightarrow J_3 \rightarrow J_1$ ,  $J_1 \rightarrow J \rightarrow \dots \rightarrow J_1$  and  $J_1 \rightarrow J_2 \rightarrow \dots \rightarrow \dots \rightarrow J \rightarrow \dots \rightarrow J_1$  of lengths  $3, m + 1$  and  $12 + m$  respectively for some  $1 \leq m \leq 13$ . Then  $35 \in \text{Per}(f)$ .*

*Proof.* — In this proof we will use the fact that a loop obtained by concatenating two different loops contains at least one non-branching interval (see Lemma 9.8).

For  $m = 1, 2, \dots, 13$ , we consider the loops  $J_1 \rightarrow J_2 \rightarrow J_3 \rightarrow J_1$  and either

- $J_1 \rightleftharpoons J$ ; or
- $J_1 \rightarrow J_2 \rightarrow \overset{11}{\dots} \rightarrow \dots \rightarrow J \rightarrow \overset{2}{\dots} \rightarrow \dots \rightarrow J_1$ ; or
- $J_1 \rightarrow J \rightarrow \overset{3}{\dots} \rightarrow \dots \rightarrow J_1$ ; or
- $J_1 \rightarrow J_2 \rightarrow \overset{11}{\dots} \rightarrow \dots \rightarrow J \rightarrow \overset{4}{\dots} \rightarrow \dots \rightarrow J_1$ ; or
- $J_1 \rightarrow J_2 \rightarrow \overset{11}{\dots} \rightarrow \dots \rightarrow J \rightarrow \overset{5}{\dots} \rightarrow \dots \rightarrow J_1$ ; or
- $J_1 \rightarrow J \rightarrow \overset{6}{\dots} \rightarrow \dots \rightarrow J_1$ ; or
- $J_1 \rightarrow J \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow J_1$ ; or
- $J_1 \rightarrow J_2 \rightarrow \overset{11}{\dots} \rightarrow \dots \rightarrow J \rightarrow \overset{8}{\dots} \rightarrow \dots \rightarrow J_1$ ; or
- $J_1 \rightarrow J \rightarrow \overset{9}{\dots} \rightarrow \dots \rightarrow J_1$ ; or
- $J_1 \rightarrow J \rightarrow \overset{10}{\dots} \rightarrow \dots \rightarrow J_1$ ; or
- $J_1 \rightarrow J_2 \rightarrow \overset{11}{\dots} \rightarrow \dots \rightarrow J \rightarrow \overset{11}{\dots} \rightarrow \dots \rightarrow J_1$ ; or
- $J_1 \rightarrow J \rightarrow \overset{12}{\dots} \rightarrow \dots \rightarrow J_1$ ; or
- $J_1 \rightarrow J \rightarrow \overset{13}{\dots} \rightarrow \dots \rightarrow J_1$

respectively. Since we can put 35 as  $7 \cdot 3 + 7 \cdot 2$ , or  $7 \cdot 3 + 14$ , or  $3 + 8 \cdot 4$ , or  $3 + 16 \cdot 2$ , or  $6 \cdot 3 + 17$ , or  $7 \cdot 3 + 2 \cdot 7$ , or  $3 + 8 \cdot 4$ , or  $5 \cdot 3 + 20$ , or  $5 \cdot 3 + 2 \cdot 10$ , or  $8 \cdot 3 + 11$ , or  $4 \cdot 3 + 23$ , or  $3 \cdot 3 + 13 \cdot 2$  or  $7 \cdot 3 + 14$ , the result follows.  $\square$

*Proof of Proposition 12.1.* — If  $C_i \rightleftharpoons C_j$  for some  $i, j \in \{1, 2, 3\}$ ,  $i \neq j$ , then from Lemma 9.7 the result follows. So from now on we can assume that we do not have  $C_i \rightleftharpoons C_j$  for  $i, j \in \{1, 2, 3\}$ ,  $i \neq j$ .

Therefore, without loss of generality, from Remarks 12.5 and 12.7 we can assume that  $C_1 \rightarrow C_2 \rightarrow C_3 \rightarrow C_1$ . We claim that there are three basic intervals  $J_i \subset C_i$ ,  $i = 1, 2, 3$  such that  $J_1 \rightarrow J_2 \rightarrow J_3 \rightarrow J_1$ . Now we prove the claim. If there exists a 3-orbit  $\{x, y, z\}$  with  $x \in C_1$ ,  $y \in C_2$  and  $z \in C_3$ , then we consider the basic intervals  $J_i \subset C_i$  for  $i = 1, 2, 3$  containing  $x, y$  and  $z$  respectively. From the linearity of  $f$  we get that  $J_1 \rightarrow J_2 \rightarrow J_3 \rightarrow J_1$ . Now we suppose that there are no 3-orbits  $\{x, y, z\}$  with  $x \in C_1, y \in C_2$

and  $z \in C_3$ . Since  $k$  is not multiple of 3 and  $C_1 \rightarrow C_2 \rightarrow C_3 \rightarrow C_1$ , without loss of generality we can assume that there is a closed subinterval  $K \subset C_1$  such that  $K$  is a crossing subset of  $C_2$ . Let  $K_3 \subset C_3$  be a minimal closed subinterval  $f$ -covering  $K$ . Let  $K_2 \subset C_2$  be a minimal closed subinterval  $f$ -covering  $K_3$ . Finally let  $K_1 \subset C_1$  be a minimal closed subinterval  $f$ -covering  $K_2$ . In particular  $K_1 \rightarrow K_2 \rightarrow K_3 \rightarrow K_1$ . Since there are no 3-orbits  $\{x, y, z\}$  with  $x \in C_1, y \in C_2$  and  $z \in C_3$ , from Lemma 3.3 it follows that  $\mathbf{0} \in K_1 \cap K_2 \cap K_3$  and the branching intervals  $J_i \subset K_i$  for  $i = 1, 2, 3$  verify  $J_1 \rightarrow J_2 \rightarrow J_3 \rightarrow J_1$ . Thus the claim is proved.

Denote by  $\gamma$  the loop  $J_1 \rightarrow J_2 \rightarrow J_3 \rightarrow J_1$ .

First suppose that  $J_i$   $f^k$ -covers itself for some  $i \in \{1, 2, 3\}$ . Thus from Lemma 9.5 there is a loop of length  $k$  containing  $J_i$ . This loop together with  $\gamma$  give us a non-repetitive loop of length  $n = ki + 3j$  for  $i \geq 1, j \geq 1$ . Since  $k$  is no divisible by 3, the loop of length  $n$  is non-repetitive. Moreover, from Remark 4.4 at least one of its intervals does not contain  $\mathbf{0}$ . Hence from Proposition 3.4 the result follows.

Now we can assume that  $J_i$  does not  $f^k$ -cover itself for  $i = 1, 2, 3$ . Thus, since  $P$  has period  $k$ , we get that  $J_i$   $f^k$ -covers  $K$  for each  $K \in S \setminus \{J_i\}, K \subset C_i, i = 1, 2, 3$ . Since  $k \neq 3$ , without loss of generality we can assume that  $J \leftarrow J_1 \rightarrow J_2 \rightarrow J_3 \rightarrow J_1$  where  $J \in S \setminus \{J_2\}$  and  $J \subset C_2$ . Consequently  $J_2$   $f^k$ -covers  $J$  and from Lemma 9.5 there exists a path of length  $k$  starting at  $J_2$  and ending at  $J$ . On the other hand, from Lemma 9.6 we can suppose that there is a path of length  $m, 1 \leq m \leq k+2$  starting at  $J$  and ending at  $J_1$ . Therefore we get the loops  $\gamma, J_1 \rightarrow J \rightarrow \overset{m}{\dots} \rightarrow \dots \rightarrow J_1$ , and  $J_1 \rightarrow J_2 \rightarrow \overset{k}{\dots} \rightarrow \dots \rightarrow J \rightarrow \overset{m}{\dots} \rightarrow \dots \rightarrow J_1$  of lengths 3,  $m + 1$  and  $k + m + 1$  respectively.

If  $k = 11$  the result follows from Lemma 12.8. So Proposition 12.1 (b) holds.

From now on we take  $k = 7$  and we will prove statement (a) of Proposition 12.1. Denote by  $\mathcal{N}$  the set  $\mathbb{N} \setminus \{2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 21, 22, 23, 28, 29, 35\}$ . In the rest of this section we will take into account the following facts. If  $L$  is a basic interval contained in a circle  $C$ , then  $L$   $f^k$ -covers  $L$  or  $L$   $f^k$ -covers  $M$  for each  $M \in S \setminus \{L\}, M \subset C$  because  $L$  has endpoints elements of  $P'$  and  $f$  has period  $k$ . Again from Lemma 9.8, if we concatenate two different loops, the new loop contains some non-branching interval.

Concatenating the three loops containing  $J_1$  of lengths 3,  $m + 1$  and

$k + m + 1 = 8 + m$  we get in a similar way to the proof of Lemma 12.8 that if  $m \in \{1, 3, 4, 6, 7\}$ , then  $\mathcal{N} \subset \text{Per}(f)$ ; if  $m = 2$ , then  $\mathcal{N} \setminus \{20\} \subset \text{Per}(f)$ ; if  $m = 5$ , then  $\mathcal{N} \setminus \{20, 26\} \subset \text{Per}(f)$ ; if  $m = 8$ , then  $\mathcal{N} \setminus \{20, 26, 32\} \subset \text{Per}(f)$ ; and if  $m = 9$ , then  $\mathcal{N} \setminus \{24\} \subset \text{Per}(f)$ .

First we suppose that  $J$   $f^7$ -covers  $J_2$ . Hence from Lemma 9.5 there is a path  $J \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow J_2$ , then we have the loops  $\gamma$  and  $J_2 \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow J \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow J_2$  of length 3 and 14 respectively. Concatenating these loops, from Lemma 9.8 and Proposition 3.4 we get that  $\{20, 26, 32\} \subset \text{Per}(f)$  and the result follows for  $m \in \{2, 5, 8\}$ . Now, if  $m = 9$ , we consider the loops  $J \rightarrow \overset{m}{\dots} \rightarrow J_1 \rightarrow J$  and  $J \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow J_2 \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow J$  of lengths  $m + 1 = 10$  and 14 respectively. In the same way as above we get that  $24 \in \text{Per}(f)$  and the proof follows.

Finally we assume that  $J$  does not  $f^7$ -cover  $J_2$ . Thus  $J$   $f^7$ -covers itself, and from Lemma 9.5 there is a loop  $J \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow J$ . Therefore we get the loops  $\gamma$  and  $J_2 \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow J \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow J \rightarrow \overset{m}{\dots} \rightarrow J_1 \rightarrow J_2$  of lengths 3 and  $15 + m$ . This last loop of length  $15 + m$  will be denoted by  $\beta$ . We note that since  $J_1$   $f$ -covers  $J_2$  and  $J$ , from Remark 4.4,  $\mathbf{0} \notin J \cap J_2$ . In particular  $\beta$  contains some non-branching interval. As before, if  $m = 2$ , then  $20 \in \text{Per}(f)$ . If  $m = 5$ , then concatenating the loops  $\gamma$  and  $\beta$  we get  $26 \in \text{Per}(f)$ . Now we show that  $20 \in \text{Per}(f)$ . Of course we can assume that  $m = 5$  is the minimal length of all paths from  $J$  to  $J_1$  (otherwise we are done). From the subpath  $J \rightarrow \overset{m}{\dots} \rightarrow J_1$  of  $\beta$ , and by the minimality of  $m$ , it follows that  $\beta$  cannot be a repetition of a loop of length smaller or equal than  $m = 5$ . So if  $\beta$  is repetitive, then it is twice a loop of length 10. Therefore the subpath  $J \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow J \rightarrow \overset{5}{\dots} \rightarrow \dots \rightarrow J_1$  of  $\beta$  must be  $J \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow J \rightarrow \overset{3}{\dots} \rightarrow \dots \rightarrow J \rightarrow \overset{2}{\dots} \rightarrow \dots \rightarrow J_1$ . Then there is a path of length 2 from  $J$  to  $J_1$  in contradiction with the minimality of  $m = 5$ . Hence we obtain that if  $m = 5$ , then  $20 \in \text{Per}(f)$ . If  $m = 9$ , then  $\beta$  has length 24. We can suppose that  $m = 9$  is the minimal length of all paths from  $J$  to  $J_1$ . From the path  $J \rightarrow \overset{m}{\dots} \rightarrow J_1$  and by the minimality of  $m$  we have that  $\beta$  is not a repetition of a loop of length smaller or equal than  $m = 9$ . Thus if  $\beta$  is repetitive, then it is twice a loop of length 12. From the path  $J \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow J \rightarrow \overset{9}{\dots} \rightarrow \dots \rightarrow J_1$ , we obtain  $J \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow J \rightarrow \overset{5}{\dots} \rightarrow \dots \rightarrow J \rightarrow \overset{4}{\dots} \rightarrow \dots \rightarrow J_1$ . This is a contradiction with the minimality of  $m$ . Consequently, if  $m = 9$ , then  $24 \in \text{Per}(f)$ . If  $m = 8$ , then concatenating  $\gamma$  and  $\beta$  we get  $\{26, 32\} \subset \text{Per}(f)$ .



So in the rest of this proof we shall assume that  $m = 8$  and we will show that  $20 \in \text{Per}(f)$ . We can suppose that  $m$  is the shortest length of all paths starting at  $J$  and ending at  $J_1$ ; otherwise we have proved that the result follows.

Let  $J \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow M_4 \rightarrow M_5 \rightarrow M_6 \rightarrow M_7 \rightarrow J_1$  be the path  $\varphi$  from  $J$  to  $J_1$  of length  $m = 8$ . We shall study the basic intervals  $M_i$  which form  $\varphi$ . Suppose that  $\varphi$  contains  $J_2$ . From the minimality of  $m$  and since  $J_2 \rightarrow J_3 \rightarrow J_1$ , we get  $J_2 \neq M_i$  for  $i \in \{1, \dots, 5\}$ . If  $J_2 = M_6$ , then we obtain the loop  $J \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow J \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow M_4 \rightarrow M_5 \rightarrow J_2 \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow J$  of length 20. By the minimality of  $m$  this loop is non-repetitive and the result follows. If  $J_2 = M_7$  then consider the loops  $\gamma$  and  $J_1 \rightleftharpoons J_2$ . We note that at least one of the intervals  $J_1, J_2, J_3$  does not contains  $\mathbf{0}$ , because  $J_1 \leftarrow J_2 \rightarrow J_3$  and a branching interval  $f$ -covers exactly one branching interval (see Remark 4.4). So the result holds. Hence we can suppose that  $\varphi$  does not contain  $J_2$ . We remark that  $\varphi$  contains 9 different basic intervals by the minimality of  $m$ . Since  $\text{Card}(S) = k + 3 = 10$ ,  $\varphi$  must contain  $J_3$ . Again from the minimality of  $m$  and since  $J_3 \rightarrow J_1$ , we get  $J_3 = M_7$ . For  $j \in \{1, 2, \dots, 6\}$  consider the basic interval  $M_j$ . Since  $M_j \in C_i$  for some  $i \in \{1, 2, 3\}$ , we get the path  $J_i \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow M_j$ . If  $M_j$   $f^7$ -covers  $J_i$ , then we obtain the loop  $J_i \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow M_j \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow J_i$ . This loop of length 14 together with  $\gamma$  give us that  $20 \in \text{Per}(f)$ . So from now on we can assume that  $M_j$  does not  $f^7$ -cover  $J_i$  and consequently we have  $M_j \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow M_j$  for  $j = 1, 2, \dots, 6$ .

Suppose that  $M_6 \subset C_2$ . Then we consider the loops  $\gamma, J_2 \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow M_6 \rightarrow \overset{3}{\dots} \rightarrow \dots \rightarrow J_2$  and  $M_6 \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow M_6$  and the result follows. Suppose that  $M_6 \subset C_3$ . Then from the loops  $\gamma$  and  $J_3 \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow M_6 \rightarrow J_3$  we get that  $20 \in \text{Per}(f)$ . So we can assume that  $M_6 \subset C_1$ .

Suppose that  $M_5 \subset C_1$ . Then we consider the loops  $\gamma, J_1 \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow M_5 \rightarrow \overset{3}{\dots} \rightarrow \dots \rightarrow J_1$  and  $M_5 \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow M_5$  and the result follows. If  $M_5 \subset C_2$ , then from the loops  $\gamma, J_2 \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow M_5 \rightarrow \overset{4}{\dots} \rightarrow \dots \rightarrow J_2$  the result holds. So we can assume that  $M_5 \subset C_3$ .

If  $M_4 \subset C_1$ , then we consider the loops  $\gamma$  and  $J_1 \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow M_4 \rightarrow \overset{4}{\dots} \rightarrow \dots \rightarrow J_1$  and the result follows. If  $M_4 \subset C_3$ , then by the loops  $\gamma, J_3 \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow M_4 \rightarrow \overset{3}{\dots} \rightarrow \dots \rightarrow J_3$  and  $M_4 \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow M_4$  we obtain  $20 \in \text{Per}(f)$ . Hence we can suppose that  $M_4 \subset C_2$ .

Suppose that  $M_3 \subset C_2$ . Then we consider the loops  $J_2 \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow M_3 \rightarrow \overset{6}{\dots} \rightarrow \dots \rightarrow J_2$  and  $M_3 \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow M_3$  and we are done. If  $M_3 \subset C_3$ , then from the loops  $\gamma$  and  $J_3 \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow M_3 \rightarrow \overset{4}{\dots} \rightarrow \dots \rightarrow J_3$  the result follows. So we can assume that  $M_3 \subset C_1$ .

Suppose that  $M_2 \subset C_1$ . Then we consider the loops  $J_1 \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow M_2 \rightarrow \overset{6}{\dots} \rightarrow \dots \rightarrow J_1$  and  $M_2 \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow M_2$  and the result holds. If  $M_2 \subset C_2$ , then from the loops  $\gamma$  and  $J_2 \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow M_2 \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow J_2$  the result follows. Therefore we can assume that  $M_2 \subset C_3$ .

If  $M_1 \subset C_1$ , then from the loops  $\gamma$  and  $J_1 \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow M_1 \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow J_1$  we are done. If  $M_1 \subset C_3$ , then we consider the loops  $J_3 \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow M_1 \rightarrow \overset{6}{\dots} \rightarrow \dots \rightarrow J_3$  and  $M_1 \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow M_1$  and the result holds. So we can suppose that  $M_1 \subset C_2$ .

Therefore the ten basic intervals of  $S$  satisfy that  $\{J_1, M_3, M_6\} \subset C_1$ ,  $\{J, J_2, M_1, M_4\} \subset C_2$  and  $\{J_3, M_2, M_5\} \subset C_3$ .

If  $J$   $f$ -covers  $J_1, J_2$  or  $J_3$  then we have the loops  $\gamma$  and  $J_1 \rightleftharpoons J$ ;  $\gamma$  and  $J_1 \rightarrow J \rightarrow J_2 \rightarrow J_3 \rightarrow J_1$ ; or  $J \rightarrow \overset{7}{\dots} \rightarrow \dots \rightarrow J$  and  $J \rightarrow J_3 \rightarrow J_1 \rightarrow J$  according with  $J \rightarrow J_1, J \rightarrow J_2$  or  $J \rightarrow J_3$  respectively. Thus we obtain that  $20 \in \text{Per}(f)$  and we are done. Clearly  $J$  does not  $f$ -cover  $M_j$  for  $j \in \{2, \dots, 6\}$  by the minimality of  $m$ . Hence we can assume that the only basic interval  $f$ -covered by  $J$  is  $M_1$ .

If  $M_1$   $f$ -covers some interval of  $\{J, J_1, J_2, J_3\}$ , then in a similar way as before we get that  $20 \in \text{Per}(f)$ . Again from the minimality of  $m$ ,  $M_1$  does not  $f$ -cover  $M_j$  for  $j > 2$ . So we can assume that the only basic interval  $f$ -covered by  $M_1$  is  $M_2$ .

If  $M_2$   $f$ -covers some interval of  $\{J, J_1, J_2, J_3, M_1\}$ , we obtain easily that  $20 \in \text{Per}(f)$ . Clearly  $M_2 \nrightarrow M_j$  for  $j > 3$ . So we can assume that the only basic interval  $f$ -covered by  $M_2$  is  $M_3$ .

If  $M_3$   $f$ -covers some interval of  $\{J, J_1, J_2, J_3, M_1, M_2\}$ , we get that  $20 \in \text{Per}(f)$  and we are done. By the minimality of  $m$   $M_3 \nrightarrow M_j$  for  $j > 4$ . Thus we can suppose that the only basic interval  $f$ -covered by  $M_3$  is  $M_4$ .

Since  $J, M_1, M_2$  and  $M_3$   $f$ -cover a unique basic interval, namely  $M_1, M_2, M_3$  and  $M_4$  respectively, it follows that if  $J$  is a branching (respectively non-branching) interval, then  $M_1, M_2, M_3$  and  $M_4$  are branching (respectively non-branching) intervals; here we are used Remark 4.4. This is a contradiction with the fact that  $J, M_1$  and  $M_4$  are contained in the

circle  $C_2$  and  $C_2$  has exactly four basic intervals. Thus statement (a) of Proposition 12.1 holds and we are done.  $\square$

### 13. Upper bounds of the full periodicity kernel.

Blokh proved in [Bk1], [Bk2] the existence of a natural number  $L(G)$  such that if a continuous self-map on a graph  $G$  verifies that  $\{1, 2, \dots, L(G)\} \subset \text{Per}(f)$  then  $\text{Per}(f) = \mathbb{N}$ . This result shows that the full periodicity kernel of  $G$  is a finite set. Of course,  $\{1, 2, \dots, L(G)\}$  contains the full periodicity kernel of  $G$ .

We consider the trefoil and its proper subspaces and we shall compare the Blokh bound  $L(E)$  with the best upper bound of the full periodicity kernel, which follows from Theorems 1.1, 1.2, 1.3, 1.4, 1.5 and 1.6. See Table 13.1. We note that  $L(E)$  is fairly good for the  $n$ -star but it is too much large for the other spaces.

### 14. About the topological entropy.

The *topological entropy* of a graph map  $f$  is a non-negative real number  $h(f)$  associated to  $f$  which increases with the complexity of  $f$ . For a definition and main properties see [ALM2].

Llibre and Misiurewicz [LM] obtain the next result for continuous self-maps of graphs.

**THEOREM 14.1.** — *Let  $f$  be a continuous self-map of a graph. Then the following statements are equivalent:*

- (a)  $h(f) > 0$ .
- (b) *There is  $m \in \mathbb{N}$  such that  $\{m \cdot n : n \in \mathbb{N}\} \subset \text{Per}(f)$ .*

We have the following result for our spaces as a corollary of the above theorem.

**COROLLARY 14.2.** — *Let  $K_E$  be the full periodicity kernel of  $E$ . Let  $f$  be an  $E$  map. If  $K_E \subset \text{Per}(f)$  then  $h(f) > 0$ .*

### BIBLIOGRAPHY

- [ALM1] L. ALSÈDÀ, J. LLIBRE and M. MISIUREWICZ, Periodic orbits of maps of  $Y$ , Trans. Amer. Math. Soc., 313 (1989), 475–538.

$E$	best upper bound	$L(E)$
$I_2$	3	8
$I_3$	7	24
$I_4$	11	32
$I_5$	23	60
$I_6$	29	72
$O$	3	153 548 648
$O_1$	7	2 643 549 795
$O_2$	11	43 419 841 302
$O_3$	23	571 949 175 609
$O_4$	29	2 650 538 105 490
$\infty$	11	43 419 841 302
$\infty_1$	23	571 949 175 609
$\infty_2$	29	2 650 538 105 490
$T$	29	2 650 538 105 490

Table 13.1. *Upper bounds for the full periodicity kernel of  $E$ .*

- [ALM2] L. ALSÈDÀ, J. LLIBRE and M. MISIUREWICZ, Combinatorial dynamics in dimension one, Advanced Series in Nonlinear Dynamics, Vol. 5, World Scientific, 1993.
- [ALMT] L. ALSÈDÀ, J. LLIBRE, M. MISIUREWICZ and C. TRESSER, Periods and entropy for Lorenz-like maps, Ann. Inst. Fourier, 39-4 (1989), 929-952.
- [AM] L. ALSÈDÀ and J.M. MORENO, Linear orderings and the full periodicity kernel for the  $n$ -star, J. Math. Anal. Appl., 180 (1993), 599-616.
- [Ba] S. BALDWIN, An extension of Sharkovskii's Theorem to the  $n$ -od, Ergod. Th. & Dynam. Sys., 11 (1991), 249-271.
- [BL] S. BALDWIN and J. LLIBRE, Periods of maps on trees with all branching points fixed, Ergodic Th. & Dynam. Sys., 15 (1995), 239-246.
- [Bc1] L. BLOCK, Periodic orbits of continuous maps of the circle, Trans. Amer. Math. Soc., 260 (1980), 553-562.
- [Bc2] L. BLOCK, Periods of periodic points of maps of the circle which have a fixed point, Proc. Amer. Math. Soc., 82 (1981), 481-486.
- [BGM] L. BLOCK, J. GUCKENHEIMER, M. MISIUREWICZ and L.S. YOUNG, Periodic points and topological entropy of one dimensional maps, Lecture Notes in Math., Springer-Verlag, Heidelberg, 819 (1980), 18-34.
- [Bk1] A.M. BLOKH, Periods implying almost all periods for tree maps, Nonlinearity, 5 (1992), 1375-1382.

- [Bk2] A.M. BLOKH, On some properties of graph maps: spectral decomposition, Misiurewicz conjecture and abstract sets of periods, preprint, Max-Plank-Institut für Mathematik, Bonn.
- [LL1] C. LESEDUARTE and J. LLIBRE, On the set of periods for  $\sigma$  maps, to appear in Trans. Amer. Math. Soc.
- [LL2] C. LESEDUARTE and J. LLIBRE, On the full periodicity kernel for one-dimensional maps, preprint, 1994,.
- [LM] J. LLIBRE and M. MISIUREWICZ, Horseshoes, entropy and periods for graph maps, Topology, 32 (1993), 649–664.
- [LPR1] J. LLIBRE, J. PARAÑOS and J. A. RODRÍGUEZ, The full periodicity kernel for  $\sigma$  maps, J. Math. Anal. and Appl., 182 (1994), 639–651.
- [LPR2] J. LLIBRE, J. PARAÑOS and J.A. RODRÍGUEZ, Sets of periods for maps on connected graphs with zero Euler characteristic having all branching points fixed, to appear in J. Math. Anal. and Appl.
- [LPR3] J. LLIBRE, J. PARAÑOS and J.A. RODRÍGUEZ, International Journal of Bifurcation and Chaos, 5 (1995), 1395–1405.
- [LR] J. LLIBRE and R. REVENTÓS, Sur le nombre d'orbites périodiques d'une application continue du cercle en lui-même, C. R. Acad. Sci. Paris, Sér. I Math., 294 (1982), 52–54.
- [LY] T. LI and J. YORKE, Period three implies chaos, Amer. Math. Monthly, 82 (1975), 985–992.
- [M] P. MUMBRÚ, Períodes 1, 2, 3, 4, 5, 7 equivalen a caos, Master Thesis, Universitat Autònoma de Barcelona, 1982.
- [Sh] A.N. SHARKOVSKII, Co-existence of the cycles of a continuous mapping of the line into itself (Russian), Ukrain. Math. Zh., 16 (1964), 61–71.
- [St] P.D. STRAFFIN, Periodic points of continuous functions, Math. Mag., 51 (1978), 99–105.

Manuscrit reçu le 4 mai 1995,  
 accepté le 25 septembre 1995.

C. LESEDUARTE,  
 Departament de Matemàtica Aplicada II  
 Universitat Politècnica de Catalunya  
 08222 Terrassa, Barcelona (Spain).  
 leseduarte@ma2.upc.es.

J. LLIBRE,  
 Departament de Matemàtiques  
 Facultat de Ciències  
 Universitat Autònoma de Barcelona  
 08193 Bellaterra, Barcelona (Spain).  
 jllibre@mat.uab.es.